mathematics

Article

n-Derivations and (n,m)-Derivations of Lattices

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Received: 25 October 2018; Accepted: 4 December 2018; Published: 6 December 2018

Abstract: In this paper, firstly, as a generalization of derivations on a lattice, the notion of n-derivation is introduced and some fundamental properties are investigated. Secondly, the concept of (n,m)-derivation-homomorphism on lattices is described and important and characteristic properties are given.

Keywords: derivation; n-derivation; (n,m)-derivation; lattice

1. Introduction

A lattice, which has many real-world applications, such as information theory [1] and cryptanalysis [2], is defined as the following [3]:

If a nonempty set L endowed with operations "$\wedge$" and "$\vee$" satisfies the following conditions, then L is called a lattice.

(A) $x \wedge x = x, x \vee x = x$;
(B) $x \wedge y = y \wedge x, x \vee y = y \vee x$;
(C) $(x \wedge y) \wedge z = x \wedge (y \wedge z), (x \vee y) \vee z = x \vee (y \vee z)$;
(D) $(x \wedge y) \vee x = x, (x \vee y) \wedge x = x$,

for all $x, y, z \in L$.

Additionally, a lattice has the following properties:

A binary relation "$\leq$" in a lattice L is defined by

$$x \leq y \text{ if and only if } x \wedge y = x, x \vee y = y.$$

Lemma 1. *Let* $(L, \wedge, \vee)$ *be a lattice. Then,* $(L, \leq)$ *is a poset and for any* $x, y \in L, x \wedge y$ *is the g.l.b. of {x,y} and* $x \vee y$ *is the* l.u.b. *of {x,y}.*

A lattice L is called distributive if the identity (E) or (F) holds:

(E) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
(F) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

In any lattice, the conditions (E) and (F) are equivalent.

The notion of derivation which comes in analogy with Leibniz's formula for derivations in a ring introduced from the analytic theory is helpful to the research of structure and property in the algebraic system. Many authors have studied derivations in rings, near-rings, BCI-algebras, lattices, and various algebraic structures [4–10]. Multiderivations (e.g., biderivation, 3-derivation, or n-derivation, in general) have been explored in (semi-) rings [11–14]. Some researchers have studied n-derivations, (n,m)-derivations, and higher derivations on various algebraic structures, such as triangular rings, von Neumann algebras, lattice ordered rings, and J-subspace lattice algebras [15–21].

The notion of lattice derivation was introduced and developed by Szasz [9] and was also employed to investigate some further properties by Ferrari [22] and Xin [10]. The definition of derivation on a lattice L is given as the following:

Let $d : L \to L$ be a function. d is called a derivation on L, if it satisfies the equality $d(x \wedge y) = (x \wedge d(y)) \vee (d(x) \wedge y)$ for all $x, y \in L$.

In [10], Xin et al. introduced the notion of derivation for a lattice and discussed some related properties. They gave some equivalent conditions under which derivation is an isotone for lattices with the greatest element, modular lattices, and distributive lattices, and characterized modular lattices and distributive lattices by isotone derivation. Moreover, they proved that if d is an isotone derivation of a lattice L, the fixed set $Fix_d(L)$ is an ideal of L and D(L) is isomorphic to L in a distributive lattice L.

Until now, researchers have studied the generalization of derivations on lattices. Some of these studies are as follows:

- A function $d : L \to L$ is called an f-derivation on L if there exists a function $f : L \to L$, such that $d\,(x \wedge y) = (d(x) \wedge f(y)) \vee (f(x) \wedge d(y))$ [23];
- Let $D : L \times L \to L$ be a symmetric mapping. D is called a symmetric bi-derivation on L, if it satisfies the equality: $D(x \wedge y,\ z) = (D(x, z) \wedge y) \vee (x \wedge D(y, z))$ [24];
- A function $D : L \to L$ is called a generalized derivation on L if there exists a derivation $d : L \to L$, such that $D\,(x \wedge y) = (D(x) \wedge y) \vee (x \wedge d(y))$ [25];
- A function $d : L \to L$ is called a generalized (f, g)-derivation of L if there exist functions $f, g : L \to L$, such that $d(x \wedge y) = (d(x) \wedge f(y)) \vee (g(x) \wedge d(y))$ [26];
- Let $D : L \times L \times L \to L$ be a permuting mapping [27]. D is called permuting tri-(f,g)-derivation of L if there exist functions $f, g\ :\ L \to L$, such that D satisfies the equation

$$D(x \wedge w,\ y,\ z) = (D(x, y, z) \wedge f(w)) \vee (g(x) \wedge D(w, y, z));$$

- Let $I = \{0, 1, 2, \ldots, t\}$ or $I = \mathbb{N} = \{0, 1, 2, \ldots\}$ (with $t \to \infty$ in this case) and $D = \{d_n\}_{n \in I}$ be a family of mappings of L such that $d_0 = id_L$. D is said to be a higher derivation of length t on L if, for every $n \in I$ and $a, b \in L$, D satisfies the equality $d_n(a \wedge b) = \bigvee_{i+j=n} (d_i(a) \wedge d_j(b))$ ([18]).

In the above studies about the generalization of the derivations on lattices, researchers discussed some related properties. Modular lattices and distributive lattices were characterized by isotone derivation. Moreover, they showed that the fixed set $Fix_D(L)$ where D is a generalized derivation is an ideal of L.

Our research was mainly motivated by the studies in [19–21] on n-derivations and (n,m)-derivations on various algebraic structures. In this paper, firstly, as a generalization of derivations on a lattice, the notion of n-derivation is introduced and some important properties are investigated. Secondly, we consider a kind of multimapping that is either a derivation or a $\wedge$-homomorphism called (n,m)-derivation-homomorphism on lattices. Furthermore, its important and characteristic properties will be described. In this paper, the investigation of the properties of derivations of lattices is considered by a purely theoretical point of view.

2. n-Derivations on Lattices

In the following, L will denote a lattice.

A mapping $D : L \times L \to L$ is called symmetric if $D(x, y) = D(y, x)$ holds for all $x, y \in L$. A mapping $d : L \to L$ defined by d(x) = D(x,x) is called a trace of D where D is a symmetric mapping.

Let $n \geq 2$ be a fixed positive integer and $L^n = \underbrace{L \times L \times \ldots \times L}_{n\text{ times}}$. A map $\Delta : L^n \to L$ is said to be symmetric (or permutting) if the equation $\Delta(x_1, x_2, \ldots, x_n) = \Delta(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$ holds for all $x_i \in L$ and for every permutation $\{\pi(1), \pi(2), \ldots, \pi(n)\}$.

The following definition introduces the notion of n-derivation for a lattice. This definition generalizes the notions of derivation, biderivation, and 3-derivation on lattices.

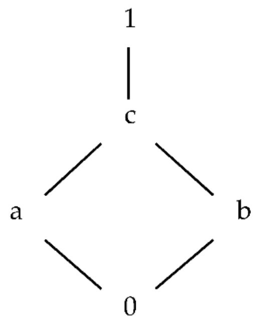
Definition 1. *The map* $\Delta : L^n \to L$ *will be called an n-derivation if* Δ *is a derivation according to all components; that is,*

$$\begin{array}{c} \Delta(x_1 \wedge a, x_2, \ldots, x_n) = (\Delta(x_1, x_2, \ldots, x_n) \wedge a) \vee (x_1 \wedge \Delta(a, x_2, \ldots, x_n) \\ \Delta(x_1, x_2 \wedge a, x_3, \ldots, x_n) = (\Delta(x_1, \ldots, x_n) \wedge a) \vee (x_2 \wedge \Delta(x_1, x_2, x_3, \ldots, a)) \\ \ldots \\ \Delta(x_1, \ldots, x_{n-1}, x_n \wedge a) = (\Delta(x_1, \ldots, x_n) \wedge a) \vee (x_n \wedge \Delta(x_1, \ldots, x_{n-1}, a)) \end{array}$$

are valid for all x_i and $a \in L$.

Remark 1. *Of course, an 1-derivation is a derivation and 2-derivation is a biderivation. If* Δ *is symmetric, then the above equalities are equivalent to each other. In this case, if* $n = 2$, Δ *is symmetric biderivation, and if* $n = 3$, Δ *is permutting tri-derivation.*

Example 1. *Let L = {0,a,b,c,1} be a lattice with the following figure:*



We define a mapping Δ *on L by* $\Delta(x_1, x_2, \ldots, x_n) = x_1 \wedge x_2 \wedge \ldots \wedge x_n \wedge a$. *It is easily seen that* Δ *is an n-derivation on L.*

Example 2. *Consider the lattice* ($\mathbb{N}$; *max,min) whose associated poset is the chain* ($\mathbb{N}$,$\leq$) *with the usual total order. We define a mapping* Δ *on* $\mathbb{N}^n$ *by* $\Delta(x_1, x_2, \ldots, x_n) = \min(x_1, \ldots, x_n, a)$ *for some* $a \in \mathbb{N}$. *It is easily seen that* Δ *is an n-derivation.*

Recently, Çeven [24] defined symmetric biderivation and its trace for a lattice and proved some results. In the following, we have extended his definitions and theorems to n-derivations of lattices.

Definition 2. *Let* Δ *be an n-derivation on L. A map* $\delta : L \to L$ *defined by* $\delta(x) = \Delta(x, x, \ldots, x)$ *is called the trace of* Δ.

Proposition 1. *Let* Δ *be an n-derivation on L with trace* δ *. Then,* $\delta(x) \leq x$ *for all* $x \in L$.

Proof. Note that

$$\begin{aligned} \delta(x) &= \Delta(x, x, \ldots, x) \\ &= \Delta(x \wedge x, x, \ldots, x) \\ &= (\Delta(x, x, \ldots, x) \wedge x) \vee (x \wedge \Delta(x, x, \ldots, x)) \\ &= \Delta(x, x, \ldots, x) \wedge x \\ &= \delta(x) \wedge x, \end{aligned}$$

and so $\delta(x) \leq x$. □

Proposition 2. *Let* Δ *be an n-derivation on L. Then,* $\Delta(x_1, x_2, \ldots, x_n) \leq x_i$ *for all* $i \in \{1, 2, \ldots, n\}$.

Proof. Since

$$\begin{aligned} \Delta(x_1, x_2, \ldots, x_n) &= \Delta(x_1 \wedge x_1, x_2, \ldots, x_n) \\ &= (\Delta(x_1, x_2, \ldots, x_n) \wedge x_1) \vee (x_1 \wedge \Delta(x_1, x_2, \ldots, x_n)) \\ &= x_1 \wedge \Delta(x_1, x_2, \ldots, x_n) \end{aligned}$$

we have $\Delta(x_1, x_2, \ldots, x_n) \leq x_1$. Similarly, it is obtained that $\Delta(x_1, x_2, \ldots, x_n) \leq x_i$ for all $i \in \{1, 2, \ldots, n\}$. □

Remark 2. *In Propositions 1 and 2, we obtained interesting properties for any n-derivation* Δ *and its trace* δ*; that is,* $\Delta(x_1, x_2, \ldots, x_n) \leq x_i$ *for all* $i \in \{1, 2, \ldots, n\}$ *and* $\delta(x) \leq x$*. This means that any n-derivation and its trace in lattices are contraction mappings. By the principle of contraction mappings, they must have fixed points. The properties of the fixed point set of n-derivation and its trace for a lattice will be discussed in future work.*

Corollary 1. *Let* Δ *be an n-derivation on L. Then,* $\Delta(x_1, x_2, \ldots, x_n) \leq \bigwedge\limits_{i \in I} x_i$ *where* $I = \{1, 2, \ldots, n\}$.

The definition of a joinitive mapping for a lattice was first given in [24]. We defined the notion of n-joinitivity for n-derivation of a lattice as the following and obtained some related results.

Definition 3. *The map* $\Delta : L^n \to L$ *is called an n-joinitive mapping* ($\vee-$*homomorphism) if* Δ *is a joinitive mapping according to all components; that is,*

$$\begin{gathered} \Delta(x_1 \vee a, x_2, \ldots, x_n) = \Delta(x_1, x_2, \ldots, x_n) \vee \Delta(a, x_2, \ldots, x_n), \\ \Delta(x_1, x_2 \vee a, \ldots, x_n) = \Delta(x_1, x_2, \ldots, x_n) \vee \Delta(x_1, a, \ldots, x_n), \\ \ldots \\ \Delta(x_1, x_2, \ldots, x_n \vee a) = \Delta(x_1, x_2, \ldots, x_n) \vee \Delta(x_1, x_2, \ldots, a) \end{gathered}$$

are valid for all x_i *and* $a \in L$.

Theorem 1. *Let* Δ *be a permutting and joinitive n-derivation and* δ *be the trace of* Δ*. Then,*

$$\delta(x \vee y) = \delta(x) \vee \Delta(\underbrace{x, x, \ldots, x}_{n-1 \text{ times}}, y) \vee \Delta(\underbrace{x, x, \ldots, x}_{n-2 \text{ times}}, y, y) \vee \ldots \vee \Delta(x, \underbrace{y, y, \ldots, y}_{n-1 \text{ times}}) \vee \delta(y).$$

Proof. Since Δ is a permutting, joinitive n-derivation and by properties of lattice, we have

$$\begin{aligned} \delta(x \vee y) &= \Delta(\underbrace{x \vee y, x \vee y, \ldots, x \vee y}_{n \text{ times}}) \\ &= \Delta(x, \underbrace{x \vee y, x \vee y, \ldots, x \vee y}_{n-1 \text{ times}}) \vee \Delta(y, \underbrace{x \vee y, x \vee y, \ldots, x \vee y}_{n-1 \text{ times}}) \\ &= \Delta(x, x, \underbrace{x \vee y, x \vee y, \ldots, x \vee y}_{n-2 \text{ times}}) \vee \Delta(x, y, \underbrace{x \vee y, x \vee y, \ldots, x \vee y}_{n-2 \text{ times}}) \vee \Delta(y, y, \underbrace{x \vee y, x \vee y, \ldots, x \vee y}_{n-2 \text{ times}}) \\ &= \ldots \\ &= \Delta(\underbrace{x, x, \ldots, x}_{n \text{ times}}) \vee \Delta(\underbrace{x, x, \ldots, x}_{n-1 \text{ times}}, y) \vee \Delta(\underbrace{x, x, \ldots, x}_{n-2 \text{ times}}, y, y) \vee \ldots \vee \Delta(x, \underbrace{y, y, \ldots, y}_{n-1 \text{ times}}) \vee \Delta(\underbrace{y, y, \ldots, y}_{n \text{ times}}) \\ &= \delta(x) \vee \Delta(\underbrace{x, x, \ldots, x}_{n-1 \text{ times}}, y) \vee \Delta(\underbrace{x, x, \ldots, x}_{n-2 \text{ times}}, y, y) \vee \ldots \vee \Delta(x, \underbrace{y, y, \ldots, y}_{n-1 \text{ times}}) \vee \delta(y) \end{aligned}$$

□

Theorem 2. *Let L be a distributive lattice, Δ be a permutting n-derivation, and δ be the trace of Δ. Then,*

$$\delta(x \wedge y) = \delta(x) \vee \Delta(\underbrace{x, x, \dots, x}_{n-1 \text{ times}}, y) \vee \Delta(\underbrace{x, x, \dots, x}_{n-2 \text{ times}}, y, y) \vee \dots \vee \Delta(x, \underbrace{y, y, \dots, y}_{n-1 \text{ times}}) \vee \delta(y)$$

Proof. Note that

$$\begin{aligned}
\delta(x \wedge y) &= \Delta(\underbrace{x \wedge y, x \wedge y, \dots, x \wedge y}_{n \text{ times}}) \\
&= (x \wedge \Delta(y, \underbrace{x \wedge y, \dots, x \wedge y}_{n-1 \text{ times}})) \vee (\Delta(x, \underbrace{x \wedge y, \dots, x \wedge y}_{n-1 \text{ times}}) \wedge y) \\
&= \Delta(y, \underbrace{x \wedge y, \dots, x \wedge y}_{n-1 \text{ times}}) \vee \Delta(x, \underbrace{x \wedge y, \dots, x \wedge y}_{n-1 \text{ times}}) \\
&= (x \wedge \Delta(y, y, \underbrace{x \wedge y, \dots, x \wedge y}_{n-2 \text{ times}})) \vee (\Delta(y, x, \underbrace{x \wedge y, \dots, x \wedge y}_{n-2 \text{ times}}) \wedge y) \\
&\quad \vee (x \wedge \Delta(x, y, \underbrace{x \wedge y, \dots, x \wedge y}_{n-2 \text{ times}})) \vee (\Delta(x, x, \underbrace{x \wedge y, \dots, x \wedge y}_{n-2 \text{ times}}) \wedge y) \\
&= \Delta(y, y, \underbrace{x \wedge y, \dots, x \wedge y}_{n-2 \text{ times}}) \vee \Delta(y, x, \underbrace{x \wedge y, \dots, x \wedge y}_{n-2 \text{ times}}) \vee \Delta(x, x, \underbrace{x \wedge y, \dots, x \wedge y}_{n-2 \text{ times}}) \\
&= \Delta(y, y, y, \underbrace{x \wedge y, \dots, x \wedge y}_{n-3 \text{ times}}) \vee \Delta(y, y, x, \underbrace{x \wedge y, \dots, x \wedge y}_{n-3 \text{ times}}) \\
&\quad \vee \Delta(y, x, x, \underbrace{x \wedge y, \dots, x \wedge y}_{n-3 \text{ times}}) \vee \Delta(y, x, y, \underbrace{x \wedge y, \dots, x \wedge y}_{n-2 \text{ times}}) \\
&\quad \vee \Delta(x, x, x, \underbrace{x \wedge y, \dots, x \wedge y}_{n-3 \text{ times}}) \vee \Delta(x, x, y, \underbrace{x \wedge y, \dots, x \wedge y}_{n-3 \text{ times}}) \\
&= \Delta(y, y, y, \underbrace{x \wedge y, \dots, x \wedge y}_{n-3 \text{ times}}) \vee \Delta(y, y, x, \underbrace{x \wedge y, \dots, x \wedge y}_{n-3 \text{ times}}) \\
&\quad \vee \Delta(y, x, x, \underbrace{x \wedge y, \dots, x \wedge y}_{n-3 \text{ times}}) \vee \Delta(x, x, x, \underbrace{x \wedge y, \dots, x \wedge y}_{n-3 \text{ times}}) \\
&= \dots \\
&= \delta(x) \vee \Delta(\underbrace{x, x, \dots, x}_{n-1 \text{ times}}, y) \vee \Delta(\underbrace{x, x, \dots, x}_{n-2 \text{ times}}, y, y) \vee \dots \vee \Delta(x, \underbrace{y, y, \dots, y}_{n-1 \text{ times}}) \vee \delta(y),
\end{aligned}$$

completing the proof. □

Corollary 2. *Let Δ be a n-derivation and δ be the trace of Δ.*

(i) $\Delta(x_1, x_2, \dots, x_n) = 0$ *if at least one of the components is 0.*
(ii) $\delta(x) \vee \delta(y) \leq \delta(x \vee y)$ *if Δ is permutting and joinitive.*
(iii) $\delta(x) \vee \delta(y) \leq \delta(x \wedge y)$ *if Δ is permutting.*
(iv) $\delta^2(x) = \delta(x)$ *if L is distributive lattice and Δ is permutting.*

Proof. (i) is clear by Proposition 2, (ii) is clear by Theorem 1, and (iii) is seen from Theorem 2.

(iv) since $\delta^2(x) = \delta(\delta(x)) \leq \delta(x)$ by Proposition 1, we get

$$\begin{aligned}
\delta^2(x) &= \delta(\delta(x)) \\
&= \delta(x \wedge \delta(x) && \text{, by Proposition 2.4} \\
&= \delta(x) \vee \Delta(\underbrace{x, x, \ldots, x}_{n-1 \text{ times}}, \delta(x)) \vee \Delta(\underbrace{x, x, \ldots, x}_{n-2 \text{ times}}, \delta(x), \delta(x)) \\
&\quad \vee \ldots \vee \Delta(x, \underbrace{\delta(x), \delta(x), \ldots, \delta(x)}_{n-1 \text{ times}}) \vee \delta^2(x) && \text{,by Theorem 2.9} \\
&= \delta(x) && \text{,by Proposition 2.5.}
\end{aligned}$$

□

3. (n,m)-Derivation-Homomorphisms on Lattices

In [19], Li and Xu introduced the notion of (n,m)-derivation-homomorphism in an associative ring. It is a kind of multimapping that is either a derivation or a homomorphism for each component when the other components are fixed by any given elements. In the following, this notion will be described for lattices and used to obtain some related results. This will give a generalization of n-derivation for lattices.

Definition 4. *Let* L *be a lattice. The map* $f : L^{n\times m} \to L$ *is called (n,m)-derivation-$\wedge-$homomorphism (shortly, (n,m)-der-$\wedge$-hom) on a lattice if f satisfies the following conditions:*

(i) *For* $i \in \{1,2, \ldots ,n\}$

$$f(a_1, \ldots, a_i \wedge x, \ldots, a_{n+m}) = (a_i \wedge f(a_1, \ldots, x, \ldots, a_{n+m})) \vee (f(a_1, \ldots, a_i, \ldots, a_{n+m})) \wedge x) \quad (1)$$

(ii) *For* $i \in \{n+1, \ldots, n+m\}$

$$f(f(a_1, \ldots, a_i \wedge y, \ldots, a_{n+m}) = f(a_1, \ldots, x, \ldots, a_{n+m}) \wedge f(a_1, \ldots, a_i, \ldots, a_{n+m}) \quad (2)$$

Remark 3. *In Definition 4, we see that* $(1,0) - \mathrm{der} - \wedge - \mathrm{hom}$ *is a derivation,* $(2,0) - \mathrm{der} - \wedge - \mathrm{hom}$ *is a bi-derivation, and* $(n,0) - \mathrm{der} - \wedge - \mathrm{hom}$ *is an n-derivation on L. In the following, our concern will focus on the case* $mn \neq 0$.

Definition 5. *Let f be a* $(n,0) - \mathrm{der} - \wedge - \mathrm{hom}$. *The map* $g : L \to L$ *defined by* $g(a) = f(a, a, \ldots, a)$ *is called the trace of f.*

Example 3. *It is easily seen that the map* $f: L \times L \to L$ defined by $f(x,y) = x \wedge y$ is a $(1,1) - der - \wedge - hom$.

Firstly, we consider $(1,1) - \mathrm{der} - \wedge - \mathrm{hom}$ and describe the properties of it.

Proposition 3. *Let* L *be a lattice with at least element 0 and the greatest element of 1, and let f be* $(1,1) - \mathrm{der} - \wedge - \mathrm{hom}$. *Then*

(i) $f(a,b) \leq a$,
(ii) $f(0,b) = 0$,
(iii) $f(a,0) \leq f(a,b)$,
(iv) $a \wedge f(1,b) \leq f(a,b) \leq f(a,1)$.

If f is also $\vee-$homomorphism, then

(v) $f(a,b) \leq f(1,b)$
for all $a,b \in L$.

Proof. Since f is $(1,1) - \mathrm{der} - \wedge - \mathrm{hom}$, it satisfies

$$f(a \wedge x, b) = (a \wedge f(x,b)) \vee (f(a,b) \wedge x), \tag{3}$$

$$f(a, b \wedge x) = f(a,b) \wedge f(a,x). \tag{4}$$

(i) Taking x = a in Equation (3), since $f(a,b) = f(a \wedge a, b) = (a \wedge f(a,b)) \vee (f(a,b) \wedge a) = a \wedge f(a,b)$, we have $f(a,b) \leq a$ for all $a,b \in L$.
(ii) Taking a = 0 in (i), we have $f(0,b) = 0$ for all $b \in L$.
(iii) Taking x = 0 in Equation (4), since $f(a,0) = f(a, b \wedge 0) = f(a,b) \wedge f(a,0)$, we get $f(a,0) \leq f(a,b)$ for all $a,b \in L$.
(iv) Taking x = 1 in Equation (3), since $f(a,b) = f(a \wedge 1, b) = (a \wedge f(1,b)) \vee (f(a,b) \wedge 1) = (a \wedge f(1,b)) \vee f(a,b)$ we have $a \wedge f(1,b) \leq f(a,b)$. Also, taking x = 1 in Equation (4), since $f(a,b) = f(a, b \wedge 1) = f(a,b) \wedge f(a,1)$, we have $f(a,b) \leq f(a,1)$.
(v) If f is $\vee-$ homomorphism, then it satisfies $f(a \vee x, b) = f(a,b) \vee f(x,b)$ and $f(a, b \vee x) = f(a,b) \vee f(a,x)$. Then, taking x = 1 in the first equality, we see that $f(a,b) \leq f(1,b)$. □

Proposition 4. *Let L be a lattice and f be* $(1,1) - \mathrm{der} - \wedge - \mathrm{hom}$ *with trace g. Then, for all* $a \in L$,

(i) $g(a) \leq a$,
(ii) $f(a, g(a)) \leq g^2(a)$,
(iii) $g^2(a) = f(a, g(a))$ *if f is* $\vee-$ *homomorphism.*

Proof. (i) Since $g(a) = f(a,a) = f(a \wedge a, a) = (a \wedge g(a)) \vee (g(a) \wedge a) = a \wedge g(a)$, we get $g(a) \leq a$ for all $a \in L$.
(ii) Since $g^2(a) = g(g(a)) \leq g(a) \leq a$ and $f(a,b) \leq a$ by Proposition 3 (i), we have

$$\begin{aligned}
g^2(a) &= g(g(a)) \\
&= f(g(a), g(a)) \\
&= f(a \wedge g(a), g(a)) \\
&= (a \wedge g^2(a)) \vee (f(a, g(a)) \wedge g(a)) \\
&= g^2(a) \vee (f(a, g(a)) \wedge f(a,a)) \\
&= g^2(a) \vee f(a, g(a) \wedge a) \\
&= g^2(a) \vee f(a, g(a)) .
\end{aligned}$$

Hence, we get $f(a, g(a)) \leq g^2(a)$.
(iii) Using (ii), we have

$$\begin{aligned}
g^2(a) &= g^2(a) \vee f(a, g(a)) \\
&= f(g(a), g(a)) \vee f(a, g(a)) \\
&= f(g(a) \vee a, g(a)) \\
&= f(a, g(a)) .
\end{aligned}$$

□

Proposition 5. *Let L be a lattice and f be* $(1,1) - \mathrm{der} - \wedge - \mathrm{hom}$ *with trace g. Then*

$$g(a) \wedge f(a,b) \wedge f(b,a) \wedge g(b) \leq g(a \wedge b).$$

If f is joinitive, then

$$g(a \vee b) = g(a) \vee f(a,b) \vee f(b,a) \vee g(b)$$

for all $a, b \in L$.

Proof. Note that

$$\begin{aligned} g(a \wedge b) &= f(a \wedge b, a \wedge b) \\ &= (a \wedge f(b, a \wedge b)) \vee (f(a, a \wedge b) \wedge b) \\ &= (a \wedge f(b,a) \wedge f(b,b)) \vee (f(a,a) \wedge f(a,b) \wedge b) \\ &= (a \wedge f(b,a) \wedge g(b)) \vee (g(a) \wedge f(a,b) \wedge b). \end{aligned}$$

Hence we have $a \wedge f(b,a) \wedge g(b) \leq g(a \wedge b)$ and $g(a) \wedge f(a,b) \wedge b \leq g(a \wedge b)$. Therefore, we get $a \wedge f(b,a) \wedge g(b) \wedge g(a) \wedge f(a,b) \wedge b \leq g(a \wedge b)$; that is, $g(a) \wedge f(a,b) \wedge f(b,a) \wedge g(b) \leq g(a \wedge b)$.

Now let f be joinitive. Then

$$\begin{aligned} g(a \vee b) &= f(a \vee b, a \vee b) \\ &= f(a,a) \vee f(a,b) \vee f(b,a) \vee f(b,b) \\ &= g(a) \vee f(a,b) \vee f(b,a) \vee g(b). \end{aligned} \tag{3}$$

Now we consider $(n, m) - der - \wedge - hom$ and describe its properties. □

Proposition 6. *Let L be a lattice with the least element 0 and the greatest element 1, and let f be* $(n, m) - der - \wedge - hom$. Then

(i) $f(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}) \leq a_i$ for $i \in \{1, 2, \ldots, n\}$,
(ii) $f(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}) = 0$ *if* $a_i = 0$ *for at least one* $i \in \{1, 2, \ldots, n\}$,
(iii) $f(a_1, \ldots, \underbrace{0}_{i \text{ th place}}, \ldots, a_{n+m}) \leq f(a_1, \ldots, a_i, \ldots, a_{n+m})$ for $i \in \{n+1, n+2, \ldots, n+m\}$,
(iv) $a_i \wedge f(a_1, \ldots, \underbrace{1}_{i \text{ th place}}, \ldots, a_{n+m}) \leq f(a_1, \ldots, a_i, \ldots, a_{n+m})$ for $i \in \{1, 2, \ldots, n\}$,
(v) $f(a_1, \ldots, a_i, \ldots, a_{n+m}) \leq f(a_1, \ldots, \underbrace{1}_{i \text{ th place}}, \ldots, a_{n+m})$ for $i \in \{n+1, n+2, \ldots, n+m\}$

where $a_i \in L$ for $i \in \{1, 2, \ldots, n+m\}$.

Proof. (**i**) Since, for $i \in \{1, 2, \ldots, n\}$

$$\begin{aligned} f(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}) &= f(a_1, \ldots, a_i \wedge a_i, \ldots, a_{n+m}) \\ &= (a_i \wedge f(a_1, \ldots, a_i, \ldots, a_{n+m})) \vee (f(a_1, \ldots, a_i, \ldots, a_{n+m}) \wedge a_i) \\ &= a_i \wedge f(a_1, \ldots, a_i, \ldots, a_{n+m}), \end{aligned} \tag{4}$$

we get $f(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}) \leq a_i$ for $i \in \{1, 2, \ldots, n\}$.
(**ii**) From (i), it is clear.
(**iii**) Writing y = 0 in the Equation (2), we have

$$\begin{aligned} f(a_1, \ldots, \underbrace{0}_{i \text{ th place}}, \ldots, a_{n+m}) &= f(a_1, \ldots, a_i \wedge 0, \ldots, a_{n+m}) \\ &= f(a_1, \ldots, a_i, \ldots, a_{n+m}) \wedge f(a_1, \ldots, \underbrace{0}_{i \text{ th place}}, \ldots, a_{n+m}). \end{aligned} \tag{5}$$

Hence we see that $f(a_1, \ldots, \underbrace{0}_{i \text{ th place}}, \ldots, a_{n+m}) \leq f(a_1, \ldots, a_i, \ldots, a_{n+m})$ for $i \in \{n+1, n+2, \ldots, n+m\}$.

(**iv**) Using Equation (1), we have

$$\begin{aligned} f(a_1,\dots,a_i,\dots,a_{n+m}) &= f(a_1,\dots,a_i\wedge 1,\dots,a_{n+m}) \\ &= (a_i\wedge f(a_1,\dots,1,\dots,a_{n+m}))\vee(f(a_1,\dots,a_i,\dots,a_{n+m})\wedge 1) \\ &= (a_i\wedge f(a_1,\dots,1,\dots,a_{n+m}))\vee(f(a_1,\dots,a_i,\dots,a_{n+m}). \end{aligned} \tag{6}$$

Hence we get the desired result.
(**v**) Using Equation (2), we have

$$\begin{aligned} f(a_1,\dots,a_i,\dots,a_{n+m}) &= f(a_1,\dots,a_i\wedge 1,\dots,a_{n+m}) \\ &= f(a_1,\dots,a_i,\dots,a_{n+m})\wedge f(a_1,\dots,1,\dots,a_{n+m}) \end{aligned} \tag{7}$$

Hence we get the desired result.

From Proposition 6, we obtain the following results: If L is a lattice with the least element 0, the greatest element is 1 and f is an $(n,m)-\mathrm{der}-\wedge-\mathrm{hom}$. Then

(**vi**) $f(a_1,\dots,a_n,a_{n+1},\dots,a_{n+m})\le a_1\wedge a_2\wedge\dots\wedge a_n$, (by (i))
(**vii**) Using (iii),

$$\begin{aligned} f(a_1,\dots,a_n,0,\dots,0) &\le f(a_1,\dots,a_n,a_{n+1},0,\dots,0) \\ &\le f(a_1,\dots,a_n,a_{n+1},a_{n+2},0,\dots,0) \\ &\le \dots \\ &\le f(a_1,\dots,a_n,a_{n+1},\dots,a_{n+m}). \end{aligned} \tag{8}$$

(**viii**) By (iv),

$$\begin{aligned} a_1\wedge a_2\wedge\dots\wedge a_n\wedge f(1,\dots,1,a_{n+1},\dots,a_{n+m}) &\le a_2\wedge\dots\wedge a_n\wedge f(a_1,1,\dots,1,a_{n+1},\dots,a_{n+m}) \\ &\le \dots \\ &\le a_n\wedge f(a_1,\dots,a_{n-1},1,a_{n+1},\dots,a_{n+m}) \\ &\le f(a_1,\dots,a_n,a_{n+m},\dots,a_{n+m}). \end{aligned} \tag{9}$$

(**ix**) By (v),

$$\begin{aligned} f(a_1,\dots,a_n,a_{n+1},\dots,a_{n+m}) &\le f(a_1,\dots,a_n,1,a_{n+2},\dots,a_{n+m}) \\ &\le f(a_1,\dots,a_n,1,1,a_{n+3},\dots,a_{n+m}) \\ &\le \dots \\ &\le f(a_1,\dots,a_n,1,\dots,1). \end{aligned} \tag{10}$$

□

Proposition 7. *Let L be a lattice and f be* $(n,m)-\mathrm{der}-\wedge-\mathrm{hom}$ $(n\ge 2)$ *with trace g. Then*

(i) $g(a)\le a$,
(ii) $f(g(a),\dots,\underbrace{a}_{\text{i th place}},\dots,g(a),\underbrace{g(a)}_{\text{n+1 th place}},\dots,g(a))\le g^2(a)$ *for* $i\in\{1,2,\dots,n\}$,
(iii) $g^2(a)=f(g(a),\dots,\underbrace{a}_{\text{i th place}},\dots,g(a),\underbrace{g(a)}_{\text{n+1 th place}},\dots,g(a))$ *for* $i\in\{1,2,\dots,n\}$

for all $a\in L$.

Proof. (i) Since $g(a)=f(a,a,\dots,a)=f(a\wedge a,a,\dots,a)=(a\wedge g(a))\vee(g(a)\wedge a)=a\wedge g(a)$, we have $g(a)\le a$.

(ii) Since

$$\begin{aligned}
g^2(a) &= g(g(a)) \\
&= f(g(a),\ldots,g(a)) \\
&= f(g(a),\ldots,\underbrace{a \wedge g(a)}_{\text{i th place}},\ldots,g(a)) \quad (1 \leq i \leq n) \\
&= (a \wedge g^2(a)) \vee (f(g(a),\ldots,\underbrace{a}_{\text{i th place}},\ldots,g(a)) \wedge g(a)) \\
&= g^2(a) \vee f(g(a),\ldots,\underbrace{a}_{\text{i th place}},\ldots,g(a)) \text{ by Proposition 3.7 (i),}
\end{aligned} \tag{11}$$

we get the desired result.
(iii) We have, by (ii),

$$\begin{aligned}
g^2(a) &= g^2(a) \vee f(g(a),\ldots,\underbrace{a}_{\text{i th place}},\ldots,g(a)) \quad (1 \leq i \leq n) \\
&= f(g(a),\ldots,g(a)) \vee f(g(a),\ldots,\underbrace{a}_{\text{i th place}},\ldots,g(a)) \\
&= f(g(a),\ldots,\underbrace{g(a) \vee a}_{\text{i th place}},\ldots,g(a)) \\
&= f(g(a),\ldots,\underbrace{a}_{\text{i th place}},\ldots,g(a))
\end{aligned} \tag{12}$$

□

4. Conclusions

Studies on derivations of algebraic structures started at the end of the 1950s. Algebraic structures were characterized by using these derivations. For example, in [8], Posner proved that a ring is commutative if the derivation defined on the ring satisfies certain conditions. Until today, many types of derivations and many generalization of these derivations have been defined on algebraic structures and the properties of this structures have been investigated by using these derivations. In this paper, we introduced the notion of n-derivation and (n,m)-derivation-homomor-phism, which are generalizations of derivation on a lattice, and proved important and characteristic properties of them. In future work, we will discuss the properties of the fixed sets of n-derivation, (n,m)-derivation-homomorphism, and their traces. Also, the lattices will be characterized by using n-derivation, (n,m)-derivation-homomorphism, and their traces.

Funding: This research received no external funding.

Conflicts of Interest: The author declares that there are no conflict of interest regarding this manuscript.

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