mathematics

MDPI

Article

A Parameter-Based Ostrowski–Grüss Type Inequalities with Multiple Points for Derivatives Bounded by Functions on Time Scales

Seth Kermausuor [1] and Eze R. Nwaeze [2,*]

[1] Department of Mathematics and Computer Science, Alabama State University, Montgomery, AL 36101, USA; skermausour@alasu.edu

[2] Department of Mathematics, Tuskegee University, Tuskegee, AL 36088, USA

* Correspondence: enwaeze@tuskegee.edu

Received: 5 November 2018; Accepted: 11 December 2018; Published: 14 December 2018

Abstract: In this paper, we present some Ostrowski–Grüss-type inequalities on time scales for functions whose derivatives are bounded by functions for k points via a parameter. The 2D versions of these inequalities are also presented. Our results generalize some of the results in the literature. As a by-product, we apply our results to the continuous and discrete calculus to obtain some interesting inequalities in this direction.

Keywords: Ostrowski–Grüss-type inequality; Montgomery identity; time scales; parameter

JEL Classification: 26D10; 26D15; 54C30

1. Introduction

In 1997, Dragomir and Wang [1] proved that if $f : [a, b] \to \mathbb{R}$ is a differentiable function such that there exist constants $\gamma, \Gamma \in \mathbb{R}$ with $\gamma \leq f'(x) \leq \Gamma$ for all $x \in [a, b]$, then we have:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(s)ds - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \right| \leq \frac{1}{4}(b-a)(\Gamma - \gamma) \tag{1}$$

for all $x \in [a, b]$. The above inequality is known in the literature as the Ostrowski–Grüss-type inequality. This inequality has been improved (see [2,3]) and generalized (see [4–8]) in several different ways. In particular, the authors in [5,6] obtained some Ostrowski–Grüss-type inequalities with the derivatives bounded by functions instead of scalars. Since the introduction of the theory of time scales by the German mathematician Stefan Hilger in his Ph.D thesis [9], several integral inequalities have been extended to time scales by many authors (see [10–12]). For some extensions of (1) to time scales, we refer the reader to the papers [13–19]. Recently, Nwaeze et al. [16] obtained the following generalization of Inequality (1) on time scales for $k + 1$ points via a parameter.

Theorem 1 ([16], Theorem 13). *Suppose that:*

1. $a, b \in \mathbb{T}$, $\lambda \in [0, 1]$, $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ *is a partition of the interval* $[a, b]$ *for* $x_0, x_1, \cdots, x_k \in \mathbb{T}$,
2. $\alpha_i \in \mathbb{T}$ $(i = 0, 1, \cdots, k+1)$ *is* $k + 2$ *points so that* $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \cdots, k)$ *and* $\alpha_{k+1} = b$,
3. $f : [a, b] \to \mathbb{R}$ *is a differentiable function;* f^Δ *is rd-continuous; and there exists* $\gamma, \Gamma \in \mathbb{R}$ *such that* $\gamma \leq f^\Delta(t) \leq \Gamma$ *for all* $t \in [a, b]$*. Then, the following inequality holds,*

$$
\begin{aligned}
\Bigg| &(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)+\lambda\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\frac{f(\alpha_i)+f(\alpha_{i+1})}{2}-\int_a^b f^\sigma(t)\Delta t \\
&-\frac{\Gamma+\gamma}{2}\sum_{i=0}^{k-1}\left[h_2\left(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\right)-h_2\left(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\right)\right. \\
&\left.-\frac{\Gamma+\gamma}{2}\sum_{i=0}^{k-1}\left[h_2\left(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\right)-h_2\left(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\right)\right]\right]\Bigg| \\
&\leq\frac{\Gamma-\gamma}{2}\sum_{i=0}^{k-1}\left[h_2\left(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\right)+h_2\left(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\right)\right. \\
&\qquad\left.+h_2\left(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\right)+h_2\left(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\right)\right],
\end{aligned}
\tag{2}
$$

where $h_2(t,s)=\int_s^t(\tau-s)\,\Delta\tau$. *Inequality (2) is sharp in the sense that the constant 1/2 on the right-hand side cannot be replaced by a smaller one.*

Motivated by the results in [5,6], the goal of this article is to provide some Ostrowski–Grüss-type inequalities on time scales for functions whose derivatives are bounded by functions. Our findings give a broader view of some established published results.

2. Time Scale Essentials

In this section, we collect basic time scale concepts that will aid in better understanding of this work. For more on this subject, we refer the interested reader to Hilger's Ph.D. thesis [9], the books [20,21], and the survey [10].

Definition 1 ([20])**.** *A time scale* $\mathbb{T}$ *is an arbitrary nonempty closed subset of the real numbers.*

We assume throughout that $\mathbb{T}$ has the topology that is inherited from the standard topology on $\mathbb{R}$. It is also assumed throughout that in $\mathbb{T}$, the interval $[a,b]$ means the set $\{t\in\mathbb{T}\colon a\leq t\leq b\}$ for the points $a<b$ in $\mathbb{T}$. Since a time scale may not be connected, we need the following concept of jump operators.

Definition 2 ([20])**.** *For each* $t\in\mathbb{T}$*, the forward jump operator* $\sigma:\mathbb{T}\to\mathbb{T}$ *is defined by* $\sigma(t)=\inf\{s\in\mathbb{T}\colon s>t\}$ *and the backward jump operator* $\rho:\mathbb{T}\to\mathbb{T}$ *is defined by* $\rho(t)=\sup\{s\in\mathbb{T}:s<t\}$.

Definition 3 ([20])**.** *If* $\sigma(t)>t$*, then we say that* t *is right-scattered, while if* $\rho(t)<t$*, then we say that* t *is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If* $\sigma(t)=t$*, then* t *is called right-dense, and if* $\rho(t)=t$*, then* t *is called left-dense. Points that are both right-dense and left-dense are called dense.*

Definition 4 ([20])**.** *The mapping* $\mu:\mathbb{T}\to[0,\infty)$ *defined by* $\mu(t)=\sigma(t)-t$ *is called the graininess function. The set* $\mathbb{T}^\kappa$ *is defined as follows: if* $\mathbb{T}$ *has a left-scattered maximum* m*, then* $\mathbb{T}^\kappa=\mathbb{T}-\{m\}$*; otherwise,* $\mathbb{T}^\kappa=\mathbb{T}$.

If $\mathbb{T}=\mathbb{R}$, then $\mu(t)=0$, and when $\mathbb{T}=\mathbb{Z}$, we have $\mu(t)=1$.

Definition 5 ([20])**.** *Let* $f:\mathbb{T}\to\mathbb{R}$ *and* $t\in\mathbb{T}^\kappa$*. Then, we define* $f^\Delta(t)$ *to be the number (provided it exists) with the property that for any given* $\epsilon>0$*, there exists a neighborhood* U *of* t *such that:*

$$
\left|f(\sigma(t))-f(s)-f^\Delta(t)\,[\sigma(t)-s]\right|\leq\epsilon\,|\sigma(t)-s|\,,\quad\forall s\in U.
$$

We call $f^{\Delta}(t)$ the delta derivative of f at t. Moreover, we say that f is delta differentiable (or in short, differentiable) on $\mathbb{T}^{\kappa}$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$. The function $f^{\Delta} : \mathbb{T}^{\kappa} \to \mathbb{R}$ is then called the delta derivative of f on $\mathbb{T}^{\kappa}$.

In the case $\mathbb{T} = \mathbb{R}$, $f^{\Delta}(t) = \frac{df(t)}{dt}$. In the case $\mathbb{T} = \mathbb{Z}$, $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$, which is the usual forward difference operator.

Theorem 2 ([20]). *If $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^{\kappa}$, then the product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t and:*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t).$$

Definition 6 ([20]). *The function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous on $\mathbb{T}$ provided it is continuous at all right-dense points $t \in \mathbb{T}$ and its left-sided limits exist at all left-dense points $t \in \mathbb{T}$. The set of all rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd}(\mathbb{T},\mathbb{R})$. Furthermore, the set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by $C^1_{rd}(\mathbb{T},\mathbb{R})$.*

It follows from ([20] Theorem 1.74) that every rd-continuous function has an anti-derivative.

Definition 7 ([20]). *Let $F : \mathbb{T} \to \mathbb{R}$ be a function. Then, $F : \mathbb{T} \to \mathbb{R}$ is called the anti-derivative of f on $\mathbb{T}$ if it satisfies $F^{\Delta}(t) = f(t)$ for any $t \in \mathbb{T}^{\kappa}$. In this case, the Cauchy integral is defined by:*

$$\int_a^b f(t)\Delta t = F(b) - F(a), \quad a, b \in \mathbb{T}.$$

Theorem 3 ([20]). *Let $f, g \in C_{rd}(\mathbb{T},\mathbb{R})$, $a, b, c \in \mathbb{T}$, and $\alpha, \beta \in \mathbb{R}$. Then:*

(1) $\int_a^b [\alpha f(t) + \beta g(t)]\, \Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t.$

(2) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t.$

(3) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t.$

(4) $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t.$

(5) If $|f(t)| \le g(t)$ on $[a, b]$, then:

$$\left| \int_a^b f(t)\Delta t \right| \le \int_a^b g(t)\Delta t.$$

Definition 8 ([20]). *The polynomials $h_k, g_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$ are defined recursively as thus: $h_0(t,s) := g_0(t,s) := 1$ for all $s, t \in \mathbb{T}$, $g_{k+1}(t,s) = \int_s^t g_k(\sigma(\tau), s)\Delta\tau$, and $h_{k+1}(t,s) = \int_s^t h_k(\tau, s)\Delta\tau$ for all $s, t \in \mathbb{T}$.*

In view of the above definition, we make the following remarks that will come handy in the sequel (see [20] Example 1.102).

- For $\mathbb{T} = \mathbb{R}$, $h_2(t,s) = \frac{(t-s)^2}{2}$.
- For $\mathbb{T} = \mathbb{Z}$, $h_2(t,s) = \frac{(t-s)(t-s-1)}{2}$.

3. Main Results

To prove our 1D results, we will need the following lemma given in [22].

Lemma 1 (Generalized Montgomery identity with a parameter)**.** *Suppose that:*

1. $a, b \in \mathbb{T},\ \lambda \in [0,1],\ I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ *is a partition of the interval* $[a,b]$ *for* $x_0, x_1, \cdots, x_k \in \mathbb{T}$,
2. $\alpha_i \in \mathbb{T}\ (i = 0, 1, \cdots, k+1)$ *is* $k+2$ *points so that* $\alpha_0 = a,\ \alpha_i \in [x_{i-1}, x_i]\ (i = 1, \cdots, k)$, *and* $\alpha_{k+1} = b$,
3. $f : [a,b] \to \mathbb{R}$ *is a differentiable function.*

Then, we have the following equation:

$$\begin{aligned}
&\int_a^b K(t, I_k) f^{\Delta}(t)\Delta t + \int_a^b f^{\sigma}(t)\Delta t \\
&= (1-\lambda)\sum_{i=0}^{k}\Big(\alpha_{i+1} - \alpha_i\Big) f(x_i) + \lambda \sum_{i=0}^{k}\Big(\alpha_{i+1} - \alpha_i\Big)\frac{f(\alpha_i) + f(\alpha_{i+1})}{2},
\end{aligned} \tag{3}$$

where:

$$K(t, I_k) = \begin{cases}
t - \Big(\alpha_1 - \lambda\frac{\alpha_1 - a}{2}\Big), & t \in [a, \alpha_1), \\
t - \Big(\alpha_1 + \lambda\frac{\alpha_2 - \alpha_1}{2}\Big), & t \in [\alpha_1, x_1), \\
t - \Big(\alpha_2 - \lambda\frac{\alpha_2 - \alpha_1}{2}\Big), & t \in [x_1, \alpha_2), \\
\vdots & \\
t - \Big(\alpha_{k-1} + \lambda\frac{\alpha_k - \alpha_{k-1}}{2}\Big), & t \in [\alpha_{k-1}, x_{k-1}), \\
t - \Big(\alpha_k - \lambda\frac{\alpha_k - \alpha_{k-1}}{2}\Big), & t \in [x_{k-1}, \alpha_k), \\
t - \Big(\alpha_k + \lambda\frac{\alpha_{k+1} - \alpha_k}{2}\Big), & t \in [\alpha_k, b],
\end{cases} \tag{4}$$

provided for each $i \in \{0, 1, 2, \ldots, k-1\}$, $\alpha_{i+1} - \lambda\frac{\alpha_{i+1} - \alpha_i}{2}$ *and* $\alpha_{i+1} + \lambda\frac{\alpha_{i+2} - \alpha_{i+1}}{2}$ *belong to* $\mathbb{T}$.

Remark 1. *It is easy to see that for* $k = 2,\ \lambda = 0,\ \alpha_1 = a,\ \alpha_2 = b$, *and* $x_1 = x$, *we recover the classical Montgomery identity on time scales.*

Theorem 4. *Suppose* f *satisfies the conditions of Lemma 1. If there exists functions* α, β *with* $\alpha(t) \le f^{\Delta}(t) \le \beta(t)$ *for all* $t \in [a,b]$, *then we have the inequality:*

$$\begin{aligned}
&\Bigg|(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1} - \alpha_i) f(x_i) + \lambda\sum_{i=0}^{k}(\alpha_{i+1} - \alpha_i)\frac{f(\alpha_i) + f(\alpha_{i+1})}{2} \\
&- \int_a^b f^{\sigma}(t)\Delta t - \int_a^b K(t, I_k)\frac{\alpha(t) + \beta(t)}{2}\Delta t\Bigg| \\
&\le \frac{1}{2}\|\beta - \alpha\|_{\infty}\sum_{i=0}^{k-1}\Bigg[h_2\Big(\alpha_{i+1}, \alpha_{i+1} - \lambda\frac{\alpha_{i+1} - \alpha_i}{2}\Big) + h_2\Big(x_i, \alpha_{i+1} - \lambda\frac{\alpha_{i+1} - \alpha_i}{2}\Big) \\
&\quad + h_2\Big(x_{i+1}, \alpha_{i+1} + \lambda\frac{\alpha_{i+2} - \alpha_{i+1}}{2}\Big) + h_2\Big(\alpha_{i+1}, \alpha_{i+1} + \lambda\frac{\alpha_{i+2} - \alpha_{i+1}}{2}\Big)\Bigg]
\end{aligned} \tag{5}$$

provided for each $i \in \{0, 1, 2, \ldots, k-1\}$, $\alpha_{i+1} - \lambda\frac{\alpha_{i+1} - \alpha_i}{2}$ *and* $\alpha_{i+1} + \lambda\frac{\alpha_{i+2} - \alpha_{i+1}}{2}$ *belong to* $\mathbb{T}$.

Proof. First, we observe that:

$$\int_a^b K(t, I_k) f^{\Delta}(t)\Delta t - \int_a^b K(t, I_k)\frac{\alpha(t) + \beta(t)}{2}\Delta t = \int_a^b K(t, I_k)\Big(f^{\Delta}(t) - \frac{\alpha(t) + \beta(t)}{2}\Big)\Delta t. \tag{6}$$

Using (6), we get:

$$\left|\int_a^b K(t,I_k)f^\Delta(t)\Delta t - \int_a^b K(t,I_k)\frac{\alpha(t)+\beta(t)}{2}\Delta t\right| \le \int_a^b \left|K(t,I_k)\right|\left|f^\Delta(t) - \frac{\alpha(t)+\beta(t)}{2}\right|\Delta t. \tag{7}$$

From the condition that $\alpha(t) \le f^\Delta(t) \le \beta(t),\ t \in [a,b]$, one obtains:

$$\left|f^\Delta(t) - \frac{\alpha(t)+\beta(t)}{2}\right| \le \frac{1}{2}\|\beta-\alpha\|_\infty \tag{8}$$

for all $t \in [a,b]$. Now, from Lemma 1, we have:

$$\begin{aligned}&\int_a^b K(t,I_k)f^\Delta(t)\Delta t\\ &= (1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i) + \lambda\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\frac{f(\alpha_i)+f(\alpha_{i+1})}{2} - \int_a^b f^\sigma(t)\Delta t\end{aligned} \tag{9}$$

and:

$$\begin{aligned}\int\limits_a^b |K(t,I_k)|\Delta t = \sum_{i=0}^{k-1}\Bigg[&h_2\left(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\right) + h_2\left(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\right)\\ &+ h_2\left(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\right) + h_2\left(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\right)\Bigg].\end{aligned} \tag{10}$$

The desired inequality in (5) is obtained from (7) by using (8)–(10). □

Theorem 5. *Suppose f satisfies the conditions of Lemma 1. If there exists functions α, β with $\alpha(t) \le f^\Delta(t) \le \beta(t)$ for all $t \in [a,b]$, then we have the inequality:*

$$\begin{aligned}&\Bigg|(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i) + \lambda\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\frac{f(\alpha_i)+f(\alpha_{i+1})}{2} - \int_a^b f^\sigma(t)\Delta t\\ &-\frac{f(b)-f(a)}{b-a}\sum_{i=0}^{k-1}\Bigg[h_2\left(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\right) - h_2\left(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\right)\\ &+h_2\left(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\right) - h_2\left(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\right)\Bigg]\\ &-\int_a^b P(t,I_k)\frac{\alpha(t)+\beta(t)}{2}\Delta t\Bigg|\\ &\le \frac{1}{2}\|\beta-\alpha\|_\infty\int_a^b \left|P(t,I_k)\right|\Delta t,\end{aligned} \tag{11}$$

where $P(t,I_k) = K(t,I_k) - \frac{1}{b-a}\int_a^b K(s,t)\Delta s$ and provided for each $i \in \{0,1,2,\ldots,k-1\}$, $\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}$ and $\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}$ belong to $\mathbb{T}$.

Proof. Using the definition of $P(\cdot,\cdot)$, we get:

$$\begin{aligned}\int_a^b P(t,I_k)\left(f^\Delta(t) - \frac{\alpha(t)+\beta(t)}{2}\right)\Delta t = \int_a^b K(t,I_k)f^\Delta(t)\Delta t - \frac{1}{b-a}\int_a^b f^\Delta(t)\Delta t\int_a^b K(s,I_k)\Delta s\\ -\int_a^b P(t,I_k)\frac{\alpha(t)+\beta(t)}{2}\Delta t.\end{aligned} \tag{12}$$

Therefore,

$$\begin{aligned}\Big|\int_a^b K(t,I_k)f^{\Delta}(t)\Delta t-\frac{1}{b-a}\int_a^b f^{\Delta}(t)\Delta t\int_a^b K(s,I_k)\Delta s\\ -\int_a^b P(t,I_k)\frac{\alpha(t)+\beta(t)}{2}\Delta t\Big|\\ \le\int_a^b |P(t,I_k)|\left|f^{\Delta}(t)-\frac{\alpha(t)+\beta(t)}{2}\right|\Delta t.\end{aligned}\tag{13}$$

By the time scale version of the fundamental theorem of calculus, one gets:

$$\int_a^b f^{\Delta}(t)\Delta t=f(b)-f(a),\tag{14}$$

and the identity:

$$\begin{aligned}\int_a^b K(t,I_k)\Delta t=\sum_{i=0}^{k-1}\Bigg[&h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)-h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)\\ &+h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)-h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Bigg].\end{aligned}\tag{15}$$

From the condition that $\alpha(t)\le f^{\Delta}(t)\le\beta(t),\ t\in[a,b]$, we have that:

$$\left|f^{\Delta}(t)-\frac{\alpha(t)+\beta(t)}{2}\right|\le\frac{1}{2}\|\beta-\alpha\|_{\infty}\tag{16}$$

for all $t\in[a,b]$. Now, by Lemma 1, we obtain:

$$\begin{aligned}&\int_a^b K(t,I_k)f^{\Delta}(t)\Delta t\\ &=(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)+\lambda\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\frac{f(\alpha_i)+f(\alpha_{i+1})}{2}-\int_a^b f^{\sigma}(t)\Delta t.\end{aligned}\tag{17}$$

By using (13)–(17), we obtain the desired inequality in (11). □

The two-variable time scale calculus and multiple integration have been introduced in [23,24]. In what follows, we provide the two-dimensional versions of Theorems 4 and 5. To do this, we need the following two-dimensional version of Lemma 1. This lemma is given in [19], and it follows quite easily by using Lemma 1, so the proof is omitted.

Lemma 2 (2D Generalized Montgomery Identity with a parameter)**.** *Let $\lambda\in[0,1]$; $a,b\in\mathbb{T}_1$; $c,d\in\mathbb{T}_2$ with $a<b,\ c<d$. Suppose that:*

1. *$I_k: a=x_0<x_1<\cdots<x_{k-1}<x_k=b$ is a partition of the interval $[a,b]_{\mathbb{T}_1}$ for $x_0,x_1,\cdots,x_k\in\mathbb{T}_1$, and $J_k: c=y_0<y_1<\cdots<y_{k-1}<y_k=d$ is a partition of the interval $[c,d]_{\mathbb{T}_2}$ for $y_0,y_1,\cdots,y_k\in\mathbb{T}_2$;*
2. *$\alpha_i\in\mathbb{T}_1,\beta_i\in\mathbb{T}_2$ $(i=0,1,\cdots,k+1)$ is $k+2$ points so that $\alpha_0=a,\ \alpha_i\in[x_{i-1},x_i]_{\mathbb{T}_1}$ $(i=1,\cdots,k)$ and $\alpha_{k+1}=b$, $\beta_0=c,\ \beta_i\in[y_{i-1},y_i]_{\mathbb{T}_2}$ $(i=1,\cdots,k)$, and $\beta_{k+1}=d$;*
3. *$f:[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}\to\mathbb{R}$ is a $\Delta_1\Delta_2$ differentiable function.*

Then, we have the identity:

$$\begin{aligned}
&\int_a^b \int_c^d K(s,t,I_k,J_k)\frac{\partial^2 f(s,t)}{\Delta_2 t \Delta_1 s}\Delta_2 t \Delta_1 s \\
&= (1-\lambda)^2 \sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)f(x_i,y_j) + \frac{(1-\lambda)\lambda}{2}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,y_j) \\
&+ f(\alpha_{i+1},y_j) + f(x_i,\beta_j) + f(x_i,\beta_{j+1})\Big) - (1-\lambda)\sum_{j=0}^{k}\int_a^b (\beta_{j+1}-\beta_j) f(\sigma(s),y_j)\Delta_1 s \\
&+ \frac{\lambda^2}{4}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,\beta_j)+f(\alpha_i,\beta_{j+1})+f(\alpha_{i+1},\beta_j)+f(\alpha_{i+1},\beta_{j+1})\Big) \\
&- \frac{\lambda}{2}\sum_{j=0}^{k}\int_a^b(\beta_{j+1}-\beta_j)\Big(f(\sigma(s),\beta_j)+f(\sigma(s),\beta_{j+1})\Big)\Delta_1 s - (1-\lambda)\sum_{i=0}^{k}\int_c^d(\alpha_{i+1}-\alpha_i)f(x_i,\sigma(t))\Delta_2 t \\
&- \frac{\lambda}{2}\sum_{i=0}^{k}\int_c^d(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,\sigma(t))+f(\alpha_{i+1},\sigma(t))\Big)\Delta_2 t + \int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s, \qquad (18)
\end{aligned}$$

where $K(s,t,I_k,J_k) = K_1(s,I_k)K_2(t,J_k)$ *and:*

$$K_1(s,I_k) = \begin{cases}
s-\left(\alpha_1-\lambda\frac{\alpha_1-a}{2}\right), & s\in[a,\alpha_1)_{\mathbb{T}_1}, \\
s-\left(\alpha_1+\lambda\frac{\alpha_2-\alpha_1}{2}\right), & s\in[\alpha_1,x_1)_{\mathbb{T}_1}, \\
s-\left(\alpha_2-\lambda\frac{\alpha_2-\alpha_1}{2}\right), & s\in[x_1,\alpha_2)_{\mathbb{T}_1}, \\
\vdots & \\
s-\left(\alpha_{k-1}+\lambda\frac{\alpha_k-\alpha_{k-1}}{2}\right), & s\in[\alpha_{k-1},x_{k-1})_{\mathbb{T}_1}, \\
s-\left(\alpha_k-\lambda\frac{\alpha_k-\alpha_{k-1}}{2}\right), & s\in[x_{k-1},\alpha_k)_{\mathbb{T}_1}, \\
s-\left(\alpha_k+\lambda\frac{\alpha_{k+1}-\alpha_k}{2}\right), & s\in[\alpha_k,b]_{\mathbb{T}_1},
\end{cases}$$

$$K_2(t,J_k) = \begin{cases}
t-\left(\beta_1-\lambda\frac{\beta_1-c}{2}\right), & t\in[c,\beta_1)_{\mathbb{T}_2}, \\
t-\left(\beta_1+\lambda\frac{\beta_2-\beta_1}{2}\right), & t\in[\beta_1,y_1)_{\mathbb{T}_2}, \\
t-\left(\beta_2-\lambda\frac{\beta_2-\beta_1}{2}\right), & t\in[y_1,\beta_2)_{\mathbb{T}_2}, \\
\vdots & \\
t-\left(\beta_{k-1}+\lambda\frac{\beta_k-\beta_{k-1}}{2}\right), & t\in[\beta_{k-1},y_{k-1})_{\mathbb{T}_2}, \\
t-\left(\beta_k-\lambda\frac{\beta_k-\beta_{k-1}}{2}\right), & t\in[y_{k-1},\beta_k)_{\mathbb{T}_2}, \\
t-\left(\beta_k+\lambda\frac{\beta_{k+1}-\beta_k}{2}\right), & t\in[\beta_k,d]_{\mathbb{T}_2}.
\end{cases}$$

Theorem 6. *Suppose the function f satisfies the conditions of Lemma 2 and there exists functions γ, Γ such that $\gamma(s,t) \leq \frac{\partial^2 f(s,t)}{\Delta_2 t \Delta_1 s} \leq \Gamma(s,t)$ for all $s \in [a,b]_{\mathbb{T}_1}, t \in [c,d]_{\mathbb{T}_2}$. Then, we have the inequality:*

$$\begin{aligned}
&\Bigg|(1-\lambda)^2 \sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)f(x_i,y_j) + \frac{(1-\lambda)\lambda}{2}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,y_j) \\
&+ f(\alpha_{i+1},y_j) + f(x_i,\beta_j) + f(x_i,\beta_{j+1})\Big) - (1-\lambda)\sum_{j=0}^{k}\int_a^b(\beta_{j+1}-\beta_j)f(\sigma(s),y_j)\Delta_1 s \\
&+ \frac{\lambda^2}{4}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,\beta_j)+f(\alpha_i,\beta_{j+1})+f(\alpha_{i+1},\beta_j)+f(\alpha_{i+1},\beta_{j+1})\Big) \\
&- \frac{\lambda}{2}\sum_{j=0}^{k}\int_a^b(\beta_{j+1}-\beta_j)\Big(f(\sigma(s),\beta_j)+f(\sigma(s),\beta_{j+1})\Big)\Delta_1 s - (1-\lambda)\sum_{i=0}^{k}\int_c^d(\alpha_{i+1}-\alpha_i)f(x_i,\sigma(t))\Delta_2 t \\
&- \frac{\lambda}{2}\sum_{i=0}^{k}\int_c^d(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,\sigma(t))+f(\alpha_{i+1},\sigma(t))\Big)\Delta_2 t + \int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s \\
&- \int_a^b\int_c^d K(s,t,I_k,J_k)\frac{\gamma(s,t)+\Gamma(s,t)}{2}\Delta_2 t\Delta_1 s\Bigg| \\
&\leq \frac{1}{2}\|\Gamma-\gamma\|_\infty \Bigg\{\sum_{i=0}^{k-1}\Big[h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)+h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big) \\
&+ h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)+h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big] \\
&\times \sum_{j=0}^{k-1}\Big[h_2\Big(y_j,\beta_{j+1}-\lambda\frac{\beta_{j+1}-\beta_j}{2}\Big)+h_2\Big(\beta_{j+1},\beta_{j+1}-\lambda\frac{\beta_{j+1}-\beta_j}{2}\Big) \\
&+ h_2\Big(\beta_{j+1},\beta_{j+1}+\lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big)+h_2\Big(y_{j+1},\beta_{j+1}+\lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big)\Big]\Bigg\}. \qquad (19)
\end{aligned}$$

Proof. We observe that:

$$\begin{aligned}
&\int_a^b\int_c^d K(s,t,I_k,J_k)\Big(\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}-\frac{\gamma(s,t)+\Gamma(s,t)}{2}\Big)\Delta_2 t\Delta_1 s \\
&= \int_a^b\int_c^d K(s,t,I_k,J_k)\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}\Delta_2 t\Delta_1 s - \int_a^b\int_c^d K(s,t,I_k,J_k)\frac{\gamma(s,t)+\Gamma(s,t)}{2}\Delta_2 t\Delta_1 s. \qquad (20)
\end{aligned}$$

Under the condition that $\gamma(s,t) \leq \frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s} \leq \Gamma(s,t)$ for all $s \in [a,b]_{\mathbb{T}_1}, t \in [c,d]_{\mathbb{T}_2}$, we deduce that:

$$\left|\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}-\frac{\gamma(s,t)+\Gamma(s,t)}{2}\right| \leq \frac{1}{2}\|\Gamma-\gamma\|_\infty \qquad (21)$$

for all $s \in [a,b]_{\mathbb{T}_1}, t \in [c,d]_{\mathbb{T}_2}$. Furthermore, we have that:

$$\begin{aligned}
&\left|\int_a^b\int_c^d K(s,t,I_k,J_k)\Big(\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}-\frac{\gamma(s,t)+\Gamma(s,t)}{2}\Big)\Delta_2 t\Delta_1 s\right| \\
&\qquad\leq \int_a^b\int_c^d \Big|K(s,t,I_k,J_k)\Big|\left|\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}-\frac{\gamma(s,t)+\Gamma(s,t)}{2}\right|\Delta_2 t\Delta_1 s \qquad (22)
\end{aligned}$$

and:

$$\begin{aligned}\int_a^b |K(s,t,I_k,J_k)|\Delta_2 t\Delta_1 s = &\sum_{i=0}^{k-1}\Big[h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)+h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)\\ &+h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)+h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big]\\ &\times\sum_{j=0}^{k-1}\Big[h_2\Big(y_j,\beta_{j+1}-\lambda\frac{\beta_{j+1}-\beta_j}{2}\Big)+h_2\Big(\beta_{j+1},\beta_{j+1}-\lambda\frac{\beta_{j+1}-\beta_j}{2}\Big)\\ &+h_2\Big(\beta_{j+1},\beta_{j+1}+\lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big)+h_2\Big(y_{j+1},\beta_{j+1}+\lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big)\Big]. \end{aligned} \tag{23}$$

The desired inequality in (19) is obtained from the inequality in (22) by using (18), (20), (21), and (23). □

Theorem 7. *Suppose the function f satisfies the conditions of Lemma 2 and there exists functions γ,Γ such that $\gamma(s,t)\le\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}\le\Gamma(s,t)$ for all $s\in[a,b]_{\mathbb{T}_1}, t\in[c,d]_{\mathbb{T}_2}$. Then, we have the inequality:*

$$\begin{aligned}&\Big|(1-\lambda)^2\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)f(x_i,y_j)+\frac{(1-\lambda)\lambda}{2}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,y_j)\\ &+f(\alpha_{i+1},y_j)+f(x_i,\beta_j)+f(x_i,\beta_{j+1})\Big)-(1-\lambda)\sum_{j=0}^{k}\int_a^b(\beta_{j+1}-\beta_j)f(\sigma(s),y_j)\Delta_1 s\\ &+\frac{\lambda^2}{4}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,\beta_j)+f(\alpha_i,\beta_{j+1})+f(\alpha_{i+1},\beta_j)+f(\alpha_{i+1},\beta_{j+1})\Big)\\ &-\frac{\lambda}{2}\sum_{j=0}^{k}\int_a^b(\beta_{j+1}-\beta_j)\Big(f(\sigma(s),\beta_j)+f(\sigma(s),\beta_{j+1})\Big)\Delta_1 s-(1-\lambda)\sum_{i=0}^{k}\int_c^d(\alpha_{i+1}-\alpha_i)f(x_i,\sigma(t))\Delta_2 t\\ &-\frac{\lambda}{2}\sum_{i=0}^{k}\int_c^d(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,\sigma(t))+f(\alpha_{i+1},\sigma(t))\Big)\Delta_2 t+\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\\ &-\frac{f(b,d)-f(b,c)-f(a,d)+f(a,c)}{(b-a)(d-c)}\\ &\times\sum_{i=0}^{k-1}\Big[h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)-h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)\\ &+h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)-h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big]\\ &\times\sum_{j=0}^{k-1}\Big[h_2\Big(\beta_{j+1},\beta_{j+1}-\lambda\frac{\beta_{j+1}-\beta_j}{2}\Big)-h_2\Big(y_j,\beta_{j+1}-\lambda\frac{\beta_{j+1}-\beta_j}{2}\Big)\\ &+h_2\Big(y_{j+1},\beta_{j+1}+\lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big)-h_2\Big(\beta_{j+1},\beta_{j+1}+\lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big)\Big]\\ &-\int_a\int_c^d P(s,t,I_k,J_k)\frac{\gamma(s,t)+\Gamma(s,t)}{2}\Delta_2 t\Delta_1 s\Big|\\ &\le\frac{1}{2}\|\Gamma-\gamma\|_\infty\int_a^b\int_c^d\Big|P(s,t,I_k,J_k)\Big|\Delta_2 t\Delta_1 s,\end{aligned} \tag{24}$$

where $P(s,t,I_k,J_k)=K(s,t,I_k,J_k)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d K(s,t,I_k,J_k)\Delta_2 t\Delta_1 s$.

Proof. We first consider the following computations.

$$\begin{aligned}
&\int_a^b \int_c^d P(s,t,I_k,J_k)\Big(\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s} - \frac{\gamma(s,t)+\Gamma(s,t)}{2}\Big)\Delta_2 t\Delta_1 s \\
&= \int_a^b \int_c^d P(s,t,I_k,J_k)\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}\Delta_2 t\Delta_1 s - \int_a^b \int_c^d P(s,t,I_k,J_k)\frac{\gamma(s,t)+\Gamma(s,t)}{2}\Delta_2 t\Delta_1 s \\
&= \int_a^b \int_c^d K(s,t,I_k,J_k)\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}\Delta_2 t\Delta_1 s - \frac{1}{(b-a)(d-c)}\int_a^b \int_c^b \frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}\Delta_2 t\Delta_1 s \int_a^b \int_c^d K(s,t,I_k,J_k)\Delta_2 t\Delta_1 s \\
&- \int_a^b \int_c^d P(s,t,I_k,J_k)\frac{\gamma(s,t)+\Gamma(s,t)}{2}\Delta_2 t\Delta_1 s, \qquad (25)
\end{aligned}$$

$$\int_a^b \int_c^b \frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}\Delta_2 t\Delta_1 s = f(b,d) - f(b,c) - f(a,d) + f(a,c), \qquad (26)$$

and:

$$\begin{aligned}
\int_a^b \int_c^d K(s,t,I_k,J_k)\Delta_2 t\Delta_1 s &= \int_a^b K_1(s,I_k)\Delta_1 s \int_c^d K_2(t,J_k)\Delta_2 t \\
&= \sum_{i=0}^{k-1}\Big[h_2\Big(\alpha_{i+1},\alpha_{i+1} - \lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big) - h_2\Big(x_i,\alpha_{i+1} - \lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big) \\
&+ h_2\Big(x_{i+1},\alpha_{i+1} + \lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big) - h_2\Big(\alpha_{i+1},\alpha_{i+1} + \lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big] \\
&\times \sum_{j=0}^{k-1}\Big[h_2\Big(\beta_{j+1},\beta_{j+1} - \lambda\frac{\beta_{j+1}-\beta_j}{2}\Big) - h_2\Big(y_j,\beta_{j+1} - \lambda\frac{\beta_{j+1}-\beta_j}{2}\Big) \\
&+ h_2\Big(y_{j+1},\beta_{j+1} + \lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big) - h_2\Big(\beta_{j+1},\beta_{j+1} + \lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big)\Big]. \qquad (27)
\end{aligned}$$

Furthermore, we have that:

$$\begin{aligned}
&\Big|\int_a^b \int_c^d P(s,t,I_k,J_k)\Big(\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s} - \frac{\gamma(s,t)+\Gamma(s,t)}{2}\Big)\Delta_2 t\Delta_1 s\Big| \\
&\qquad \le \int_a^b \int_c^d \Big|P(s,t,I_k,J_k)\Big|\Big|\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s} - \frac{\gamma(s,t)+\Gamma(s,t)}{2}\Big|\Delta_2 t\Delta_1 s. \qquad (28)
\end{aligned}$$

The desired inequality in (24) is obtained from the inequality in (28) by using (18), (21), (25)–(27). □

4. Applications

In this section, we apply Theorems 4 and 6 to the continuous and discrete calculus to obtain some interesting inequalities. Similar results can be obtained from Theorems 5 and 7.

Corollary 1. *If we take $\mathbb{T} = \mathbb{R}$ in Theorem 4, then the following inequality holds:*

$$\begin{aligned}
&\Bigg|(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i) + \lambda\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\frac{f(\alpha_i)+f(\alpha_{i+1})}{2} \\
&- \int_a^b f(t)dt - \int_a^b K(t,I_k)\frac{\alpha(t)+\beta(t)}{2}dt\Bigg| \\
&\le \frac{1}{2}\|\beta-\alpha\|_\infty \sum_{i=0}^{k-1}\Bigg[\frac{\lambda^2\Big(\alpha_{i+1}-\alpha_i\Big)^2}{8} + \frac{\Big(2x_i-\lambda\alpha_i+(\lambda-2)\alpha_{i+1}\Big)^2}{8} \\
&\qquad + \frac{\Big(2x_{i+1}-\lambda\alpha_{i+2}+(\lambda-2)\alpha_{i+1}\Big)^2}{8} + \frac{\lambda^2\Big(\alpha_{i+2}-\alpha_{i+1}\Big)^2}{8}\Bigg]. \qquad (29)
\end{aligned}$$

Remark 2. *Putting $\lambda = 0$, then Corollary 1 reduces to Theorem 2.1 in* [5].

Corollary 2. *Let $\mathbb{T} = \mathbb{Z}, a = 0, b = n$ in Theorem 4, and suppose:*

(1) $\mathbb{I}_k := \{j_0, j_1, \cdots, j_k\} \subset \mathbb{Z}$, where $a = j_0 < j_1 < \cdots < j_k = b$, is a partition of the set $[0, n] \cap \mathbb{Z}$
(2) $\{\alpha_0, \alpha_1, \cdots, \alpha_{k+1}\} \subset \mathbb{Z}$ is a set of $k+2$ points such that $\alpha_0 = 0, \alpha_i \in [j_{i-1}, j_i]$ for $i = 1, 2, \cdots, k$, and $\alpha_{k+1} = n$;
(3) $f(k) = x_k$.

Then, we have the inequality,

$$\begin{aligned}
&\Bigg|(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)x_{j_i} + \lambda\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\frac{x_{\alpha_i}+x_{\alpha_{i+1}}}{2} \\
&- \sum_{j=1}^{n}x_j - \sum_{j=0}^{n-1}K(j,I_k)\frac{\alpha(j)+\beta(j)}{2}\Bigg| \\
&\le \frac{1}{4}\|\beta-\alpha\|_\infty\sum_{i=0}^{k-1}\Bigg[j_i\Big(j_i+(\lambda-2)\alpha_{i+1}-\lambda\alpha_i-1\Big) + 2\Big(\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}+\frac{1}{2}\Big)^2 - 1 \\
&+ j_{i+1}\Big(j_{i+1}+(\lambda-2)\alpha_{i+1}-\lambda\alpha_{i+2}-1\Big) + 2\Big(\lambda\frac{\alpha_{i+1}-\alpha_i}{2}-\frac{1}{2}\Big)^2 \\
&+ 2\alpha_{i+1}\Big((1-\lambda)\alpha_{i+1}+\lambda\frac{\alpha_{i+1}+\alpha_i}{2}+1\Big)\Bigg]. \qquad (30)
\end{aligned}$$

Corollary 3. *If we let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ in Theorem 6, then we have the inequality:*

$$\begin{aligned}
&\Bigg|(1-\lambda)^2 \sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)f(x_i,y_j) + \frac{(1-\lambda)\lambda}{2}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,y_j) \\
&+ f(\alpha_{i+1},y_j) + f(x_i,\beta_j) + f(x_i,\beta_{j+1})\Big) - (1-\lambda)\sum_{j=0}^{k}\int_a^b(\beta_{j+1}-\beta_j)f(s,y_j)ds \\
&+ \frac{\lambda^2}{4}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,\beta_j)+f(\alpha_i,\beta_{j+1})+f(\alpha_{i+1},\beta_j)+f(\alpha_{i+1},\beta_{j+1})\Big) \\
&- \frac{\lambda}{2}\sum_{j=0}^{k}\int_a^b(\beta_{j+1}-\beta_j)\Big(f(s,\beta_j)+f(s,\beta_{j+1})\Big)ds - (1-\lambda)\sum_{i=0}^{k}\int_c^d(\alpha_{i+1}-\alpha_i)f(x_i,t)dt \\
&- \frac{\lambda}{2}\sum_{i=0}^{k}\int_c^d(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,\sigma(t))+f(\alpha_{i+1},\sigma(t))\Big)\Delta_2 t + \int_a^b\int_c^d f(s,t)dtds \\
&- \int_a^b\int_c^d K(s,t,I_k,J_k)\frac{\gamma(s,t)+\Gamma(s,t)}{2}dtds\Bigg| \\
&\le \frac{1}{128}\|\Gamma-\gamma\|_\infty\Bigg\{\sum_{i=0}^{k-1}\Big[\lambda^2\Big(\alpha_{i+1}-\alpha_i\Big)^2 + \Big(2x_i-\lambda\alpha_i+(\lambda-2)\alpha_{i+1}\Big)^2 \\
&+ \Big(2x_{i+1}-\lambda\alpha_{i+2}+(\lambda-2)\alpha_{i+1}\Big)^2 + \lambda^2\Big(\alpha_{i+2}-\alpha_{i+1}\Big)^2\Big] \\
&\times \sum_{j=0}^{k-1}\Big[\lambda^2\Big(\beta_{j+1}-\beta_j\Big)^2 + \Big(2y_j-\lambda\beta_j+(\lambda-2)\beta_{j+1}\Big)^2 \\
&+ \Big(2y_{j+1}-\lambda\beta_{j+2}+(\lambda-2)\beta_{j+1}\Big)^2 + \lambda^2\Big(\beta_{j+2}-\beta_{j+1}\Big)^2\Big]\Bigg\}. \qquad (31)
\end{aligned}$$

Proof. The proof follows by setting $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ in Theorem 6 and using the fact that:

$$h_2(s,t) = \frac{(s-t)^2}{2}.$$

□

Remark 3. *By substituting $\lambda = 0$, then the inequality in Corollary 3 reduces to the inequality (with some minor error in the left-hand side) in ([5] Theorem 2.3).*

Corollary 4. *Let $f : \{a, a+1, \cdots, b-1, b\} \times \{c, c+1, \cdots, d-1, d\} \to \mathbb{R}$ be a function, and suppose there exist functions γ, Γ such that*

$$\gamma(s,t) \le f(s+1,t+1) - f(s+1,t) - f(s,t+1) + f(s,t) \le \Gamma(s,t)$$

for all $(s,t) \in \{a, a+1, \cdots, b-1, b\} \times \{c, c+1, \cdots, d-1, d\}$. *Then, the following inequality holds.*

$$\begin{aligned}
&\Bigg|(1-\lambda)^2 \sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)f(x_i,y_j) + \frac{(1-\lambda)\lambda}{2}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,y_j) \\
&+ f(\alpha_{i+1},y_j) + f(x_i,\beta_j) + f(x_i,\beta_{j+1})\Big) - (1-\lambda)\sum_{j=0}^{k}\sum_{s=a}^{b-1}(\beta_{j+1}-\beta_j)f(s+1,y_j) \\
&+ \frac{\lambda^2}{4}\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_{j+1}-\beta_j)(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,\beta_j) + f(\alpha_i,\beta_{j+1}) + f(\alpha_{i+1},\beta_j) + f(\alpha_{i+1},\beta_{j+1})\Big) \\
&- \frac{\lambda}{2}\sum_{j=0}^{k}\sum_{s=a}^{b-1}(\beta_{j+1}-\beta_j)\Big(f(s+1,\beta_j) + f(s+1,\beta_{j+1})\Big) - (1-\lambda)\sum_{i=0}^{k}\sum_{t=c}^{d-1}(\alpha_{i+1}-\alpha_i)f(x_i,t+1) \\
&- \frac{\lambda}{2}\sum_{i=0}^{k}\sum_{t=c}^{d-1}(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i,t+1) + f(\alpha_{i+1},t+1)\Big) + \sum_{s=a}^{b-1}\sum_{t=c}^{d-1}f(s+1,t+1) \\
&- \sum_{s=a}^{b-1}\sum_{t=c}^{d-1}K(s,t,I_k,J_k)\frac{\gamma(s,t)+\Gamma(s,t)}{2}\Delta_2 t\Delta_1 s\Bigg| \\
&\le \frac{1}{2}\|\Gamma-\gamma\|_\infty \Bigg\{\sum_{i=0}^{k-1}\Bigg[h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big) + h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big) \\
&+ h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big) + h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Bigg] \\
&\times \sum_{j=0}^{k-1}\Bigg[h_2\Big(y_j,\beta_{j+1}-\lambda\frac{\beta_{j+1}-\beta_j}{2}\Big) + h_2\Big(\beta_{j+1},\beta_{j+1}-\lambda\frac{\beta_{j+1}-\beta_j}{2}\Big) \\
&+ h_2\Big(\beta_{j+1},\beta_{j+1}+\lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big) + h_2\Big(y_{j+1},\beta_{j+1}+\lambda\frac{\beta_{j+2}-\beta_{j+1}}{2}\Big)\Bigg]\Bigg\}, \qquad (32)
\end{aligned}$$

where $h_2(s,t) = \frac{(s-t)(s-t-1)}{2}$ *for all* $s, t \in \mathbb{Z}$.

Proof. The result follows by letting $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ in Theorem 6. □

5. Conclusions

Some one-dimensional and two-dimensional Ostrowski–Grüss-type inequalities for functions whose derivatives are bounded by functions on time scales involving multiple points and a parameter have been presented. Some particular inequalities are obtained by applying Theorems 4 and 6 to the continuous and discrete calculus. Several other inequalities could be obtained by choosing different time scales with different values of the parameter λ. Some related results on the Ostrowski-type inequalities can be found in [25–35].

Author Contributions: S.K. and E.R.N contributed equally to this work.

Funding: This research received no external funding.

Acknowledgments: Many thanks to the three anonymous referees for their valuable comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Dragomir, S.S.; Wang, S. An inequality of Ostrowski–Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules. *Comput. Math. Appl.* **1997**, *33*, 15–20. [CrossRef]
2. Cheng, X.L. Improvement of some Ostrowski–Grüss type inequalities. *Comput. Math. Appl.* **2001**, *42*, 109–114. [CrossRef]
3. Matić, M.; Pečarić, J.; Ujević, N. Improvement and further generalization of inequalities of Ostrowski–Grüss type. *Comput. Math. Appl.* **2000**, *39*, 161–175. [CrossRef]

4. Feng, Q.; Meng, F. Some generalized Ostrowski–Grüss type integral inequalities. *Comput. Math. Appl.* **2012**, *63*, 652–659. [CrossRef]
5. Feng, Q.; Meng, F. New Ostrowski–Grüss type inequalities with the derivatives bounded by functions. *J. Ineq. Appl.* **2013**. [CrossRef]
6. Niezgoda, M. A new inequality of Ostrowski–Grüss type and applications some numerical quadrature rules. *Comput. Math. Appl.* **2009**, *58*, 589–596. [CrossRef]
7. Pearce, C.E.M.; Pečarić, J.; Ujević, N.; Varošanec, S. Generalizations of some inequalities of Ostrowski–Grüss type. *Math. Inequal. Appl.* **2000**, *3*, 25–34. [CrossRef]
8. Ujević, N. New bounds for the first inequality of Ostrowski–Grüss type and applications. *Comput. Math. Appl.* **2003**, *46*, 421–427. [CrossRef]
9. Hilger, S. Ein Maβkettenkalkül Mit Anwendung auf Zentrumsmannigfaltigkeiten. Ph.D. Thesis, Universität Würzburg, Würzburg, Germany, 1988.
10. Agarwal, R.; Bohner, M.; Peterson, A. Inequalities on time scales: A survey. *Math. Inequal. Appl.* **2001**, *4*, 535–557. [CrossRef]
11. Bohner, M.; Matthews, T. The Grüss inequality on time scales. *Commun. Math. Anal.* **2007**, *3*, 1–8.
12. Bohner, M.; Matthews, T. Ostrowski inequalities on time. *J. Inequal. Pure Appl. Math.* **2008**, *9*, 6.
13. Liu, W.J.; Ngô, Q.-A. An Ostrowski–Grüss type inequality on time scales. *Comput. Math. Appl.* **2009**, *58*, 1207–1210. [CrossRef]
14. Ngô, Q.-A.; Liu, W.J. A sharp Grüss type inequality on time scales and application to the sharp Ostrowski–Grüss inequality. *Commun. Math. Anal.* **2009**, *6*, 33–41.
15. Nwaeze, E.R.; Kermausuor, S. New bounds of Ostrowski–Grüss type inequality for $(k+1)$ points on time scales. *Int. J. Anal. Appl.* **2017**, *15*, 211–221.
16. Nwaeze, E.R.; Kermausuor, S.; Tameru, A.M. New time scale generalizations of the Ostrowski–Grüss type inequality for k points. *J. Ineq. Appl.* **2017**, *2017*, 245. [CrossRef] [PubMed]
17. Nwaeze, E.R.; Tameru, A.M. On weighted Montgomery identity for k points and its associates on time scales. *Abstr. Appl. Anal.* **2017**. [CrossRef]
18. Tuna, A.; Daghan, D. Generalization of Ostrowski and Ostrowski–Grüss type inequalities on time scales. *Comput. Math. Appl.* **2010**, *60*, 803–811. [CrossRef]
19. Kermausuor, S.; Nwaeze, E.R. New Generalized 2D Ostrowski type inequalities on time scales with k^2 points using a parameter. *FILOMAT* **2018**, *32*, 9.
20. Bohner, M.; Peterson, A. *Dynamic Equations on Time Scales*; Birkhäuser Boston: Boston, MA, USA, 2001.
21. Bohner, M.; Peterson, A. *Advances in Dynamic Equations on Time Series*; Birkhäuser Boston: Boston, MA, USA, 2003.
22. Xu, G.; Fang, Z.B. A Generalization of Ostrowski type inequality on time scales with k points. *J. Math. Inequal.* **2017**, *11*, 41–48.
23. Bohner, M.; Guseinov, G.S. Partial differentiation on time scales. *Dyn. Syst. Appl.* **2004**, *13*, 351–379.
24. Bohner, M.; Guseinov, G.S. Multiple integration on time scales. *Dyn. Syst. Appl.* **2005**, *14*, 579–606.
25. Choi, J.; Set, E.; Tomar, M. Certain generalized Ostrowski type inequalities for local fractional integrals. *Commun. Korean Math. Soc.* **2017**, *32*, 601–617.
26. El-Deeb, A.; Elsennary, H.A.; Nwaeze, E.R. Generalized Weighted Ostrowski, Trapezoid and Grüss Type Inequalities on Time Scales. *Fasc. Math.* **2018**, *60*, 123–144. [CrossRef]
27. Liu, W.J.; Park, J.K. A companion of Ostrowski like inequality and applications to composite quadrature rules. *J. Comput. Anal. Appl.* **2017**, *22*, 19–24.
28. Liu, W.; Gao, X.; Wen, Y. Approximating the finite Hilbert transform via some companions of Ostrowski's inequalities. *Bull. Malays. Math. Sci. Soc.* **2016**, *39*, 1499–1513. [CrossRef]
29. Liu, W.; Tuna, A. Diamond-α weighted Ostrowski type and Grüss type inequalities on time scales. *Appl. Math. Comput.* **2015**, *270*, 251–260. [CrossRef]
30. Kermausuor, S.; Nwaeze, E.R.; Torres, D.F.M. Generalized weighted Ostrowski and Ostrowski–Grüss type inequalities on time scale via a parameter function. *J. Math. Inequal.* **2017**, *11*, 1185–1199. [CrossRef]
31. Nwaeze, E.R. New integral inequalities on time scales with applications to the continuous and discrete calculus. *Commun. Appl. Anal.* **2018**, *22*, 1–17.
32. Nwaeze, E.R. Time scale versions of the Ostrowski–Grüss type inequality with a parameter function. *J. Math. Inequal.* **2018**, *12*, 531–543. [CrossRef]

33. Tuna, A.; Liu, W. New weighted Čebyšev–Ostrowski type integral inequalities on time scales. *J. Math. Inequal.* **2016**, *10*, 327–356. [CrossRef]
34. Wang, S.; Xue, Q.; Liu, W. Some new perturbed generalizations of Ostrowski-Grüss type inequalities for bounded differentiable mappings and applications. *Appl. Math. Inf. Sci.* **2013**, *7*, 2077–2081. [CrossRef]
35. Jiang, Y.; Ruzgar, H.; Liu, W.; Tuna, A. Some new generalizations of Ostrowski type inequalities on time scales involving combination of Delta-integral means. *J. Nonlinear Sci. Appl.* **2014**, *7*, 311–324. [CrossRef]

© 2018 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).