mathematics

Article

A Sharp Rellich Inequality on the Sphere

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Received: 25 October 2018; Accepted: 26 November 2018; Published: 27 November 2018

Abstract: We obtain a Rellich type inequality on the sphere and give the corresponding best constant. The result complements some related inequalities in recent literatures.

Keywords: rellich inequality; sphere; sharp constant

MSC: Primary 26D10; Secondary 46E36

1. Introduction

The classical Rellich inequality states that [1], for $n \geq 5$ and all $f \in C_c^\infty(\mathbb{R}^n \backslash \{0\})$,

$$\int_{\mathbb{R}^n} |\Delta f|^2 dx \geq \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} \frac{f^2}{|x|^4} dx.$$

The constant $\frac{n^2(n-4)^2}{16}$ is optimal and never archived. Under additional conditions there are also versions for lower dimensions. There has been a lot of research concerning the Rellich inequality on the Euclidean space due to its applications to spectral theory, harmonic analysis, geometry and partial differential equations. We see [2–6] and the references therein.

The validity of the Rellich inequality on a manifold and its best constants allows people to obtain qualitative properties on the manifold. For complete noncompact Riemannian manifolds, under some geometric assumptions on the weight function ρ, Kome and Özaydin [7] proved that for $f \in C_c^\infty(M - \rho^{-1}\{0\})$ (where $\alpha < 2, C + \alpha - 3 > 0$)

$$\int_M \rho^\alpha |\Delta f|^2 dV \geq \frac{(C+\alpha-3)^2(C-\alpha+1)^2}{16} \int_M \rho^\alpha \frac{f^2}{\rho^4} dV.$$

Particularly, they also obtained the improved versions of a Rellich-type inequality which involves both first and second order derivatives in the Poincaré conformal disc model ($n > 2$, $\frac{8-n}{3} < \alpha < 2$)

$$\int_{\mathbb{B}^n} r^\alpha |\Delta f|^2 dV \geq \frac{(n-\alpha)^2}{4} \int_{\mathbb{B}^n} r^\alpha \frac{|\nabla f|^2}{r^4} dV,$$

where $f \in C_c^\infty(\mathbb{B}^n)$ and $r = \log \frac{1+|x|}{1-|x|}$ is the geodesic distance. Furthermore, the constant $\frac{(n-\alpha)^2}{4}$ is sharp. Along this line, we refer to [4,7–11] and so on.

However, there are not many literatures discussing the Rellich inequality on the sphere so far. See [12–14] for details. In [14] Xiao derived the following inequality

$$C \int_{\mathbb{S}^n} \frac{f^2}{\sin^2 d(p,x)} dV + \int_{\mathbb{S}^n} |\Delta f|^2 dV \geq \frac{n^2(n-4)^2}{16} \left(\int_{\mathbb{S}^n} \frac{f^2}{d(p,x)^4} dV + \int_{\mathbb{S}^n} \frac{f^2}{(\pi - d(p,x))^4} dV \right)$$

for $f \in C_c^\infty(\mathbb{S}^n - d^{-1}(0) \cup d^{-1}(\pi))$, where $d(p,x)$ is the geodesic distance from p to x on $\mathbb{S}^n$ and C is some positive constant. Moreover, the constant $\frac{n^2(n-4)^2}{16}$ is sharp.

In this short note we will obtain another type of Rellich inequality on the sphere and also give the corresponding sharp constant. Our main theorem is as follows:

Theorem 1. *Let $(\mathbb{S}^n, g)$ $(n \geq 5)$ be the n-sphere with sectional curvature 1 and p be a fixed point in $\mathbb{S}^n$. Then for any function $f \in C_c^\infty(\mathbb{S}^n - d^{-1}(0) \cup d^{-1}(\pi))$,*

$$C(n)\int_{\mathbb{S}^n} \frac{f^2}{\sin^2 d(p,x)} dV + \int_{\mathbb{S}^n} |\Delta f|^2 dV \geq \frac{n^2(n-4)^2}{16}\int_{\mathbb{S}^n} \frac{f^2}{\sin^4 d(p,x)} dV,$$

where $C(n) = \frac{n(n-4)(n^2-2n-4)}{8}$ and the constant $\frac{n^2(n-4)^2}{16}$ is sharp.

Remark 1. *In Euclidean spaces (resp. a Riemannian manifold, the Poincaré conformal disc model), the Laplacian of the distance function (resp. some weighted function) equals to $\frac{n-1}{|x|}$ (resp. is not less than $\frac{C}{\rho}$, $\frac{n-1}{r}$). Thus the Rellich inequality certainly contains the term $\frac{f^2}{|x|^4}$ (resp. $\frac{f^2}{\rho^4}$, $\frac{f^2}{r^4}$). Since on the sphere the Laplacian of the distance function is $\Delta d(p,x) = (n-1)\cot d(p,x)$ when d is smooth (see [15] p. 207), the terms $\frac{f^2}{\sin^2 d(p,x)}$ and $\frac{f^2}{\sin^4 d(p,x)}$ are naturally involved. So, it is a bit different in form from that in Euclidean spaces and some other type of Rellich inequalities. It is interesting that, even though the coefficient $C(n)$ is replaced by an arbitrary number, the constant $\frac{n^2(n-4)^2}{16}$ is still sharp. To prove the result, we give some modifications in constructing the auxiliary function, and then do calculations in two hemispheres by using the antipodal points. The remainder of the approaches used are similar to Xiao's paper [14]. See also in [7,16].*

2. The Proof of the Main Result

Proof of Theorem 1. Denote by $r_p(x) = d(p,x)$ the distance function from the fixed point $p \in \mathbb{S}^n$. Let f be a smooth function in $C_c^\infty(\mathbb{S}^n \backslash \{p,q\})$, where q is the antipodal point of p. Then

$$\Delta f^2 = 2f\Delta f + 2|\nabla f|^2,$$

and thus

$$-2\int_{\mathbb{S}^n} \frac{f\Delta f}{\sin^2 r_p} dV = -\int_{\mathbb{S}^n} \frac{\Delta f^2}{\sin^2 r_p} dV + 2\int_{\mathbb{S}^n} \frac{|\nabla f|^2}{\sin^2 r_p} dV. \tag{1}$$

Compute

$$\begin{aligned}
\int_{\mathbb{S}^n} \frac{\Delta f^2}{\sin^2 r_p} dV &= \int_{\mathbb{S}^n} f^2 \Delta \sin^{-2} r_p dV = \int_{\mathbb{S}^n} f^2 \mathrm{div}(\nabla \sin^{-2} r_p) dV \\
&= -2\int_{\mathbb{S}^n} f^2 \mathrm{div}(\sin^{-3} r_p \cos r_p \nabla r_p) dV \\
&= -2\int_{\mathbb{S}^n} f^2 \left[\sin^{-3} r_p \cos r_p \Delta r_p + (\sin^{-3} r_p \cos r_p)'\right] dV \\
&= -2\int_{\mathbb{S}^n} f^2 \left[\frac{n-4}{\sin^4 r_p} - \frac{n-3}{\sin^2 r_p}\right] dV,
\end{aligned} \tag{2}$$

where we have used $\Delta r_p = (n-1)\cot r_p$ in the sphere. To estimate $\int_{\mathbb{S}^n} \frac{|\nabla f|^2}{\sin^2 r_p} dV$, we put $f = (\sin r_p)^{-\frac{n-4}{2}} \varphi$. Then

$$\nabla f = \varphi \nabla (\sin r_p)^{-\frac{n-4}{2}} + (\sin r_p)^{-\frac{n-4}{2}} \nabla \varphi,$$

and

$$
\begin{aligned}
|\nabla f|^2 &= \varphi^2|\nabla(\sin r_p)^{-\frac{n-4}{2}}|^2 + (\sin r_p)^{-(n-4)}|\nabla\varphi|^2 + 2(\sin r_p)^{-\frac{n-4}{2}}\varphi\langle\nabla(\sin r_p)^{-\frac{n-4}{2}}, \nabla\varphi\rangle \\
&\geq \frac{(n-4)^2}{4}\varphi^2(\sin r_p)^{-(n-2)}\cos^2 r_p + \frac{1}{2}\langle\nabla(\sin r_p)^{-(n-4)}, \nabla\varphi^2\rangle.
\end{aligned}
$$

This gives

$$
\int_{\mathbb{S}^n}\frac{|\nabla f|^2}{\sin^2 r_p}dV \geq \frac{(n-4)^2}{4}\int_{\mathbb{S}^n}\frac{\varphi^2}{(\sin r_p)^n}\cos^2 r_p dV - \frac{n-4}{2(n-2)}\int_{\mathbb{S}^n}\varphi^2\Delta(\sin r_p)^{-(n-2)}dV.
$$

A direct computation shows that

$$
\begin{aligned}
\Delta(\sin r_p)^{-(n-2)} &= \operatorname{div}(\nabla(\sin r_p)^{-(n-2)}) = -(n-2)\operatorname{div}((\sin r_p)^{-(n-1)}\cos r_p\nabla r_p) \\
&= -(n-2)(\sin r_p)^{-(n-1)}\cos r_p\Delta r_p - (n-2)\left((\sin r_p)^{-(n-1)}\cos r_p\right)' \\
&= (n-2)(\sin r_p)^{2-n}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_{\mathbb{S}^n}\frac{|\nabla f|^2}{\sin^2 r_p}dV \geq &\frac{(n-4)^2}{4}\int_{\mathbb{S}^n}\frac{\varphi^2}{(\sin r_p)^n}\cos^2 r_p dV - \frac{n-4}{2}\int_{\mathbb{S}^n}\varphi^2(\sin r_p)^{2-n}dV \\
= &\frac{(n-4)^2}{4}\int_{\mathbb{S}^n}\frac{\varphi^2}{(\sin r_p)^n}dV - \left[\frac{(n-4)^2}{4} + \frac{n-4}{2}\right]\int_{\mathbb{S}^n}\frac{\varphi^2}{(\sin r_p)^{n-2}}dV \\
= &\frac{(n-4)^2}{4}\int_{\mathbb{S}^n}\frac{f^2}{\sin^4 r_p}dV - \left[\frac{(n-4)^2}{4} + \frac{n-4}{2}\right]\int_{\mathbb{S}^n}\frac{f^2}{\sin^2 r_p}dV.
\end{aligned} \tag{3}
$$

By the Cauchy-Schwarz inequality, one has

$$
-2\int_{\mathbb{S}^n}\frac{f\Delta f}{\sin^2 r_p}dV \leq \frac{n(n-4)}{4}\int_{\mathbb{S}^n}\frac{f^2}{\sin^4 r_p}dV + \frac{4}{n(n-4)}\int_{\mathbb{S}^n}|\Delta f|^2 dV. \tag{4}
$$

Finally, combining (1)∼(4), we obtain

$$
\frac{n(n-4)(n^2-2n-4)}{8}\int_{\mathbb{S}^n}\frac{f^2}{\sin^2 d(p,x)}dV + \int_{\mathbb{S}^n}|\Delta f|^2 dV \geq \frac{n^2(n-4)^2}{16}\int_{\mathbb{S}^n}\frac{f^2}{\sin^4 d(p,x)}dV.
$$

In what follows, we show the constant $\frac{n^2(n-4)^2}{16}$ is sharp. The skill is borrowed from [16] (see also [14]). Let $\eta : R \to [0,1]$ be a smooth function such that $0 \leq \eta \leq 1$ and

$$
\eta(t) = \begin{cases} 1, & t \in [-1,1]; \\ 0, & |t| \geq 2. \end{cases}
$$

Let $H(t) = 1 - \eta(t)$. For sufficient small $\varepsilon > 0$, Set

$$
f_\varepsilon(r_p) = \begin{cases} 0, & r_p = 0; \\ H(\frac{r_p}{\varepsilon})\sin^{\frac{2-n}{2}} r_p, & 0 < r_p \leq \frac{\pi}{2}; \\ H(\frac{\pi-r_p}{\varepsilon})\sin^{\frac{2-n}{2}}(\pi - r_p), & \frac{\pi}{2} \leq r_p < \pi; \\ 0, & r_p = \pi. \end{cases}
$$

Observe that $f_\varepsilon(r_p)$ can be approximated by smooth functions on the sphere $\mathbb{S}^n$.

Let q be the antipodal point of p. Then $d(p,q)=\pi$ and for any point $x\in\mathbb{S}^n$ we have $r_p+r_q=\pi$. Since the constructed function f_ε possesses a fair degree of bilateral symmetry on the sphere, it is easier to compute in the following by using the antipodal points p and q.

$$\begin{aligned}\int_{\mathbb{S}^n}\frac{f_\varepsilon^2}{\sin^2 r_p}dV=&\int_{B_p(\frac{\pi}{2})}\frac{f_\varepsilon^2}{\sin^2 r_p}dV+\int_{B_q(\frac{\pi}{2})}\frac{f_\varepsilon^2}{\sin^2 r_p}dV\\ =&\mathrm{Vol}(\mathbb{S}^{n-1})\int_\varepsilon^{\frac{\pi}{2}}H^2(\frac{r_p}{\varepsilon})\sin r_p dr+\mathrm{Vol}(\mathbb{S}^{n-1})\int_{\frac{\pi}{2}}^{\pi-\varepsilon}H^2(\frac{\pi-r_p}{\varepsilon})\sin(\pi-r_p)dr\\ =&2\mathrm{Vol}(\mathbb{S}^{n-1})\int_\varepsilon^{\frac{\pi}{2}}H^2(\frac{r_p}{\varepsilon})\sin r_p dr\\ \le&\pi\mathrm{Vol}(\mathbb{S}^{n-1}),\end{aligned}\tag{5}$$

$$\begin{aligned}\int_{\mathbb{S}^n}\frac{f_\varepsilon^2}{\sin^4 r_p}dV=&\int_{B_p(\frac{\pi}{2})}\frac{f_\varepsilon^2}{\sin^4 r_p}dV+\int_{B_q(\frac{\pi}{2})}\frac{f_\varepsilon^2}{\sin^4 r_p}dV\\ =&\mathrm{Vol}(\mathbb{S}^{n-1})\int_\varepsilon^{\frac{\pi}{2}}H^2(\frac{r_p}{\varepsilon})\sin^{-1} r_p dr+\mathrm{Vol}(\mathbb{S}^{n-1})\int_{\frac{\pi}{2}}^{\pi-\varepsilon}H^2(\frac{\pi-r_p}{\varepsilon})\sin^{-1}(\pi-r_p)dr\\ =&2\mathrm{Vol}(\mathbb{S}^{n-1})\int_\varepsilon^{\frac{\pi}{2}}H^2(\frac{r_p}{\varepsilon})\sin^{-1} r_p dr\\ \ge&2\mathrm{Vol}(\mathbb{S}^{n-1})\int_{2\varepsilon}^{\frac{\pi}{2}}\sin^{-1} r_p dr.\end{aligned}\tag{6}$$

Next we are to estimate $\int_{\mathbb{S}^n}|\Delta f_\varepsilon|^2dV$. When $0<r_p<\frac{\pi}{2}$, the distance function $d(p,x)$ is smooth, and thus

$$\begin{aligned}\Delta f_\varepsilon=&\mathrm{div}(\nabla f_\varepsilon)\\ =&\mathrm{div}\left(\left(\frac{1}{\varepsilon}H'(\frac{r_p}{\varepsilon})(\sin r)^{-\frac{n-4}{2}}-\frac{n-4}{2}H(\frac{r_p}{\varepsilon})(\sin r_p)^{-\frac{n-2}{2}}\cos r_p\right)\nabla r_p\right)\end{aligned}$$

$$\begin{aligned}=&\left(\frac{1}{\varepsilon}H'(\frac{r_p}{\varepsilon})(\sin r_p)^{-\frac{n-4}{2}}-\frac{n-4}{2}H(\frac{r_p}{\varepsilon})(\sin r_p)^{-\frac{n-2}{2}}\cos r_p\right)(n-1)\cot r_p\\ &+\left(\frac{1}{\varepsilon}H'(\frac{r_p}{\varepsilon})(\sin r_p)^{-\frac{n-4}{2}}-\frac{n-4}{2}H(\frac{r_p}{\varepsilon})(\sin r_p)^{-\frac{n-2}{2}}\cos r_p\right)'\\ =&\frac{1}{\varepsilon^2}(\sin r_p)^{-\frac{n-4}{2}}H''(\frac{r_p}{\varepsilon})+\frac{3}{\varepsilon}(\sin r_p)^{-\frac{n-2}{2}}\cos r_p H'(\frac{r_p}{\varepsilon})\\ &-\frac{n-4}{2}H(\frac{r_p}{\varepsilon})\left[\frac{n}{2}(\sin r_p)^{-\frac{n}{2}}\cos^2 r_p-(\sin r_p)^{-\frac{n-4}{2}}\right],\end{aligned}$$

and when $\frac{\pi}{2}<r_p<\pi$, one can get the same formula as above by letting $r_q=\pi-r_p$. Therefore,

$$\int_{\mathbb{S}^n}|\Delta f_\varepsilon|^2dV=\int_{B_p(\frac{\pi}{2})}|\Delta f_\varepsilon|^2dV+\int_{B_q(\frac{\pi}{2})}|\Delta f_\varepsilon|^2dV=2\int_{B_p(\frac{\pi}{2})}|\Delta f_\varepsilon|^2dV,$$

and thus by Minkowski inequality,

$$
\begin{aligned}
&\frac{1}{\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}}\left(\int_{\mathbb{S}^n}|\Delta f_\varepsilon|^2 dV\right)^{\frac{1}{2}}\\
=&\frac{\sqrt{2}}{\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}}\left(\int_{B_p(\frac{\pi}{2})}|\Delta f_\varepsilon|^2 dV\right)^{\frac{1}{2}}\\
\leq&\sqrt{2}\left(\int_\varepsilon^{\frac{\pi}{2}} H^2(\frac{r_p}{\varepsilon})\left|\frac{-n(n-4)}{4}(\sin r_p)^{-\frac{n}{2}}\cos^2 r_p+\frac{n-4}{2}(\sin r_p)^{-\frac{n-4}{2}}\right|^2(\sin r_p)^{n-1}dr\right)^{\frac{1}{2}} \qquad (7)\\
&+\frac{3\sqrt{2}}{\varepsilon}\left(\int_\varepsilon^{2\varepsilon}\left|(\sin r_p)^{-\frac{n-2}{2}}\cos r_p H'(\frac{r_p}{\varepsilon})\right|^2(\sin r_p)^{n-1}dr\right)^{\frac{1}{2}}\\
&+\frac{\sqrt{2}}{\varepsilon^2}\left(\int_\varepsilon^{2\varepsilon}\left|(\sin r_p)^{-\frac{n-4}{2}}H''(\frac{r_p}{\varepsilon})\right|^2(\sin r_p)^{n-1}dr\right)^{\frac{1}{2}}\\
=&I+II+III.
\end{aligned}
$$

A straightforward calculation yields

$$
\begin{aligned}
I\leq&\sqrt{2}\left(\int_\varepsilon^{\frac{\pi}{2}}\left|\frac{-n(n-4)}{4}(\sin r_p)^{-\frac{n}{2}}+\frac{(n-4)(n+2)}{4}(\sin r_p)^{-\frac{n-4}{2}}\right|^2(\sin r_p)^{n-1}dr\right)^{\frac{1}{2}},\\
II\leq&\frac{3\sqrt{2}}{\varepsilon}\max_{t\in[0,2]}H'(t)\left(\int_\varepsilon^{2\varepsilon}\left|(\sin r_p)^{-\frac{n-2}{2}}\cos r_p\right|^2(\sin r_p)^{n-1}dr\right)^{\frac{1}{2}}\\
\leq&\frac{3\sqrt{2}}{\varepsilon}\max_{t\in[0,2]}H'(t)\left(\int_\varepsilon^{2\varepsilon}r_p dr\right)^{\frac{1}{2}}=3\sqrt{3}\max_{t\in[0,2]}H'(t),\\
III\leq&\frac{2}{\varepsilon^2}\max_{t\in[0,2]}H''(t)\left(\int_\varepsilon^{2\varepsilon}\sin^3 r_p dr\right)^{\frac{1}{2}}\leq\frac{2}{\varepsilon^2}\max_{t\in[0,2]}H''(t)\left(\int_\varepsilon^{2\varepsilon}r_p^3 dr\right)^{\frac{1}{2}}=\sqrt{15}\max_{t\in[0,2]}H''(t).
\end{aligned}
$$

Since $f_\varepsilon(r_p)$ can be approximated by smooth functions on the sphere $\mathbb{S}^n$, then, by (5)–(7), it holds that

$$
\begin{aligned}
&\inf_{f\in C^\infty(\mathbb{S}^n)\setminus\{0\}}\frac{\int_{\mathbb{S}^n}|\Delta f|^2 dV+C(n)\int_{\mathbb{S}^n}\frac{f^2}{\sin^2 r_p}dV}{\int_{\mathbb{S}^n}\frac{f^2}{\sin^4 r_p}dV}\\
\leq&\frac{\int_{\mathbb{S}^n}|\Delta f_\varepsilon|^2 dV+C(n)\int_{\mathbb{S}^n}\frac{f_\varepsilon^2}{\sin^2 r_p}dV}{\int_{\mathbb{S}^n}\frac{f_\varepsilon^2}{\sin^4 r_p}dV}\\
\leq&\frac{\pi C(n)}{2\int_{2\varepsilon}^{\frac{\pi}{2}}\sin^{-1}r_p dr}+\left(\frac{I+II+III}{\sqrt{2}\sqrt{\int_{2\varepsilon}^{\frac{\pi}{2}}\sin^{-1}r_p dr}}\right)^2.
\end{aligned}
$$

Letting $\varepsilon\to 0^+$, we have

$$
\begin{aligned}
&\inf_{f\in C^\infty(\mathbb{S}^n)\setminus\{0\}} \frac{\int_{\mathbb{S}^n} |\Delta f|^2 dV + C(n)\int_{\mathbb{S}^n} \frac{f^2}{\sin^2 r_p} dV}{\int_{\mathbb{S}^n} \frac{f^2}{\sin^4 r_p} dV} \\
\leq &\lim_{\varepsilon\to 0^+} \frac{\int_\varepsilon^{\frac{\pi}{2}} \left| \frac{-n(n-4)}{4} (\sin r_p)^{-\frac{n}{2}} + \frac{(n-4)(n+2)}{4} (\sin r_p)^{-\frac{n-4}{2}} \right|^2 (\sin r_p)^{n-1} dr}{\int_{2\varepsilon}^{\frac{\pi}{2}} \sin^{-1} r_p dr} \\
= &\lim_{\varepsilon\to 0^+} \frac{-\left| \frac{-n(n-4)}{4} (\sin\varepsilon)^{-\frac{n}{2}} + \frac{(n-4)(n+2)}{4} (\sin\varepsilon)^{-\frac{n-4}{2}} \right|^2 (\sin\varepsilon)^{n-1}}{-2\sin(2\varepsilon)^{-1}} \\
= &\frac{n^2(n-4)^2}{16}.
\end{aligned}
$$

This completes the proof. □

Funding: This project is supported by AHNSF (1608085MA03), NNSFC(11471246) and KLAMFJPU (No. SX201805).

Conflicts of Interest: The authors declare no conflict of interest.

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