Variants on Andrica's Conjecture with and without the Riemann Hypothesis

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Abstract: The gap between what we can explicitly prove regarding the distribution of primes and what we suspect regarding the distribution of primes is enormous. It is (reasonably) well-known that the Riemann hypothesis is not sufficient to prove Andrica's conjecture: $\forall n \geq 1$, is $\sqrt{p_{n+1}} - \sqrt{p_n} \leq 1$? However, can one at least get tolerably close? I shall first show that with a logarithmic modification, provided one assumes the Riemann hypothesis, one has

$$\sqrt{p_{n+1}}/\ln p_{n+1} - \sqrt{p_n}/\ln p_n < 11/25; \qquad (n \geq 1).$$

Then, by considering more general m^{th} roots, again assuming the Riemann hypothesis, I show that

$$\sqrt[m]{p_{n+1}} - \sqrt[m]{p_n} < 44/(25\,e\,[m-2]); \qquad (n \geq 3;\ m > 2).$$

In counterpoint, if we limit ourselves to what we can currently prove unconditionally, then the only explicit Andrica-like results seem to be variants on the relatively weak results below:

$$\ln^2 p_{n+1} - \ln^2 p_n < 9; \qquad \ln^3 p_{n+1} - \ln^3 p_n < 52; \qquad \ln^4 p_{n+1} - \ln^4 p_n < 991; \qquad (n \geq 1).$$

I shall also update the region on which Andrica's conjecture is unconditionally verified.
SEQUENCES: A005250, A002386, A005669, A000101, A107578.

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MSC: 11A41 (primes); 11N05 (distribution of primes)

1. Introduction

The Riemann hypothesis continues to defeat all attempts to prove or disprove it [1–6]. On the other hand assuming the Riemann hypothesis provides a wealth of intriguing tentative suggestions regarding the distribution of the prime numbers [1–6]. However, even the Riemann hypothesis is insufficient to prove Andrica's conjecture [7–9]:

$$\forall n \geq 1, \quad \text{is} \quad \sqrt{p_{n+1}} - \sqrt{p_n} \leq 1? \tag{1}$$

(In Section 6 below, by considering the known maximal prime gaps, I shall numerically [and unconditionally] verify Andrica's conjecture up to just below the 81st maximal prime gap; certainly for all primes less than 1.836×10^{19}.) Somewhat weaker results that have been unconditionally proved include Sandor's 1985 result [10]

$$\liminf_{n \to \infty} \sqrt[4]{p_n} \left(\sqrt{p_{n+1}} - \sqrt{p_n} \right) = 0, \tag{2}$$

and the more recent 2017 result by Lowry–Duda [11] that for $\alpha > 0$, $\beta > 0$, and $\alpha + \beta < 1$:

$$\liminf_{n \to \infty} \sqrt[\beta]{p_n} \left(\sqrt[\alpha]{p_{n+1}} - \sqrt[\alpha]{p_n} \right) = 0. \tag{3}$$

Other recent articles on the distribution of primes include [12,13].

In the current article I shall seek to derive results as close to Andrica's conjecture as possible. The basic tools we use are based on the behaviour of prime gaps under the Riemann hypothesis. I shall primarily make use of the recent fully explicit bound due to Carneiro, Milinovich, and Soundararajan [14]:

Theorem 1 (Prime gaps; Carneiro–Milinovich–Soundararajan). *Assuming the Riemann hypothesis,*

$$\forall n \geq 3, \ p_n \geq 5, \qquad g_n := p_{n+1} - p_n < \frac{22}{25} \sqrt{p_n} \ln p_n. \tag{4}$$

(In Section 7 below, by considering the known maximal prime gaps, I shall numerically [and unconditionally] verify this result up to just below the 81st maximal prime gap; certainly for all primes less than 1.836×10^{19}.) The "close to Andrica" results I will prove below are these:

Theorem 2 (Logarithmic modification of Andrica). *Assuming the Riemann hypothesis,*

$$\forall n \geq 1, \qquad \frac{\sqrt{p_{n+1}}}{\ln p_{n+1}} - \frac{\sqrt{p_n}}{\ln p_n} \leq \frac{11}{25}. \tag{5}$$

Theorem 3 (Higher root modification of Andrica). *Assuming the Riemann hypothesis,*

$$\forall n \geq 3, \ \forall m > 2, \qquad \sqrt[m]{p_{n+1}} - \sqrt[m]{p_n} \leq \frac{44}{25 \, e \, (m - 2)}. \tag{6}$$

2. Known Results on Prime Gaps Assuming Riemann Hypothesis

Two older theorems addressing the issue of prime gaps are "ineffective" (meaning one or more implicit constants are known to be finite but are otherwise undetermined):

Theorem 4 (Cramer 1919 [15,16]). *Assuming the Riemann hypothesis,*

$$g_n := p_{n+1} - p_n = O(\sqrt{p_n} \ln(p_n)). \tag{7}$$

Unfortunately, this particular theorem only gives qualitative, but not quantitative, information.

Theorem 5 (Goldston 1982 [17]). *Assuming the Riemann hypothesis,*

$$g_n := p_{n+1} - p_n \leq 4\sqrt{p_n} \ln(p_n); \qquad n \text{ sufficiently large.} \tag{8}$$

This particular theorem gives quantitative information about the size of prime gaps, but only qualitative information as to the domain of validity. The considerably more recent Carneiro, Milinovich, and Soundararajan theorem [14], presented as Theorem 1 above, is fully explicit. It is that theorem that will be my primary tool.

3. Logarithmic Modification of Andrica: Proof of Theorem 2

The proof of Theorem 2, given our input assumptions, is straightforward.
Consider the function $\sqrt{p}/\ln p$ and note

$$\left(\frac{\sqrt{p}}{\ln p}\right)' = \frac{\ln p - 2}{2\sqrt{p} \ln^2 p} > 0 \qquad (p > e^2 \approx 7.39); \tag{9}$$

$$\left(\frac{\sqrt{p}}{\ln p}\right)'' = -\frac{\ln^2 p - 8}{4p^{3/2} \ln^3 p} < 0 \qquad (p > e^{\sqrt{8}} \approx 16.92). \tag{10}$$

So $\sqrt{p}/\ln p$ is certainly monotone and convex for $p \geq 17$. Thence for $p_n \geq 17$, that is $n \geq 7$, we have

$$\frac{\sqrt{p_{n+1}}}{\ln p_{n+1}} - \frac{\sqrt{p_n}}{\ln p_n} < \frac{\ln p_n - 2}{2\sqrt{p_n}\ln^2 p_n} \, g_n < \frac{11}{25}\left(\frac{\ln p_n - 2}{\ln p_n}\right) = \frac{11}{25}\left(1 - \frac{2}{p_n}\right) < \frac{11}{25}. \tag{11}$$

(The first step in this chain of inequalities is based on convexity, the second step on the Carneiro–Milinovich–Soundararajan result; the remaining steps are trivial.) A quick verification shows that this inequality also holds for $n \in \{1, 2, 3, 4, 5, 6\}$ and so we have

$$\frac{\sqrt{p_{n+1}}}{\ln p_{n+1}} - \frac{\sqrt{p_n}}{\ln p_n} < \frac{11}{25}; \qquad (n \geq 1). \tag{12}$$

This by no means an optimal bound, but it is, given the Riemann hypothesis and the Carneiro–Milinovich–Soundararajan prime gap result it implies, both easy to establish and easy to work with.

4. Higher-Root Modification Of Andrica: Proof of Theorem 3

The proof of Theorem 3, given our input assumptions, is straightforward.

Consider the function $\sqrt[m]{p} = p^{1/m}$ and restrict attention to $p > 0$ and $m > 1$. Then

$$(p^{1/m})' = \frac{1}{m} p^{1/m-1} > 0; \qquad (p^{1/m})'' = -\frac{m-1}{m^2} p^{1/m-2} < 0; \tag{13}$$

Thus the function $\sqrt[m]{p} = p^{1/m}$ is monotone and convex for $p > 0$ and $m > 1$. We have

$$\sqrt[m]{p_{n+1}} - \sqrt[m]{p_n} < \frac{1}{m} p_n^{1/m-1} \, g_n < \frac{22}{25\,m} p_n^{1/m-1/2} \, \ln(p_n); \qquad (n \geq 3, m > 1). \tag{14}$$

(The first step in this chain of inequalities is based on convexity, the second step on the Carneiro–Milinovich–Soundararajan result.) If $m \leq 2$ this result is true but not particularly interesting. For $m > 2$ the function $p^{1/m-1/2} \ln p$ rises from zero (at $p = 1$) to a maximum, and subsequently dies back to zero asymptotically as $p \to \infty$. The maximum occurs at $p_{\text{critical}} = \exp(2m/(m-2))$ where the function takes on the value $(p^{1/m-1/2} \ln p)_{\max} = 2me^{-1}/(m-2)$, thereby implying

$$\sqrt[m]{p_{n+1}} - \sqrt[m]{p_n} < \frac{44}{25} \frac{e^{-1}}{m-2}; \qquad (n \geq 3; \; m > 2). \tag{15}$$

This is the result we were seeking to establish. This is again by no means an optimal bound, but it is, given the Riemann hypothesis and the Carneiro–Milinovich–Soundararajan prime gap result it implies, both easy to establish and easy to work with.

Note that for $m = 2$, the situation relevant to the standard Andrica conjecture, we merely have

$$\sqrt{p_{n+1}} - \sqrt{p_n} < \frac{11}{25} \, \ln(p_n); \qquad (n \geq 3). \tag{16}$$

For $m = 2$ this is not enough to conclude anything useful regarding the standard Andrica conjecture. In contrast for $m = 3$ and $m = 4$ we certainly have:

Corollary 1 (Cube root modification of Andrica). *Assuming the Riemann hypothesis,*

$$\forall n \geq 1, \qquad \sqrt[3]{p_{n+1}} - \sqrt[3]{p_n} \leq \frac{44}{25e} < \frac{13}{20}. \tag{17}$$

Corollary 2 (Fourth root modification of Andrica). *Assuming the Riemann hypothesis,*

$$\forall n \geq 1, \qquad \sqrt[4]{p_{n+1}} - \sqrt[4]{p_n} \leq \frac{22}{25e} < \frac{13}{40}. \tag{18}$$

5. Unconditional Results for Andrica–Like Variants

I shall now focus on some much weaker but unconditional results. A number of explicit unconditional theorems on the occurrence of primes in small gaps are of the following general form.

Theorem 6 (Primes in short intervals). *For $x > x_0$, $m \geq 0$, and some explicit $C > 0$, there is always at least one prime in the interval:*

$$\left(x, x + C \frac{x}{\ln^m x} \right). \tag{19}$$

Specifically, we have:

x_0	m	C	Reference	arXiv
396738	2	$\frac{1}{25}$	Dusart 2010 [18]	1002.0442
2898329	2	$\frac{1}{111}$	Trudgian 2014 [19]	1401.2689
468991632	2	$\frac{1}{5000}$	Dusart 2018 [20]	—
89693	3	1	Dusart 2018 [20]	—
6034256	3	0.087	Axler 2017 [21]	1703.08032
1	4	198.2	Axler 2017 [21]	1703.08032

Taking $x = p_n$ in Theorem 6, this becomes a bound on the prime gaps.

Theorem 7 (Prime gaps). *For $p_n > x_0$, that is $n > \pi(x_0)$, and taking $m \geq 0$ and $C > 0$ from Theorem 6, the prime gaps are unconditionally bounded by:*

$$g_n := p_{n+1} - p_n < C \frac{p_n}{\ln^m p_n}. \tag{20}$$

Now consider the function $(\ln^{m+1} p)$ and note

$$(\ln^{m+1} p)' = \frac{(m+1)\ln^m p}{p} > 0; \qquad (m+1 > 0, \ p > 1); \tag{21}$$

$$(\ln^{m+1} p)'' = -\frac{(m+1)\ln^{m-1} p \, (\ln p - m)}{p^2} < 0; \qquad (m+1 > 0, \ p > e^m). \tag{22}$$

Then, in view of the convexity of $\ln^m p$ on the specified domain, we have

$$(\ln^{m+1} p_{n+1}) - (\ln^{m+1} p_n) \quad < \quad \frac{(m+1)\ln^m p_n}{p_n} g_n \tag{23}$$

$$< \quad \frac{(m+1)\ln^m p_n}{p_n} C \frac{p_n}{\ln^m p_n} \tag{24}$$

$$= \quad (m+1) C. \tag{25}$$

That is, we have demonstrated:

Theorem 8 (Explicit unconditional bounds). *For $p_n > \max\{x_0, e^m\}$, that is, $n > \pi(\max\{x_0, e^m\})$, and taking $m \geq 0$ and $C > 0$ from the table in Theorem 6, we have the explicit unconditional bounds:*

$$(\ln^{m+1} p_{n+1}) - (\ln^{m+1} p_n) < (m+1)C. \tag{26}$$

While weak, these bounds are both unconditional and fully explicit. I shall now explore several corollaries.

Corollary 3.

$$(\ln^5 p_{n+1}) - (\ln^5 p_n) < 991; \qquad (n \geq 1). \tag{27}$$

Proof. The general result above already implies this for $p_n > e^4$, that is $n > 16$, the remaining cases can be checked by explicit calculation. □

Corollary 4.

$$(\ln^4 p_{n+1}) - (\ln^4 p_n) < 0.348; \qquad (p_n > 6034256; \; n > 415069). \tag{28}$$

This bound, for the specified value 0.348, is sharp — it fails by 5% for $p_n = 6034256$, $n = 415069$. I shall now present a more relaxed bound with a wider range of validity.

Corollary 5.

$$(\ln^4 p_{n+1}) - (\ln^4 p_n) < 52; \qquad (p_n \geq 2; \; n \geq 1). \tag{29}$$

Proof. Use the previous corollary for $n > 415069$ and check $n \leq 415069$ by explicit calculation. □

Corollary 6.

$$(\ln^3 p_{n+1}) - (\ln^3 p_n) < \frac{2}{111}; \qquad (p_n > 2898329; \; n > 209989). \tag{30}$$

This bound, for the specified value $\frac{2}{111}$, is sharp—it fails by over 50% for $p_n = 2898329$, $n = 209989$. I now present a more relaxed bound with a wider range of validity.

Corollary 7.

$$(\ln^3 p_{n+1}) - (\ln^3 p_n) < \frac{2}{25}; \qquad (p_n > 396738; \; n > 33608). \tag{31}$$

This bound, for the specified value $\frac{2}{25}$, is sharp — it fails by over 57% for $p_n = 396738$, $n = 33608$. I now present an even more relaxed bound with a wider range of validity.

Corollary 8.

$$(\ln^3 p_{n+1}) - (\ln^3 p_n) < 9; \qquad (p_n \geq 2; \; n \geq 1). \tag{32}$$

Proof. Use the previous corollary for $n > 33608$ and check $n \leq 33608$ by explicit calculation. □

While all relatively weak, all these bounds have the pronounced virtue of being both fully explicit, and completely unconditional. Furthermore is is now clear how to systematically turn prime gap results of the type considered in Theorem 6 into Andrica-like bounds of the type considered in Theorem 8.

6. Unconditional Numerical Results for the Standard Andrica Conjecture

Andrica's conjecture can be rearranged to be equivalent to

$$g_n \leq 2\sqrt{p_n} + 1. \tag{33}$$

In an unpublished note (see comments in References [8,9]) Imran Ghory rephrased this in terms of maximal prime gaps. Let the triplet (i, g_i^*, p_i^*) denote the ith maximal prime gap; of width g_i^*, starting at the prime p_i^*. (See see the sequences A005250, A002386, A005669, A000101, A107578.) 80 such maximal

prime gaps are currently known [22], up to $g_{80}^* = 1550$ and $p_{80}^* = 18,361,375,334,787,046,697 > 1.836 \times 10^{19}$. Imran Ghory observed the equivalent of

$$\forall p_n \in [p_i^*, p_j^*] \qquad g_n \leq g_j^*; \qquad 2\sqrt{p_n} + 1 \geq 2\sqrt{p_i^*} + 1. \tag{34}$$

That is, Andrica's conjecture certainly holds on the interval $p_n \in [p_i^*, p_j^*]$ if one has

$$g_j^* \leq 2\sqrt{p_i^*} + 1; \qquad \text{that is} \qquad \left(\frac{g_j^*}{2} - 1\right)^2 < p_i^*. \tag{35}$$

But this is easily checked to hold on the intervals $[p_{19}^*, p_{80}^*]$, $[p_{11}^*, p_{19}^*]$, $[p_7^*, p_{11}^*]$, $[p_5^*, p_7^*]$, $[p_4^*, p_5^*]$, $[p_3^*, p_4^*]$, $[p_2^*, p_3^*]$, and $[p_1^*, p_2^*]$. Thus Andrica's conjecture certainly holds up to the 80th maximal prime gap, $p_{80}^* = 18,361,375,334,787,046,697 > 1.836 \times 10^{19}$. This bound will certainly be improved as additional maximal prime gaps are identified.

As a slight variant on this argument, consider the interval $[p_i^*, p_{i+1}^* - 1]$, from the lower end of the ith maximal prime gap to just below the beginning of the $(i+1)^{th}$ maximal prime gap. Then everywhere in this interval

$$\forall p_n \in [p_i^*, p_{i+1}^* - 1] \qquad g_n \leq g_i^*; \qquad 2\sqrt{p_n} + 1 \geq 2\sqrt{p_i^*} + 1. \tag{36}$$

That is, Andrica's conjecture certainly holds on the entire interval $p_n \in [p_i^*, p_{i+1}^* - 1]$ if it holds at the beginning of this interval.

Consequently, explicitly checking the inequality for p_{80}^*, Andrica's conjecture holds unconditionally up to just before the beginning of the 81st maximal prime gap, $p_{81}^* - 1$, even if we do not yet know the value of p_{81}^*.

7. Unconditional Numerical Results for the Carneiro–Milinovich–Soundararajan Inequality

Consider the Carneiro–Milinovich–Soundararajan inequality

$$g_n := p_{n+1} - p_n < \frac{22}{25}\sqrt{p_n}\ln p_n, \tag{37}$$

and note that the right-hand side is monotone increasing. Again consider the interval $[p_i^*, p_{i+1}^* - 1]$, from the lower end of the ith maximal prime gap to just below the beginning of the $(i+1)^{th}$ maximal prime gap. Then the Carneiro–Milinovich–Soundararajan inequality certainly holds on the entire interval $p_n \in [p_i^*, p_{i+1}^* - 1]$ if it holds at the beginning of this interval. Explicitly checking the inequality for all the known maximal prime gaps up to p_{80}^*, see reference [22], the Carneiro–Milinovich–Soundararajan inequality holds unconditionally up to just before the beginning of the 81st maximal prime gap, $p_{81}^* - 1$, even if we do not yet know the value of p_{81}^*. Certainly the the Carneiro–Milinovich–Soundararajan inequality holds for all primes less than 1.836×10^{19}. (Of course, assuming the Riemann hypothesis, Carneiro–Milinovich–Soundararajan have theorem; so this numerical exercise is only a partial numerical check on the input to our arguments.) This also implies that theorems 2 and 3 are unconditionally numerically verified for all primes less than 1.836×10^{19}.

8. Discussion

While the Riemann hypothesis provides (among very many other things) a nice explicit bound on prime gaps, it is still not quite sufficient to prove Andrica's conjecture—though as seen above, one can get tolerably close. There are a number of places where the argument might be tightened—the presentation above was designed to be simple and direct, not necessarily optimal. Of course the really big improvement in theorems 1–2–3 would be if any of these results could be made unconditional. While the numerical evidence certainly suggests this, a proof seems impossible with current techniques.

In contrast what we can currently prove unconditionally is rather weak; so improving the constants in theorems 6–7–8 would also be of some considerable interest.

Conflicts of Interest: The author declares no conflict of interest.

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    For all of the maximal prime gaps up to $(75, g_{75}^*, p_{75}^*)$ see (as of 27 November 2018): http://primerecords.dk/primegaps/maximal.htm. https://primes.utm.edu/notes/GapsTable.html.
    (See also the sequences A005250, A002386, A005669, A000101, A107578.)