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# A Lichnerowicz–Obata–Cheng Type Theorem on Finsler Manifolds

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**Abstract:** Let  $(M, F, d\mu)$  be a Finsler manifold with the Ricci curvature bounded below by a positive number and constant  $S$ -curvature. We prove that, if the first eigenvalue of the Finsler–Laplacian attains its lower bound, then  $M$  is isometric to a Finsler sphere. Moreover, we establish a comparison result on the Hessian trace of the distance function.

**Keywords:** the first eigenvalue; Ricci curvature;  $S$ -curvature; Finsler sphere

**MSC:** Primary 53C60; Secondary 53C24

## 1. Introduction

Lichnerowicz [1] proved if  $(M, g)$  is a complete connected Riemannian  $n$ -manifold such that  $\text{Ric} \geq (n - 1)k > 0$ , then the first closed eigenvalue of the Laplacian is not less than  $nk$ . Soon after, the Obata rigidity theorem [2] further shows that if the first closed eigenvalue attains its lower bound, then the manifold is isometric to the Euclidean sphere  $\mathbb{S}^n(\frac{1}{\sqrt{k}})$ . Under the same curvature condition, Cheng’s maximum diameter theorem [3] states that, if the diameter  $\text{Diam}(M) = \frac{\pi}{\sqrt{k}}$ , then the manifold is isometric to  $\mathbb{S}^n(\frac{1}{\sqrt{k}})$ . Therefore, the first closed eigenvalue attaining its lower bound and the diameter attaining its maximum characterize the standard sphere.

A Finsler structure  $F$  on a manifold  $M$  is called reversible if  $F(x, y) = F(x, -y)$  for all  $x \in M$  and all  $y \in T_x M$ . Kim-Yim [4] proved Cheng’s maximum diameter theorem for reversible Finsler manifolds. Based on this work, the author [5] obtained a Lichnerowicz–Obata type theorem. On the general Finsler manifolds, Cheng’s maximum diameter theorem and the Obata type rigidity theorem were given in [6] recently.

To obtain such theorems, it is assumed that the weighted Ricci curvature satisfies  $\text{Ric}_N \geq K$  for some positive constant  $K$  and  $N \in [n, +\infty)$  in the above articles. See [5,6] for details. In this paper, we change the conditions and obtain a Lichnerowicz–Obata–Cheng type result in the following.

**Theorem 1.** *Let  $(M, F, d\mu)$  be a complete Finsler  $n$ -manifold with the Busemann–Hausdorff volume form. Assume that the Ricci curvature  $\text{Ric} \geq (n - 1)k > 0$  and the  $S$ -curvature  $S = (n + 1)cF$  for some constant  $c$ . Then, the first eigenvalue of the Finsler–Laplacian satisfies*

$$\lambda_1 \geq \frac{n(n - 1)k + (n + 1)^2 c^2 - (n + 1)c}{n - 1 + (n + 1)c}.$$

Moreover, the diameter of  $M$  attains its maximum  $\frac{\pi}{\sqrt{k}}$  and the  $S$ -curvature vanishes if the equality holds. In this case,  $(M, F, d\mu)$  is isometric to a standard Finsler sphere.

**Remark 1.** The result shows that both the Ricci curvature and the S-curvature have some influence on the first eigenvalue. However, the lower bound can not be attained unless S-curvature vanishes and the flag curvature is constant. A Finsler sphere is defined to be a complete Finsler manifold with Busemann Hausdorff volume form, constant flag curvature  $k$  and vanishing S-curvature (see [6]). We remark that there are infinitely many Finsler spheres, and, if  $F$  is reversible, the sphere is just the Euclidean sphere  $S^n(\frac{1}{\sqrt{k}})$ .

The paper is organized as follows. In Section 2, some fundamental concepts and formulas which are necessary for the present paper are given. The main theorem and the comparison theorem are then proved in Sections 3 and 4, respectively.

### 2. Preliminaries

Let  $M$  be an  $n$ -dimensional smooth manifold and  $\pi : TM \rightarrow M$  be the natural projection from the tangent bundle  $TM$ . Let  $(x, y)$  be a point of  $TM$  with  $x \in M, y \in T_xM$ , and let  $(x^i, y^j)$  be the local coordinates on  $TM$  with  $y = y^j \partial / \partial x^i$ . A Finsler metric on  $M$  is a function  $F : TM \rightarrow [0, +\infty)$  satisfying the following properties:

- (i) *Regularity:*  $F(x, y)$  is smooth in  $TM \setminus 0$ ;
- (ii) *Positive homogeneity:*  $F(x, \lambda y) = \lambda F(x, y)$  for  $\lambda > 0$ ;
- (iii) *Strong convexity:* The fundamental quadratic form

$$g := g_{ij}(x, y) dx^i \otimes dx^j, \quad g_{ij} := \frac{\partial^2 F^2}{\partial y^i \partial y^j}$$

is positive definite.

Given two linearly independent vectors  $V, W \in T_xM \setminus 0$ , the flag curvature is defined by

$$K(V, W) := \frac{g_V(R^V(V, W)W, V)}{g_V(V, V)g_V(W, W) - g_V(V, W)^2},$$

where  $R^V$  is the Chern curvature:

$$R^V(X, Y)Z = D_X^V D_Y^V Z - D_Y^V D_X^V Z - D_{[X, Y]}^V Z.$$

Then, the Ricci curvature for  $(M, F)$  is defined as

$$\text{Ric}(V) = \sum_{\alpha=1}^{n-1} K(V, e_\alpha),$$

where  $e_1, \dots, e_{n-1}, \frac{V}{F(V)}$  form an orthonormal basis of  $T_xM$  with respect to  $g_V$ .

Let  $(M, F, d\mu)$  be a Finsler  $n$ -manifold. Given a vector  $V \in T_xM$ , let  $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$  be a geodesic with  $\gamma(0) = x, \dot{\gamma}(0) = V$ . Define

$$\dot{S}(V) := F^{-2}(V) \frac{d}{dt} [S(\gamma(t), \dot{\gamma}(t))]_{t=0},$$

where  $S(V)$  denotes the S-curvature at  $(x, V)$ . The weighted Ricci curvature of  $(M, F, d\mu)$  is defined by (see [7])

$$\begin{cases} \text{Ric}_n(V) := \begin{cases} \text{Ric}(V) + \dot{S}(V), & \text{for } S(V) = 0, \\ -\infty, & \text{otherwise,} \end{cases} \\ \text{Ric}_N(V) := \text{Ric}(V) + \dot{S}(V) - \frac{S(V)^2}{(N-n)F(V)^2}, \quad \forall N \in (n, \infty), \\ \text{Ric}_\infty(V) := \text{Ric}(V) + \dot{S}(V). \end{cases}$$

For a smooth function  $f$ , the *gradient vector* of  $f$  at  $x$  is defined by  $\nabla f(x) := \mathcal{L}^{-1}(df)$ , where  $\mathcal{L} : T_x M \rightarrow T_x^* M$  is the Legendre transform. Let  $V = V^i \frac{\partial}{\partial x^i}$  be a smooth vector field on  $M$ . The *divergence* of  $V$  with respect to an arbitrary volume form  $d\mu = \sigma(x)dx$  is defined by

$$\operatorname{div} V := \sum_{i=1}^n \left( \frac{\partial V^i}{\partial x^i} + V^i \frac{\partial \log \sigma}{\partial x^i} \right).$$

Then, the *Finsler–Laplacian* of  $f$  can be defined by

$$\Delta f := \operatorname{div}(\nabla f).$$

Set  $\mathcal{U} = \{x \in M \mid \nabla f(x) \neq 0\}$ . The Hessian of  $f$  is defined by (see [8])

$$H(f)(X, Y) := XY(f) - \nabla_X^{\nabla f} Yf, \quad \forall X, Y \in TM|_{\mathcal{U}}.$$

We also denote  $H(f) \triangleq \nabla^2 f$ .

### 3. The Proof of the Main Theorem

In this section, we shall prove the main result of the paper. Let us first give the following theorem.

**Theorem 2.** *Let  $(M, F, d\mu)$  be a complete Finsler  $n$ -manifold with Ricci curvature  $\operatorname{Ric} \geq (n - 1)k > 0$  and constant  $S$ -curvature  $S = (n + 1)cF$ . Then, the first eigenvalue of the Finsler–Laplacian satisfies*

$$\lambda_1 \geq \frac{n(n - 1)k + (n + 1)^2 c^2 - (n + 1)c}{n - 1 + (n + 1)c}.$$

Moreover, the diameter of  $M$  attains its maximum  $\frac{\pi}{\sqrt{k}}$  and the  $S$ -curvature vanishes if the equality holds.

**Proof.** Recall that, for a smooth function  $f$ , the following equation (see [9]) holds pointwise on  $\mathcal{U}$

$$\Delta^{\nabla f} \left( \frac{F(\nabla f)^2}{2} \right) - D(\Delta f)(\nabla f) = F(\nabla f)^2 \operatorname{Ric}_{\infty}(\nabla f) + \|\nabla^2 f\|_{HS(\nabla f)}^2.$$

Since  $S$ -curvature is constant,  $\dot{S} = 0$  and  $\operatorname{Ric}_{\infty}(\nabla f) = \operatorname{Ric}(\nabla f)$ , and thus

$$\Delta^{\nabla f} \left( \frac{F(\nabla f)^2}{2} \right) - D(\Delta f)(\nabla f) = F(\nabla f)^2 \operatorname{Ric}(\nabla f) + \|\nabla^2 f\|_{HS(\nabla f)}^2. \tag{1}$$

Now, let  $f$  be a first eigenfunction of the Finsler–Laplacian with the first eigenvalue  $\lambda_1$ , namely  $\Delta f = -\lambda_1 f$ . Integrating Equation (1) and using the divergence lemma on  $M$ , we obtain

$$\int_M \lambda_1 F(\nabla f)^2 d\mu \geq \int_M \left( F(\nabla f)^2 \operatorname{Ric}(\nabla f) + \|\nabla^2 f\|_{HS(\nabla f)}^2 \right) d\mu. \tag{2}$$

For  $x \in \mathcal{U}$ , choose a  $g_{\nabla f}$ -orthogonal basis  $e_1, e_2, \dots, e_n \in T_x M$  such that  $(f_{ij}) := (H(f)(e_i, e_j))$  is a diagonal matrix. If the  $S$ -curvature is constant, it follows from [8] that

$$\Delta f = \operatorname{tr}_{\nabla f} H(f) - (n + 1)cF(\nabla f). \tag{3}$$

Then, by Equation (3), we have

$$\begin{aligned} \|\nabla^2 f\|_{HS(\nabla f)}^2 &= \sum_{i=1}^n f_{ii}^2 \geq \frac{1}{n} \left( \sum_{i=1}^n f_{ii} \right)^2 = \frac{1}{n} \left( \text{tr}_{\nabla f} H(f) \right)^2 \\ &= \frac{1}{n} (\Delta f + (n+1)cF(\nabla f))^2 \\ &= \frac{1}{n} [(\Delta f)^2 + 2(n+1)c\Delta f F(\nabla f) + (n+1)^2 c^2 F(\nabla f)^2] \\ &\geq \frac{1}{n} [(\Delta f)^2 - (n+1)c\lambda_1(f^2 + F(\nabla f)^2) + (n+1)^2 c^2 F(\nabla f)^2]. \end{aligned} \tag{4}$$

From the fact

$$\Delta^{\nabla f} f^2 = \text{div}(\nabla^{\nabla f} f^2) = \text{div}(2f\nabla f) = 2f\Delta f + 2F(\nabla f)^2,$$

we get

$$(\Delta f)^2 = -\lambda_1 f \Delta f = \lambda_1 \left( F(\nabla f)^2 - \frac{1}{2} \Delta^{\nabla f} f^2 \right),$$

and thus, by using (4), we have

$$\begin{aligned} \|\nabla^2 f\|_{HS(\nabla f)}^2 &\geq \frac{1}{n} \left[ \lambda_1 \left( F(\nabla f)^2 - \frac{1}{2} \Delta^{\nabla f} f^2 \right) \right. \\ &\quad \left. - (n+1)c\lambda_1(f^2 + F(\nabla f)^2) + (n+1)^2 c^2 F(\nabla f)^2 \right]. \end{aligned} \tag{5}$$

Therefore, Equation (2) and Equation (5) and the assumption of Theorem 2 yield

$$\int_M \left( \frac{n-1+(n+1)c}{n} \lambda_1 - (n-1)k - \frac{(n+1)^2 c^2 - (n+1)c}{n} \right) F(\nabla u)^2 d\mu \geq 0,$$

which means that

$$\lambda_1 \geq \frac{n(n-1)k + (n+1)^2 c^2 - (n+1)c}{n-1+(n+1)c}.$$

Here, we use  $\lambda_1 = \frac{\int_M F(\nabla f)^2 d\mu}{\int_M f^2 d\mu}$  for a first eigenfunction,  $f$ . If the equality holds, then all of the relevant inequalities should become the equalities. Particularly, from (4), we have  $S = 0$ . In addition,  $f_{ii} = -kf$  for any  $1 \leq i \leq n$ .

Let

$$\varphi(x) = F(\nabla f)^2 + kf^2.$$

Write  $f_i = df(e_i)$ . Then,

$$\nabla f = \sum_{i=1}^n g_{\nabla f}(\nabla f, e_i) e_i = \sum_{i=1}^n f_i e_i,$$

and thus

$$\begin{aligned} d\varphi(e_i) &= dg_{\nabla f}(\nabla f, \nabla f)(e_i) + 2kfd f(e_i) \\ &= 2g_{\nabla f}(\nabla_{e_i}^{\nabla f} \nabla f, \nabla f) + 2C_{\nabla f}(\nabla f, \nabla f, \nabla_{e_i}^{\nabla f} \nabla f) + 2kff_i \\ &= 2H(f)(\nabla f, e_i) + 2kff_i \\ &= 2f_i(f_{ii} + kf) = 0, \end{aligned}$$

which implies that  $\varphi$  is constant on  $\mathcal{U}$ . Now, if  $M \setminus \mathcal{U}$  has zero measure, then, by continuity, we conclude that  $\varphi$  is constant on  $M$ . If there is an open set  $\mathcal{V} \subset M \setminus \mathcal{U}$ , then  $f$  is constant on  $\mathcal{V}$  and so is  $\varphi$ . In this case, we can expand  $\mathcal{V}$  such that  $M \setminus \{\mathcal{U} \cup \mathcal{V}\}$  is of zero measure. Then, by continuity, again we obtain that  $\varphi$  is constant on  $M$ .

Next, we suppose that  $f$  takes its maximum and minimum at  $p$  and  $q$ , respectively. Then,  $\nabla f(p) = \nabla f(q) = 0$ , and thus  $\varphi(p) = kf^2(p) = \varphi(q) = kf^2(q)$ . Notice that  $f$  is not constant on  $M$ . Therefore, we have  $f(p) = -f(q)$ . Without loss of generality, we assume  $f(p) = 1$  and  $f(q) = -1$ . In this case,  $\varphi \equiv k$  on  $M$ . Let  $\gamma(s)$  be the minimal regular geodesic of  $(M, F)$  from  $p$  to  $q$  with the tangent vector  $\dot{\gamma}(s)$ . Then, we have

$$\frac{F(\nabla f)}{\sqrt{1-f^2}} = \sqrt{k}$$

along  $\gamma(s)$ . Let  $d_M$  denote the diameter of  $(M, F)$ . We then obtain

$$\sqrt{k}d_M \geq \sqrt{k} \int_{\gamma} F(\dot{\gamma}) ds = \int_{\gamma} F(\dot{\gamma}) \frac{F(\nabla f)}{\sqrt{1-f^2}} ds.$$

From  $|\frac{df}{ds}| = |g_{\nabla f}(\nabla f, \dot{\gamma})| \leq F(\dot{\gamma})F(\nabla f)$ , one gets

$$\int_{\gamma} F(\dot{\gamma}) \frac{F(\nabla f)}{\sqrt{1-f^2}} ds \geq \int_{-1}^1 \frac{df}{\sqrt{1-f^2}} = \pi.$$

Combining two inequalities above, it follows that  $d_M \geq \frac{\pi}{\sqrt{k}}$ . On the other hand, by Myers' Lemma, we obtain  $d_M \leq \frac{\pi}{\sqrt{k}}$ . Thus,  $d_M = \frac{\pi}{\sqrt{k}}$ .  $\square$

If we remove the assumption on the  $S$ -curvature, and strengthen the other condition, the diameter can also attain its maximum  $\frac{\pi}{\sqrt{k}}$ . This means that the restriction on the  $S$ -curvature is not a necessary condition.

**Proposition 1.** *Let  $(M, F)$  be a complete Finsler  $n$ -manifold with the Ricci curvature  $\text{Ric} \geq (n - 1)k > 0$ . If there exists a nonzero differential function  $f$  such that*

$$H(f)(X, X) = -kf g_{\nabla f}(X, X)$$

for any  $X \in T_{\mathcal{U}}M$ , then the diameter of  $(M, F)$  attains its maximum  $\frac{\pi}{\sqrt{k}}$ .

**Proof.** Using the condition, we get  $f_{ii} = -kf$  by a straightforward computation. The remaining proof is the same as above.  $\square$

**Remark 2.** *In a Finsler manifold with constant flag curvature  $k$ , we choose  $f = -\cos(\sqrt{k}r)$ . Then,  $H(f)(X, X) = -kf g_{\nabla f}(X, X)$ . By Proposition 1, the diameter attains its maximum  $\frac{\pi}{\sqrt{k}}$ . Proof of Theorem 1. It follows from Theorem 2 and the following lemma directly.*

**Lemma 1.** *Let  $(M, F, d\mu)$  be a complete connected Finsler  $n$ -manifold with the Busemann–Hausdorff volume form. If the weighted Ricci curvature satisfies  $\text{Ric}_n \geq (n - 1)k > 0$  and  $\text{Diam}(M) = \frac{\pi}{\sqrt{k}}$ , then  $(M, F)$  is isometric to a standard Finsler sphere (Ref. [6]).*

#### 4. A Comparison Theorem on the Hessian

Under different curvature conditions, the Laplacian comparison theorem in Finsler geometry was obtained by Wu-Xin [8], Ohta [10] and other geometers; however, none of these geometers have studied the rigidity phenomenon if the equality holds.

In this section, we will make a study of comparison theorem on the Hessian trace because it has important relationship with the Laplacian. We not only give an estimate on the Hessian trace of the distance function, but we also characterize the properties when the equality holds.

**Theorem 3.** *Let  $(M, F)$  be a Finsler  $n$ -manifold. If the Ricci curvature satisfies  $\text{Ric} \geq (n - 1)k$ , then the Hessian trace of the distance function  $r(x) = d_F(p, x)$  from any given point  $p \in M$  can be estimated as follows, whenever  $r$  is smooth:*

$$\text{tr}_{\nabla r} H(r) \leq (n - 1)\text{ct}_k(r).$$

The equality holds if and only if the radial flag curvature  $K(\dot{\gamma}(t); \cdot) \equiv k$  along the geodesic  $\gamma(t)$ , satisfying  $\gamma(0) = p$ . In this case, any Jacobi field  $J(t)$  orthogonal to  $\dot{\gamma}(t)$  can be written as  $J(t) = s_k(t)E(t)$ , where  $E(t) \perp \dot{\gamma}$  is a parallel vector field along  $\gamma$ .

**Remark 3.** *To get such a comparison result, we do not need to choose any volume form. If  $(M, F)$  is a Riemannian manifold or a Finsler manifold with vanishing  $S$ -curvature, then  $\text{tr}_{\nabla r} H(r) = \Delta r$ . In this situation, Theorem 3 is nothing but the Laplacian comparison theorem. Here,*

$$s_k = \begin{cases} \frac{1}{\sqrt{k}} \sin(\sqrt{kt}), & k > 0, \\ t, & k=0, \\ \frac{1}{\sqrt{-k}} \sinh(\sqrt{-kt}), & k < 0, \end{cases} \quad \text{ct}_k(r) = \begin{cases} \sqrt{k} \cot(\sqrt{kr}) & k > 0, \\ \frac{1}{r} & k = 0, \\ \sqrt{-k} \coth(\sqrt{-kr}) & k < 0. \end{cases}$$

**Proof.** The first part follows the classical proof. See also [8] in the Finsler setting. Suppose that  $r(x) = d_F(p, x)$  is smooth at  $q$ , and  $e_1, e_2, \dots, e_n = \nabla r(q)$  is a  $g_{\nabla r}$ -orthogonal frame at  $q$ . Then, we get local vector fields  $E_1, \dots, E_{n-1}, E_n = \nabla r$  by parallel transport along the geodesic  $\gamma$  such that  $E_i(q) = e_i$  for any  $1 \leq i \leq n$ . Let  $J_i$  be the unique Jacobi field satisfying  $J_i(0) = 0, J_i(r(q)) = E_i(q)$ . Set

$$W_i(t) = \frac{s_k(t)}{s_k(r(q))} E_i(t).$$

Then,

$$W_i(0) = J_i(0) = 0, W_i(q) = J_i(r(q)).$$

By the Index Lemma, we have

$$\begin{aligned} \text{tr}_{\nabla r} H(r)|_q &= \sum_{i=1}^n H(r)(e_i, e_i) = \sum_{i=1}^{n-1} I_\gamma(J_i, J_i) \leq \sum_{i=1}^{n-1} I_\gamma(W_i, W_i) \\ &= \frac{1}{s_k(r(q))^2} \int_0^{r(q)} \{(n - 1)s'_k(t)^2 - \text{Ric}(\dot{\gamma}(t))s_k(t)^2\} dt \\ &\leq \frac{(n - 1)}{s_k(r(q))^2} \int_0^{r(q)} \{s'_k(t)^2 - ks_k(t)^2\} dt \\ &= (n - 1)\text{ct}_k(r(q)). \end{aligned}$$

If the equality holds, then

$$I_\gamma(J_i, J_i) = I_\gamma(W_i, W_i)$$

for any  $1 \leq i \leq n - 1$ , which means that  $J_i = W_i$ . Thus, any Jacobi field orthogonal to  $\dot{\gamma}(t)$  can be written as

$$J(t) = s_k(t) \sum_{i=1}^{n-1} a_i E_i(t) =: s_k(t)E(t).$$

Substituting it into the Jacobi equality

$$\ddot{J} + R(\dot{\gamma}, \dot{J})\dot{\gamma} = 0,$$

and noting that  $E(t)$  is parallel along  $\gamma$ , we obtain

$$s_k(t)''E(t) + s_k(t)R(\dot{\gamma}, E(t))\dot{\gamma} = 0.$$

This gives that

$$K(\dot{\gamma}; E) = \frac{g_{\dot{\gamma}}(R(\dot{\gamma}, E(t))\dot{\gamma}, E(t))}{F(E(t))^2} = -\frac{s_k''(t)^2}{s_k(t)} = k$$

for any  $E \perp \dot{\gamma}(t)$ .  $\square$

From Theorem 3, it is easy to obtain the following result.

**Corollary 1.** *The conditions are the same as in Theorem 3. Let  $f : [a, b] \rightarrow \mathbb{R}$  be a smooth function with  $f' \geq 0$ . Then,*

$$\text{tr}_{\nabla_r} H(f(r)) \leq f''(r) + (n-1)f'(r)\text{ct}_k(r).$$

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