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Nonlinear Stability of ρ-Functional Equations in Latticetic Random Banach Lattice Spaces

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Abstract: In this paper, we prove the generalized nonlinear stability of the first and second of the following ρ-functional equations, $G(|a|\Delta^*_{\mathcal{A}}|b|)\Delta^*_{\mathcal{B}}G(|a|\Delta^{**}_{\mathcal{A}}|b|) - G(|a|)\Delta^{**}_{\mathcal{B}}G(|b|) = \rho(2\left[G\left(\frac{|a|\Delta^*_{\mathcal{A}}|b|}{2}\right)\Delta^*_{\mathcal{B}}G\left(\frac{|a|\Delta^{**}_{\mathcal{A}}|b|}{2}\right)\right] - G(|a|)\Delta^{**}_{\mathcal{B}}G(|b|))$, and $2\left[G\left(\frac{|a|\Delta^*_{\mathcal{A}}|b|}{2}\right)\Delta^*_{\mathcal{B}}G\left(\frac{|a|\Delta^{**}_{\mathcal{A}}|b|}{2}\right)\right] - G(|a|)\Delta^{**}_{\mathcal{B}}G(|b|) = \rho\big(G(|a|\Delta^*_{\mathcal{A}}|b|)\Delta^*_{\mathcal{B}}G(|a|\Delta^{**}_{\mathcal{A}}|b|) - G(|a|)\Delta^{**}_{\mathcal{B}}G(|b|)\big)$ in latticetic random Banach lattice spaces, where ρ is a fixed real or complex number with $\rho \neq 1$.

Keywords: stability; fixed point; lattictic random Banach space; ρ-functional equations

1. Introduction

In 1986 [1], Alsina investigated the stability of functional equations in random normed spaces. This was a milestone that revealed the role of random theory as a powerful tool for studying stability of functional equations, and many mathematicians attempted to develop and generalize the problem of stability in random normed spaces with a practical approach. In 2011 [2], Saadati et al. proved the nonlinear stability of a cubic functional equation in non-Archimedean random normed space. They also proved nonlinear stability of a $\mathcal{L}$-random additive-cubic-quartic (ACQ) functional equation [3].

For the first time, in 2012 [4], Agbeko investigated the stability of a maximum preserving functional equation in latticetic environments. He presented a generalization of the Hyers–Ulam–Aoki stability problem in Banach lattice spaces, by replacing the supremum operation with additive operation in Cauchy's equation, which is called the maximum preserving functional equation. In addition to supremum and infimum operations, in 2015, Agbeko [5] developed nonlinear stability for different combinations of these two operations, and proved it using the core of the direct method presented by Forti [6]. On the other hand, in 2017, Park and Jang [7] introduced ρ-functional equations, and proved the stability of the equations in various spaces.

In the present work, we introduce a latticetic operation-preserving ρ-functional equations and prove the stability of the first and second latticetic operation-preserving ρ-functional equations by the direct method and fixed-point method in latticetic random Banach lattice spaces, which is a generalization of research by Agbeko, Park and Jang.

2. Preliminaries

At first, we describe some known concepts and results, which will be useful in the next section of this study.

Theorem 1 (see [8,9]). *Assume that $(\mathcal{A}, d_c)$ is a complete generalized metric space. Assume that $\beta : \mathcal{A} \to \mathcal{A}$ is a strictly contractive mapping with Lipschitz constant $L < 1$. If there exist non-negative integers n_0 such that*

(C1) $d_c\left(\beta^n a, \beta^{n+1} a\right) < \infty, \forall a \in \mathcal{A}$; *then we have*

(C2) *the sequence $\{\beta^n a\}$ convergence to a fixed point b^* of β;*

(C3) *b^* is the unique fixed point of β in the set $\mathcal{B} = \{b \in \mathcal{A} \mid d_c(\beta^n a, b) < \infty\}$;*

(C4) $d_c(b, b^*) \leq (1/(1-L)) d_c(b, \beta b), \forall b \in \mathcal{B}$.

Definition 1. *An ordered set $\mathcal{M} = (M, \geq)$ is called a complete lattice if*

(CL1) $\forall (\emptyset \neq) A \subset M$, *$A$ admits supremum and infimum,*

(CL2) $0_{\mathcal{M}} = \inf M$, $1_{\mathcal{M}} = \sup M$.

Suppose that Δ_M^+ be the space of lattice random distribution function, i.e.,

$$\Delta_M^+ = \{G \mid G : R \cup \{-\infty, +\infty\} \to M, G(0) = 0_{\mathcal{M}}, G(+\infty) = 1_{\mathcal{M}},$$
$$G \text{ is non-decreasing and left-continuous on } R\}.$$

It is clear that the space $(\Delta_M^+, \geq_M)$ is an ordered set (i.e., $F \geq G$ if and only if (iff) $F(t) \geq_M G(t)$ for all $t \in R$).

Also, the distribution function $\delta_0(t)$ given by

$$\delta_0(t) = \begin{cases} 0_{\mathcal{M}}, & \text{if } t \leq 0, \\ 1_{\mathcal{M}}, & \text{if } t > 0. \end{cases}$$

is the maximal element for D_M^+.

Moreover, if $l^- G(a)$ denotes the left limit of the function G at the point a and $D_M^+ = \{M \in \Delta_M^+ : l^- G(+\infty) = 1_{\mathcal{M}}\}$, then obviously $D_M^+ \subset \Delta_M^+$.

Definition 2 (see [10]). *Assume that $\mathcal{T} : M \times M \to M$. $\mathcal{T}$ is a triangular norm brifly (t-norm), iff for all $a, b, c \in M$:*

(TN1) $\mathcal{T}(a, 1_{\mathcal{M}}) = a$ *(boundary condition);*

(TN2) $\mathcal{T}(a, b) = \mathcal{T}(b, a)$ *(commutativity);*

(TN3) $\mathcal{T}(a, \mathcal{T}(b, c)) = \mathcal{T}(\mathcal{T}(a, b), c)$ *(associativity);*

(TN4) $\mathcal{T}(a, b) \leq_M \mathcal{T}(a', b')$ *if $a \leq_M a'$ and $b \leq_M b'$ (monotonicity).*

For example, $\mathcal{T}_M(a, b) = \min\{a, b\}$, for all $a, b \in [0, 1]$ is a t-norm on $[0, 1]^2$.

If

$$\lim_{n \to \infty} \mathcal{T}(a_n, b) = \mathcal{T}(a, b), \tag{1}$$

for all $b \in M$, then $\mathcal{T}$ is called a continous t-norm, where $a_n \to a \in M$.

Definition 3 (see [9,11]). *If there exist a continuous t-norm $\diamond$ and a continuous t-conorm $\square$ on $[0, 1]$, we define, for all $a = (a_1, a_2), b = (b_1, b_2) \in M$,*

$$\mathcal{T}(a, b) = (a_1 \diamond ^{\circ \circ \circ \circ \circ \circ \circ \circ \circ} b_1, a_2 \square b_2). \tag{2}$$

Then, $\mathcal{T}$ is called t-representable on $M = [0, 1]^2$.

For example,

$$\begin{aligned} \mathcal{T}(a, b) &= (a_1 b_1, \min\{a_2 + b_2, 1\}), \\ \mathcal{T}'(a, b) &= (\min\{a_1, b_1\}, \max\{a_2, b_2\}) \end{aligned} \tag{3}$$

for all $a = (a_1, a_2), b = (b_1, b_2) \in [0,1]^2$ are continuous t-representables.

Define the mapping $\mathcal{T}_\wedge$ from M^2 to M by

$$\mathcal{T}_\wedge(a,b) = \begin{cases} a & \text{if } b \geq_M a, \\ b & \text{if } a \geq_M b. \end{cases} \tag{4}$$

If $\{a_n\}$ is a given sequence in M, then $(\mathcal{T}_\wedge)_{i=1}^n a_i$ is defined recurrently by $(\mathcal{T}_\wedge)_{i=1}^1 a_i = a_1$ and $(\mathcal{T}_\wedge)_{i=1}^n a_i = \mathcal{T}_\wedge\left((\mathcal{T}_\wedge)_{i=1}^{n-1} a_i, a_n\right)$ for $n \geq 2$ (see [9]).

Definition 4. *Assume that $\mathcal{N}_\mathcal{F} : M \to M$. $\mathcal{N}_\mathcal{F}$ is called a negation function, iff*

(NF1) $\mathcal{N}_\mathcal{F}(0_\mathcal{M}) = 1, \mathcal{N}_\mathcal{F}(1_\mathcal{M}) = 0$,
(NF2) $\mathcal{N}_\mathcal{F}(a) \leq \mathcal{N}_\mathcal{F}(b)$, if $a \geq b$ (monotonically).

A negation function is involutive, iff

(NF3) $\mathcal{N}_\mathcal{F}(\mathcal{N}_\mathcal{F}(a)) = a$, $\forall a \in [0,1]$.

Definition 5. *A triple $(\mathcal{A}, \mu, \mathcal{T}_\wedge)$ is called a latticetic random normed space (briefly, LRN-space) if $\mathcal{A}$ is a vector space and $\mu : \mathcal{A} \to D_M^+$ such that the following conditions hold:*

(L1) $\mu_a(t) = \delta_0(t)$ for all $t > 0$ iff $a = 0$;
(L2) $\mu_{\eta a}(t) = \mu_a(1/|\eta|)$ for all a in $\mathcal{A}$, $\eta \neq 0$ and $t \geq 0$;
(L3) $\mu_{a+b}(t+s) \geq_M \mathcal{T}_\wedge(\mu_a(t), \mu_b(s))$ for all $a, b \in \mathcal{A}$ and $t, s \geq 0$.

We note that from (L2) it follows that $\mu_{-a}(t) = \mu_a(t)$ for all $a \in \mathcal{A}$ and $t > 0$.

Example 1. *Assume that $M = [0,1]^2$ and operation $\leq_M$ are defined by*

$$\begin{gathered} M = \{(a_1, a_2) : (a_1, a_2) \in [0,1]^2, a_1 + a_2 \leq 1\}, \\ (a_1, a_2) \leq_M (b_1, b_2) \Longleftrightarrow a_1 \leq b_1, \quad a_2 \geq b_2, \quad \forall a = (a_1, a_2), b = (b_1, b_2) \in M. \end{gathered} \tag{5}$$

Then, $(M, \leq_M)$ is a complete lattice (see [9]). In this complete lattice, we denote its units by $0_M = (0,1)$ and $1_M = (1,0)$. Let $(\mathcal{A}, \|\cdot\|)$ be a normed space, $\mathcal{T}(a,b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$ for all $a = (a_1, a_2), b = (b_1, b_2) \in [0,1]^2$ and μ be a mapping defined by

$$\mu_a(t) = \left(\frac{t}{t + \|a\|}, \frac{\|a\|}{t + \|a\|}\right), \quad \forall t \in \mathbb{R}^+. \tag{6}$$

Then, $(\mathcal{A}, \mu, \mathcal{T})$ is an LRN-space.

If $(\mathcal{A}, \mu, \mathcal{T}_\wedge)$ be an LRN-space, then

$$\Omega(U) = \{U(\varepsilon, r) : \varepsilon >_M 0_M, r \in M \setminus \{0_M, 1_M\}\}, \quad U(\varepsilon, r) = \{a \in \mathcal{A} : G_a(\varepsilon) >_M \mathcal{N}_\mathcal{F}(r)\} \tag{7}$$

are neighborhoods of null vector for linear topology on $\mathcal{A}$ generalized by the norm G.

Definition 6. *Assume that $(\mathcal{A}, \mu, \mathcal{T}_\wedge)$ is an LRN-space.*

(1) We say $a_n \to a \in M$ if, for every $t > 0$ and $r \in M \setminus \{0_M\}$, there exists a positive integer N such that $\mu_{a_n - a}(t) >_M \mathcal{N}_\mathcal{F}(r)$ whenever $n \geq N$.
(2) We say $\{a_n\} \in M$ is a Cauchy sequence if, for every $t > 0$ and $r \in M \setminus \{0_M\}$, there exists a positive integer N such that $\mu_{a_n - a_m}(t) >_M \mathcal{N}_\mathcal{F}(r)$ whenever $n \geq m > N$.
(3) A LRN-space $(\mathcal{A}, \mu, \mathcal{T}_\wedge)$ is said to be complete if every Cauchy sequence in $\mathcal{A}$ is convergent to a point in $\mathcal{A}$.

Theorem 2. *If $(\mathcal{A}, \mu, \mathcal{T}_\wedge)$ is an LRN-space and $a_n \to a$, then $\lim_{n\to\infty} \mu_{a_n}(t) = \mu_a(t)$.*

Proof. The proof is the same as classical RN-spaces, see [11]. □

Lemma 1. *Let $(\mathcal{A}, \mu, \mathcal{T}_\wedge)$ be an LRN-space and $a \in \mathcal{A}$. If*

$$\mu_a(t) = C, \qquad \forall t > 0, \tag{8}$$

then $C = 1_{\mathcal{M}}$ and $a = 0$.

Proof. Assume that $\mu_a(t) = C$ for all $t > 0$. Since $\mathrm{Ran}(\mu) \subseteq D_M^+$, we have $C = 1_{\mathcal{M}}$ and by (L1) we conclude that $a = 0$. □

Definition 7. *Suppose that triple $(\mathcal{A}, \mu, \mathcal{T})$ is an LRN-space. Then, $(\mathcal{A}, \mu, \mathcal{T})$ is called latticetic random Banach space (briefly, LRB-space) if $\mathcal{A}$ is complete with respect to the random metric included by random norm.*

Definition 8. *Suppose that $(\mathcal{A}, \wedge_{\mathcal{A}}, \vee_{\mathcal{A}})$ is a vector lattice and $(\mathcal{B}, \wedge_{\mathcal{B}}, \vee_{\mathcal{B}})$ is a Banach lattice with $\mathcal{A}^+$ and $\mathcal{B}^+$ their respective positive cones. A map $G : \mathcal{A} \to \mathcal{B}$ is cone-related if*

$$G(\mathcal{A}^+) = \{G(|a|) : \ a \in \mathcal{A}\} \subset \mathcal{B}^+. \tag{9}$$

(For more about this notion see [4,5].)

3. Stability of the first ρ-Functional Equation: Direct Method

In this section, using a direct method, we prove nonlinear stability of the first ρ-functional equation in latticetic random Banach lattice space (briefly, LRBL-space).

Definition 9. *Assume that $(\mathcal{A}, \wedge_{\mathcal{A}}, \vee_{\mathcal{A}})$ is a vector lattice space, $(\mathcal{B}, \mu, \mathcal{T}_\wedge)$ is an LRN-space and $G : \mathcal{A} \to \mathcal{B}$ is a cone-related mapping. Then, the following operator equation is called a Cauchy latticetic operation-preserving functional equation if:*

$$G(|a|\Delta^*_{\mathcal{A}}|b|)\Delta^*_{\mathcal{B}}G(|a|\Delta^{**}_{\mathcal{A}}|b|) = G(|a|)\Delta^{**}_{\mathcal{B}}G(|b|) \tag{10}$$

for all $a, b \in \mathcal{A}$, where $\Delta^*_{\mathcal{A}}, \Delta^{**}_{\mathcal{A}} \in \{\wedge_{\mathcal{A}}, \vee_{\mathcal{A}}\}$ and $\Delta^*_{\mathcal{B}}, \Delta^{**}_{\mathcal{B}} \in \{\wedge_{\mathcal{B}}, \vee_{\mathcal{B}}\}$ are fixed lattice operations.

Note that if the above four lattice operators are all the supremum (join) or the infimum (meet), then the functional Equation (10) is just the definition of a join-homomorphism or a meet-homomorphism. Moreover, if $\Delta^*_{\mathcal{A}}$ and $\Delta^{**}_{\mathcal{A}}$ are the same, then the left-hand side of Equation (10) is the map of the meets or the joints.

Lemma 2. *Assume that $(\mathcal{A}, \wedge_{\mathcal{A}}, \vee_{\mathcal{A}})$ and $(\mathcal{B}, \wedge_{\mathcal{B}}, \vee_{\mathcal{B}})$ are vector lattice spaces. If a mapping $G : \mathcal{A} \to \mathcal{B}$ satisfies*

$$G(|a|\Delta^*_{\mathcal{A}}|b|)\Delta^*_{\mathcal{B}}G(|a|\Delta^{**}_{\mathcal{A}}|b|) - G(|a|\Delta^{**}_{\mathcal{B}}|b|) = \rho\left(2\left[G\left(\frac{|a|\Delta^*_{\mathcal{A}}|b|}{2}\right)\Delta^*_{\mathcal{B}}G\left(\frac{|a|\Delta^{**}_{\mathcal{A}}|b|}{2}\right)\right] - G(|a|)\Delta^{**}_{\mathcal{B}}G(|b|)\right) \tag{11}$$

for all $a, b \in \mathcal{A}$, then functional Equation (11) is a Cauchy latticetic operation-preserving functional equation.

Proof. Assume that $G : \mathcal{A} \to \mathcal{B}$ satisfies Equation (11). Letting $a = b = 0$ in Equation (11), then we have $G(0) = 0$.

Letting $b = a$ in Equation (11), we get $2G(\frac{|a|}{2}) - G(|a|) = 0$ and so

$$G(\frac{|a|}{2}) = \frac{1}{2}G(|a|) \tag{12}$$

for all $a \in \mathcal{A}$.

It follows from Equations (11) and (12) that

$$\begin{aligned} G(|a|\Delta^*_{\mathcal{A}}|b|)\Delta^*_{\mathcal{B}}G(|a|\Delta^{**}_{\mathcal{A}}|b|) - G(|a|\Delta^{**}_{\mathcal{B}}|b|) &= \rho\left(2\left[G\left(\frac{|a|\Delta^*_{\mathcal{A}}|b|}{2}\right)\Delta^*_{\mathcal{B}}G\left(\frac{|a|\Delta^{**}_{\mathcal{A}}|b|}{2}\right)\right] - G(|a|\Delta^{**}_{\mathcal{B}}|b|)\right) \\ &= \rho\left([G(|a|\Delta^*_{\mathcal{A}}|b|)\Delta^*_{\mathcal{B}}G(|a|\Delta^{**}_{\mathcal{A}}|b|)] - G(|a|\Delta^{**}_{\mathcal{B}}|b|)\right), \end{aligned}$$

and so

$$G(|a|\Delta^*_{\mathcal{A}}|b|)\Delta^*_{\mathcal{B}}G(|a|\Delta^{**}_{\mathcal{A}}|b|) = G(|a|\Delta^{**}_{\mathcal{B}}|b|)$$

for all $a, b \in \mathcal{A}$. $\square$

Now, we prove nonlinear stability of the first ρ-functional Equation in latticetic random Banach lattice spaces.

For a given mapping $G : \mathcal{A} \to \mathcal{B}$, we define

$$\begin{aligned} D_{\rho_{0.1}}G(a,b) &= G(|a|\Delta^*_{\mathcal{A}}|b|)\Delta^*_{\mathcal{B}}G(|a|\Delta^{**}_{\mathcal{A}}|b|) - G(|a|\Delta^{**}_{\mathcal{B}}|b|) \\ &\quad - \rho\left(2\left[G\left(\frac{|a|\Delta^*_{\mathcal{A}}|b|}{2}\right)\Delta^*_{\mathcal{B}}G\left(\frac{|a|\Delta^{**}_{\mathcal{A}}|b|}{2}\right)\right] - G(|a|)\Delta^{**}_{\mathcal{B}}G(|b|)\right), \end{aligned} \tag{13}$$

$$\begin{aligned} D_{\rho_{0.2}}G(a,b) &= 2\left[G\left(\frac{|a|\Delta^*_{\mathcal{A}}|b|}{2}\right)\Delta^*_{\mathcal{B}}G\left(\frac{|a|\Delta^{**}_{\mathcal{A}}|b|}{2}\right)\right] - G(|a|\Delta^{**}_{\mathcal{B}}|b|) \\ &\quad - \rho\left(G(|a|\Delta^*_{\mathcal{A}}|b|)\Delta^*_{\mathcal{B}}G(|a|\Delta^{**}_{\mathcal{A}}|b|) - G(|a|)\Delta^{**}_{\mathcal{B}}G(|b|)\right) \end{aligned} \tag{14}$$

for all $a, b \in \mathcal{A}$.

Theorem 3. *Assume that $(\mathcal{A}, \wedge_{\mathcal{A}}, \vee_{\mathcal{A}})$ is a vector lattice space and $(\mathcal{B}, \mu', \mathcal{T}_\wedge)$ is an LRBL-space in which $\mathcal{T}_\wedge = \mathcal{T}_M$ and $\varphi : \mathcal{A}^2 \to D^+_M$ ($\phi(a,b)$ is denoted by $\phi_{a,b}$), such that there exists $0 < \alpha < \frac{1}{2}$ with*

$$\mu'_{\varphi(2a,2b)}(t) \geq \mu'_{2^\alpha\varphi(a,b)}(t) \tag{15}$$

and

$$\lim_{n\to\infty} \mu'_{\varphi(2^n a, 2^n b)}(2^n t) = 1 \tag{16}$$

for all $a, b \in \mathcal{A}$ and $t > 0$. If $G : \mathcal{A} \to \mathcal{B}$ is a cone-related functional such that

$$\mu_{D_{\rho_{0.1}}G(a,b)}(t) \geq \mu'_{\varphi(a,b)}(t) \tag{17}$$

for all $a, b \in \mathcal{A}$ and $t > 0$, then there is a unique cone-related mapping $H : \mathcal{A} \to \mathcal{B}$ which satisfies the Cauchy latticetic operation-preserving functional equation such that

$$\mu_{G(|a|)-H(|a|)}(t) \geq \mu'_{\varphi(a,b)}\left(\frac{2-2^\alpha}{2^\alpha}t\right) \tag{18}$$

for all $a \in \mathcal{A}$ and $t > 0$.

Proof. Putting $b = a$ in Equation (17), we see that

$$\mu_{2G(\frac{|a|}{2})-G(a)}(t) \geq \mu'_{\varphi(a,a)}(t). \tag{19}$$

Replacing a by $2a$ in Equation (19), we obtain that

$$\begin{aligned}\mu_{\frac{1}{2}G(2|a|)-G(|a|)}(t) &= \mu_{\frac{1}{2}[G(2|a|)-2G(|a|)]}(t)\\ &= \mu_{G(2|a|)-2G(|a|)}(2t)\\ &\geq \mu'_{\varphi(2a,2a)}(2t)\\ &\geq \mu'_{2^{\alpha}\varphi(a,a)}(2t) = \mu'_{\varphi(a,a)}\left(\frac{t}{2^{\alpha-1}}\right).\end{aligned}$$

Therefore,

$$\mu_{\frac{1}{2}G(2|a|)-G(|a|)}(2^{\alpha-1}t) \geq \mu'_{\varphi(a,a)}(t) \tag{20}$$

holds. It follows that

$$\mu_{\frac{1}{2^{n+1}}G(2^{n+1}|a|)-\frac{1}{2^n}G(2^n|a|)}(t) \geq \mu'_{\varphi(a,a)}\left(\frac{t}{2^{n(\alpha-1)}}\right),$$

and so

$$\begin{aligned}\mu_{\frac{1}{2^n}G(2^n|a|)-G(|a|)}\left(t\sum_{k=0}^{n-1}2^{k(\alpha-1)}\right) &= \mu_{\sum_{k=0}^{n-1}\frac{1}{2^{k+1}}G(2^{k+1}|a|)-\frac{1}{2^k}G(2^k|a|)}\left(t\sum_{k=0}^{n-1}2^{k(\alpha-1)}\right)\\ &\geq T_{M_{k=0}^{n-1}}\left(\mu_{\frac{1}{2^{k+1}}G(2^{k+1}|a|)-\frac{1}{2^k}G(2^k|a|)}(t2^{k(\alpha-1)})\right)\\ &\geq T_{M_{k=0}^{n-1}}\left(\mu'_{\varphi(a,a)}(t)\right).\end{aligned}$$

This implies that

$$\mu_{\frac{1}{2^n}G(2^n|a|)-G(|a|)}(t) \geq \mu'_{\varphi_{a,a}}\left(\frac{t}{\sum_{k=0}^{n-1}2^{k(\alpha-1)}}\right). \tag{21}$$

Replacing a by $2^p a$ in (21), we have

$$\begin{aligned}\mu_{\frac{1}{2^{n+p}}G(2^{n+p}|a|)-\frac{1}{2^p}G(2^p|a|)}(t) &\geq \mu'_{\varphi_{a,a}}\left(\frac{t}{\sum_{k=0}^{n+p-1}2^{k(\alpha-1)}}\right)\\ &\to 1, \qquad \text{when} \quad n\to+\infty,\end{aligned} \tag{22}$$

so $\left\{\frac{1}{2^n}G(2^n|a|)\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{B},\mu,\mathcal{T}_\wedge)$. Since $(\mathcal{B},\mu,\mathcal{T}_\wedge)$ is an LRBL-space, that is, $(\mathcal{B},\mu,\mathcal{T}_\wedge)$ is complete with respect to the randomness induced by random norm, there exists a point $H(|a|)\in\mathcal{B}$ such that

$$\lim_{n\to\infty}\frac{1}{2^n}G(2^n|a|) = H(|a|).$$

Fix $a\in\mathcal{A}$ and put $p=0$ in Equation (21), then we obtain

$$\mu_{\frac{1}{2^n}G(2^n|a|)-G(|a|)}(t) \geq \mu'_{\varphi_{a,a}}\left(\frac{t}{\sum_{k=0}^{n-1}2^{k(\alpha-1)}}\right),$$

and so, for any $\delta'>0$,

$$\begin{aligned}\mu_{H(|a|)-G(|a|)}(t+\delta') &\geq \mathcal{T}_M\left(\mu_{H(|a|)-\frac{1}{2^n}G(2^n|a|)}(\delta'),\mu_{\frac{1}{2^n}G(2^n|a|)-G(|a|)}(t)\right)\\ &\geq \mathcal{T}_M\left(\mu_{H(|a|)-\frac{1}{2^n}G(2^n|a|)},\mu'_{\varphi_{(a,a)}}\left(\frac{t}{\sum_{k=0}^{n-1}2^{k(\alpha-1)}}\right)\right).\end{aligned} \tag{23}$$

Taking $n \to \infty$ in Equation (23), we get

$$\mu_{H(|a|)-G(|a|)}(t+\delta') \geq \mu'_{\varphi(a,a)}\left(t\frac{2-2^{\alpha}}{2^{\alpha}}\right). \tag{24}$$

δ' is arbitrary, by taking $\delta' \to 0$ in Equation (24), we get

$$\mu_{H(|a|)-G(|a|)}(t) \geq \mu'_{\varphi(a,a)}\left(t\frac{2-2^{\alpha}}{2^{\alpha}}\right). \tag{25}$$

Replacing a and b by $2^n a, 2^n b$ in Equation (17), respectively, we get

$$\mu_{D_{\rho_{0.1}}G(2^n a,2^n b)}(t) \geq \mu'_{\varphi_{2^n a,2^n b}}(2^n t) \tag{26}$$

for all $a, b \in X$ and $t > 0$. Since $\lim_{n\to\infty} \mu'_{\varphi(2^n a,2^n b)}(2^n t) = 1$, we conclude that H satisfies Equation (11), so by Lemma 2, $H : \mathcal{A} \to \mathcal{B}$ satisfies the Cauchy latticetic operation-preserving functional equation.

Assume that there exists another cone-related mapping $H' : \mathcal{A} \to \mathcal{B}$ which satisfies Equation (17). Then, we obtain that

$$\begin{aligned}
\mu_{H(|a|)-H'(|a|)}(t) &= \lim_{n\to\infty} \mu_{\frac{1}{2^n}H(2^n|a|)-\frac{1}{2^n}H'(2^n|a|)}(t) \\
&\geq \lim_{n\to\infty} \mathcal{T}_M\left\{\mu_{\frac{1}{2^n}H(2^n|a|)-\frac{1}{2^n}G(2^n|a|)}(t/2), \mu_{\frac{1}{2^n}G(2^n|a|)-\frac{1}{2^n}H'(2^n|a|)}(t/2)\right\} \\
&= \lim_{n\to\infty} \min\left\{\mu_{\frac{1}{2^n}H(2^n|a|)-\frac{1}{2^n}G(2^n|a|)}(t/2), \mu_{\frac{1}{2^n}G(2^n|a|)-\frac{1}{2^n}H'(2^n|a|)}(t/2)\right\} \\
&\geq \lim_{n\to\infty} \mu'_{\varphi(2^n a,2^n a)}\left(\frac{2^{n+1}(2-2^{\alpha})}{2^{\alpha}}t\right) \\
&\geq \lim_{n\to\infty} \mu'_{\varphi(a,a)}\left(\frac{2^{n+1}(2-2^{\alpha})}{2^{\alpha}.2^{n\alpha}}t\right).
\end{aligned}$$

On the other hand, we have $\lim_{n\to\infty} \mu'_{\varphi_{a,a}}\left(\frac{2^{n+1}(2-2^{\alpha})}{2^{\alpha}.2^{n\alpha}}\right) = 1$. Therefore, it follows that $\mu_{H(|a|)-H'(|a|)}(t) = 1$ for all $t > 0$ and so $H(|a|) = H'(|a|)$ (i.e., H is unique). This completes the proof. □

Corollary 1. *Assume that $v \geq 0$ and $(\mathcal{A}, \wedge_{\mathcal{A}}, \vee_{\mathcal{A}})$ is a vector lattice space and $(\mathcal{B}, \mu', \mathcal{T}_{\wedge})$ is an LRBL-space in which $\mathcal{T}_{\wedge} = \mathcal{T}_M$ and $0 < p < 1$. Assume that $G : \mathcal{A} \to \mathcal{B}$ is a cone-related mapping satisfying*

$$\mu_{D_{\rho_{0.1}}G(a,b)}(t) \geq \mu'_{(\|a\|^p+\|b\|^p)v}(t) \tag{27}$$

for all $a \in \mathcal{A}$ and $t > 0$. Then, the limit $H(|a|) = \lim_{n\to\infty} \frac{1}{2^n}G(2^n|a|)$ exists for all $a \in \mathcal{A}$ and defines a unique cone-related mapping $H : \mathcal{A} \to \mathcal{B}$ such that

$$\mu_{G(|a|)-H(|a|)} \geq \mu'_{\|a\|^p v}\left(\frac{(2-2^p)t}{2^p}\right). \tag{28}$$

Proof. Assume that $\varphi : \mathcal{A}^2 \to [0, \infty)$ is a mapping by $\varphi(a, b) = (\|a\|^p + \|b\|^p)v$. Then, from Theorem 3, the conclusion follows. □

4. Stability of the first ρ-Functional Equation: A Different Method

Throughout this section, using the fixed-point method, we prove the nonlinear stablility of the first ρ-functional Eqaution in LRBL-spaces.

Theorem 4. *Assume that $(\mathcal{A}, \wedge_{\mathcal{A}}, \vee_{\mathcal{A}})$ is a vector lattice space and $(\mathcal{B}, \mu, \mathcal{T}_{\wedge})$ is an LRBL-space in which $\mathcal{T}_{\wedge} = \mathcal{T}_M$ and $\varphi : \mathcal{A}^2 \to D_M^+$ ($\phi(a,b)$ is denoted by $\phi_{a,b}$) such that there exists $0 < \alpha < \frac{1}{2}$,*

$$\varphi_{2a,2b}(t) \leq \varphi_{a,b}\left(\frac{t}{2^\alpha}\right) \tag{29}$$

for all $a, b \in \mathcal{A}$ and $t > 0$. Assume that $G : \mathcal{A} \to \mathcal{B}$ is a cone-related mapping such that

$$\mu_{D_{\rho_{0,1}}G(a,b)}(t) \geq \varphi_{a,b}(t) \tag{30}$$

for all $a, b \in \mathcal{A}$ and $t > 0$. Then, for all $a \in \mathcal{A}$,

$$H(|a|) := \lim_{n\to\infty} \frac{G(2^n|a|)}{2^n}$$

exists and $H : \mathcal{A} \to \mathcal{B}$ is a unique cone-related functional that satisfies the Cauchy latticetic operation-preserving function equation such that

$$\mu_{G(|a|)-H(|a|)}(t) \geq \varphi_{a,a}\left(\frac{2-2^\alpha}{2^\alpha}t\right) \tag{31}$$

for all $a \in \mathcal{A}$ and $t > 0$.

Proof. Putting $b = a$ in Equation (30), we have

$$\mu_{2G(\frac{|a|}{2})-G(|a|)}(t) \geq \varphi_{a,b}\,(t)\,. \tag{32}$$

Replace a by $2a$ in Equation (32), we obtain that

$$\begin{aligned}\mu_{2G(|a|)-G(2|a|)}(t) = \mu_{G(2|a|)-2G(|a|)}(t) &\geq \varphi_{2a,2a}(t)\\ &\geq \varphi_{a,a}\left(\frac{t}{2^\alpha}\right).\end{aligned} \tag{33}$$

Then,

$$\mu_{\frac{1}{2}G(2|a|)-G(|a|)}(t) \geq 2^{\alpha-1}\varphi_{(a,a)}(t). \tag{34}$$

Consider the set $S := \{g : \mathcal{A} \to \mathcal{B}, g(0) = 0\}$ and the generalized metric d_c in S defined by

$$d_c(g,h) = \inf_{u\in(0,\infty)} \left\{\mu_{g(a)-h(a)}(ut) \geq \varphi_{a,a}(t), \quad \forall a \in \mathcal{A}, \forall t > 0\right\},$$

where as usual, $\inf\emptyset = \infty$.

It is easy to show that (S, d_c) is a complete generalized metric space that defines the operator $\beta : (S, d_c) \to (S, d_c)$ such that

$$\beta g(a) = \frac{1}{2}g(2|a|) \tag{35}$$

for all $a \in \mathcal{A}$. Given $g, h \in S$, let $u \in [0, \infty]$ be an orbitrary constant with $d_c(g,h) \leq u$, that is,

$$\mu_{g(|a|)-h(|a|)}(t) \geq u\varphi(a,a).$$

$$\begin{aligned}\mu_{\beta g(|a|)-\beta h(|a|)}(t) = \mu_{\frac{1}{2}[g(2|a|)-h(2|a|)]}(t) &\geq \frac{1}{2}u\varphi(2a,2a)\\ &\geq \frac{1}{2}u2^\alpha\varphi(a,a) = 2^{\alpha-1}u\varphi(a,a)\end{aligned} \tag{36}$$

for all $a \in \mathcal{A}$, that is, $d_c(\beta g, \beta h) < 2^{\alpha-1}u$. Thus, we have

$$d_c(\beta g, \beta h) \leq 2^{\alpha-1} d_c(g, h) \tag{37}$$

for all $g, h \in S$.

So, β is a strictly contractive mapping with constant $2^{\alpha-1} < 1$ on S and $\alpha \in [0, 1/2)$. By Equation (36) we have

$$d_c(\beta G, G) \leq 2^{\alpha-1} < \infty.$$

By Theorem 1, there exists a cone-related functional $H : \mathcal{A} \to \mathcal{B}$ satisfying the following:
1. H is a fixed point of β, that is,

$$H(2|a|) = 2H(|a|)$$

for all $a \in \mathcal{A}$. Also, the mapping H is a unique fixed point of β in the set

$$M = \{g \in S : d_c(g, h) < \infty\}.$$

This implies that

$$H(c|a|) = cH(|a|), \ \ \forall a \in \mathcal{A}, \ \forall c \in \mathbb{R}^+.$$

2. $d_c(\beta^n G, H) \to 0$ as $n \to \infty$. So,

$$\lim_{n\to\infty} \frac{1}{2^n} G(2^n|a|) = H(|a|)$$

for all $a \in \mathcal{A}$.
3. $d_c(G, H) \leq \frac{1}{1-L} d(G, JG)$, so we have

$$\mu_{H(|a|)-G(|a|)}(t) \geq \varphi_{a,a}\left(\frac{2^{\alpha-1}t}{1-2^{\alpha-1}}\right) = \varphi_{a,a}\left(\frac{2^\alpha}{2-2^\alpha}t\right),$$

which implies that inequality (30) holds.

Now we show that H satisfies Lemma 2; replacing a and b with $2^n a$ and $2^n b$, in Equation (30), we get

$$\begin{aligned}\mu_{D_{\rho_{0.1}} G(2^n a, 2^n b)}(t) &\geq \varphi_{2^n a, 2^n b}(t) \\ &\geq 2^{n\alpha} \varphi_{2^n a, 2^n b}(t).\end{aligned}$$

Then,

$$\mu_{\frac{1}{2^n} G(2^n|a|) - G(|a|)}\left(\frac{t}{2^n}\right) \geq 2^{n\alpha} \varphi_{2^n a, 2^n b}(t) \to 1.$$

Since $\lim_{n\to\infty} 2^{n(\alpha-1)} = \infty$, then

$$\lim_{n\to\infty} \mu_{\frac{1}{2^n} G(2^n|a|) - G(|a|)}(t) = 1.$$

Therefore, it follows that $\mu_{D_{\rho_{0.1}} G(a,b)}(t) = 1$ for all $t > 0$ and so $D_{\rho_{0.1}} G(a, b) = 0$.
Thus, a cone-related functional $H : \mathcal{A} \to \mathcal{B}$ satisfies the first ρ-functional Equation. □

5. Stability of the Second ρ-Functional Equation: Direct Method

In this section, using a direct method, we prove nonlinear stability the second ρ-functional Equation in latticetic random Banach lattice space.

Lemma 3. *Assume that $(\mathcal{A}, \wedge_{\mathcal{A}}, \vee_{\mathcal{A}})$ and $(\mathcal{B}, \wedge_{\mathcal{B}}, \vee_{\mathcal{B}})$ are vector lattice spaces. If a mapping $G : \mathcal{A} \to \mathcal{B}$ satisfies*

$$2\left[G\left(\frac{|a|\Delta^{*}_{\mathcal{A}}|b|}{2}\right)\Delta^{*}_{\mathcal{B}}G\left(\frac{|a|\Delta^{**}_{\mathcal{A}}|b|}{2}\right)\right] - G(|a|)\Delta^{**}_{\mathcal{B}}G(|b|) = \rho\left(G(|a|\Delta^{*}_{\mathcal{A}}|b|)\Delta^{*}_{\mathcal{B}}G(|a|\Delta^{**}_{\mathcal{A}}|b|) - G(|a|)\Delta^{**}_{\mathcal{B}}G(|b|)\right), \quad (38)$$

for all $a, b \in \mathcal{A}$, then functional Equation (38) is a Cauchy latticetic operation-preserving functional equation.

Proof. Assume that $G : \mathcal{A} \to \mathcal{B}$ satisfies Equation (38), letting $b = a$ in Equation (38), we get

$$2G\left(\frac{|a|}{2}\right) - G(|a|) = \rho(G(|a|) - G(|a|) = 0.$$

Then, $2G\left(\frac{|a|}{2}\right) - G(|a|) = 0$, and so

$$G\left(\frac{|a|}{2}\right) = \frac{1}{2}G(|a|), \quad \forall \in \mathcal{A}. \quad (39)$$

It follows from Equations (38) and (39) that

$$\begin{aligned} G(|a|\Delta^{*}_{\mathcal{A}}|b|)\Delta^{*}_{\mathcal{B}}G(|a|\Delta^{**}_{\mathcal{A}}|b|) - G(|a|)\Delta^{**}_{\mathcal{B}}G(|b|) &= \rho\left(2\left[G\left(\frac{|a|\Delta^{*}_{\mathcal{A}}|b|}{2}\right)\Delta^{*}_{\mathcal{B}}G\left(\frac{|a|\Delta^{**}_{\mathcal{A}}|b|}{2}\right)\right] - G(|a|)\Delta^{**}_{\mathcal{B}}G(|b|)\right) \\ &= \rho\left(G(|a|\Delta^{*}_{\mathcal{A}}|b|)\Delta^{*}_{\mathcal{B}}G(|a|\Delta^{**}_{\mathcal{A}}|b|) - G(|a|)\Delta^{**}_{\mathcal{B}}G(|b|)\right), \end{aligned}$$

and so

$$G(|a|\Delta^{*}_{\mathcal{A}}|b|)\Delta^{*}_{\mathcal{B}}G(|a|\Delta^{**}_{\mathcal{A}}|b|) = G(|a|)\Delta^{**}_{\mathcal{B}}G(|b|)$$

for all $a \in \mathcal{A}$. □

Now, we prove nonlinear stability of the second ρ-functional Equation in latticetic random Banach lattice spaces.

Theorem 5. *Assume that $(\mathcal{A}, \wedge_{\mathcal{A}}, \vee_{\mathcal{A}})$ is a vector lattice space and $(\mathcal{B}, \mu', \mathcal{T}_{\wedge})$ is an LRBL-space in which $\mathcal{T}_{\wedge} = \mathcal{T}_M$ and $\varphi : \mathcal{A}^2 \to D^+_M$ ($\phi(a, b)$ is denoted by $\phi_{a,b}$) such that there exists $0 < \alpha < 2$,*

$$\mu'_{\varphi(\frac{a}{2},\frac{a}{2})}(t) \geq \mu'_{2^{\alpha}\varphi(a,a)}(t)$$

and

$$\lim_{n\to\infty} \mu'_{\varphi(\frac{a}{2^n},\frac{a}{2^n})}(t/2^n) = 1$$

for all $a, b \in \mathcal{A}$ and $t > 0$. If $G : \mathcal{A} \to \mathcal{B}$ is a cone-related functional such that

$$\mu_{D_{\rho_{0.2}}G(a,b)}(t) \geq \mu'_{\varphi(a,b)}(t)$$

for all $a, b \in \mathcal{A}$ and $t > 0$, then there is a unique cone-related mapping $H : \mathcal{A} \to \mathcal{B}$ which satisfies the Cauchy latticetic operation-preserving functional equation such that

$$\mu_{G(|a|)-H(|a|)}(t) \geq \mu'_{\varphi(a,a)}\left(\frac{2^{\alpha}-2}{2^{\alpha}}t\right)$$

for all $a \in \mathcal{A}$ and $t > 0$.

Corollary 2. *Assume that $v \geq 0$ and $(\mathcal{A}, \wedge_{\mathcal{A}}, \vee_{\mathcal{A}})$ is a vector lattice space and $(\mathcal{B}, \mu', \mathcal{T}_{\wedge})$ is an LRBL-space in which $\mathcal{T}_{\wedge} = \mathcal{T}_M$ and $P > 1$. Let $G : \mathcal{A} \to \mathcal{B}$ be a cone-related mapping satisfying*

$$\mu_{D_{\rho_{0.2}}G(a,b)}(t) \geq \mu'_{\|a\|^p+\|b\|^p v}(t)$$

for all $a \in \mathcal{A}$ *and* $t > 0$*. Then, the limit* $H(|a|) = \lim_{n\to\infty} 2^n G\left(\frac{|a|}{2^n}\right)$ *exists for all* $a \in \mathcal{A}$ *and defines a unique cone-related mapping* $H : \mathcal{A} \to \mathcal{B}$ *such that*

$$\mu_{G(|a|)-H(|a|)}(t) \geq \mu'_{\|a\|^p v}\left(\frac{(2^p - 2)}{2^p} t\right). \tag{40}$$

Proof. Assume that $\varphi : \mathcal{A}^2 \to [0, \infty)$ is a mapping by $\varphi(a, b) = (\|a\|^p + \|b\|^p)v$, then from Theorem 5, the conclusion follows. □

6. Stability of the Second ρ-Functional Equation: Different Method

In this section, using the fixed-point method, we prove nonlinear stability of the second ρ-functional Equation in LRBL-spaces.

Theorem 6. *Assume that* $(\mathcal{A}, \wedge_{\mathcal{A}}, \vee_{\mathcal{A}})$ *is a vector lattice space,* $(\mathcal{B}, \mu, \mathcal{T}_\wedge)$ *is an LRBL-space in which* $\mathcal{T}_\wedge = \mathcal{T}_M$ *and* $\varphi : \mathcal{A}^2 \to D_M^+$ *(*$\phi(a, b)$ *is denoted by* $\phi_{a,b}$*) such that there exists* $\alpha < 1$,

$$\varphi_{\frac{a}{2},\frac{b}{2}}(t) \leq \varphi_{a,b}(t/2\alpha)$$

for all $a, b \in \mathcal{A}$ *and* $t > 0$*. Assume that* $G : \mathcal{A} \to \mathcal{B}$ *is a cone-related mapping such that*

$$\mu_{D_{\rho_{0.2}} G(a,b)}(t) \geq \varphi_{a,b}(t)$$

for all $a, b \in \mathcal{A}$ *and* $t > 0$*; then, for all* $a \in \mathcal{A}$,

$$H(|a|) := \lim_{n\to\infty} 2^n G\left(\frac{|a|}{2^n}\right)$$

exists and $H : \mathcal{A} \to \mathcal{B}$ *is a unique cone-related function that satisfies the Cauchy latticetic operation-preserving functional equation such that*

$$\mu_{G(|a|)-H(|a|)}(t) \geq \varphi_{a,a}\left(\frac{2^\alpha - 2}{2^\alpha} t\right)$$

for all $a \in \mathcal{A}$ *and* $t > 0$.

Proof. Similar to Theorem 4. □

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