mathematics

Article

A Note on the Classical Gauss Sums

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Received: 17 November 2018; Accepted: 6 December 2018; Published: 8 December 2018

Abstract: The main purpose of this paper is to study the computational problem of one kind rational polynomials of the classical Gauss sums, and using the purely algebraic methods and the properties of the character sums mod p (a prime with $p \equiv 1 \bmod 12$) to give an exact evaluation formula for it.

Keywords: Twelfth-order character mod p; classical Gauss sums; rational polynomials; analytic method; evaluation formula

MSC: 11L05; 11L07

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1. Introduction

As usual, let $q \geq 3$ be an integer, χ be any Dirichlet character mod q. Then the classical Gauss sum $\tau(\chi)$ is defined as follows:

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) e\left(\frac{a}{q}\right),$$

where $e(y) = e^{2\pi i y}$.

This sum occupies a very vital position in the research of analytic number theory, and plenty of famous number theory problems are closely related to it. Because of this reason, many number theory experts have studied the properties of the classical Gauss sums, and obtained a series of important conclusions. For example, Z. Y. Chen and W. P. Zhang [1] provided the following result:

Let p be an odd prime with $p \equiv 1 \bmod 4$, λ be any fourth-order character mod p. Then one has the identity

$$\tau^2(\lambda) + \tau^2\left(\overline{\lambda}\right) = \sqrt{p} \cdot \sum_{a=1}^{p-1} \left(\frac{a + \overline{a}}{p}\right) = 2\sqrt{p} \cdot \alpha, \tag{1}$$

where $\left(\frac{*}{p}\right) = \chi_2$ indicates the the Legendre symbol mod p, and α denotes $\frac{1}{2}\sum_{a=1}^{p-1}\left(\frac{a+\overline{a}}{p}\right)$ for convenience.

H. Bai and J. Y. Hu [2] used identity (1) to obtain a second-order linear recursive formula for one kind rational polynomials involving the classical Gauss sums. That is, let p be a prime with $p \equiv 1 \bmod 8$, ψ be any eighth-order character mod p, and

$$F_k(p) = \frac{\tau^k(\psi)}{\tau^k(\psi^3)} + \frac{\tau^k(\psi^3)}{\tau^k(\psi)}.$$

Then we have the second-order linear recursive formula

$$F_{2k}(p) = \frac{2\alpha}{\sqrt{p}} \cdot F_{2k-2}(p) - F_{2k-4}(p),$$

where $F_0(p) = 2$, $F_2(p) = \frac{2\alpha}{\sqrt{p}}$, and α is the same as in (1).

W. P. Zhang and J. Y. Hu [3] (the different form can also be found in B. C. Berndt and R. J. Evans [4]) proved that for any prime p with $p \equiv 1 \bmod 3$ and any third-order character $\psi \bmod p$, one has the equation

$$\tau^3(\psi) + \tau^3(\overline{\psi}) = dp, \tag{2}$$

where $\tau(\psi)$ denotes the classical Gauss sums, d is uniquely determined by $4p = d^2 + 27b^2$ and $d \equiv 1 \bmod 3$.

Chen Li [5] used the identity (2) to prove the following conclusions. Let

$$U_k(p, \chi) = \frac{\tau^{3k}(\chi)}{\tau^{3k}(\overline{\chi})} + \frac{\tau^{3k}(\overline{\chi})}{\tau^{3k}(\chi)}.$$

Then for any prime p with $p \equiv 1 \bmod 12$ and any third-order character $\psi \bmod p$, one has the second-order linear recursive formula

$$U_{k+1}(p, \psi) = \frac{d^2 - 2p}{p} \cdot U_k(p, \psi) - U_{k-1}(p, \psi), \tag{3}$$

where the initial values $U_0(p, \psi) = 2$ and $U_1(p, \psi) = \frac{d^2-2p}{p}$, d is the same as in (1).

Therefore, from (3) one can deduce the general term

$$U_k(p, \psi) = \left(\frac{d^2 - 2p + 3dbi\sqrt{3}}{2p}\right)^k + \left(\frac{d^2 - 2p - 3dbi\sqrt{3}}{2p}\right)^k, \quad i^2 = -1.$$

If p be a prime with $p \equiv 7 \bmod 12$, then one has

$$U_{k+1}(p, \psi) = \frac{i\left(2p - d^2\right)}{p} \cdot U_k(p, \psi) - U_{k-1}(p, \psi), \tag{4}$$

where the initial values $U_0(p, \psi) = 2$, $U_1(p, \psi) = \frac{i\left(2p-d^2\right)}{p}$.

Similarly, from (4) one can also deduce the general term

$$U_k(p, \psi) = i^k \left(\frac{2p - d^2 + \sqrt{8p^2 - 4pd^2 + d^4}}{2p}\right)^k + i^k \left(\frac{2p - d^2 - \sqrt{8p^2 - 4pd^2 + d^4}}{2p}\right)^k.$$

Other works related to the classical Gauss sums and trigonometric sums can also be found in the references [6–15] . Here we will not list them one by one.

In this paper, as a note of [2,5], we will study a similar problem for prime p is a prime with $p \equiv 1 \bmod 12$ and any twelfth-order character $\chi \bmod p$. More specifically, let χ be any twelfth-order character mod p and

$$W_k(p, \chi) = \frac{\tau^{6k}(\chi)}{\tau^{6k}(\chi^5)} + \frac{\tau^{6k}(\overline{\chi})}{\tau^{6k}(\overline{\chi}^5)}. \tag{5}$$

Then we can use the purely algebraic methods and the properties of the classical Gauss sums to give a evaluation formula for (5). That is, we have the following:

Theorem 1. *Let p be a prime with $p \equiv 1 \bmod 12$, χ be any twelfth-order character* mod p. *Then for any positive integer k, we have the second-order linear recursive formula*

$$W_{k+1}(p,\chi) = \frac{2p^2 - 4pd^2 + d^4}{p^2} \cdot W_k(p,\chi) - W_{k-1}(p,\chi),$$

where the initial values $W_0(p,\chi) = 2$ and $W_1(p,\chi) = \frac{2p^2-4pd^2+d^4}{p^2}$, d is uniquely determined by $4p = d^2 + 27b^2$ and $d \equiv 1 \bmod 3$.

From this recursive formula we may immediately deduce the general term

$$W_k(p,\chi) = \left(\frac{\beta + \sqrt{\beta^2 - 4}}{2}\right)^k + \left(\frac{\beta - \sqrt{\beta^2 - 4}}{2}\right)^k,$$

where $\beta = \frac{2p^2-4pd^2+d^4}{p^2}$.

2. Two Simple Lemmas

In this part, two necessary lemmas in the proof process of our theorem will be given. Hereafter, we will need use many properties of the classical Gauss sums, the third-order character and the fourth-order character mod p, all of which can be found reference [16], so we will not repeat them here. First we have the following:

Lemma 1. *Let p be a prime with $p \equiv 1 \bmod 6$, χ be any sixth-order character* mod p. *Then about the classical Gauss sums $\tau(\chi)$, the following identity holds,*

$$\tau^3(\chi) + \tau^3(\overline{\chi}) = \begin{cases} p^{\frac{1}{2}}\left(d^2 - 2p\right) & \text{if } p = 12h+1, \\ -i \cdot p^{\frac{1}{2}}\left(d^2 - 2p\right) & \text{if } p = 12h+7, \end{cases}$$

where $i^2 = -1$, d is uniquely determined by $4p = d^2 + 27b^2$ and $d \equiv 1 \bmod 3$.

Proof. In fact, this is Lemma 3 of [5], so its proof is omitted. □

Lemma 2. *Let p be a prime with $p \equiv 1 \bmod 12$. Then for any twelfth-order character χ* mod p, *we obtain the identity*

$$\frac{\tau^6(\chi)}{\tau^6(\chi^5)} + \frac{\tau^6(\overline{\chi})}{\tau^6(\overline{\chi}^5)} = \frac{2p^2 - 4pd^2 + d^4}{p^2},$$

where d is the same as in (2).

Proof. Let ψ be any third-order character mod p, λ be any fourth-order character mod p. Then for any twelfth-order character χ mod p, we must have $\chi = \psi\lambda$, $\overline{\psi}\lambda$, $\psi\overline{\lambda}$ or $\overline{\psi}\overline{\lambda}$. Without loss of generality we suppose that $\chi = \psi\lambda$, then for any integer b with $(b,p) = 1$, note that we have the identity (see Theorem 7.5.4 in [17])

$$\sum_{a=0}^{p-1} e\left(\frac{ba^2}{p}\right) = \chi_2(b) \cdot \sqrt{p},$$

where $\chi_2 = \left(\frac{*}{p}\right)$ denotes the Legendre's symbol mod p. □

From the properties of Gauss sums we have

$$\begin{aligned}
&\sum_{a=0}^{p-1} \chi\left(a^2-1\right) = \frac{1}{\tau\left(\overline{\chi}\right)} \sum_{b=1}^{p-1} \overline{\chi}(b) \sum_{a=0}^{p-1} e\left(\frac{b(a^2-1)}{p}\right) \\
= &\frac{1}{\tau\left(\overline{\chi}\right)} \sum_{b=1}^{p-1} \overline{\chi}(b) e\left(\frac{-b}{p}\right) \sum_{a=0}^{p-1} e\left(\frac{ba^2}{p}\right) = \frac{\sqrt{p}}{\tau\left(\overline{\chi}\right)} \sum_{b=1}^{p-1} \overline{\chi}(b)\chi_2(b) e\left(\frac{-b}{p}\right) \\
= &\frac{\overline{\chi}(-1)\chi_2(-1)\sqrt{p}\,\tau\left(\overline{\chi}\chi_2\right)}{\tau\left(\overline{\chi}\right)} = \frac{\overline{\lambda}(-1)\sqrt{p}\,\tau\left(\overline{\psi}\lambda\right)}{\tau\left(\overline{\chi}\right)},
\end{aligned} \tag{6}$$

where we have used the identities $\psi(-1) = \chi_2(-1) = 1$ and $\overline{\lambda}\chi_2 = \lambda$.

On the other hand, note that $\overline{\chi}^2 = \overline{\psi}^2\overline{\lambda}^2 = \psi\chi_2$, we can also deduce that

$$\begin{aligned}
&\sum_{a=0}^{p-1} \chi\left(a^2-1\right) = \sum_{a=0}^{p-1} \chi\left((a+1)^2-1\right) = \sum_{a=1}^{p-1} \chi(a)\chi(a+2) \\
= &\frac{1}{\tau\left(\overline{\chi}\right)} \sum_{b=1}^{p-1} \overline{\chi}(b) \sum_{a=1}^{p-1} \chi(a) e\left(\frac{b(a+2)}{p}\right) = \frac{\tau(\chi)}{\tau\left(\overline{\chi}\right)} \sum_{b=1}^{p-1} \overline{\chi}^2(b) e\left(\frac{2b}{p}\right) \\
= &\frac{\overline{\psi}(2)\chi_2(2)\tau(\chi)\tau(\psi\chi_2)}{\tau\left(\overline{\chi}\right)}.
\end{aligned} \tag{7}$$

Combining (6) and (7) we can deduce that

$$\overline{\lambda}(-1)\sqrt{p}\,\tau\left(\overline{\psi}\lambda\right) = \overline{\psi}(2)\chi_2(2)\tau(\chi)\tau(\psi\chi_2)$$

or

$$\frac{\tau^6\left(\psi\lambda\right)}{\tau^6\left(\overline{\psi}\lambda\right)} = \frac{1}{p^3} \cdot \tau^6\left(\overline{\psi}\chi_2\right). \tag{8}$$

Similarly, we also have

$$\frac{\tau^6\left(\overline{\psi\lambda}\right)}{\tau^6\left(\psi\overline{\lambda}\right)} = \frac{1}{p^3} \cdot \tau^6\left(\psi\chi_2\right). \tag{9}$$

Note that $p \equiv 1 \bmod 12$, $\overline{\psi}\chi_2 = \overline{\psi\chi_2}$ is a sixth-order character $\bmod\, p$ and $\tau(\overline{\psi}\chi_2)\tau\left(\psi\chi_2\right) = p$, from (8), (9) and Lemma 1 we may instantly deduce the identity

$$\begin{aligned}
&\tfrac{\tau^6(\psi\lambda)}{\tau^6(\overline{\psi}\lambda)} + \tfrac{\tau^6(\overline{\psi\lambda})}{\tau^6(\psi\overline{\lambda})} = \tfrac{1}{p^3}\left(\tau^3(\overline{\psi}\chi_2) + \tau^3\left(\psi\chi_2\right)\right)^2 - \tfrac{2}{p^3}\tau^3(\overline{\psi}\chi_2)\tau^3\left(\psi\chi_2\right) \\
= &\tfrac{p(d^2-2p)^2}{p^3} - \tfrac{2}{p^3} \cdot p^3 = \tfrac{2p^2-4pd^2+d^4}{p^2}.
\end{aligned} \tag{10}$$

It is clear that if $\chi = \psi\lambda$, then $\chi^5 = \overline{\psi}\lambda$, $\overline{\chi} = \overline{\psi\lambda}$ and $\overline{\chi}^5 = \psi\overline{\lambda}$. Now our result follows from (10). This proves Lemma 2.

3. Proof of the Theorem

In this section, we complete the proof of our theorem. In fact for any prime p with $p \equiv 1 \bmod 12$ and any twelfth-order character χ mod p, from the definition of $W_k(p,\chi)$ and Lemma 2 we have $W_0(p,\chi) = 2$ and $W_1(p,\chi) = \frac{2p^2-4pd^2+d^4}{p^2} = \beta$. If integer $k \geq 1$, then we have

$$\begin{aligned} & \beta \cdot W_k(p,\chi) = \tfrac{2p^2-4pd^2+d^4}{p^2} \cdot W_k(p,\chi) \\ = & \left(\tfrac{\tau^{6k}(\chi)}{\tau^{6k}(\chi^5)} + \tfrac{\tau^{6k}(\overline{\chi})}{\tau^{6k}(\overline{\chi}^5)} \right) \cdot \left(\tfrac{\tau^{6}(\chi)}{\tau^{6}(\chi^5)} + \tfrac{\tau^{6}(\overline{\chi})}{\tau^{6}(\overline{\chi}^5)} \right) \\ = & \tfrac{\tau^{6k+6}(\chi)}{\tau^{6k+6}(\chi^5)} + \tfrac{\tau^{6k+6}(\overline{\chi})}{\tau^{6k+6}(\overline{\chi}^5)} + \tfrac{\tau^{6k-6}(\chi)}{\tau^{6k-6}(\chi^5)} + \tfrac{\tau^{6k-6}(\overline{\chi})}{\tau^{6k-6}(\overline{\chi}^5)} \\ = & W_{k+1}(p,\chi) + W_{k-1}(p,\chi). \end{aligned} \tag{11}$$

The formula (11) implies that

$$W_{k+1}(p,\chi) = \frac{2p^2-4pd^2+d^4}{p^2} \cdot W_k(p,\chi) - W_{k-1}(p,\chi)$$

with initial values $W_0(p,\chi) = 2$ and $W_1(p,\chi) = \frac{2p^2-4pd^2+d^4}{p^2} = \beta$.

This completes the proof of our theorem.

4. Conclusions

The main purpose of this paper is to use the properties of the classical Gauss sums to give an interesting second-order linear recurrence formula for one kind special Gauss sums, so we obtained an exact evaluation formula for the general term of this kind sums. Of course, the sums in our paper is quite different from the sums in references [2,5]. This provides a new way of thinking, direction and method for us to further study on more general problems.

It is obvious that if p has a twelfth-order character χ mod p, then p just has $\phi(12) = 4$ different twelfth-order characters χ, $\overline{\chi}$, χ^5 and $\overline{\chi}^5$ mod p, they all appear in (5), and conjugated configuration.

For any prime p with $p \equiv 1 \bmod 5$, it also just has $\phi(5) = 4$ different fifth-order characters χ mod p, we define $H_k(p,\chi)$ as follows:

$$H_k(p,\chi) = \frac{\tau^{rk}(\chi)}{\tau^{rk}(\chi^2)} + \frac{\tau^{rk}(\chi^4)}{\tau^{rk}(\chi^3)}.$$

Then we think there must be a positive integer r so that $H_k(p,\chi)$ has a second-order linear recurrence formula as in our theorem. In fact, for a certain positive integer r, we only need to know $H_1(p,\chi)$.

This still is an open problem. We will continue to study.

Author Contributions: All authors have equally contributed to this work. All authors read and approved the final manuscript.

Funding: This work is supported by Hainan Provincial N. S. F. (118MS041) and the N. S. F. (11771351, 11501452) of P. R. China.

Acknowledgments: The authors would like to thank the referees for their very helpful and detailed comments, which have significantly improved the presentation of this paper.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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