mathematics

MDPI

Article

Planar Graphs under Pythagorean Fuzzy Environment

Muhammad Akram [1,*], Jawaria Mohsan Dar [1] and Adeel Farooq [2]

[1] Department of Mathematics, University of the Punjab, New Campus, Lahore 54590, Pakistan; jiadar4@gmail.com

[2] Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Lahore 54590, Pakistan; adeelfarooq@cuilahore.edu.pk

* Correspondence: m.akram@pucit.edu.pk; Tel.: +92-99231241

Received: 18 October 2018; Accepted: 21 November 2018; Published: 25 November 2018

Abstract: Graph theory plays a substantial role in structuring and designing many problems. A number of structural designs with crossings can be found in real world scenarios. To model the vagueness and uncertainty in graphical network problems, many extensions of graph theoretical ideas are introduced. To deal with such uncertain situations, the present paper proposes the concept of Pythagorean fuzzy multigraphs and Pythagorean fuzzy planar graphs with some of their eminent characteristics by investigating Pythagorean fuzzy planarity value with strong, weak and considerable edges. A close association is developed between Pythagorean fuzzy planar and dual graphs. This paper also includes a brief discussion on non-planar Pythagorean fuzzy graphs and explores the concepts of isomorphism, weak isomorphism and co-weak isomorphism for Pythagorean fuzzy planar graphs. Moreover, it presents a problem that shows applicability of the proposed concept.

Keywords: Pythagorean fuzzy planar graphs; Pythagorean fuzzy planarity value; Pythagorean fuzzy dual graphs; weak and co-weak isomorphism

1. Introduction

Graph theory is rapidly moving into the core of mathematics due to its applications in various fields, including physics, biochemistry, biology, electrical engineering, astronomy, operations research and computer science. The theory of planar graphs is based on Euler's polyhedral formula, which is related to the polyhedron edges, vertices and faces. In modern era, the applications of planar graphs occur naturally such as designing and structuring complex radio electronic circuits, railway maps, planetary gearbox and chemical molecules. While modeling an urban city, pipelines, railway lines, subway tunnels, electric transmission lines and metro lines are extremely important. Crossing is beneficial as it helps in utilizing less space and is inexpensive, but there are some drawbacks too. As the crossing of such lines is quite dangerous for human lives, but, by taking certain amount of security measures, it can be made. The crossing between the uncrowded route and crowded route is less risky as compared to the crossing between two crowded routes. In fuzzy graphs, the terms' uncrowded route and crowded route referred to as weak edge and strong edge. The allowance of such crossings leads to fuzzy planar graph theory [1–3].

In the long-established mathematical models, the information about the complex phenomena is very precise. However, it is an impractical supposition that the exact information is sufficient to model the real world problems that involve inherent haziness. Fuzzy set theory, originally proposed by Zadeh [4], is the most efficient tool having the capability to deal with imprecise and incomplete information. To cope with imprecise and incomplete information, consisting of doubts in human judgement, the fuzzy set shows some restrictions. Hence, for characterizing the hesitancy more explicitly, fuzzy sets were extended to intuitionistic fuzzy sets (IFSs) by Atanassov [5], which assigns a membership grade μ and a nonmembership grade ν to the objects, satisfying the condition $\mu + \nu \leq 1$ and the hesitancy part $\pi = 1 - \mu - \nu$. The IFSs have gained extensive attention and have been broadly applied in different areas of real life. The limitation $\mu + \nu \leq 1$ confines the choice of the membership and nonmembership grades in IFS. To evade this situation, Yager [6–8] initiated the idea of Pythagorean fuzzy set (PFS), depicted by a membership grade μ and a nonmembership grade ν with the condition $\mu^2 + \nu^2 \leq 1$. Zhang and Xu [9] introduced the concept of Pythagorean fuzzy number (PFN) for interpreting the dual aspects of an element. The motivation of PFSs can be described as, in a decision-making environment, a specialist gives the preference information about an alternative with the membership grade 0.6 and the non-membership grade 0.5. It is noted that the IFN fails to address this situation, as $0.6 + 0.5 > 1$. However, $(0.6)^2 + (0.5)^2 \leq 1$. Thus, PFSs comprise more uncertainties than IFSs and are usually capable of accommodating greater degrees of uncertainty. The comparison between intuitionistic fuzzy number space and Pythagorean fuzzy number space is shown in Figure 1.

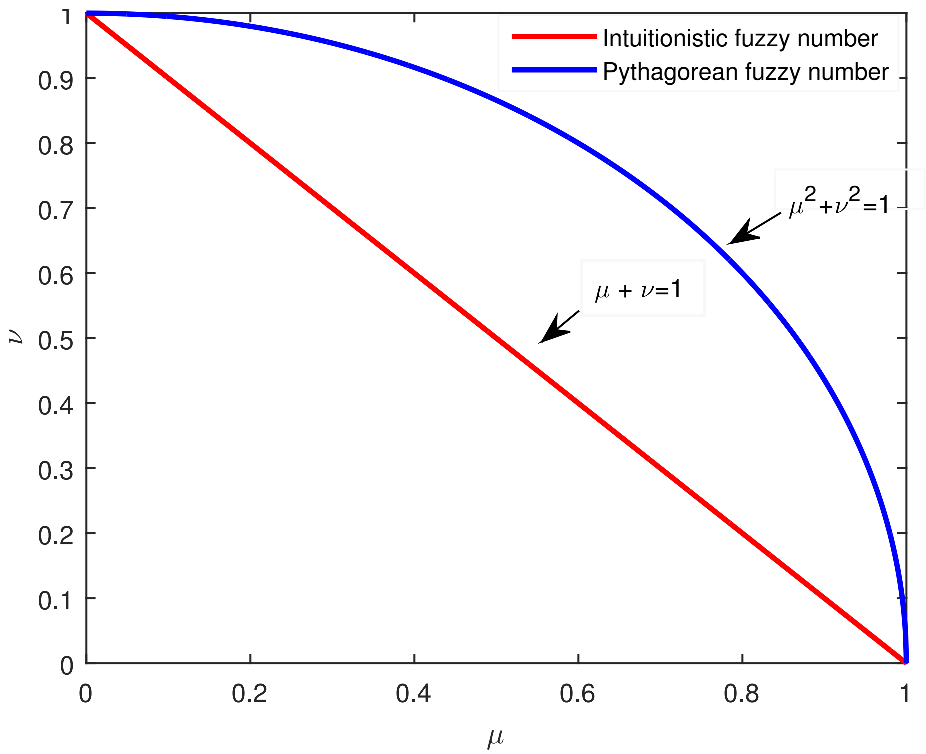


Figure 1. Comparison of spaces of the IFN and the PFN.

Graphs are the pictorial representation that bond the objects and highlight their information. To emphasis a real-world problem, the bondedness between the objects occurs due to some relations. However, when there exists uncertainty and haziness in the bonding, then the corresponding graph model can be taken as a fuzzy graph model. In 1973, Kaufmann [10] presented the idea of fuzzy graphs, based on Zadeh's fuzzy relation in 1971. Afterwards, Rosenfeld [11] discussed several basic graphs' theoretical concepts in fuzzy graphs. Some remarks on fuzzy graphs were explored by Bhattacharya [12]. Mordeson and Peng [13] discussed fuzzy graphs' operations and their properties. The concept of intuitionistic fuzzy relations and intuitionistic fuzzy graphs was initiated by Shannon and Atanassov [14] and some of their eminent properties were explored in [15]. Parvathi et al. [16] described operations on intuitionistic fuzzy graphs. Many new concepts involving intuitionistic fuzzy hypergraphs and strong intuitionistic fuzzy graphs were given by Akram et al. [17,18]. On the basis of Akram and Davvaz's IFGs [17], Naz et al. [19] gave the idea of PFGs along with applications. Some results related to PFGs have been discussed in [20]. Pythagorean fuzzy graph energy was studied by Naz and Akram [21]. Dhavudh and Srinivasan [22,23] coped with IFGs2k. Verma et al. and Akram et al. [24] proposed some operations of PFGs. Recently, Akram et al. [25] introduced certain graphs under Pythagorean fuzzy environment. Abdul-Jabbar et al. [26] put forward the idea of a fuzzy dual graph and investigated some of its crucial properties. Yager [27] used the notation of fuzzy bags to define fuzzy multiset. Pal et al. [1] and Samanta et al. [2] developed the notion of fuzzy planar graphs and studied its properties. Pramanik et al. [3] discussed special planar fuzzy graphs. Furthermore, some extensions of planar fuzzy graph were studied [28–30]. For other terminologies and applications, one can see [31–37]. Under the Pythagorean fuzzy environment, the graph theoretical results have been extended in this paper. The structure and applicability of planar graphs are full of surprises. For example, in the designing of complex radioelectronic circuits, elements can be arranged in such a manner that the conductors connecting each other do not intersect. This problem can be solved by using the concept of planar graphs. This research paper describes the concept of Pythagorean fuzzy multigraphs (PFMGs), Pythagorean fuzzy planar graphs (PFPGs) and Pythagorean fuzzy dual graphs (PFDGs) that allow the mathematical structuring of a road or communication network. By using these graphs, several real world problems can be analyzed and designed. The work explores a significant property known as planarity. Meanwhile, a critical analysis is done on nonplanar PFGs. A close association is developed between Pythagorean fuzzy planar graphs and Pythagorean fuzzy dual graphs. Furthermore, the concept of isomorphism, co-weak isomorphism and weak isomorphism are established between PFPGs. Some substantial results are investigated. In the end, an application of PFPG is discussed.

2. Pythagorean Fuzzy Multigraph

Definition 1. *A Pythagorean fuzzy multiset (PFMS) $\mathscr{A}$ taken from nonempty set $\mathcal{X}$ is classified by two functions, 'count membership' and 'count non-membership' of $\mathscr{A}$ denoted by $\mathcal{CM}_{\mathscr{A}}$ and $\mathcal{CN}_{\mathscr{A}}$ and given as $\mathcal{CM}_{\mathscr{A}} : \mathcal{X} \to \mathcal{Q}$ and $\mathcal{CN}_{\mathscr{A}} : \mathcal{X} \to \mathcal{Q}$, where $\mathcal{Q}$ is the set of all crisp multisets taken from the unit interval $[0, 1]$, such that, for each $r \in \mathcal{X}$, the degree of membership sequence is described as a decreasingly ordered sequence of objects in $\mathcal{CM}_{\mathscr{A}}(r)$, represented as $(\mu^1_{\mathscr{A}}(r), \mu^2_{\mathscr{A}}(r), \ldots, \mu^p_{\mathscr{A}}(r))$, where $\mu^1_{\mathscr{A}}(r) \geq \mu^2_{\mathscr{A}}(r) \geq \ldots \geq \mu^p_{\mathscr{A}}(r)$ and the corresponding degree of non-membership sequence will be represented as $(\nu^1_{\mathscr{A}}(r), \nu^2_{\mathscr{A}}(r), \ldots, \nu^p_{\mathscr{A}}(r))$ such that $(\mu^j_{\mathscr{A}}(r))^2 + (\nu^j_{\mathscr{A}}(r))^2 \leq 1$ for all $r \in \mathcal{X}$ and $j = 1, 2, \ldots, p$. A PFMS $\mathscr{A}$ is denoted by*

$$\{\langle r, (\mu^1_{\mathscr{A}}(r), \mu^2_{\mathscr{A}}(r), \ldots, \mu^p_{\mathscr{A}}(r)), (\nu^1_{\mathscr{A}}(r), \nu^2_{\mathscr{A}}(r), \ldots, \nu^p_{\mathscr{A}}(r))\rangle | r \in \mathcal{X}\}.$$

Multigraphs play a crucial role for any kind of network design where multiedges are involved. Likewise, in Pythagorean fuzzy graph theory, Pythagorean fuzzy multigraphs have vast usage. As Pythagorean fuzzy planar graph can not be defined without Pythagorean fuzzy multigraph, hence, on the basis of Pythagorean fuzzy multiset, we propose the idea of Pythagorean fuzzy multigraph.

Definition 2. *Let $\mathscr{A} = (\mu_{\mathscr{A}}, \nu_{\mathscr{A}})$ be a PFS on $\mathcal{X}$ and let $\mathscr{B} = \{(rs, \mu_{\mathscr{B}}(rs)_j, \nu_{\mathscr{B}}(rs)_j), j = 1, 2, \ldots, n \mid rs \in \mathscr{V} \times \mathscr{V}\}$ be a PFMS on $\mathscr{V} \times \mathscr{V}$ such that*

$$\begin{aligned} \mu_{\mathscr{B}}(rs)_j &\leq \min\{\mu_{\mathscr{A}}(r), \mu_{\mathscr{A}}(s)\}, \\ \nu_{\mathscr{B}}(rs)_j &\leq \max\{\nu_{\mathscr{A}}(r), \nu_{\mathscr{A}}(s)\}, \end{aligned}$$

$\forall j = 1, 2, \ldots, n$. Then, $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ is known as Pythagorean fuzzy multigraph.

Example 1. *Consider a multigraph $\mathscr{G}^* = (\mathscr{V}, \mathscr{E})$, where $\mathscr{V} = \{r_1, r_2, r_3, r_4\}$ and $\mathscr{E} = \{r_1r_2, r_2r_3, r_3r_4, r_3r_4, r_2r_4, r_2r_4, r_1r_4, r_1r_4\}$. Let $\mathscr{A}$ and $\mathscr{B}$ be PF vertex set and PF multiedge set defined on $\mathscr{V}$ and $\mathscr{V} \times \mathscr{V}$, respectively,*

$$\mathscr{A} = \left\langle \left(\frac{r_1}{0.4}, \frac{r_2}{0.2}, \frac{r_3}{0.6}, \frac{r_4}{0.3}\right), \left(\frac{r_1}{0.7}, \frac{r_2}{0.9}, \frac{r_3}{0.5}, \frac{r_4}{0.8}\right) \right\rangle \text{ and}$$

$$\mathscr{B} = \left\langle \left(\frac{r_1r_2}{0.2}, \frac{r_2r_3}{0.15}, \frac{r_3r_4}{0.3}, \frac{r_3r_4}{0.3}, \frac{r_2r_4}{0.1}, \frac{r_2r_4}{0.2}, \frac{r_1r_4}{0.2}, \frac{r_1r_4}{0.3} \right), \left(\frac{r_1r_2}{0.9}, \frac{r_2r_3}{0.45}, \frac{r_3r_4}{0.8}, \frac{r_3r_4}{0.75}, \frac{r_2r_4}{0.9}, \frac{r_2r_4}{0.9}, \frac{r_1r_4}{0.8}, \frac{r_1r_4}{0.75} \right) \right\rangle.$$

By direct calculation, one can look from Figure 2 that it is a PFMG.

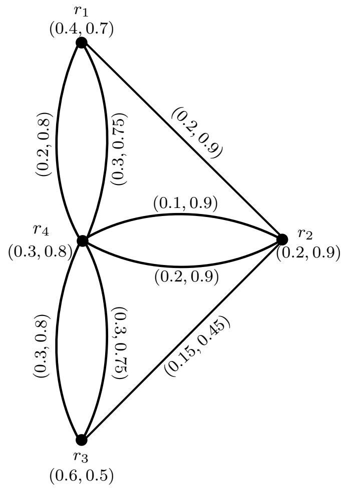


Figure 2. Pythagorean fuzzy multigraph.

Definition 3. *Let* $\mathscr{B} = \{(rs, \mu_{\mathscr{B}}(rs)_j, \nu_{\mathscr{B}}(rs)_j), j = 1, 2, \dots, n \mid rs \in \mathscr{V} \times \mathscr{V}\}$ *be a PF multiedge set in PFMG* $\mathscr{G}$*; then,*

1. *The order of* $\mathscr{G}$ *is represented by* $\mathscr{O}(\mathscr{G})$ *and defined as* $\mathscr{O}(\mathscr{G}) = (\sum_{r \in \mathscr{V}} \mu_{\mathscr{A}}(r), \sum_{r \in \mathscr{V}} \nu_{\mathscr{A}}(r))$.
2. *The size of* $\mathscr{G}$ *is represented by* $\mathscr{S}(\mathscr{G})$ *and defined as* $\mathscr{S}(\mathscr{G}) = (\sum_{j=1}^{n} \mu_{\mathscr{B}}(rs)_j, \sum_{j=1}^{n} \nu_{\mathscr{B}}(rs)_j)$ *for all* $rs \in \mathscr{V} \times \mathscr{V}$.
3. *The degree of vertex* $r \in \mathscr{V}$ *is represented by* $deg_{\mathscr{G}}(r)$ *and defined as* $deg_{\mathscr{G}}(r) = (\sum_{j=1}^{n} \mu_{\mathscr{B}}(rs)_j, \sum_{j=1}^{n} \nu_{\mathscr{B}}(rs)_j)$ *for all* $s \in \mathscr{V}$.
4. *The total degree of vertex* $r \in \mathscr{V}$ *is represented by* $tdeg_{\mathscr{G}}(s)$ *and defined as* $tdeg_{\mathscr{G}}(r) = (\sum_{j=1}^{n} \mu_{\mathscr{B}}(rs)_j + \mu_{\mathscr{A}}(r), \sum_{j=1}^{n} \nu_{\mathscr{B}}(rs)_j + \nu_{\mathscr{A}}(r))$ *for all* $s \in \mathscr{V}$.

Definition 4. *Let* $\mathscr{G}$ *be a Pythagorean fuzzy multigraph on* $\mathscr{G}^*$*. If each vertex has the same degree of membership and nonmembership values in* $\mathscr{G}$*, then* $\mathscr{G}$ *is known as a regular Pythagorean fuzzy multigraph.*

Example 2. *In Example 1, by direct calculation, one can see*

1. *The order of* $\mathscr{G}$ $\mathscr{O}(\mathscr{G}) = (\sum_{r \in \mathscr{V}} \mu_{\mathscr{A}}(r), \sum_{r \in \mathscr{V}} \nu_{\mathscr{A}}(r)) = (1.5, 2.9)$.
2. *The size of* $\mathscr{G}$ $\mathscr{S}(\mathscr{G}) = (\sum_{j=1}^{n} \mu_{\mathscr{B}}(rs)_j, \sum_{j=1}^{n} \nu_{\mathscr{B}}(rs)_j) = (1.75, 6.25)$.
3. *The degree of the vertices are* $deg_{\mathscr{G}}(r_1) = (0.7, 2.45)$, $deg_{\mathscr{G}}(r_2) = (0.65, 3.15)$, $deg_{\mathscr{G}}(r_3) = (0.75, 2)$, $deg_{\mathscr{G}}(r_4) = (1.4, 4.9)$.
4. *The total degree of the vertices are* $tdeg_{\mathscr{G}}(r_1) = (1.1, 3.15)$, $tdeg_{\mathscr{G}}(r_2) = (0.85, 4.05)$, $tdeg_{\mathscr{G}}(r_3) = (1.35, 2.5)$, $tdeg_{\mathscr{G}}(r_4) = (1.7, 5.7)$.

In addition, $\mathscr{G}$ *is not regular as degree of membership and nonmembership values of the vertices are not equal.*

Definition 5. *Let* $\mathscr{G}$ *be a PFMG such that* $\mathscr{B} = \{(rs, \mu_{\mathscr{B}}(rs)_j, \nu_{\mathscr{B}}(rs)_j), j = 1, 2, \dots, n \mid rs \in \mathscr{V} \times \mathscr{V}\}$*. Then,*

1. *The degree of an edge* $rs \in \mathscr{V} \times \mathscr{V}$ *is represented by* $\mathscr{D}_{\mathscr{G}}(rs)$ *and defined as* $\mathscr{D}_{\mathscr{G}}((rs)) = ((deg_{\mu})_{\mathscr{G}}(r) + (deg_{\mu})_{\mathscr{G}}(s) - 2\mu_{\mathscr{B}}(rs)_j, (deg_{\nu})_{\mathscr{G}}(r) + (deg_{\nu})_{\mathscr{G}}(s) - 2\nu_{\mathscr{B}}(rs)_j)$.

2. *The total degree of an edge* $rs \in \mathscr{V} \times \mathscr{V}$ *is represented by* $\mathscr{D}_{\mathscr{G}}(rs)$ *and defined as* $t\mathscr{D}_{\mathscr{G}}((rs)) = ((deg_{\mu})_{\mathscr{G}}(r) + (deg_{\mu})_{\mathscr{G}}(s) - \mu_{\mathscr{B}}(rs)_j, (deg_{\nu})_{\mathscr{G}}(r) + (deg_{\nu})_{\mathscr{G}}(s) - \nu_{\mathscr{B}}(rs)_j)$,

where $(rs)_j$ *is the jth edge between r and s.*

Definition 6. *A Pythagorean fuzzy multigraph* $\mathscr{G}$ *is known as edge regular, if the degree of membership and nonmembership values of all edges in* $\mathscr{G}$ *are equal.*

Example 3. *In Example 1, the degree of edges are*
$\mathscr{D}_{\mathscr{G}}(r_1r_2) = (0.95, 3.8)$, $\mathscr{D}_{\mathscr{G}}(r_2r_3) = (1.1, 4.25)$, $\mathscr{D}_{\mathscr{G}}(r_3r_4) = (1.55, 5.3)$, $\mathscr{D}_{\mathscr{G}}(r_3r_4) = (1.55, 5.4)$,
$\mathscr{D}_{\mathscr{G}}(r_2r_4) = (1.85, 6.25)$, $\mathscr{D}_{\mathscr{G}}(r_2r_4) = (1.65, 6.25)$, $\mathscr{D}_{\mathscr{G}}(r_1r_4) = (0.95, 4.1)$, $\mathscr{D}_{\mathscr{G}}(r_1r_4) = (0.75, 4.1)$,

whereas the total degree of edges are
$t\mathscr{D}_{\mathscr{G}}(r_1r_2) = (1.15, 4.7)$, $t\mathscr{D}_{\mathscr{G}}(r_2r_3) = (1.25, 4.7)$, $t\mathscr{D}_{\mathscr{G}}(r_3r_4) = (1.85, 6.1)$, $t\mathscr{D}_{\mathscr{G}}(r_3r_4) = (1.85, 6.15)$,
$t\mathscr{D}_{\mathscr{G}}(r_2r_4) = (1.95, 7.15)$, $t\mathscr{D}_{\mathscr{G}}(r_2r_4) = (1.85, 7.15)$, $t\mathscr{D}_{\mathscr{G}}(r_1r_4) = (1.15, 4.8)$, $t\mathscr{D}_{\mathscr{G}}(r_1r_4) = (1.05, 4.85)$.

In addition, $\mathscr{G}$ *is not an edge regular Pythagorean multigraph as degree of the membership and nonmembership values are not the same.*

Theorem 1. *Let* $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ *be a Pythagorean fuzzy multigraph. If* $\mathscr{G}$ *is regular and edge regular Pythagorean fuzzy multigraph, then the membership values* $\mu_{\mathscr{B}}(rs)_j$ *and nonmembership values* $\nu_{\mathscr{B}}(rs)_j$ *for each edge* $rs \in \mathscr{V} \times \mathscr{V}$ *are constant.*

Proof. Let $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ be a Pythagorean fuzzy multigraph. Assume that $\mathscr{G}$ is regular and edge regular Pythagorean fuzzy multigraph, then there exist constants p_1, p_2 and q_1, q_2, respectively, such that, for each vertex,

$$deg_{\mathscr{G}}(r) = ((deg_{\mu})_{\mathscr{G}}(r), (deg_{\nu})_{\mathscr{G}}(r)) = (p_1, p_2).$$

For each edge $rs \in \mathscr{V} \times \mathscr{V}$,

$$\begin{aligned} \mathscr{D}_{\mathscr{G}}(rs) &= ((\mathscr{D}_{\mu})_{\mathscr{G}}(rs), (\mathscr{D}_{\nu})_{\mathscr{G}}(rs)) \\ &= ((deg_{\mu})_{\mathscr{G}}(r) + (deg_{\mu})_{\mathscr{G}}(s) - 2\mu_{\mathscr{B}}(rs)_j, (deg_{\nu})_{\mathscr{G}}(r) + (deg_{\nu})_{\mathscr{G}}(s) - 2\nu_{\mathscr{B}}(rs)_j) \\ &= (q_1, q_2). \end{aligned}$$

Hence, for the membership and nonmembership values,

$$\begin{aligned} p_1 + p_1 - 2\mu_{\mathscr{B}}(rs)_j &= 2q_1, \\ 2p_1 - 2\mu_{\mathscr{B}}(rs)_j &= 2q_1, \\ 2p_1 - 2q_1 &= 2\mu_{\mathscr{B}}(rs)_j, \\ p_1 - q_1 &= \mu_{\mathscr{B}}(rs)_j, \end{aligned}$$

$$\begin{aligned} p_2 + p_2 - 2\mu_{\mathscr{B}}(rs)_j &= 2q_2, \\ 2p_2 - 2\mu_{\mathscr{B}}(rs)_j &= 2q_2, \\ 2p_2 - 2q_2 &= 2\mu_{\mathscr{B}}(rs)_j, \\ p_2 - q_2 &= \mu_{\mathscr{B}}(rs)_j. \end{aligned}$$

Thus, we conclude that the membership and nonmembership values of a regular Pythagorean fuzzy multigraph with edge regular are constant. □

Theorem 2. *Let $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ be a Pythagorean fuzzy multigraph on a crisp graph $\mathscr{G}^* = (\mathscr{V}, \mathscr{E})$. If $\mathscr{G}^*$ is p-regular multigraph, $\mu_{\mathscr{B}}(rs)_j$ and $\nu_{\mathscr{B}}(rs)_j$ are constant for each edge $rs \in \mathscr{V} \times \mathscr{V}$, then $\mathscr{G}$ is regular and edge regular Pythagorean fuzzy multigraph.*

Proof. Assume that $\mathscr{G}^* = (\mathscr{V}, \mathscr{E})$ is a p - regular multigraph. Let $\mu_{\mathscr{B}}(rs)_j = q_1$ and $\nu_{\mathscr{B}}(rs)_j = q_2$. Then, for each vertex $r \in \mathscr{V}$,

$$\begin{aligned}
deg_{\mathscr{G}}(r) &= ((deg_\mu)_{\mathscr{G}}(r), (deg_\nu)_{\mathscr{G}}(r)) \\
&= (\sum_{s\neq r}\mu_{\mathscr{B}}(rs)_j, \sum_{y\neq r}\nu_{\mathscr{B}}(rs)_j) \\
&= (p \times q_1, p \times q_2) \\
&= (\sum_{r\neq y}\mu_{\mathscr{B}}(sr)_j, \sum_{r\neq s}\nu_{\mathscr{B}}(sr)_j) \\
&= ((deg_\mu)_{\mathscr{G}}(s), (deg_\nu)_{\mathscr{G}}(s)) \\
&= deg_{\mathscr{G}}(s).
\end{aligned}$$

For each edge $rs \in \mathscr{V} \times \mathscr{V}$,

$$\begin{aligned}
\mathscr{D}_{\mathscr{G}}(rs) &= ((\mathscr{D}_\mu)_{\mathscr{G}}(rs), (\mathscr{D}_\nu)_{\mathscr{G}}(rs)) \\
&= ((deg_\mu)_{\mathscr{G}}(r) + (deg_\mu)_{\mathscr{G}}(s) - 2\mu_{\mathscr{B}}(rs)_j, (deg_\nu)_{\mathscr{G}}(r) + (deg_\nu)_{\mathscr{G}}(s) - 2\nu_{\mathscr{B}}(rs)_j) \\
&= ((p \times q_1) + (p \times q_1) - 2(q_1), (p \times q_2) + (p \times q_2) - 2(q_2)) \\
&= (2q_1(p-1), 2q_2(p-1)).
\end{aligned}$$

Hence, $\mathscr{G}$ is regular and edge regular Pythagorean fuzzy multigraph. □

Definition 7. *Let $\mathscr{B} = \{(rs, \mu_{\mathscr{B}}(rs)_j, \nu_{\mathscr{B}}(rs)_j), j = 1, 2, \ldots, n \mid rs \in \mathscr{V} \times \mathscr{V}\}$ be a PF multiedge set in PFMG $\mathscr{G}$. A multiedge rs of $\mathscr{G}$ is said to be strong if*

$$\mu_{\mathscr{B}}(rs)_j \geq \frac{1}{2}\{\mu_{\mathscr{A}}(r) \wedge \mu_{\mathscr{A}}(s)\},$$

$$\nu_{\mathscr{B}}(rs)_j \leq \frac{1}{2}\{\nu_{\mathscr{A}}(r) \vee \nu_{\mathscr{A}}(s)\},$$

$\forall j = 1, 2, \ldots, n.$

Example 4. *In Example 1, $(\mu_{\mathscr{B}}(r_2r_3), \nu_{\mathscr{B}}(r_2r_3))$ is a strong edge as*

$$0.15 > \frac{1}{2}\{0.6 \wedge 0.2\} \text{ and } 0.45 = \frac{1}{2}\{0.5 \vee 0.9\}.$$

Definition 8. *Let $\mathscr{B} = \{(rs, \mu_{\mathscr{B}}(rs)_j, \nu_{\mathscr{B}}(rs)_j), j = 1, 2, \ldots, n \mid rs \in \mathscr{V} \times \mathscr{V}\}$ be a PF multiedge set in PFMG $\mathscr{G}$. A multiedge rs of $\mathscr{G}$ is said to be effective if*

$$\mu_{\mathscr{B}}(rs)_j = \{\mu_{\mathscr{A}}(r) \wedge \mu_{\mathscr{A}}(s)\},$$

$$\nu_{\mathscr{B}}(rs)_j = \{\nu_{\mathscr{A}}(r) \vee \nu_{\mathscr{A}}(s)\},$$

where j is fixed integer.

Example 5. *In Example 1, $(\mu_{\mathscr{B}}(r_1r_2), \nu_{\mathscr{B}}(r_1r_2))$ is an effective edge as*

$$0.2 = \{0.4 \wedge 0.2\} \text{ and } 0.9 = \{0.7 \vee 0.9\}.$$

Definition 9. *Let $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ be a Pythagorean fuzzy multigraph and $\mathscr{B} = \{(rs, \mu_{\mathscr{B}}(rs)_j, \nu_{\mathscr{B}}(rs)_j),\ j = 1, 2, \ldots, n \mid rs \in \mathscr{V} \times \mathscr{V}\}$ be a Pythagorean fuzzy multiedge set. A PFMG $\mathscr{G}$ is said to be complete if*

$$\mu_{\mathscr{B}}(rs)_j = \{\mu_{\mathscr{A}}(r) \wedge \mu_{\mathscr{A}}(s)\},$$
$$\nu_{\mathscr{B}}(rs)_j = \{\nu_{\mathscr{A}}(r) \vee \nu_{\mathscr{A}}(s)\},$$

$\forall j = 1, 2, \ldots, n$ and $\forall r, s \in \mathscr{V}$.

Example 6. *Consider a multigraph $\mathscr{G}^* = (\mathscr{V}, \mathscr{E})$ where $\mathscr{V} = \{r_1, r_2, r_3, r_4\}$ and $\mathscr{E} = \{r_1r_2, r_1r_4, r_1r_4, r_1r_3, r_2r_3, r_2r_4, r_3r_4\}$. Let $\mathscr{A}$ and $\mathscr{B}$ be Pythagorean fuzzy vertex set and Pythagorean fuzzy multiedge set defined on $\mathscr{V}$ and $\mathscr{V} \times \mathscr{V}$, respectively.*

$$\mathscr{A} = \left\langle \left(\frac{r_1}{0.8}, \frac{r_2}{0.55}, \frac{r_3}{0.35}, \frac{r_4}{0.55} \right), \left(\frac{r_1}{0.6}, \frac{r_2}{0.6}, \frac{r_3}{0.8}, \frac{r_4}{0.7} \right) \right\rangle \text{ and}$$
$$\mathscr{B} = \left\langle \left(\frac{r_1r_2}{0.55}, \frac{r_1r_4}{0.55}, \frac{r_1r_4}{0.55}, \frac{r_1r_3}{0.35}, \frac{r_2r_3}{0.35}, \frac{r_2r_4}{0.55}, \frac{r_3r_4}{0.35} \right), \left(\frac{r_1r_2}{0.6}, \frac{r_1r_4}{0.7}, \frac{r_1r_4}{0.7}, \frac{r_1r_3}{0.8}, \frac{r_2r_3}{0.8}, \frac{r_2r_4}{0.7}, \frac{r_3r_4}{0.8} \right) \right\rangle.$$

Directly, one can see from Figure 3 that it is a complete Pythagorean fuzzy multigraph.

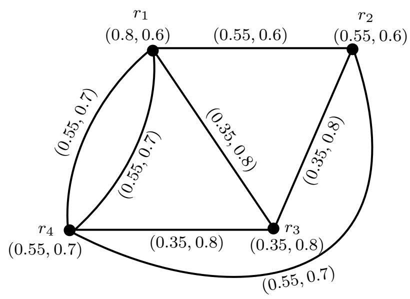


Figure 3. Complete Pythagorean fuzzy multigraph.

3. Pythagorean Fuzzy Planar Graphs

In planar graph, the intersection between edges is not acceptable. However, in this section, we determine a Pythagorean fuzzy planar graph in an interesting manner with a parameter called 'Pythagorean fuzzy Planarity'. Planarity is an amount that measures how much a graph is planar. It is very useful in connecting different networking models, structuring websites containing many pages, designing electronic chip, etc. Sometimes, crossing between edges can not be avoided so for this purpose we only consider minimum number of crossing. Hence, Pythagorean fuzzy planar graphs are important for these kinds of connections.

Some correlated terms are discussed below before going to the main definition.

Definition 10. *The strength of the Pythagorean fuzzy edge rs is defined as*

$$\mathscr{S}_{rs} = (\mathscr{M}_{rs}, \mathscr{N}_{rs}) = \left(\frac{\mu_{\mathscr{B}}(rs)_j}{\mu_{\mathscr{A}}(r) \wedge \mu_{\mathscr{A}}(s)}, \frac{\nu_{\mathscr{B}}(rs)_j}{\nu_{\mathscr{A}}(r) \vee \nu_{\mathscr{A}}(s)} \right).$$

An edge rs of PFMG is known as strong if $\mathscr{M}_{rs} \geq 0.5$ and $\mathscr{N}_{rs} \leq 0.5$ otherwise, known as weak.

Example 7. *Consider a multigraph $\mathscr{G}^* = (\mathscr{V}, \mathscr{E})$, where $\mathscr{V} = \{r_1, r_2, r_3\}$ and $\mathscr{E} = \{r_1r_2, r_1r_2, r_2r_3, r_2r_3, r_1r_3\}$. Let $\mathscr{A}$ and $\mathscr{B}$ be PF vertex set and PF multiedge set defined on $\mathscr{V}$ and $\mathscr{V} \times \mathscr{V}$, respectively.*

$$\mathscr{A} = \left\langle \left(\frac{r_1}{0.4}, \frac{r_2}{0.7}, \frac{r_3}{0.25} \right), \left(\frac{r_1}{0.65}, \frac{r_2}{0.35}, \frac{r_3}{0.8} \right) \right\rangle \text{ and}$$
$$\mathscr{B} = \left\langle \left(\frac{r_1r_2}{0.4}, \frac{r_1r_2}{0.4}, \frac{r_2r_3}{0.15}, \frac{r_2r_3}{0.2}, \frac{r_1r_3}{0.2} \right), \left(\frac{r_1r_2}{0.6}, \frac{r_1r_2}{0.3}, \frac{r_2r_3}{0.8}, \frac{r_2r_3}{0.75}, \frac{r_1r_3}{0.8} \right) \right\rangle.$$

The PFMG as shown in Figure 4 have edges $(r_1r_3, 0.2, 0.8)$, $(r_1r_2, 0.4, 0.6)$, $(r_1r_2, 0.4, 0.3)$, $(r_2r_3, 0.2, 0.75)$ *and* $(r_2r_3, 0.15, 0.8)$ *with strength* $\mathscr{S}_{r_1r_3} = (0.8, 1)$, $\mathscr{S}_{r_1r_2} = (1, 0.92)$, $\mathscr{S}_{r_1r_2} = (1, 0.46)$, $\mathscr{S}_{r_2r_3} = (0.8, 0.93)$ *and* $\mathscr{S}_{r_2r_3} = (0.6, 1)$, *respectively. Since* $\mathscr{S}_{r_1r_2} = (1 > 0.5, 0.46 < 0.5)$, *hence edge* r_1r_2 *is strong and the others are weak.*

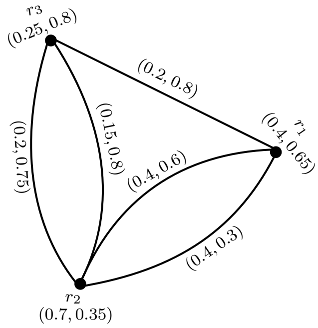


Figure 4. Pythagorean fuzzy multigraph.

Definition 11. *Let* $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ *be a Pythagorean fuzzy multigraph, where* $\mathscr{B}$ *contain two edges* $(uv, \mu_{\mathscr{B}}(uv)_j, \nu_{\mathscr{B}}(uv)_j)$ *and* $(rs, \mu_{\mathscr{B}}(rs)_k, \nu_{\mathscr{B}}(rs)_k)$ *intersecting at a point* $\mathscr{C}$ *(j and k are fixed integers). The intersecting value at the point (or cut point)* $\mathscr{C}$ *can be obtained as*

$$\mathscr{S}_{\mathscr{C}} = (\mathscr{M}_{\mathscr{C}}, \mathscr{N}_{\mathscr{C}}) = \left(\frac{\mathscr{M}_{uv} + \mathscr{M}_{rs}}{2}, \frac{\mathscr{N}_{uv} + \mathscr{N}_{rs}}{2} \right).$$

In Pythagorean fuzzy multigraph, $\mathscr{S}_{\mathscr{C}}$ *is inversely proportional to planarity i.e., if the number of points of intersections increases, planarity decreases.*

Definition 12. *Let* $\mathscr{G}$ *be a Pythagorean fuzzy multigraph. Let* $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_k$ *be the crossing points between the edges for geometric insight. Then,* $\mathscr{G}$ *is known as Pythagorean fuzzy planar graph with Pythagorean fuzzy planarity value* $\mathscr{F}$*, defined as*

$$\mathscr{F} = (\mathscr{F}_{\mathscr{M}}, \mathscr{F}_{\mathscr{N}}) = \left(\frac{1}{1 + \{\mathscr{M}_{\mathscr{C}_1} + \mathscr{M}_{\mathscr{C}_2} + \ldots + \mathscr{M}_{\mathscr{C}_k}\}}, \frac{1}{1 + \{\mathscr{N}_{\mathscr{C}_1} + \mathscr{N}_{\mathscr{C}_2} + \ldots + \mathscr{N}_{\mathscr{C}_k}\}} \right).$$

It is clear that $\mathscr{F} = (\mathscr{F}_{\mathscr{M}}, \mathscr{F}_{\mathscr{N}})$ *is bounded and* $0 < \mathscr{F}_{\mathscr{M}} \leq 1$, $0 < \mathscr{F}_{\mathscr{N}} \leq 1$. *If geometric representation of a PFPG has no intersecting point, then its Pythagorean fuzzy planarity value is considered as* $(1, 1)$ *with the underlying crisp graph as a crisp planar graph.*

Remark 1. *Every Pythagorean fuzzy graph is a PFPG with some definite planarity value.*

Example 8. *Consider a multigraph* $\mathscr{G}^* = (\mathscr{V}, \mathscr{E})$, *where* $\mathscr{V} = \{r_1, r_2, r_3, r_4, r_5, r_6\}$ *and* $\mathscr{E} = \{r_1r_2, r_1r_6, r_1r_4, r_2r_5, r_2r_5, r_2r_3, r_3r_4, r_3r_6, r_4r_5, r_6r_5\}$. *Let* $\mathscr{A}$ *and* $\mathscr{B}$ *be the PF vertex set and PF multiedge set defined on* $\mathscr{V}$ *and* $\mathscr{V} \times \mathscr{V}$, *respectively.*

$$\mathscr{A} = \left\langle \left(\frac{r_1}{0.75}, \frac{r_2}{0.6}, \frac{r_3}{0.85}, \frac{r_4}{0.9}, \frac{r_5}{0.45}, \frac{r_6}{0.2} \right), \left(\frac{r_1}{0.45}, \frac{r_2}{0.55}, \frac{r_3}{0.3}, \frac{r_4}{0.4}, \frac{r_5}{0.69}, \frac{r_6}{0.85} \right) \right\rangle \text{ and}$$

$$\mathscr{B} = \left\langle \left(\frac{r_1r_2}{0.6}, \frac{r_1r_6}{0.2}, \frac{r_1r_4}{0.6}, \frac{r_2r_5}{0.45}, \frac{r_2r_5}{0.3}, \frac{r_2r_3}{0.5}, \frac{r_3r_4}{0.8}, \frac{r_5r_6}{0.1}, \frac{r_4r_5}{0.4}, \frac{r_3r_6}{0.2} \right), \right.$$
$$\left. \left(\frac{r_1r_2}{0.5}, \frac{r_1r_6}{0.7}, \frac{r_1r_4}{0.4}, \frac{r_2r_5}{0.3}, \frac{r_2r_5}{0.6}, \frac{r_2r_3}{0.5}, \frac{r_3r_4}{0.4}, \frac{r_5r_6}{0.7}, \frac{r_4r_5}{0.69}, \frac{r_3r_6}{0.6} \right) \right\rangle.$$

There are two crossings $\mathscr{C}_1$ and $\mathscr{C}_2$ in PFMG as shown in Figure 5. $\mathscr{C}_1$ is the crossing between the edges $(r_2r_5, 0.45, 0.3)$ and $(r_3r_6, 0.2, 0.6)$ and $\mathscr{C}_2$ is the crossing between the edges $(r_2r_5, 0.3, 0.6)$ and $(r_3r_6, 0.2, 0.5)$. For the edges $(r_2r_5, 0.45, 0.3)$, $(r_3r_6, 0.2, 0.6)$ and $(r_2r_5, 0.3, 0.6)$, the strength is $\mathscr{S}_{r_2r_5} = (1, 0.43)$, $\mathscr{S}_{r_3r_6} = (1, 0.71)$ and $\mathscr{S}_{r_2r_5} = (0.67, 0.86)$, respectively.

The intersecting value of the first crossing is $\mathscr{S}_{\mathscr{C}_1} = (1, 0.57)$ and for the second crossing is $\mathscr{S}_{\mathscr{C}_2} = (0.83, 0.79)$. Hence, the Pythagorean fuzzy planarity value for PFMG is $\mathscr{F} = (0.35, 0.42)$.

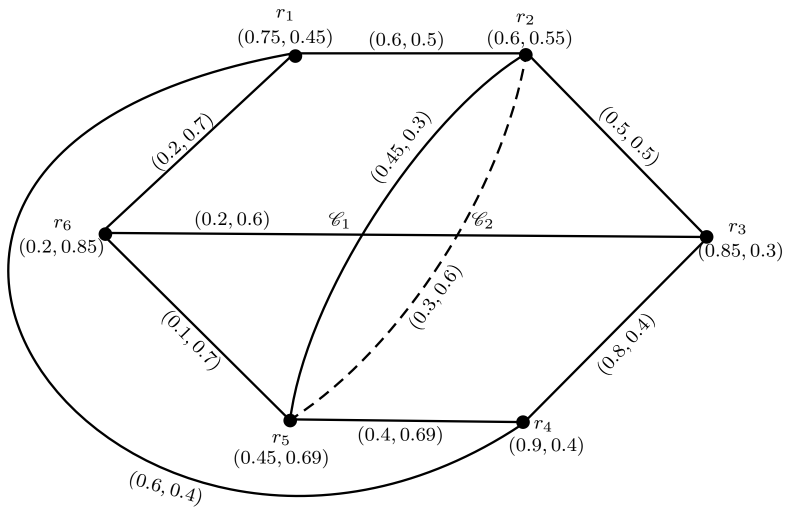


Figure 5. Pythagorean fuzzy planar graph.

Theorem 3. *Let $\mathscr{G}$ be a Pythagorean fuzzy multigraph with an effective intersecting edge. Then, Pythagorean fuzzy planarity value $\mathscr{F} = (\mathscr{F}_{\mathscr{M}}, \mathscr{F}_{\mathscr{N}})$ of $\mathscr{G}$ is stated as*

$$\mathscr{F} = (\mathscr{F}_{\mathscr{M}}, \mathscr{F}_{\mathscr{N}}) = \left(\frac{1}{1+m_{\mathscr{C}}}, \frac{1}{1+m_{\mathscr{C}}}\right),$$

where $\mathscr{F}^2{}_{\mathscr{M}} + \mathscr{F}^2{}_{\mathscr{N}} \leq 1$ and $m_{\mathscr{C}}$ is the quantity of crossing between the edges in $\mathscr{G}$.

Proof. Assume that $\mathscr{G}$ is a PFMG with an effective intersecting edge that is

$$\mu_{\mathscr{B}}(rs)_j = \{\mu_{\mathscr{A}}(r) \wedge \mu_{\mathscr{A}}(s)\},$$

$$\nu_{\mathscr{B}}(rs)_j = \{\nu_{\mathscr{A}}(r) \vee \nu_{\mathscr{A}}(s)\}.$$

Let $\mathscr{C}_1$,$\mathscr{C}_2$,...,$\mathscr{C}_k$ be the crossings between the edges in $\mathscr{G}$ where k is an integer. For each crossing edge rs in $\mathscr{G}$,

$$\mathscr{S}_{rs} = (\mathscr{M}_{rs}, \mathscr{N}_{rs}) = \left(\frac{\mu_{\mathscr{B}}(rs)_j}{\mu_{\mathscr{A}}(r) \wedge \mu_{\mathscr{A}}(s)}, \frac{\nu_{\mathscr{B}}(rs)_j}{\nu_{\mathscr{A}}(r) \vee \nu_{\mathscr{A}}(s)}\right) = (1, 1).$$

Therefore, the point of intersection $\mathscr{C}_1$ between the edges uv and rs,

$$\mathscr{S}_{\mathscr{C}_1} = (\mathscr{M}_{\mathscr{C}_1}, \mathscr{N}_{\mathscr{C}_1}) = \left(\frac{\mathscr{M}_{uv} + \mathscr{M}_{rs}}{2}, \frac{\mathscr{N}_{uv} + \mathscr{N}_{rs}}{2}\right) = \left(\frac{1+1}{2}, \frac{1+1}{2}\right) = (1, 1).$$

Hence, $\mathscr{S}_{\mathscr{C}_j} = (1, 1)$ for $j = 1, 2, \ldots, k$. Now, Planarity value of PFMG is

$$\begin{aligned}\mathscr{F} &= (\mathscr{F}_{\mathscr{M}}, \mathscr{F}_{\mathscr{N}}),\\ &= \left(\frac{1}{1+\{\mathscr{M}_{\mathscr{C}_1}+\mathscr{M}_{\mathscr{C}_2}+\ldots+\mathscr{M}_{\mathscr{C}_k}\}}, \frac{1}{1+\{\mathscr{N}_{\mathscr{C}_1}+\mathscr{N}_{\mathscr{C}_2}+\ldots+\mathscr{N}_{\mathscr{C}_k}\}}\right),\\ &= \left(\frac{1}{1+\{1+1+\ldots+1\}}, \frac{1}{1+\{1+1+\ldots+1\}}\right),\\ &= \left(\frac{1}{1+m_{\mathscr{C}}}, \frac{1}{1+m_{\mathscr{C}}}\right)\end{aligned}$$

such that $\mathscr{F}^2{}_{\mathscr{M}} + \mathscr{F}^2{}_{\mathscr{N}} \leq 1$ and $m_{\mathscr{C}}$ is the quantity of crossings between the edges in $\mathscr{G}$. □

Definition 13. *A Pythagorean fuzzy planar graph $\mathscr{G}$ is said to be strong if Pythagorean fuzzy planarity value $\mathscr{F} = (\mathscr{F}_{\mathscr{M}}, \mathscr{F}_{\mathscr{N}})$ of the graph $\mathscr{G}$ is such that $\mathscr{F}_{\mathscr{M}} > 0.5$ and $\mathscr{F}_{\mathscr{N}} < 0.86$.*

Example 9. *In Example 8, the PFPG $\mathscr{G}$ has PF planarity value $\mathscr{F} = (\mathscr{F}_{\mathscr{M}}, \mathscr{F}_{\mathscr{N}}) = (0.35, 0.42)$. Hence, $\mathscr{G}$ is not strong.*

Theorem 4. *If $\mathscr{G}$ is a strong Pythagorean fuzzy planar graph, then there is at most one crossing between strong edges.*

Proof. Assume that $\mathscr{G}$ is a strong Pythagorean fuzzy planar graph. Suppose, on the contrary, $\mathscr{G}$ contains at least two crossings $\mathscr{C}_1$ and $\mathscr{C}_2$ between the strong edges. Then, for any strong edge $(rs, \mu_{\mathscr{B}}(rs), \nu_{\mathscr{B}}(rs))$,

$$\mu_{\mathscr{B}}(rs)_j \geq \frac{1}{2}\{\mu_{\mathscr{A}}(r) \wedge \mu_{\mathscr{A}}(s)\},$$

$$\nu_{\mathscr{B}}(rs)_j \leq \frac{1}{2}\{\nu_{\mathscr{A}}(r) \vee \nu_{\mathscr{A}}(s)\}.$$

As $\mathscr{G}$ is strong PFPG, thus $\mathscr{M}_{rs} \geq 0.5$ and $\mathscr{N}_{rs} \leq 0.5$. Thus, if two strong edges $(rs, \mu_{\mathscr{B}}(rs), \nu_{\mathscr{B}}(rs))$ and $(uv, \mu_{\mathscr{B}}(uv), \nu_{\mathscr{B}}(uv))$ intersect, then

$$\tfrac{\mathscr{M}_{uv}+\mathscr{M}_{rs}}{2} \geq 0.5 \text{ and } \tfrac{\mathscr{N}_{uv}+\mathscr{N}_{rs}}{2} \leq 0.5.$$

That is, $\mathscr{M}_{\mathscr{C}_1} \geq 0.5$ and $\mathscr{N}_{\mathscr{C}_1} \leq 0.5$. Similarly, $\mathscr{M}_{\mathscr{C}_2} \geq 0.5$ and $\mathscr{N}_{\mathscr{C}_2} \leq 0.5$. This implies that $1 + \mathscr{M}_{\mathscr{C}_1} + \mathscr{M}_{\mathscr{C}_2} \geq 2$ and $1 + \mathscr{N}_{\mathscr{C}_1} + \mathscr{N}_{\mathscr{C}_2} \leq 2$. Therefore, $\mathscr{F}_{\mathscr{M}} = \frac{1}{1+\mathscr{M}_{\mathscr{C}_1}+\mathscr{M}_{\mathscr{C}_2}} \leq 0.5$ and $\mathscr{F}_{\mathscr{N}} = \frac{1}{1+\mathscr{N}_{\mathscr{C}_1}+\mathscr{N}_{\mathscr{C}_2}} \geq 0.5$.—a contradiction because $\mathscr{G}$ is strong PFPG such that $\mathscr{F}_{\mathscr{M}} > 0.5$ and $\mathscr{F}_{\mathscr{N}} < 0.86$. Thus, the crossings between two strong edges can not be two. Likewise, if the number of crossings between strong edges is one, then $1 + \mathscr{M}_{\mathscr{C}_1} \geq 1.5$ and $1 + \mathscr{N}_{\mathscr{C}_1} \leq 1.5$. Therefore, $\mathscr{F}_{\mathscr{M}} = \frac{1}{1+\mathscr{M}_{\mathscr{C}_1}} \leq 0.67$ and $\mathscr{F}_{\mathscr{N}} = \frac{1}{1+\mathscr{N}_{\mathscr{C}_1}} \geq 0.67$. Since $\mathscr{G}$ is strong, thus Pythagorean fuzzy planarity value for one point of intersection ranges from $0.5 < \mathscr{F}_{\mathscr{M}} \leq 0.67$ and $0.86 > \mathscr{F}_{\mathscr{N}} \geq 0.67$. Hence, any PFPG without crossing is a strong PFPG. Therefore, we deduce that the maximum number of crossings between strong edges is one. □

Furthermore, the validity of the above theorem is checked in the example given below.

Example 10. *Consider two strong Pythagorean fuzzy planar graphs $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$. Let $\mathscr{A}_1$ and $\mathscr{B}_1$ be PF vertex set and PF multiedge set defined on $\mathscr{V}$ and $\mathscr{V} \times \mathscr{V}$, respectively, as shown in Figure 6*

$$\mathscr{A}_1 = \left\langle \left(\frac{r_1}{0.5}, \frac{r_2}{0.8}, \frac{r_3}{0.9}, \frac{r_4}{0.6}, \frac{r_5}{0.7}, \frac{r_6}{0.6}\right), \left(\frac{r_1}{0.7}, \frac{r_2}{0.4}, \frac{r_3}{0.2}, \frac{r_4}{0.6}, \frac{r_5}{0.4}, \frac{r_6}{0.5}\right) \right\rangle \textit{ and}$$

$$\mathscr{B}_1 = \left\langle \left(\frac{r_1r_2}{0.45}, \frac{r_1r_5}{0.45}, \frac{r_1r_5}{0.44}, \frac{r_3r_6}{0.5}, \frac{r_2r_3}{0.7}, \frac{r_1r_6}{0.5}, \frac{r_2r_4}{0.45}, \frac{r_3r_5}{0.65}, \frac{r_4r_5}{0.5} \right), \left(\frac{r_1r_2}{0.2}, \frac{r_1r_5}{0.32}, \frac{r_1r_5}{0.33}, \frac{r_3r_6}{0.2}, \frac{r_2r_3}{0.19}, \frac{r_1r_6}{0.35}, \frac{r_2r_4}{0.25}, \frac{r_3r_5}{0.15}, \frac{r_4r_5}{0.15} \right) \right\rangle.$$

In addition, let $\mathscr{A}_2$ and $\mathscr{B}_2$ be PF vertex set and PF multiedge set defined on $\mathscr{V}$ and $\mathscr{V} \times \mathscr{V}$, respectively, as shown in Figure 7

$$\mathscr{A}_1 = \left\langle \left(\frac{r_1}{0.5}, \frac{r_2}{0.8}, \frac{r_3}{0.9}, \frac{r_4}{0.6}, \frac{r_5}{0.7}, \frac{r_6}{0.6} \right), \left(\frac{r_1}{0.7}, \frac{r_2}{0.4}, \frac{r_3}{0.2}, \frac{r_4}{0.6}, \frac{r_5}{0.4}, \frac{r_6}{0.5} \right) \right\rangle \text{ and}$$

$$\mathscr{B}_1 = \left\langle \left(\frac{r_1r_2}{0.45}, \frac{r_1r_5}{0.45}, \frac{r_1r_5}{0.44}, \frac{r_3r_6}{0.5}, \frac{r_2r_3}{0.7}, \frac{r_1r_6}{0.5} \frac{r_2r_4}{0.45}, \frac{r_3r_5}{0.65}, \frac{r_4r_5}{0.5}, \frac{r_2r_6}{0.56} \right), \left(\frac{r_1r_2}{0.2}, \frac{r_1r_5}{0.32}, \frac{r_1r_5}{0.33}, \frac{r_3r_6}{0.2}, \frac{r_2r_3}{0.19}, \frac{r_1r_6}{0.35}, \frac{r_2r_4}{0.25}, \frac{r_3r_5}{0.15}, \frac{r_4r_5}{0.15}, \frac{r_2r_6}{0.24} \right) \right\rangle.$$

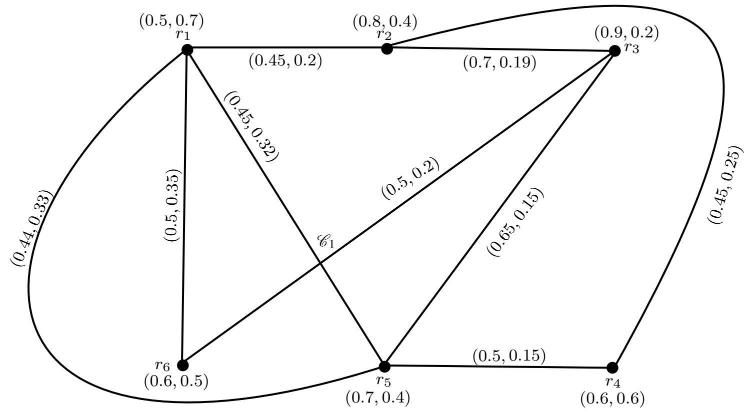


Figure 6. PFPG with one crossing.

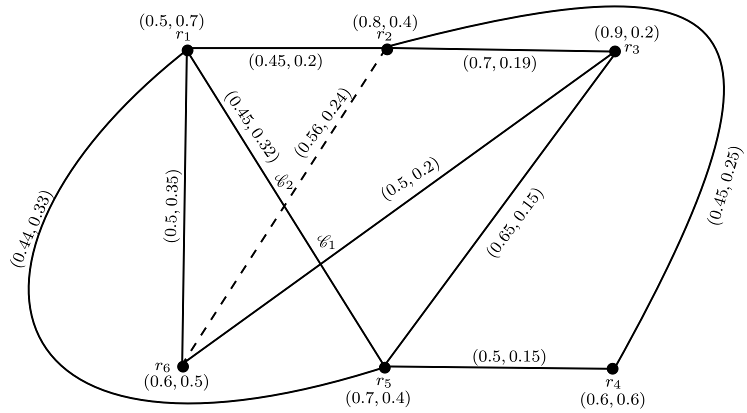


Figure 7. PFPG with two crossings.

A Pythagorean fuzzy planar graph $\mathscr{G}_1$ and $\mathscr{G}_2$ with one and two crossing between strong edges $(r_1r_5, 0.45, 0.32)$, $(r_3r_6, 0.5, 0.2)$ *and* $(r_1r_5, 0.45, 0.32)$, $(r_2r_6, 0.56, 0.24)$ *have Pythagorean fuzzy planarity value* $\mathscr{F}_1 = (0.53, 0.70)$ *and* $\mathscr{F}_2 = (0.36, 0.53)$ *that satisfies* $\mathscr{F}_{\mathscr{M}_1} \leq 0.67$, $\mathscr{F}_{\mathscr{N}_1} \geq 0.67$ *and* $\mathscr{F}_{\mathscr{M}_2} \leq 0.5$,

$\mathscr{F}_{\mathscr{N}_2} \geq 0.5$, respectively. Moreover, it is easy to see that, between strong edges, if there is no intersection, then PF planarity value $\mathscr{F}_{\mathscr{M}} > 0.67$ and $\mathscr{F}_{\mathscr{N}} < 0.74$. Hence, this analysis and the two examples above justify the statement of Theorem 4.

A fundamental theorem of PFPG is as follows.

Theorem 5. *If $\mathscr{G}$ has Pythagorean fuzzy planarity value $\mathscr{F} = (\mathscr{F}_{\mathscr{M}}, \mathscr{F}_{\mathscr{N}})$ such that $\mathscr{F}_{\mathscr{M}} > 0.67$ and $\mathscr{F}_{\mathscr{N}} < 0.74$, then, between the strong edges of PFPG $\mathscr{G}$, there is no crossing.*

Proof. Assume that $\mathscr{G}$ is a PFPG with PF planarity value $\mathscr{F}_{\mathscr{M}} > 0.67$ and $\mathscr{F}_{\mathscr{N}} < 0.74$. Suppose, on the contrary, $\mathscr{G}$ has crossing $\mathscr{C}_1$ between two strong edges $(uv, \mu_{\mathscr{B}}(uv), \nu_{\mathscr{B}}(uv))$ and $(rs, \mu_{\mathscr{B}}(rs), \nu_{\mathscr{B}}(rs))$. For any strong edge,

$$\mu_{\mathscr{B}}(rs)_j \geq \frac{1}{2}\{\mu_{\mathscr{A}}(r) \wedge \mu_{\mathscr{A}}(s)\},$$

$$\nu_{\mathscr{B}}(rs)_j \leq \frac{1}{2}\{\nu_{\mathscr{A}}(r) \vee \nu_{\mathscr{A}}(s)\}.$$

That means, $\mathscr{M}_{rs} \geq 0.5$, $\mathscr{N}_{rs} \leq 0.5$. Likewise, $\mathscr{M}_{uv} \geq 0.5$, $\mathscr{N}_{uv} \leq 0.5$. Furthermore, for the minimum value of $\mathscr{M}_{rs}$, $\mathscr{M}_{uv}$ and maximum value of $\mathscr{N}_{rs}$, $\mathscr{N}_{uv}$,

$$\mathscr{S}_{\mathscr{C}_1} = \left(\frac{\mathscr{M}_{rs} + \mathscr{M}_{uv}}{2}, \frac{\mathscr{N}_{rs} + \mathscr{N}_{uv}}{2}\right) = (\frac{0.5 + 0.5}{2}, \frac{0.5 + 0.5}{2}) = (0.5, 0.5).$$

Therefore, $\mathscr{F}_{\mathscr{M}} = \frac{1}{1+\mathscr{M}_{\mathscr{C}_1}} \leq 0.67$, $\mathscr{F}_{\mathscr{N}} = \frac{1}{1+\mathscr{N}_{\mathscr{C}_1}} \geq 0.67$—a contradiction; thus, between the strong edges of $\mathscr{G}$, there is no crossing. □

To design any type of networking model, the strength of a Pythagorean fuzzy edge plays a vital role. For such networking designs, the edge with minimum strength is not as useful as the edge with maximum strength. Hence, the edge with maximum strength is called the considerable edge. The standard definition is stated below.

Definition 14. *Let $\mathscr{G}$ be a Pythagorean fuzzy graph. An edge rs in $\mathscr{G}$ is known as considerable if*

$$\frac{\mu_{\mathscr{B}}(rs)_j}{\mu_{\mathscr{A}}(r) \wedge \mu_{\mathscr{A}}(s)} \geq \mathcal{C} \text{ and } \frac{\nu_{\mathscr{B}}(rs)_j}{\nu_{\mathscr{A}}(r) \vee \nu_{\mathscr{A}}(s)} \leq \mathcal{C},$$

whereas $0 < \mathcal{C} < 0.5$ is a rational number. If an edge is not considerable, then it is known as a nonconsiderable edge. Furthermore, an edge rs in Pythagorean fuzzy multigraph is considerable if $\mathscr{M}_{rs} \geq \mathcal{C}$ and $\mathscr{N}_{rs} \leq \mathcal{C}$, for each edge rs in $\mathscr{G}$.

Remark 2. *The rational number $0 < \mathcal{C} < 0.5$ is a pre-assigned value that may not be unique, as, for a distinct value of $\mathcal{C}$, one can acquire distinct sets of considerable edges, but it is countable. This rational number $\mathcal{C}$ is called a considerable number of a Pythagorean fuzzy graph.*

Theorem 6. *If $\mathscr{G}$ is a strong PFPG with considerable number $\mathcal{C}$, then, between considerable edges in $\mathscr{G}$, there is at most $[\frac{1}{\mathcal{C}}]$(or $\frac{1}{\mathcal{C}} - 1$) crossings.*

Proof. Assume that $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ is a strong PFPG and $\mathscr{B} = \{(rs, \mu_{\mathscr{B}}(rs)_j, \nu_{\mathscr{B}}(rs)_j), j = 1, 2, \ldots, n \mid r s \in \mathscr{V} \times \mathscr{V}\}$. Let $\mathcal{C}$ be considerable number and $\mathscr{F} = (\mathscr{F}_{\mathscr{M}}, \mathscr{F}_{\mathscr{N}})$ be the PF planarity value. Then, for any considerable edge $(rs, \mu_{\mathscr{B}}(rs), \nu_{\mathscr{B}}(rs))$,

$$\mu_{\mathscr{B}}(rs)_j \geq \mathcal{C} \times \{\mu_{\mathscr{A}}(r) \wedge \mu_{\mathscr{A}}(s)\},$$

$$\nu_{\mathscr{B}}(rs)_j \leq \mathcal{C} \times \{\nu_{\mathscr{A}}(r) \vee \nu_{\mathscr{A}}(s)\}.$$

That is, $\mathscr{M}_{rs} \geq \mathcal{C}$ and $\mathscr{N}_{rs} \leq \mathcal{C}$. Let $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n$ be crossings between considerable edges. Therefore, if two considerable edges $(uv, \mu_{\mathscr{B}}(uv), \nu_{\mathscr{B}}(uv))$ and $(rs, \mu_{\mathscr{B}}(rs), \nu_{\mathscr{B}}(rs))$ intersect, then

$$\frac{\mathscr{M}_{uv}+\mathscr{M}_{rs}}{2} \geq \mathcal{C} \text{ and } \frac{\mathscr{N}_{uv}+\mathscr{N}_{rs}}{2} \leq \mathcal{C}.$$

Thus, $\sum_{j=1}^{n} \mathscr{M}_{\mathscr{C}_j} \geq n \times \mathcal{C}$ and $\sum_{j=1}^{n} \mathscr{N}_{\mathscr{C}_j} \leq n \times \mathcal{C}$. Hence, $\mathscr{F}_{\mathscr{M}} \leq \frac{1}{1+n\mathcal{C}}$ and $\mathscr{F}_{\mathscr{N}} \geq \frac{1}{1+n\mathcal{C}}$. As $\mathscr{G}$ is strong PFPG, $0.5 < \mathscr{F}_{\mathscr{M}} \leq \frac{1}{1+n\mathcal{C}}$ and $0.86 > \mathscr{F}_{\mathscr{N}} \geq \frac{1}{1+n\mathcal{C}}$. Therefore, $0.5 < \frac{1}{1+n\mathcal{C}}$, which implies that $n < \frac{1}{\mathcal{C}}$. This inequality will be justified for some integral values n, obtained from following expression:

$$n = \begin{cases} \frac{1}{\mathcal{C}} - 1, & \text{if } \frac{1}{\mathcal{C}} \text{ is an integer}, \\ [\frac{1}{\mathcal{C}}], & \text{if } \frac{1}{\mathcal{C}} \text{ is not an integer}. \end{cases}$$

□

4. Kuratowski's Graphs and Pythagorean Fuzzy Planar Graphs

Kuratowski presented 'Kuratowski's Theorem' in 1930, by using the concept of graph homomorphism to characterize planar graphs. According to this theorem, a graph is planar if and only if it does not contain kuratowski graph as a subgraph. A kuratowski graph is basically, a subdivision of either a complete bipartite graph $\mathcal{K}_{3,3}$ or a complete graph with five vertices $\mathcal{K}_5$ where $\mathcal{K}_{3,3}$ and $\mathcal{K}_5$ are nonplanar as they cannot be drawn without intersection between edges. However, in this section, we will see that nonplanar Pythagorean fuzzy graphs are Pythagorean fuzzy planar graphs with some definite Pythagorean fuzzy planarity value.

Theorem 7. *A Pythagorean fuzzy complete graph $\mathcal{K}_5$ or $\mathcal{K}_{3,3}$ is not a strong Pythagorean fuzzy planar graph.*

Proof. Assume that $\mathscr{G} = (\mathscr{V}, \mathscr{A}, \mathscr{B})$ is a Pythagorean fuzzy complete graph with five vertices $\mathscr{V} = \{r, s, t, u, v\}$ and $\mathscr{B} = \{(rs, \mu_{\mathscr{B}}(rs), \nu_{\mathscr{B}}(rs)) | rs \in \mathscr{V} \times \mathscr{V}\}$. Since $\mathscr{G}$ is complete, then, for all $r, s \in \mathscr{V}$,

$$\mu_{\mathscr{B}}(rs) = \{\mu_{\mathscr{A}}(r) \wedge \mu_{\mathscr{A}}(s)\},$$

$$\nu_{\mathscr{B}}(rs) = \{\nu_{\mathscr{A}}(r) \vee \nu_{\mathscr{A}}(s)\}.$$

The Pythagorean fuzzy planarity value of Pythagorean fuzzy complete graph is $\mathscr{F} = (\mathscr{F}_{\mathscr{M}}, \mathscr{F}_{\mathscr{N}}) = \left(\frac{1}{1+m_{\mathscr{C}}}, \frac{1}{1+m_{\mathscr{C}}}\right)$, where $m_{\mathscr{C}}$ is the number of crossings between edges in $\mathscr{G}$.

Since the geometric insight of an underlying crisp graph of $\mathscr{G}$ is non planar and, for any representation, one crossing can not be excluded. Therefore, $\mathscr{F} = (\mathscr{F}_{\mathscr{M}}, \mathscr{F}_{\mathscr{N}}) = \left(\frac{1}{1+1}, \frac{1}{1+1}\right) = (0.5, 0.5)$. As $\mathscr{F}_{\mathscr{M}} = 0.5$, so $\mathscr{G}$ is not a strong Pythagorean fuzzy planar graph. Likewise, $\mathcal{K}_{3,3}$ has only one crossing that cannot be avoided, so it is not a strong Pythagorean fuzzy planar graph. □

Remark 3. *A Pythagorean fuzzy planar graph with five vertices and each pair of vertices connected by an edge may or may not be a strong Pythagorean fuzzy planar graph.*

Example 11. *Considering a PFPG as displayed in Figure 8, there is one crossing between two edges* $(r_1r_4, 0.4, 0.34)$ *and* $(r_5r_3, 0.5, 0.4)$. *Then, the Pythagorean fuzzy planarity value* $(0.54, 0.61)$. *Hence, it is a strong PFPG.*

Remark 4. *A Pythagorean fuzzy bipartite planar graph with six vertices, partitioned into two subsets containing three vertices each, is a strong Pythagorean fuzzy planar graph.*

Example 12. *Considering a PFPG as displayed in Figure 9, there is one crossing between two edges* $(r_1r_5, 0.55, 0.3)$ *and* $(r_24_6, 0.6, 0.19)$. *Then, the Pythagorean fuzzy planarity value* $(0.53, 0.65)$. *Hence, it is a strong PFPG.*

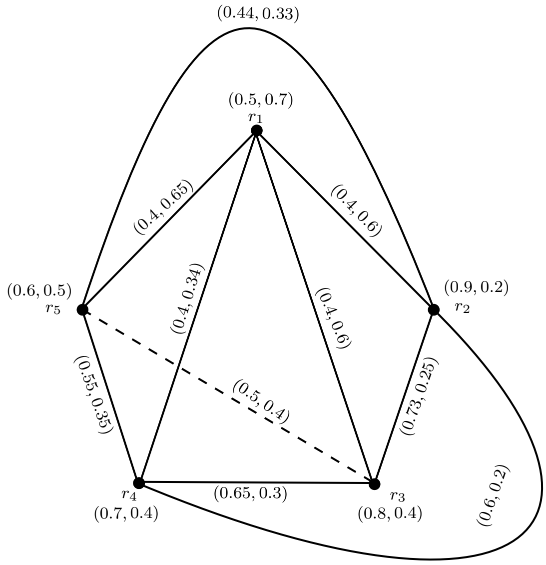


Figure 8. Pythagorean fuzzy planar graph with PF planarity (0.54,0.61).

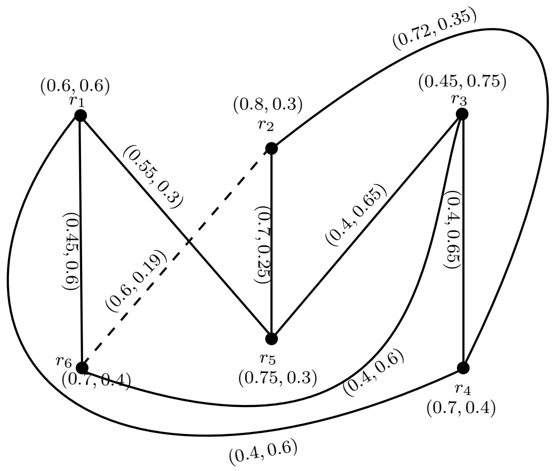


Figure 9. Pythagorean fuzzy planar graph with PF planarity (0.53,0.65).

From Theorem 7, Remarks 3 and 4, it is concluded that a complete PFG is not a strong Pythagorean fuzzy planar graph, whereas a complete PFPG may or may not be a strong Pythagorean fuzzy planar graph as justified in Examples 11 and 12.

5. Pythagorean Fuzzy Face and Pythagorean Fuzzy Dual Graphs

In Pythagorean fuzzy sense, the face of a PFPG has a significant role. It is a flat surface, enclosed by Pythagorean fuzzy edges. If all the edges in the surrounding of a Pythagorean fuzzy face have degree of membership and nonmembership $(1,0)$, then it is known as crisp face. The Pythagorean fuzzy face does not exist, if one of such edge is removed with degree of membership and nonmembership $(0,1)$. Hence, the occurrence of Pythagorean fuzzy face based on the minimum strength of Pythagorean fuzzy edge.

We consider Pythagorean fuzzy planar graph that do not carry any pair of intersecting edge. That is, its planarity value is $(1,1)$ to define Pythagorean fuzzy face.

Definition 15. *Let* $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ *be a PFPG with planarity* $(1,1)$ *and* $\mathscr{B} = \{(rs, \mu_{\mathscr{B}}(rs)_j, \nu_{\mathscr{B}}(rs)_j), j = 1,2,\ldots,n \mid rs \in \mathscr{V} \times \mathscr{V}\}$. *A region enclosed by the Pythagorean fuzzy* *edge set $\mathscr{E}' \subset \mathscr{E}$ of the geometrical representation of $\mathscr{G}$ is known as Pythagorean fuzzy face of $\mathscr{G}$. The membership and nonmembership value of Pythagorean fuzzy face are defined as*

$$\min\left\{\frac{\mu_{\mathscr{B}}(rs)_j}{\mu_{\mathscr{A}}(r) \wedge \mu_{\mathscr{A}}(s)}, j = 1, 2, \ldots, n \mid rs \in \mathscr{E}'\right\},$$

$$\max\left\{\frac{\nu_{\mathscr{B}}(rs)_j}{\nu_{\mathscr{A}}(r) \vee \nu_{\mathscr{A}}(s)}, j = 1, 2, \ldots, n \mid rs \in \mathscr{E}'\right\}.$$

Definition 16. *A Pythagorean fuzzy face is called strong if its membership value is greater than or equal to* 0.5 *and nonmembership is less than or equal to* 0.5*, otherwise weak. Moreover, an infinite region of PFPG is known as outer Pythagorean fuzzy face, while others are known as inner Pythagorean fuzzy face.*

Remark 5. *Every Pythagorean strong fuzzy face has a membership value greater than or equal to* 0.5 *and nonmembership less than or equal to* 0.5*. Thus, a strong Pythagorean fuzzy face has a strong Pythagorean fuzzy edge.*

Example 13. *Consider a PFPG $\mathscr{G}$ as displayed in Figure 10. Let $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_4$ be the Pythagorean fuzzy faces:*

- *Pythagorean fuzzy inner face $\mathcal{F}_1$ is enclosed by the edges* $(r_1r_3, 0.4, 0.6)$, $(r_1r_4, 0.4, 0.33)$, $(r_3r_4, 0.52, 0.3)$.
- *Pythagorean fuzzy inner face $\mathcal{F}_2$ is bounded by the strong edges* $(r_1r_4, 0.40, 0.33)$, $(r_1r_2, 0.38, 0.38)$, $(r_2r_4, 0.49, 0.38)$.
- *Pythagorean fuzzy inner face $\mathcal{F}_3$ is surrounded by the strong edges* $(r_2r_4, 0.49, 0.38)$, $(r_3r_4, 0.52, 0.3)$, $(r_2r_3, 0.45, 0.31)$.
- *Pythagorean fuzzy outer face $\mathcal{F}_4$ is enclosed by the edges* $(r_1r_2, 0.38, 0.38)$,$(r_1r_3, 0.4, 0.6)$,$(r_2r_3, 0.45, 0.31)$.

The membership and nonmembership value of Pythagorean fuzzy faces $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_4$ are $(0.86, 0.85)$, $(1, 0.47)$, $(0.86, 0.5)$ *and* $(0.9, 0.85)$*, respectively. Here, $\mathcal{F}_1$ and $\mathcal{F}_4$ are weak faces and $\mathcal{F}_2$ and $\mathcal{F}_3$ are strong Pythagorean fuzzy faces.*

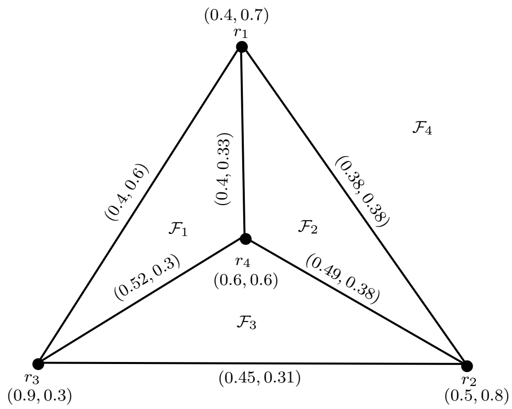


Figure 10. Faces in Pythagorean fuzzy planar graph.

In graph theory, duality is very helpful in explaining various structures like drainage system of basins, etc. It has been widely applied in computational geometry, design of integrated circuits and mesh generation. A mathematician Whitney described planarity in terms of occurrence of dual graph i.e., a graph is planar if and only it has a dual graph. This concept is very effective in solving many critical problems. Motivated from this concept, we introduce a Pythagorean fuzzy dual graph of a Pythagorean fuzzy planar graph.

Definition 17. *Let $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ be a PFPG where*

$$\mathscr{B} = \{(rt, \mu_{\mathscr{B}}(rt)_j, \nu_{\mathscr{B}}(rt)_j), j = 1, 2, \ldots, n \mid rt \in \mathscr{V} \times \mathscr{V}\}.$$

Let $\mathcal{F}_1$,$\mathcal{F}_2$,...,$\mathcal{F}_k$ be strong Pythagorean fuzzy faces of $\mathscr{G}$. Then, the Pythagorean fuzzy dual graph of $\mathscr{G}$ is a PFPG $\mathscr{G}' = (\mathscr{V}', \mathscr{A}', \mathscr{B}')$, where $\mathscr{V}' = \{r_j, j = 1, 2, \ldots, k\}$ and the vertex r_j of $\mathscr{G}'$ is taken for $\mathcal{F}_j$ of $\mathscr{G}$. Furthermore, the membership grades and nonmembership grades of vertices are given by mapping $\mathscr{A}' = (\mu_{\mathscr{A}'}, \nu_{\mathscr{A}'}) : \mathscr{V}' \to [0,1] \times [0,1]$ such that

$$\mu_{\mathscr{A}'}(r_j) = \max\{\mu_{\mathscr{B}'}(pu)_j, j = 1, 2, \ldots, m | pu \text{ is an edge in the surrounding of strong PF face } \mathcal{F}_j\},$$

$$\nu_{\mathscr{A}'}(r_j) = \min\{\nu_{\mathscr{B}'}(pu)_j, j = 1, 2, \ldots, m | pu \text{ is an edge in the surrounding of strong PF face } \mathcal{F}_j\}.$$

Meanwhile, between two faces $\mathcal{F}_i$ and $\mathcal{F}_j$ of $\mathscr{G}$, there may occur more than one common edge. Thus, between two vertices, there may exist more than one edge r_i and r_j in PFDG $\mathscr{G}'$. The membership and nonmembership values of Pythagorean fuzzy edges of PFDG are $\mu_{\mathscr{B}'}(r_i r_j)_s = \mu^s_{\mathscr{B}}(pu)_i$, $\nu_{\mathscr{B}'}(r_i r_j)_s = \nu^s_{\mathscr{B}}(pu)_i$ where $(pu)^s$ is an edge in the surrounding between strong PF faces $\mathcal{F}_i$ and $\mathcal{F}_j$ and $s = 1, 2, \ldots, l$, is the number of common edges in the surrounding of $\mathcal{F}_i$ and $\mathcal{F}_j$.

The Pythagorean fuzzy dual graph $\mathscr{G}'$ of PFPG $\mathscr{G}$ has no crossing between edges for some definite geometric representation; thus, it is PFPG of PF planarity $(1, 1)$.

Example 14. *Consider a PFPG $\mathscr{G} = (\mathscr{V}, \mathscr{A}, \mathscr{B})$ as displayed in Figure 11 such that $\mathscr{V} = \{s_1, s_2, s_3, s_4, s_5\}$. Let $\mathscr{A}$ and $\mathscr{B}$ be a PF vertex set and PF edge set defined on $\mathscr{V}$ and $\mathscr{V} \times \mathscr{V}$, respectively.*

$$\mathscr{A} = \left\langle \left(\frac{s_1}{0.70}, \frac{s_2}{0.69}, \frac{s_3}{0.35}, \frac{s_4}{0.76}, \frac{s_5}{0.79} \right), \left(\frac{s_1}{0.69}, \frac{s_2}{0.55}, \frac{s_3}{0.85}, \frac{s_4}{0.55}, \frac{s_5}{0.33} \right) \right\rangle \text{ and}$$

$$\mathscr{B} = \left\langle \left(\frac{s_1 s_2}{0.60}, \frac{s_1 s_4}{0.48}, \frac{s_2 s_4}{0.60}, \frac{s_2 s_5}{0.60}, \frac{s_2 s_3}{0.30}, \frac{s_4 s_5}{0.65}, \frac{s_3 s_4}{0.30}, \frac{s_3 s_5}{0.25} \right), \right.$$
$$\left. \left(\frac{s_1 s_2}{0.15}, \frac{s_1 s_4}{0.15}, \frac{s_2 s_4}{0.15}, \frac{s_2 s_5}{0.15}, \frac{s_2 s_3}{0.15}, \frac{s_4 s_5}{0.15}, \frac{s_3 s_4}{0.15}, \frac{s_3 s_5}{0.15} \right) \right\rangle.$$

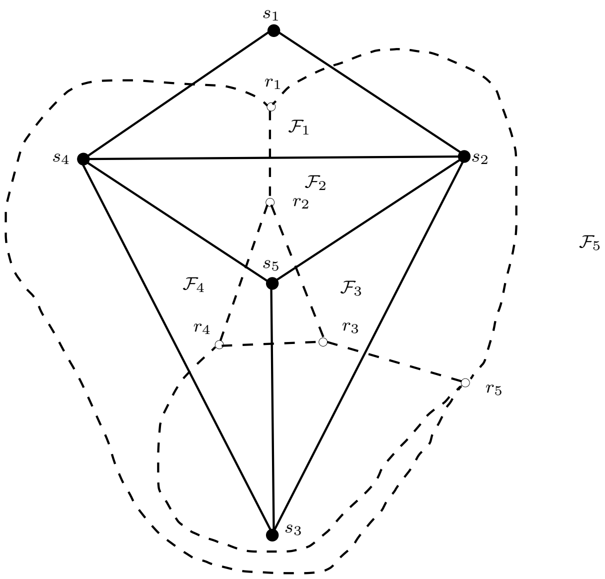


Figure 11. Pythagorean fuzzy dual graph.

The Pythagorean fuzzy faces of a Pythagorean fuzzy planar graph are given below:

- *Pythagorean fuzzy face* $\mathcal{F}_1$ *is enclosed by the edges* $(s_1s_2, 0.60, 0.15)$,$(s_1s_4, 0.48, 0.15)$,$(s_2s_4, 0.60, 0.15)$.
- *Pythagorean fuzzy face* $\mathcal{F}_2$ *is bounded by the edges* $(s_2s_4, 0.60, 0.15)$,$(s_2s_5, 0.60, 0.15)$,$(s_4s_5, 0.65, 0.15)$.
- *Pythagorean fuzzy face* $\mathcal{F}_3$ *is surrounded by the edges* $(s_2s_3, 0.30, 0.15)$,$(s_2s_5, 0.60, 0.15)$,$(s_3s_5, 0.25, 0.15)$.
- *Pythagorean fuzzy face* $\mathcal{F}_4$ *is bounded by the edges* $(s_4s_5, 0.65, 0.15)$,$(s_3s_5, 0.25, 0.15)$,$(s_3s_4, 0.30, 0.15)$.
- *Pythagorean fuzzy face* $\mathcal{F}_5$ *is enclosed by the edges* $(s_1s_2, 0.60, 0.15)$,$(s_2s_3, 0.30, 0.15)$,$(s_3s_4, 0.30, 0.15)$, $(s_1s_4, 0.48, 0.15)$.

By direct calculation, one can see that these five faces are strong Pythagorean fuzzy faces. We represent the vertices of Pythagorean fuzzy dual graph (PFDG) by small white circles and the edges by dashed lines. For each strong Pythagorean fuzzy face (SPFF), we take a vertex for the PFDG. Therefore, the vertex set $\mathscr{V}' = \{r_1, r_2, r_3, r_4, r_5\}$, *where the vertex* r_j *is extracted parallel to the SPFF* $\mathcal{F}_j$, $j = 1, 2, .., 5$. *Hence,*

$$\begin{aligned}
\mu_{\mathscr{A}'}(r_1) &= \max\{0.60, 0.48, 0.60\} = 0.60, & \nu_{\mathscr{A}'}(r_1) &= \min\{0.15, 0.15, 0.15\} = 0.15.\\
\mu_{\mathscr{A}'}(r_2) &= \max\{0.60, 0.60, 0.65\} = 0.65, & \nu_{\mathscr{A}'}(r_2) &= \min\{0.15, 0.15, 0.15\} = 0.15.\\
\mu_{\mathscr{A}'}(r_3) &= \max\{0.30, 0.60, 0.20\} = 0.60, & \nu_{\mathscr{A}'}(r_3) &= \min\{0.15, 0.15, 0.15\} = 0.15.\\
\mu_{\mathscr{A}'}(r_4) &= \max\{0.65, 0.25, 0.30\} = 0.65, & \nu_{\mathscr{A}'}(r_4) &= \min\{0.15, 0.15, 0.15\} = 0.15.\\
\mu_{\mathscr{A}'}(r_5) &= \max\{0.60, 0.30, 0.30, 0.48\} = 0.60, & \nu_{\mathscr{A}'}(r_5) &= \min\{0.15, 0.15, 0.15.0.15\} = 0.15.
\end{aligned}$$

There is one common edge bd between the faces $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *in* $\mathscr{G}$. *Hence, there exists one edge between the vertices* r_1 *and* r_2 *in PFDG of* $\mathscr{G}$. *The membership grade and nonmembership grade of the edges of PFDG are obtained as*

$$\begin{aligned}
\mu_{\mathscr{B}'}(r_1r_2) &= \mu_{\mathscr{B}}(s_2s_4) = 0.60, & \nu_{\mathscr{B}'}(r_1r_2) &= \nu_{\mathscr{B}}(s_2s_4) = 0.15.\\
\mu_{\mathscr{B}'}(r_2r_4) &= \mu_{\mathscr{B}}(s_4s_5) = 0.65, & \nu_{\mathscr{B}'}(r_2r_4) &= \nu_{\mathscr{B}}(s_4s_5) = 0.15.\\
\mu_{\mathscr{B}'}(r_2r_3) &= \mu_{\mathscr{B}}(s_2s_5) = 0.60, & \nu_{\mathscr{B}'}(r_2r_3) &= \nu_{\mathscr{B}}(s_2s_5) = 0.15.\\
\mu_{\mathscr{B}'}(r_3r_4) &= \mu_{\mathscr{B}}(s_5s_3) = 0.25, & \nu_{\mathscr{B}'}(r_3r_4) &= \nu_{\mathscr{B}}(s_5s_3) = 0.15.\\
\mu_{\mathscr{B}'}(r_1r_5) &= \mu_{\mathscr{B}}(s_1s_2) = 0.60, & \nu_{\mathscr{B}'}(r_1r_5) &= \nu_{\mathscr{B}}(s_1s_2) = 0.15.\\
\mu_{\mathscr{B}'}(r_3r_5) &= \mu_{\mathscr{B}}(s_2s_3) = 0.30, & \nu_{\mathscr{B}'}(r_3r_5) &= \nu_{\mathscr{B}}(s_2s_3) = 0.15.\\
\mu_{\mathscr{B}'}(r_4r_5) &= \mu_{\mathscr{B}}(s_4s_3) = 0.30, & \nu_{\mathscr{B}'}(r_4r_5) &= \nu_{\mathscr{B}}(s_4s_3) = 0.15.\\
\mu_{\mathscr{B}'}(r_1r_5) &= \mu_{\mathscr{B}}(s_1s_4) = 0.48, & \nu_{\mathscr{B}'}(r_1r_5) &= \nu_{\mathscr{B}}(s_1s_4) = 0.15.
\end{aligned}$$

Thus, the Pythagorean fuzzy dual graph edge set is

$$\begin{aligned}
\mathscr{B}' = \Big\langle &\left(\frac{r_1r_2}{0.60}, \frac{r_2r_4}{0.65}, \frac{r_2r_3}{0.60}, \frac{r_3r_4}{0.25}, \frac{r_1r_5}{0.60}, \frac{r_3r_5}{0.30}, \frac{r_4r_5}{0.30}, \frac{r_1r_5}{0.48}\right),\\
&\left(\frac{r_1r_2}{0.15}, \frac{r_2r_4}{0.15}, \frac{r_2r_3}{0.15}, \frac{r_3r_4}{0.15}, \frac{r_1r_5}{0.15}, \frac{r_3r_5}{0.15}, \frac{r_4r_5}{0.15}, \frac{r_1r_5}{0.15}\right)\Big\rangle.
\end{aligned}$$

Hence, $\mathscr{G}' = (\mathscr{V}', \mathscr{A}', \mathscr{B}')$ *is a PFDG of* $\mathscr{G} = (\mathscr{V}, \mathscr{A}, \mathscr{B})$.

In the Pythagorean fuzzy dual graph, we will not consider weak edges. The following theorems are given below.

Theorem 8. *Let* $\mathscr{G}$ *be a Pythagorean fuzzy planar graph without weak edges, r strong faces, q Pythagorean fuzzy edges and p vertices. Let* $\mathscr{G}'$ *be a Pythagorean fuzzy dual graph of* $\mathscr{G}$ *with* r' *faces,* q' *Pythagorean fuzzy edges and* p' *vertices, then* $p' = r$, $q' = q$ *and* $r' = p$.

Proof. The proof is easily perceived by the definition of the Pythagorean fuzzy dual graph. □

Theorem 9. *Let* $\mathscr{G}'$ *be a Pythagorean fuzzy dual graph of PFPG* $\mathscr{G}$. *The number of strong PF faces in* $\mathscr{G}'$ *is less than or equal to the number of vertices of* $\mathscr{G}$.

Proof. Assume that $\mathscr{G}'$ is a PFDG of PFPG $\mathscr{G}$ with r' strong PF faces and $\mathscr{G}$ has p vertices. Since $\mathscr{G}$ has both weak and strong PF edges, and, to develop PFDG, weak PF edges are eliminated. Hence, if $\mathscr{G}$ has some weak PF edges, then some vertices may have all its adjoining PF edges as weak PF edges.

Suppose that such vertices are in number l. These vertices are not enclosed by any strong PF faces. By eliminating these vertices and adjoining edges, the number of vertices become $p-l$. Moreover, from Theorem 8, $r' = p-l$. Hence, $r' \leq p$. This concludes that the number of strong PF faces in $\mathscr{G}'$ is less than or equal to the number of vertices of $\mathscr{G}$. □

Example 15. *The above statement is justified from Example 14, as one can see that the number of strong PF faces in* $\mathscr{G}'$ *is* 4, *which is less than* 5 (number of vertices of $\mathscr{G}$).

Theorem 10. *If* $\mathscr{G}'$ *be a Pythagorean fuzzy dual graph of a PFPG* $\mathscr{G}$ *without weak edges. Then, the membership grade and nonmembership grade of Pythagorean fuzzy edge of* $\mathscr{G}'$ *are equivalent to the membership grade and nonmembership grade of Pythagorean fuzzy edge of* $\mathscr{G}$.

Proof. Let $\mathscr{G}$ be a PFPG without weak edges. The PFDG of $\mathscr{G}$ is $\mathscr{G}'$ in which there is no crossing between any edges. Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$ be SPFF of $\mathscr{G}$. By the definition of PF dual graph, the membership grade and nonmembership grade of Pythagorean fuzzy edges of Pythagorean fuzzy dual graph are

$$\mu_{\mathscr{B}'}(r_i r_j)_s = \mu^s_{\mathscr{B}}(pu)_i, \qquad \nu_{\mathscr{B}'}(r_i r_j)_s = \nu^s_{\mathscr{B}}(pu)_i,$$

where $(pu)^s$ is an edge in the surrounding between strong PF faces $\mathcal{F}_i$ and $\mathcal{F}_j$. The common edges in the surrounding between $\mathcal{F}_i$ and $\mathcal{F}_j$ are s in number, where $s = 1, 2, \ldots, l$. The number of PF edges of two PFGs $\mathscr{G}$ and $\mathscr{G}'$ are similar as $\mathscr{G}$ has no weak edges. Hence, for every Pythagorean fuzzy edge of $\mathscr{G}$, there is a Pythagorean fuzzy edge in $\mathscr{G}'$ with similar membership grade and nonmembership grade. □

6. Isomorphism between Pythagorean Fuzzy Planar Graphs

Isomorphism is a formal mapping that propagates knowledge and better understanding between different graphs. It can be defined between complex models where the two models have equal division. If there is isomorphism between two models such that the property of one is known and the other is unknown. Then, due to isomorphism, we are able to know the property of an unknown model. By using this concept, we define isomorphism between two Pythagorean fuzzy planar graphs.

Definition 18. *An isomorphism* $F : \mathscr{G}_1 \to \mathscr{G}_2$ *of two Pythagorean fuzzy planar graphs* $\mathscr{G}_1$ *and* $\mathscr{G}_2$ *is a bijective mapping* $F : \mathscr{V}_1 \to \mathscr{V}_2$ *that satisfies*

1. $\mu_{\mathscr{A}_1}(r) = \mu_{\mathscr{A}_2}(F(r)), \nu_{\mathscr{A}_1}(r) = \nu_{\mathscr{A}_2}(F(r))$,
2. $\mu_{\mathscr{B}_1}(rs) = \mu_{\mathscr{B}_2}(F(r)F(s)), \nu_{\mathscr{B}_1}(rs) = \nu_{\mathscr{B}_2}(F(r)F(s))$,

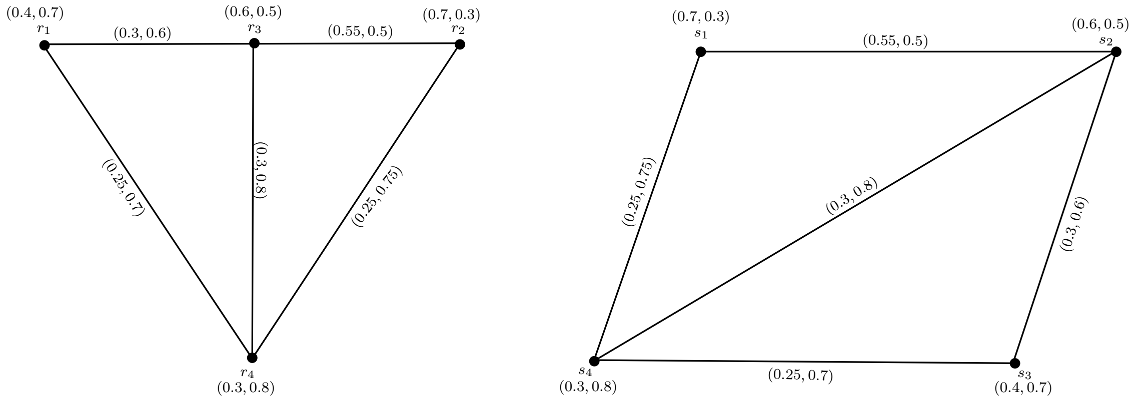
for all $r \in \mathscr{V}_1, rs \in \mathscr{E}_1$.

Example 16. *Consider two Pythagorean fuzzy planar graph* $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ *and* $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ *as shown in Figure 12 such that*

$$\mathscr{A}_1 = \left\langle \left(\frac{r_1}{0.40}, \frac{r_2}{0.7}, \frac{r_3}{0.6}, \frac{r_4}{0.3} \right), \left(\frac{r_1}{0.7}, \frac{r_2}{0.3}, \frac{r_3}{0.5}, \frac{r_4}{0.8} \right) \right\rangle \text{ and}$$
$$\mathscr{B}_1 = \left\langle \left(\frac{r_1 r_3}{0.3}, \frac{r_3 r_2}{0.55}, \frac{r_2 r_4}{0.25}, \frac{r_1 r_4}{0.25}, \frac{r_3 r_4}{0.3} \right), \left(\frac{r_1 r_3}{0.6}, \frac{r_3 r_2}{0.5}, \frac{r_2 r_4}{0.75}, \frac{r_1 r_4}{0.70}, \frac{r_3 r_4}{0.8} \right) \right\rangle.$$

$$\mathscr{A}_2 = \left\langle \left(\frac{s_1}{0.7}, \frac{s_2}{0.6}, \frac{s_3}{0.40}, \frac{s_4}{0.3} \right), \left(\frac{s_1}{0.3}, \frac{s_2}{0.5}, \frac{s_3}{0.7}, \frac{s_4}{0.8} \right) \right\rangle \text{ and}$$
$$\mathscr{B}_2 = \left\langle \left(\frac{s_1 s_2}{0.55}, \frac{s_2 s_3}{0.3}, \frac{s_3 s_4}{0.25}, \frac{s_1 s_4}{0.25}, \frac{s_2 s_4}{0.3} \right), \left(\frac{s_1 s_2}{0.55}, \frac{s_2 s_3}{0.6}, \frac{s_3 s_4}{0.70}, \frac{s_1 s_4}{0.75}, \frac{s_2 s_4}{0.8} \right) \right\rangle.$$

Since a mapping $F : \mathscr{V}_1 \to \mathscr{V}_2$ *defined by* $F(r_1) = s_3$, $F(r_2) = s_1$, $F(r_3) = s_2$, $F(r_4) = s_4$ *satisfies* $\mu_{\mathscr{A}_1}(r_i) = \mu_{\mathscr{A}_2}(F(r_i)), \nu_{\mathscr{A}_1}(r_i) = \nu_{\mathscr{A}_2}(F(r_i))$, $\mu_{\mathscr{B}_1}(r_i r_j) = \mu_{\mathscr{B}_2}(F(r_i)F(r_j)), \nu_{\mathscr{B}_1}(r_i r_j) = \nu_{\mathscr{B}_2}(F(r_i)F(r_j))$, *for all* $r_i \in \mathscr{V}_1, r_i r_j \in \mathscr{E}_1$, *where* $i, j = 1, 2, 3, 4$. *Therefore,* $\mathscr{G}_1$ *is isomorphic to* $\mathscr{G}_2$.



(**a**) $\mathscr{G}_1$ (**b**) $\mathscr{G}_2$

Figure 12. Pythagorean fuzzy planar graphs.

Definition 19. *A weak isomorphism $F : \mathscr{G}_1 \to \mathscr{G}_2$ of two Pythagorean fuzzy planar graphs $\mathscr{G}_1$ and $\mathscr{G}_2$ is a bijective mapping $F : \mathscr{V}_1 \to \mathscr{V}_2$ that satisfies*

1. *F is homomorphism,*
2. $\mu_{\mathscr{A}_1}(r) = \mu_{\mathscr{A}_2}(F(r)), \nu_{\mathscr{A}_1}(r) = \nu_{\mathscr{A}_2}(F(r)),$

for all $r \in \mathscr{V}_1$.

Example 17. *Consider two Pythagorean fuzzy planar graph $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ as shown in Figure 13 such that*

$$\mathscr{A}_1 = \left\langle \left(\frac{r_1}{0.8}, \frac{r_2}{0.7}, \frac{r_3}{0.4}, \frac{r_4}{0.5}, \frac{r_5}{0.7} \right), \left(\frac{r_1}{0.3}, \frac{r_2}{0.4}, \frac{r_3}{0.8}, \frac{r_4}{0.6}, \frac{r_5}{0.5} \right) \right\rangle \text{ and}$$
$$\mathscr{B}_1 = \left\langle \left(\frac{r_1r_2}{0.6}, \frac{r_2r_3}{0.3}, \frac{r_2r_4}{0.4}, \frac{r_3r_4}{0.2}, \frac{r_3r_5}{0.35}, \frac{r_4r_5}{0.4}, \frac{r_1r_5}{0.4} \right), \left(\frac{r_1r_2}{0.4}, \frac{r_2r_3}{0.7}, \frac{r_2r_4}{0.6}, \frac{r_3r_4}{0.7}, \frac{r_3r_5}{0.8}, \frac{r_4r_5}{0.55}, \frac{r_1r_5}{0.5} \right) \right\rangle.$$

$$\mathscr{A}_2 = \left\langle \left(\frac{s_1}{0.7}, \frac{s_2}{0.8}, \frac{s_3}{0.7}, \frac{s_4}{0.5}, \frac{s_5}{0.4} \right), \left(\frac{s_1}{0.4}, \frac{s_2}{0.3}, \frac{s_3}{0.5}, \frac{s_4}{0.6}, \frac{s_5}{0.8} \right) \right\rangle \text{ and}$$
$$\mathscr{B}_2 = \left\langle \left(\frac{s_1s_2}{0.7}, \frac{s_2s_3}{0.5}, \frac{s_1s_4}{0.5}, \frac{s_3s_4}{0.5}, \frac{s_4s_5}{0.3}, \frac{s_3s_5}{0.39}, \frac{s_1s_5}{0.4} \right), \left(\frac{s_1s_2}{0.35}, \frac{s_2s_3}{0.4}, \frac{s_1s_4}{0.5}, \frac{s_3s_4}{0.5}, \frac{s_4s_5}{0.6}, \frac{s_3s_5}{0.7}, \frac{s_1s_5}{0.69} \right) \right\rangle.$$

Since a mapping $F : \mathscr{V}_1 \to \mathscr{V}_2$ defined by $F(r_1) = s_2$, $F(r_2) = s_1$, $F(r_3) = s_5$, $F(r_4) = s_4$, $F(r_5) = s_3$ satisfies $\mu_{\mathscr{A}_1}(r_i) = \mu_{\mathscr{A}_2}(F(r_i)), \nu_{\mathscr{A}_1}(r_i) = \nu_{\mathscr{A}_2}(F(r_i))$ for all $r_i \in \mathscr{V}_1$, where $i, j = 1, 2, 3, 4$ but $\mu_{\mathscr{B}_1}(r_ir_j) \neq \mu_{\mathscr{B}_2}(F(r_i)F(r_j)), \nu_{\mathscr{B}_1}(r_ir_j) \neq \nu_{\mathscr{B}_2}(F(r_i)F(r_j))$. Therefore, $\mathscr{G}_1$ is a weak isomorphic to $\mathscr{G}_2$.

Definition 20. *A co-weak isomorphism $F : \mathscr{G}_1 \to \mathscr{G}_2$ of two Pythagorean fuzzy planar graphs $\mathscr{G}_1$ and $\mathscr{G}_2$ is a bijective mapping $F : \mathscr{V}_1 \to \mathscr{V}_2$ that satisfies*

1. *F is homomorphism,*
2. $\mu_{\mathscr{B}_1}(rs) = \mu_{\mathscr{B}_2}(F(r)F(s)), \nu_{\mathscr{B}_1}(rs) = \nu_{\mathscr{B}_2}(F(r)F(s)),$

for all $rs \in \mathscr{E}_1$.

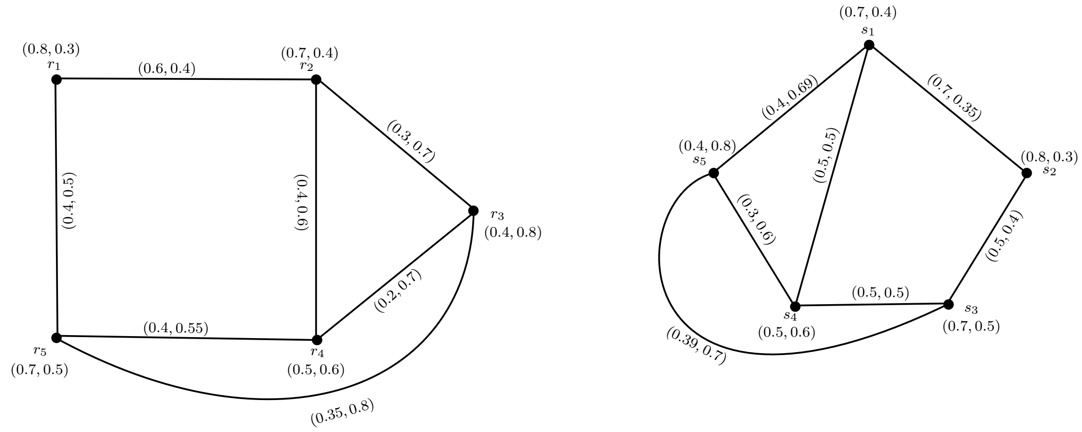
Example 18. *Consider two Pythagorean fuzzy planar graph $\mathscr{G}_1 = (\mathscr{A}_1, \mathscr{B}_1)$ and $\mathscr{G}_2 = (\mathscr{A}_2, \mathscr{B}_2)$ as shown in Figure 14 such that*

$$\mathscr{A}_1 = \left\langle \left(\frac{r_1}{0.3}, \frac{r_2}{0.7}, \frac{r_3}{0.4}, \frac{r_4}{0.8}, \frac{r_5}{0.7}, \frac{r_6}{0.8} \right), \left(\frac{r_1}{0.85}, \frac{r_2}{0.45}, \frac{r_3}{0.69}, \frac{r_4}{0.35}, \frac{r_5}{0.73}, \frac{r_6}{0.35} \right) \right\rangle \text{ and}$$

$$\mathscr{B}_1 = \left\langle \left(\frac{r_1r_2}{0.2}, \frac{r_2r_3}{0.4}, \frac{r_3r_4}{0.3}, \frac{r_2r_4}{0.7}, \frac{r_2r_6}{0.6}, \frac{r_1r_6}{0.2}, \frac{r_4r_6}{0.7}, \frac{r_5r_6}{0.4}, \frac{r_4r_5}{0.4}, \frac{r_2r_5}{0.4} \right), \left(\frac{r_1r_2}{0.7}, \frac{r_2r_3}{0.62}, \frac{r_3r_4}{0.54}, \frac{r_2r_4}{0.4}, \frac{r_2r_6}{0.35}, \frac{r_1r_6}{0.75}, \frac{r_4r_6}{0.3}, \frac{r_5r_6}{0.65}, \frac{r_4r_5}{0.6}, \frac{r_2r_5}{0.65} \right) \right\rangle.$$

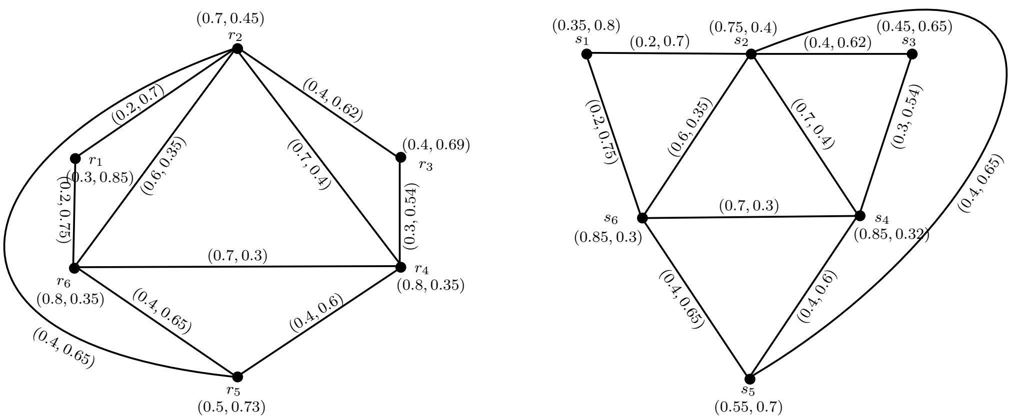
$$\mathscr{A}_2 = \left\langle \left(\frac{s_1}{0.35}, \frac{s_2}{0.75}, \frac{s_3}{0.45}, \frac{s_4}{0.85}, \frac{s_5}{0.55}, \frac{s_6}{0.85} \right), \left(\frac{s_1}{0.8}, \frac{s_2}{0.4}, \frac{s_3}{0.65}, \frac{s_4}{0.32}, \frac{s_5}{0.7}, \frac{s_6}{0.3} \right) \right\rangle \textit{ and}$$

$$\mathscr{B}_2 = \left\langle \left(\frac{s_1s_2}{0.2}, \frac{s_2s_3}{0.4}, \frac{s_3s_4}{0.3}, \frac{s_2s_4}{0.7}, \frac{s_2s_6}{0.6}, \frac{s_1s_6}{0.2}, \frac{s_4s_6}{0.7}, \frac{s_5s_6}{0.4}, \frac{s_4s_5}{0.4}, \frac{s_2s_5}{0.4} \right), \left(\frac{s_1s_2}{0.7}, \frac{s_2s_3}{0.62}, \frac{s_3s_4}{0.54}, \frac{s_2s_4}{0.4}, \frac{s_2s_6}{0.35}, \frac{s_1s_6}{0.75}, \frac{s_4s_6}{0.3}, \frac{s_5s_6}{0.65}, \frac{s_4s_5}{0.6}, \frac{s_2s_5}{0.65} \right) \right\rangle.$$



(**a**) $\mathscr{G}_1$ (**b**) $\mathscr{G}_2$

Figure 13. Pythagorean fuzzy planar graphs.



(**a**) $\mathscr{G}_1$ (**b**) $\mathscr{G}_2$

Figure 14. Pythagorean fuzzy planar graphs.

Since a mapping $F : \mathscr{V}_1 \to \mathscr{V}_2$ *defined by* $F(r_1) = s_1$, $F(r_2) = s_2$, $F(r_3) = s_3$, $F(r_4) = s_4$, $F(r_5) = s_5$, $F(r_6) = s_6$ *satisfies* $\mu_{\mathscr{B}_1}(r_ir_j) = \mu_{\mathscr{B}_2}(F(r_i)F(r_j)), \nu_{\mathscr{B}_1}(r_ir_j) = \nu_{\mathscr{B}_2}(F(r_i)F(r_j))$, *for all* $r_ir_j \in \mathscr{E}_1$, *where* $i, j = 1, 2, 3, 4$ *but* $\mu_{\mathscr{A}_1}(r_i) \neq \mu_{\mathscr{A}_2}(F(r_i)), \nu_{\mathscr{A}_1}(r_i) \neq \nu_{\mathscr{A}_2}(F(r_i))$. *Therefore,* $\mathscr{G}_1$ *is a co-weak isomorphic to* $\mathscr{G}_2$.

Some correlated results have been discussed below.

Theorem 11. *If $F : \mathscr{G} \to \mathscr{L}$ is an isomorphism from PFPG $\mathscr{G}$ to Pythagorean fuzzy graph $\mathscr{L}$. Then, $\mathscr{L}$ can be considered as PFPG with equivalent PF planarity value of $\mathscr{G}$.*

Proof. Suppose $F : \mathscr{G} \to \mathscr{L}$ is an isomorphism. As an isomorphism retains the membership and nonmembership value of vertex and edge of Pythagorean fuzzy graphs. Thus, membership and nonmembership value of $\mathscr{L}$ will be equivalent to the membership and nonmembership value of $\mathscr{G}$. Drawing and structure of $\mathscr{L}$ and $\mathscr{G}$ are similar. Hence, the crossings number between edges and Pythagorean fuzzy planarity value of $\mathscr{L}$ will be similar to $\mathscr{G}$. Thus, $\mathscr{L}$ can be considered as PFPG $\mathscr{G}$ with equivalent Pythagorean fuzzy planarity value as that of the Pythagorean fuzzy graph $\mathscr{L}$. □

Theorem 12. *Two isomorphism Pythagorean fuzzy graphs $\mathscr{G}_1$ and $\mathscr{G}_2$ have equivalent planarity value.*

Theorem 13. *Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two Pythagorean fuzzy graphs with Pythagorean fuzzy planarity $\mathscr{F}_1 = (\mathscr{F}_{\mathscr{M}_1}, \mathscr{F}_{\mathscr{N}_1})$ and $\mathscr{F}_2 = (\mathscr{F}_{\mathscr{M}_2}, \mathscr{F}_{\mathscr{N}_2})$, respectively. If $\mathscr{G}_1$ is weak isomorphic to $\mathscr{G}_2$, then $\mathscr{F}_{\mathscr{M}_1} \geq \mathscr{F}_{\mathscr{M}_2}$ and $\mathscr{F}_{\mathscr{N}_1} \leq \mathscr{F}_{\mathscr{N}_2}$.*

Proof. Let $\mathscr{G}_1$ is weak isomorphic to $\mathscr{G}_2$. Then, for any edge $xy \in \mathscr{E}_1$, there exists $F(x)F(y) \in \mathscr{E}_2$. The strength of an edge $\mathscr{S}_{rs} = (\mathscr{M}_{rs}, \mathscr{N}_{rs})$ is given as

$$\mathscr{M}_{rs} = \frac{\mu_{\mathscr{B}1}(rs)_j}{\mu_{\mathscr{A}_1}(r) \wedge \mu_{\mathscr{A}_1}(s)} \leq \frac{\mu_{\mathscr{B}2}(F(r)F(s))_j}{\mu_{\mathscr{A}_2}(F(r)) \wedge \mu_{\mathscr{A}_2}(F(s))} = \mathscr{M}_{F(r)F(s)},$$

$$\mathscr{N}_{rs} = \frac{\nu_{\mathscr{B}1}(rs)_j}{\nu_{\mathscr{A}_1}(r) \vee \nu_{\mathscr{A}_1}(s)} \geq \frac{\nu_{\mathscr{B}2}(F(r)F(s))_j}{\nu_{\mathscr{A}_2}(F(r)) \vee \nu_{\mathscr{A}_2}(F(s))} = \mathscr{N}_{F(r)F(s)}.$$

The intersecting value $\mathscr{S}_{\mathscr{C}_1} = (\mathscr{M}_{\mathscr{C}_1}, \mathscr{N}_{\mathscr{C}_1})$ between two edges uv and rs is

$$\mathscr{M}_{\mathscr{C}_1} = \frac{\mathscr{M}_{uv} + \mathscr{M}_{rs}}{2} \leq \frac{\mathscr{M}_{F(u)F(v)} + \mathscr{M}_{F(r)F(s)}}{2} = \mathscr{M}_{\mathscr{C}_{1F}},$$

$$\mathscr{N}_{\mathscr{C}_1} = \frac{\mathscr{N}_{uv} + \mathscr{N}_{rs}}{2} \geq \frac{\mathscr{N}_{F(u)F(v)} + \mathscr{N}_{F(r)F(s)}}{2} = \mathscr{N}_{\mathscr{C}_{1F}},$$

where $\mathscr{C}_{1F}$ is the intersection point between two edges $F(u)F(v)$ and $F(r)F(s)$ in $\mathscr{G}_2$. Since $\mathscr{G}_1$ is weak isomorphic to $\mathscr{G}_2$, the number of the intersecting points in the certain geometric representation in $\mathscr{G}_1$ and $\mathscr{G}_2$ are equal which are n. Hence,

$$\mathscr{F}_{\mathscr{M}_1} = \frac{1}{1 + \mathscr{M}_{\mathscr{C}_1} + \mathscr{M}_{\mathscr{C}_2} + \ldots + \mathscr{M}_{\mathscr{C}_n}} \geq \frac{1}{1 + \mathscr{M}_{\mathscr{C}_{1F}} + \mathscr{M}_{\mathscr{C}_{2F}} + \ldots + \mathscr{M}_{\mathscr{C}_{nF}}} = \mathscr{F}_{\mathscr{M}_2},$$

$$\mathscr{F}_{\mathscr{N}_1} = \frac{1}{1 + \mathscr{N}_{\mathscr{C}_1} + \mathscr{N}_{\mathscr{C}_2} + \ldots + \mathscr{N}_{\mathscr{C}_n}} \leq \frac{1}{1 + \mathscr{N}_{\mathscr{C}_{1F}} + \mathscr{N}_{\mathscr{C}_{2F}} + \ldots + \mathscr{N}_{\mathscr{C}_{nF}}} = \mathscr{F}_{\mathscr{N}_2}.$$

Thus, we conclude that, if $\mathscr{G}_1$ is weak isomorphic to $\mathscr{G}_2$, then $\mathscr{F}_{\mathscr{M}_1} \geq \mathscr{F}_{\mathscr{M}_2}$ and $\mathscr{F}_{\mathscr{N}_1} \leq \mathscr{F}_{\mathscr{N}_2}$. □

Theorem 14. *Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two Pythagorean fuzzy graphs with Pythagorean fuzzy planarity $\mathscr{F}_1 = (\mathscr{F}_{\mathscr{M}_1}, \mathscr{F}_{\mathscr{N}_1})$ and $\mathscr{F}_2 = (\mathscr{F}_{\mathscr{M}_2}, \mathscr{F}_{\mathscr{N}_2})$, respectively. If $\mathscr{G}_1$ is co-weak isomorphic to $\mathscr{G}_2$, then $\mathscr{F}_{\mathscr{M}_1} \leq \mathscr{F}_{\mathscr{M}_2}$ and $\mathscr{F}_{\mathscr{N}_1} \geq \mathscr{F}_{\mathscr{N}_2}$.*

Proof. Assume that $\mathscr{G}_1$ is co-weak isomorphic to $\mathscr{G}_2$ satisfying the conditions

$$\mu_{\mathscr{A}_1}(r) \leq \mu_{\mathscr{A}_2}(F(r)), \nu_{\mathscr{A}_1}(r) \geq \nu_{\mathscr{A}_2}(F(r)),$$

$$\mu_{\mathscr{B}_1}(rs) = \mu_{\mathscr{B}_2}(F(r)F(s)) \text{ and } \nu_{\mathscr{B}_1}(rs) = \nu_{\mathscr{B}_2}(F(r)F(s)),$$

for all $r, s \in V_1$ and $F(r), F(s) \in V_2$. Then, the strength of an edge $\mathscr{S}_{rs} = (\mathscr{M}_{rs}, \mathscr{N}_{rs})$ is given as

$$\mathscr{M}_{rs} = \frac{\mu_{\mathscr{B}1}(rs)_j}{\mu_{\mathscr{A}_1}(r) \wedge \mu_{\mathscr{A}_1}(s)} \geq \frac{\mu_{\mathscr{B}2}(F(r)F(s))_j}{\mu_{\mathscr{A}_2}(F(r)) \wedge \mu_{\mathscr{A}_2}(F(s))} = \mathscr{M}_{F(r)F(s)},$$

$$\mathscr{N}_{rs} = \frac{\nu_{\mathscr{B}1}(rs)_j}{\nu_{\mathscr{A}_1}(r) \vee \nu_{\mathscr{A}_1}(s)} \leq \frac{\nu_{\mathscr{B}2}(F(r)F(s))_j}{\nu_{\mathscr{A}_2}(F(r)) \vee \nu_{\mathscr{A}_2}(F(s))} = \mathscr{N}_{F(r)F(s)}.$$

The intersecting value $\mathscr{S}_{\mathscr{C}_1} = (\mathscr{M}_{\mathscr{C}_1}, \mathscr{N}_{\mathscr{C}_1})$ between two edges uv and rs is

$$\mathscr{M}_{\mathscr{C}_1} = \frac{\mathscr{M}_{uv} + \mathscr{M}_{rs}}{2} \geq \frac{\mathscr{M}_{F(u)F(v)} + \mathscr{M}_{F(r)F(s)}}{2} = \mathscr{M}_{\mathscr{C}_{1F}},$$

$$\mathscr{N}_{\mathscr{C}_1} = \frac{\mathscr{N}_{uv} + \mathscr{N}_{rs}}{2} \leq \frac{\mathscr{N}_{F(u)F(v)} + \mathscr{N}_{F(r)F(s)}}{2} = \mathscr{N}_{\mathscr{C}_{1F}},$$

where $\mathscr{C}_{1F}$ is the intersection point between two edges $F(u)F(v)$ and $F(r)F(s)$ in $\mathscr{G}_2$. Since $\mathscr{G}_1$ is co-weak isomorphic to $\mathscr{G}_2$, the number of the intersecting points in the certain geometric representation in $\mathscr{G}_1$ and $\mathscr{G}_2$ are equal, which are n. Hence,

$$\mathscr{F}_{\mathscr{M}_1} = \frac{1}{1 + \mathscr{M}_{\mathscr{C}_1} + \mathscr{M}_{\mathscr{C}_2} + \ldots + \mathscr{M}_{\mathscr{C}_n}} \leq \frac{1}{1 + \mathscr{M}_{\mathscr{C}_{1F}} + \mathscr{M}_{\mathscr{C}_{2F}} + \ldots + \mathscr{M}_{\mathscr{C}_{nF}}} = \mathscr{F}_{\mathscr{M}_2},$$

$$\mathscr{F}_{\mathscr{N}_1} = \frac{1}{1 + \mathscr{N}_{\mathscr{C}_1} + \mathscr{N}_{\mathscr{C}_2} + \ldots + \mathscr{N}_{\mathscr{C}_n}} \geq \frac{1}{1 + \mathscr{N}_{\mathscr{C}_{1F}} + \mathscr{N}_{\mathscr{C}_{2F}} + \ldots + \mathscr{N}_{\mathscr{C}_{nF}}} = \mathscr{F}_{\mathscr{N}_2}.$$

Thus, we conclude that, if $\mathscr{G}_1$ is co-weak isomorphic to $\mathscr{G}_2$, then $\mathscr{F}_{\mathscr{M}_1} \leq \mathscr{F}_{\mathscr{M}_2}$ and $\mathscr{F}_{\mathscr{N}_1} \geq \mathscr{F}_{\mathscr{N}_2}$. □

Theorem 15. *Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two weak isomorphic PFGs with PF planarity values $\mathscr{F}_1 = (\mathscr{F}_{\mathscr{M}_1}, \mathscr{F}_{\mathscr{N}_1})$ and $\mathscr{F}_2 = (\mathscr{F}_{\mathscr{M}_2}, \mathscr{F}_{\mathscr{N}_2})$, respectively. If the edge membership and nonmembership grades of the parallel crossing edges are equivalent, then $(\mathscr{F}_{\mathscr{M}_1}, \mathscr{F}_{\mathscr{N}_1}) = (\mathscr{F}_{\mathscr{M}_2}, \mathscr{F}_{\mathscr{N}_2})$.*

Proof. Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two weak isomorphic PFGs with PF planarity values $\mathscr{F}_1$ and $\mathscr{F}_2$, respectively. Since two PFGs are weak isomorphic, $\mu_{\mathscr{A}_1}(a) = \mu_{\mathscr{A}_2}(u), \nu_{\mathscr{A}_1}(a) = \nu_{\mathscr{A}_2}(u)$, for all $a \in \mathscr{G}_1$ and $u \in \mathscr{G}_2$. Let the Pythagorean fuzzy graphs have one crossing. Let two crossing edges in $\mathscr{G}_1$ and $\mathscr{G}_2$ are bc, de and vw, rs, respectively. Then, the cut point in $\mathscr{G}_1$ is defined by

$$\left(\frac{\frac{\mu_{\mathscr{B}_1}(bc)}{\mu_{\mathscr{A}_1}(b) \wedge \mu_{\mathscr{A}_1}(c)} + \frac{\mu_{\mathscr{B}_1}(de)}{\mu_{\mathscr{A}_1}(d) \wedge \mu_{\mathscr{A}_1}(e)}}{2}, \frac{\frac{\nu_{\mathscr{B}_1}(bc)}{\nu_{\mathscr{A}_1}(b) \vee \nu_{\mathscr{A}_1}(c)} + \frac{\nu_{\mathscr{B}_1}(de)}{\nu_{\mathscr{A}_1}(d) \vee \nu_{\mathscr{A}_1}(e)}}{2} \right).$$

Likewise, the cut point in $\mathscr{G}_2$ is defined by

$$\left(\frac{\frac{\mu_{\mathscr{B}_2}(vw)}{\mu_{\mathscr{A}_2}(v) \wedge \mu_{\mathscr{A}_2}(w)} + \frac{\mu_{\mathscr{B}_2}(vw)}{\mu_{\mathscr{A}_2}(v) \wedge \mu_{\mathscr{A}_2}(w)}}{2}, \frac{\frac{\nu_{\mathscr{B}_2}(rs)}{\nu_{\mathscr{A}_2}(r) \vee \nu_{\mathscr{A}_2}(s)} + \frac{\nu_{\mathscr{B}_2}(rs)}{\nu_{\mathscr{A}_2}(r) \vee \nu_{\mathscr{A}_2}(s)}}{2} \right).$$

Now, $\mathscr{F}_1 = \mathscr{F}_2$, if $\mu_{\mathscr{B}_1}(bc) = \mu_{\mathscr{B}_2}(vw)$ and $\nu_{\mathscr{B}_1}(bc) = \nu_{\mathscr{B}_2}(vw)$. The number of intersecting points increases. However, if the sum of crossing values of $\mathscr{G}_1$ are equivalent to that of $\mathscr{G}_2$, then Pythagorean fuzzy planarity values must be equal. Hence, for $\mathscr{F}_1 = \mathscr{F}_2$, the edge membership and nonmembership grades of crossing edges of $\mathscr{G}_1$ are equivalent to the edge membership and nonmembership grades of the parallel crossing edges in $\mathscr{G}_2$. □

Theorem 16. *Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two co-weak isomorphic PFGs with PF planarity values $\mathscr{F}_1 = (\mathscr{F}_{\mathscr{M}_1}, \mathscr{F}_{\mathscr{N}_1})$ and $\mathscr{F}_2 = (\mathscr{F}_{\mathscr{M}_2}, \mathscr{F}_{\mathscr{N}_2})$, respectively. If the minimum membership and maximum nonmembership grade of end vertices of the parallel crossing edges are equivalent, then $(\mathscr{F}_{\mathscr{M}_1}, \mathscr{F}_{\mathscr{N}_1}) = (\mathscr{F}_{\mathscr{M}_2}, \mathscr{F}_{\mathscr{N}_2})$.*

Proof. Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two co-weak isomorphic PFGs with PF planarity values $\mathscr{F}_1$ and $\mathscr{F}_2$, respectively. Since two PFGs are co-weak isomorphic, so $\mu_{\mathscr{B}_1}(ab) = \mu_{\mathscr{B}_2}(uv), \nu_{\mathscr{B}_1}(ab) = \nu_{\mathscr{B}_2}(uv)$, for all $ab \in \mathscr{G}_1$ and $uv \in \mathscr{G}_2$. Let the Pythagorean fuzzy graphs have one crossing. Let two crossing edges in $\mathscr{G}_1$ and $\mathscr{G}_2$ are cd, ef and wx, yz, respectively. Then, the cut point in $\mathscr{G}_1$ is defined by

$$\left(\frac{\frac{\mu_{\mathscr{B}_1}(cd)}{\mu_{\mathscr{A}_1}(c)\wedge\mu_{\mathscr{A}_1}(d)}+\frac{\mu_{\mathscr{B}_1}(ef)}{\mu_{\mathscr{A}_1}(e)\wedge\mu_{\mathscr{A}_1}(f)}}{2},\frac{\frac{\nu_{\mathscr{B}_1}(cd)}{\nu_{\mathscr{A}_1}(c)\vee\nu_{\mathscr{A}_1}(d)}+\frac{\nu_{\mathscr{B}_1}(ef)}{\nu_{\mathscr{A}_1}(e)\vee\nu_{\mathscr{A}_1}(f)}}{2}\right).$$

Likewise, the cut point in $\mathscr{G}_2$ is defined by

$$\left(\frac{\frac{\mu_{\mathscr{B}_2}(wx)}{\mu_{\mathscr{A}_2}(w)\wedge\mu_{\mathscr{A}_2}(x)}+\frac{\mu_{\mathscr{B}_2}(yz)}{\mu_{\mathscr{A}_2}(y)\wedge\mu_{\mathscr{A}_2}(z)}}{2},\frac{\frac{\nu_{\mathscr{B}_2}(wx)}{\nu_{\mathscr{A}_2}(w)\vee\nu_{\mathscr{A}_2}(x)}+\frac{\nu_{\mathscr{B}_1}(yz)}{\nu_{\mathscr{A}_2}(y)\vee\nu_{\mathscr{A}_2}(z)}}{2}\right).$$

Now, $\mathscr{F}_1 = \mathscr{F}_2$, if $\mu_{\mathscr{A}_1}(c) \wedge \mu_{\mathscr{A}_1}(d) = \mu_{\mathscr{A}_2}(w) \wedge \mu_{\mathscr{A}_2}(x)$ and $\nu_{\mathscr{A}_1}(c) \vee \nu_{\mathscr{A}_1}(d) = \nu_{\mathscr{A}_2}(w) \vee \nu_{\mathscr{A}_2}(x)$. The number of intersecting points increase. However, if the sum of crossing value of $\mathscr{G}_1$ is equivalent to that of $\mathscr{G}_2$, then Pythagorean fuzzy planarity values must be equal. Hence, for $\mathscr{F}_1 = \mathscr{F}_2$, the minimum membership and maximum nonmembership grades of end vertices of an edge in $\mathscr{G}_1$ is equivalent to the minimum membership and maximum nonmembership grades of parallel edge in $\mathscr{G}_2$. □

In a crisp sense, we know that double dual of planar graph is also planar. We call it self-duality of planar graph. However, this concept does not hold in a Pythagorean fuzzy planar graph as the vertex membership and nonmembership grade of Pythagorean fuzzy planar graph are not preserved in its dual graph. However, the edge membership and nonmembership grade of Pythagorean fuzzy planar graph are preserved. The following theorem illustrates this fact.

Theorem 17. *If $\mathscr{G}_2$ is the PFDG of PFDG $\mathscr{G}_1$ of a PFPG $\mathscr{G}$ without weak edges, then a co-weak isomorphism occurs between $\mathscr{G}$ and $\mathscr{G}_2$.*

Proof. Suppose that $\mathscr{G}$ is a PFPG without weak edges. Suppose that $\mathscr{G}_1$ is a PFDG of $\mathscr{G}$ and $\mathscr{G}_2$ is the PFDG of $\mathscr{G}_1$. For establishing co-weak isomorphism between $\mathscr{G}$ and $\mathscr{G}_2$. We know that the number of vertices of $\mathscr{G}_2$ is equivalent to the strong Pythagorean fuzzy faces of $\mathscr{G}_1$. Similarly, the number of strong Pythagorean fuzzy faces of $\mathscr{G}_1$ is equivalent to the number of vertices of $\mathscr{G}$. Hence, the number of vertices of $\mathscr{G}_2$ and $\mathscr{G}$ are similar. Furthermore, by definition of PFDG, the membership and nonmembership grade of an edge in PFDG is equivalent to the membership and nonmembership grade of an edge in PFPG. Thus, it is concluded that a co-weak isomorphism occurs between $\mathscr{G}$ and $\mathscr{G}_2$. □

The following example justifies the above theorem.

Example 19. *Consider a PFPG $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ without weak edges, as displayed in Figure 15 such that*

$$\mathscr{A} = \left\langle \left(\frac{r_1}{0.55}, \frac{r_2}{0.4}, \frac{r_3}{0.8}, \frac{r_4}{0.9}, \frac{r_5}{0.6}, \frac{r_6}{0.7}\right), \left(\frac{r_1}{0.69}, \frac{r_2}{0.7}, \frac{r_3}{0.3}, \frac{r_4}{0.2}, \frac{r_5}{0.5}, \frac{r_6}{0.5}\right) \right\rangle \textit{ and}$$
$$\mathscr{B} = \left\langle \left(\frac{r_1r_2}{0.25}, \frac{r_2r_3}{0.3}, \frac{r_3r_4}{0.75}, \frac{r_4r_5}{0.55}, \frac{r_5r_6}{0.55}, \frac{r_1r_6}{0.45}, \frac{r_3r_6}{0.56}\right), \left(\frac{r_1r_2}{0.14}, \frac{r_2r_3}{0.14}, \frac{r_3r_4}{0.14}, \frac{r_4r_5}{0.14}, \frac{r_5r_6}{0.14}, \frac{r_1r_6}{0.14}, \frac{r_3r_6}{0.14}\right) \right\rangle.$$

The Pythagorean fuzzy dual graph $\mathscr{G}_1$ of $\mathscr{G}$ is displayed in Figure 16, where

$$\mathscr{A}_1 = \left\langle \left(\frac{s_1}{0.7}, \frac{s_2}{0.75}, \frac{s_3}{0.75}\right), \left(\frac{s_1}{0.14}, \frac{s_2}{0.14}, \frac{s_3}{0.14}\right) \right\rangle \textit{ and}$$
$$\mathscr{B}_1 = \left\langle \left(\frac{s_1s_2}{0.56}, \frac{s_1s_3}{0.25}, \frac{s_1s_3}{0.3}, \frac{s_1s_3}{0.45}, \frac{s_2s_3}{0.55}, \frac{s_2s_3}{0.55}, \frac{s_2s_3}{0.75}\right), \left(\frac{s_1s_2}{0.14}, \frac{s_1s_3}{0.14}, \frac{s_1s_3}{0.14}, \frac{s_1s_3}{0.14}, \frac{s_2s_3}{0.14}, \frac{s_2s_3}{0.14}, \frac{s_2s_3}{0.14}\right) \right\rangle.$$

Again, constructing the dual of $\mathscr{G}_1$ as displayed in Figure 17, where

$$\mathscr{A}_2 = \left\langle \left(\frac{t_1}{0.7}, \frac{t_2}{0.7}, \frac{t_3}{0.75}, \frac{t_4}{0.75}, \frac{t_5}{0.65}, \frac{t_6}{0.56} \right), \left(\frac{t_1}{0.14}, \frac{t_2}{0.14}, \frac{t_3}{0.14}, \frac{t_4}{0.14}, \frac{t_5}{0.14}, \frac{t_6}{0.14} \right) \right\rangle \text{ and}$$
$$\mathscr{B}_2 = \left\langle \left(\frac{t_1t_2}{0.3}, \frac{t_2t_3}{0.25}, \frac{t_3t_4}{0.75}, \frac{t_4t_5}{0.55}, \frac{t_5t_6}{0.55}, \frac{t_3t_6}{0.56}, \frac{t_1t_6}{0.45} \right), \left(\frac{t_1t_2}{0.14}, \frac{t_2t_3}{0.14}, \frac{t_3t_4}{0.14}, \frac{t_4t_5}{0.14}, \frac{t_5t_6}{0.14}, \frac{t_3t_6}{0.14}, \frac{t_1t_6}{0.14} \right) \right\rangle.$$

It is easy to see that the edge membership and nonmembership grades of $\mathscr{G}_2$ are equal to the edge membership and nonmembership grades of $\mathscr{G}$, but the vertex membership and nonmembership grades of $\mathscr{G}_2$ are not equal to the vertex membership and nonmembership grades of $\mathscr{G}$, which shows that the self-duality of Pythagorean fuzzy planar graph is not satisfied. Hence, we conclude that there is co-weak isomorphism between $\mathscr{G}_2$ and $\mathscr{G}$.

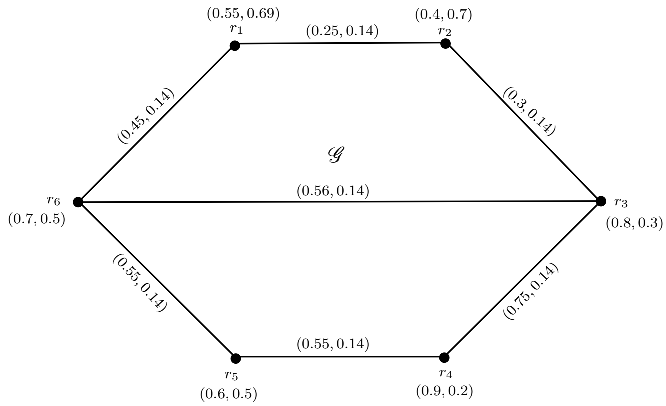


Figure 15. Pythagorean fuzzy planar graph.

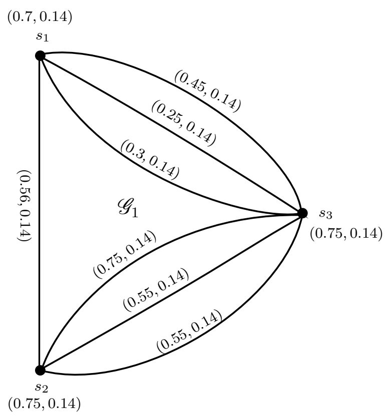


Figure 16. Pythagorean fuzzy dual graph.

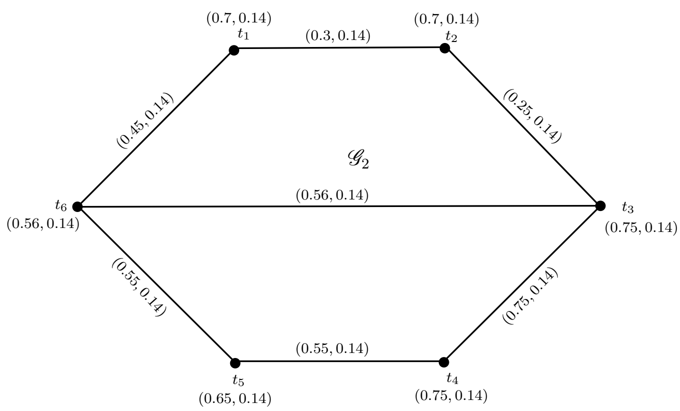


Figure 17. Pythagorean fuzzy dual graph.

7. Application

From the power plants to our houses, the potent power lines that are zigzagging our countryside or city streets carry numerously high voltage electricity. For reducing such high voltage electricity to lower voltage, an equipment is used, called a transformer. A transformer works in a very simple way, consisting of different units in which electric current flows through tiny wires. While connecting the units with each other, crossing between tiny wires may occur. Sometimes, crossing between wires is beneficial as it helps in utilizing less space and makes it inexpensive, but, on the other hand, due to crossing, the transformer heats up and there is a chance of an explosion that is quite dangerous for human life. To overcome this problem, a crossing between such wires needs to be minimized or good quality wires are needed for installation. The practical approach of Pythagorean fuzzy planar graphs can be utilized to structure this kind of situation for reducing the rate of destruction.

Consider an electric transformer in which units are connected as shown in Figure 18. Each unit $U_1, U_2, \ldots, U_7$ is represented by a vertex and each electric connection between units through tiny wire is represented by an edge. The membership grade of the vertex depicted the chances of electric spark, whereas the nonmembership grade interpreted the chances of no electric spark in the unit. The membership grade of the edge depicted the intensity of electrical hazard between two units, whereas the nonmembership grade interpreted no intensity of electrical hazard.

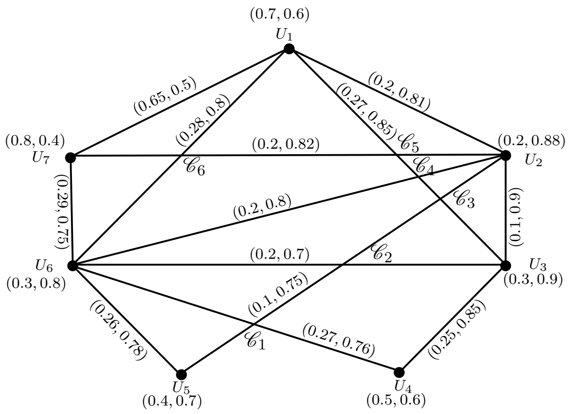


Figure 18. Transformer units connection.

As the number of crossings increase, the rate of destruction increases. Hence, the measurement of the planarity value is necessary. There are six crossings $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4, \mathscr{C}_5$ and $\mathscr{C}_6$ between the pair of wires (U_4U_6, U_2U_5), (U_3U_6, U_2U_5), (U_2U_5, U_1U_3), (U_2U_6, U_1U_3), (U_2U_7, U_1U_3) and (U_2U_7, U_1U_6), respectively. The strength of the wire $U_4U_6 = (0.9, 0.95)$, $U_2U_5 = (0.5, 0.85)$, $U_3U_6 = (0.67, 0.78)$, $U_1U_3 = (0.9, 0.94)$, $U_2U_6 = (1, 0.91)$, $U_2U_7 = (1, 0.93)$ and $U_1U_6 = (0.93, 1)$. For crossings, the point of intersections are $\mathscr{S}_{\mathscr{C}_1} = (0.7, 0.9)$, $\mathscr{S}_{\mathscr{C}_2} = (0.59, 0.82)$, $\mathscr{S}_{\mathscr{C}_3} = (0.7, 0.9)$, $\mathscr{S}_{\mathscr{C}_4} = (0.95, 0.93)$, $\mathscr{S}_{\mathscr{C}_5} = (0.95, 0.94)$ and $\mathscr{S}_{\mathscr{C}_6} = (0.97, 0.97)$. Thus, Pythagorean fuzzy planarity value $\mathscr{F} = (0.17, 0.15)$. Since the planarity value is at a minimum, it indicates the possibility of high destruction. To reduce crossing, we can change the graphical representation as shown in Figure 19.

We know that the number of intersecting points is inversely proportional to planarity. Since the number of intersecting points decrease, the Pythagorean fuzzy planarity value $\mathscr{F} = (0.63, 0.55)$ increases and rate of destruction decreases. Moreover, from the representation shown in Figure 19, it is noted that $\mathscr{C}_1$ is the only crossing left that can not be reduced, but the chance of electric hazard and rate of destruction through it can be minimized by using good quality electrical wires between U_3 and U_6, U_2 and U_5. Thus, this crossing will become less harmful. Hence, we conclude that the Pythagorean fuzzy electric connection model can be used for tracking and detecting the rate of destruction. By examining and taking extra special security measures, the percentage of destruction can be reduced and many human lives can be saved.

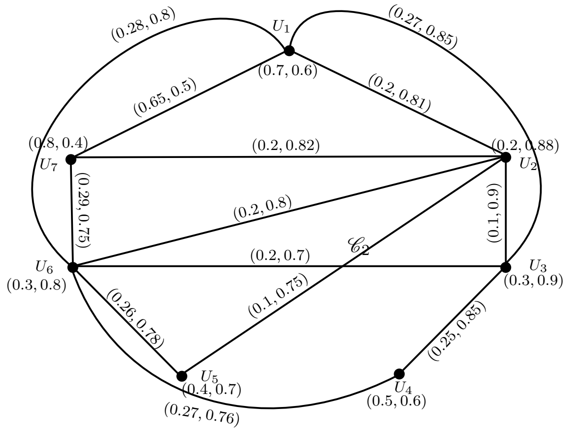


Figure 19. Pythagorean fuzzy electric connection model.

We present our proposed method for checking planarity and minimizing crossings between electric connections in the following Algorithm 1.

Algorithm 1: Planarity and minimizing crossings between electric connections.

INPUT: A discrete set of units $U = \{U_1, U_2, \ldots, U_n\}$, a set of electric connections $E = \{E_1, E_2, \ldots, E_m\}$ between the units and a set of point of intersections $C = \{\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_r\}$.
OUTPUT: Minimized crossing and increased planarity value.

1. **begin**
2. Compute the strength of the edge E_i, where $i = 1, 2, \ldots, m$ and $U_j, U_k \in U$ by using
$$\mathscr{S}_{E_i} = (\mathscr{M}_{E_i}, \mathscr{N}_{E_i}) = \left(\frac{\mu_{\mathscr{B}}(E_i)}{\mu_{\mathscr{A}}(U_j) \wedge \mu_{\mathscr{A}}(U_k)}, \frac{\nu_{\mathscr{B}}(E_i)}{\nu_{\mathscr{A}}(U_j) \vee \nu_{\mathscr{A}}(U_k)} \right).$$
3. Calculate the value of intersecting points $\mathscr{C}_l$ between the edges E_j and E_k by using the formula
$$\mathscr{S}_{\mathscr{C}_l} = (\mathscr{M}_{\mathscr{C}_l}, \mathscr{N}_{\mathscr{C}_l}) = \left(\frac{\mathscr{M}_{E_j} + \mathscr{M}_{E_k}}{2}, \frac{\mathscr{N}_{E_j} + \mathscr{N}_{E_k}}{2} \right), \text{where } l = 1, 2, \ldots, r \text{ and } E_j, E_k \in E.$$
4. Determine the Pythagorean fuzzy planarity value defined as
$$\mathscr{F} = (\mathscr{F}_{\mathscr{M}}, \mathscr{F}_{\mathscr{N}}) = \left(\frac{1}{1 + \{\mathscr{M}_{\mathscr{C}_1} + \mathscr{M}_{\mathscr{C}_2} + \ldots + \mathscr{M}_{\mathscr{C}_r}\}}, \frac{1}{1 + \{\mathscr{N}_{\mathscr{C}_1} + \mathscr{N}_{\mathscr{C}_2} + \ldots + \mathscr{N}_{\mathscr{C}_r}\}} \right).$$
5. Keep the graphical representation of the edges E_j and E_k, if $\mathscr{F}_{\mathscr{M}} > 0.5$ and $\mathscr{F}_{\mathscr{N}} < 0.86$ otherwise change the graphical representation.
6. While changing the graphical representation of the edges E_j and E_k, if no new crossing occur in this representation then Change it otherwise keep the previous representation.
7. By changing the graphical representation of the edges E_j and E_k, the crossing and planarity value will be minimized and increased, respectively.
8. **end.**

8. Conclusions

Graph theory has a vast range of applications in designing various networking problems encountered in different fields such as image capturing, transportation and data mining. To model uncertainties in graphical networking problems, numerous generalization of graph theoretical concepts have established. Pythagorean fuzzy graphs, as an extension of fuzzy graphs and intuitionistic fuzzy graphs, have better ability due to the increment of spaces in membership and nonmembership grades, for modeling the obscurity in practical world problems. This paper has utilized the idea of Pythagorean fuzzy graphs and initiated the concept of Pythagorean fuzzy multigraphs and Pythagorean fuzzy planar graphs. It has investigated the Pythagorean fuzzy planarity value by considering strong, weak and considerable edges. Moreover, a critical analysis has been done on a nonplanar Pythagorean fuzzy graph. A close association has been developed between Pythagorean fuzzy planar graphs and Pythagorean fuzzy dual graphs. Furthermore, the concept of isomorphism, weak isomorphism and co-weak isomorphism have been elaborated between Pythagorean fuzzy planar graphs and some substantial results have been investigated. In the end, it has explored an important result that there exists a co-weak isomorphism between the Pythagorean fuzzy planar graph and dual of a dual Pythagorean fuzzy planar graph. The purpose of this research work is the applicability of Pythagorean fuzzy planar graphs in the field of neural networks and geographical information systems. With the help of these graphs, many problems related to crossing including designing golf holes in a golf club, linking different houses with each other and structuring road or communication networks can be easily solved. Further studies can focus on (1) Interval-valued Pythagorean fuzzy graphs; (2) Hesitant Pythagorean fuzzy graphs and; (3) Simplified interval-valued Pythagorean fuzzy graphs.

Author Contributions: M.A., J.M.D. and A.F. conceived of the presented idea. J.M.D. and A.F. developed the theory and performed the computations. M.A. and J.M.D. verified the analytical methods.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare that they have no conflict of interest regarding the publication of the research article.

References

1. Pal, A.; Samanta, S.; Pal, M. Concept of Fuzzy Planar Graphs. In Proceedings of the Science and Information Conference, London, UK, 14 November 2013; pp. 557–563.
2. Pal, A.; Samanta, S.; Pal, M. New concept of fuzzy planar graphs. *Int. J. Adv. Res. Artif. Intell.* **2014**, *3*, 52–59.
3. Pramanik, T.; Samanta, S.; Pal, M. Special planar fuzzy graph configurations. *Int. J. Sci. Res. Publ.* **2012**, *2*, 1–4.
4. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
5. Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
6. Yager, R.R. Pythagorean fuzzy subsets. In Proceedings of the Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, AB, Canada, 24–28 June 2013; pp. 57–61.
7. Yager, R.R.; Abbasov, A.M. Pythagorean membership grades, complex numbers, and decision making. *Int. J. Intell. Syst.* **2013**, *28*, 436–452. [CrossRef]
8. Yager, R.R. Pythagorean membership grades in multi-criteria decision making. *IEEE Trans. Fuzzy Syst.* **2014**, *22*, 958–965. [CrossRef]
9. Zhang, X.; Xu, Z. Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets. *Int. J. Intell. Syst.* **2014**, *29*, 1061–1078. [CrossRef]
10. Kaufmann, A. *Introduction a la Theorie des Sour-Ensembles Flous*; Masson et Cie 1: Paris, France, 1973.
11. Rosenfeld, A. Fuzzy graphs. In *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*; Zadeh, L.A., Fu, K.S., Shimura, M., Eds.; Academic Press: Cambridge, MA, USA, 1975; pp. 77–95.
12. Bhattacharya, P. Some remarks on fuzzy graphs. *Pattern Recognit. Lett.* **1987**, *5*, 297–302. [CrossRef]
13. Mordeson, J.N.; Peng, C.S. Operations on fuzzy graphs. *Inf. Sci.* **1994**, *79*, 159–170. [CrossRef]
14. Shannon, A.; Atanassov, K.T. A first step to a theory of intuitioistic fuzzy graphs. In Proceedings of the Fuzzy Based Expert Systems, Sofia, Bulgaria, 26–29 June 1994; Lakov, D., Ed.; pp. 59–61.
15. Shannon, A.; Atanassov, K.T. Intuitioistic fuzzy graphs from α-, β-, (α, β)-levels. *Notes Intuit. Fuzzy Sets* **1995**, *1*, 32–35.
16. Parvathi, R.; Karunambigai, M.G.; Atanassov, K.T. Operations on intuitioistic fuzzy graphs. In Proceedings of the IEEE International Conference on Fuzzy Systems, Jeju Island, Korea, 20–24 August 2009; pp. 1396–1401.
17. Akram, M.; Davvaz, B. Strong intuitionistic fuzzy graphs. *Filomat* **2012**, *26*, 177–196. [CrossRef]

18. Akram, M.; Dudek, W.A. Intuitionistic fuzzy hypergraphs with applications. *Inf. Sci.* **2013**, *218*, 182–193. [CrossRef]
19. Naz, S.; Ashraf, S.; Akram, M. A novel approach to decision-making with Pythagorean fuzzy information. *Mathematics* **2018**, *6*, 1–28. [CrossRef]
20. Verma, R.; Merigo, J.M.; Sahni, M. Pythagorean fuzzy graphs: Some results. *arXiv* **2018**, arXiv:1806.06721v1.
21. Akram, M.; Naz, S. Energy of Pythagorean fuzzy graphs with applications. *Mathematics* **2018**, *6*, 136. [CrossRef]
22. Dhavudh, S.S.; Srinivasan, R. Intuitionistic fuzzy graphs of second type. *Adv. Fuzzy Math.* **2017**, *12*, 197–204.
23. Dhavudh, S.S.; Srinivasan, R. Properties of intuitionistic fuzzy graphs of second type. *Int. J. Comput. Appl. Math.* **2017**, *12*, 815–823.
24. Akram, M.; Habib, A.; Ilyas, F.; Dar, J.M. Specific types of Pythagorean fyzzy graphs and application to decision-making. *Math. Comput. Appl.* **2018**, *23*, 42. [CrossRef]
25. Akram, M.; Dar, J.M.; Naz, S. Certain Graphs under Pythagorean Fuzzy Environment. *Complex Intell. Syst.* in press.
26. Abdul-Jabbar, N.; Naoom, J.H.; Ouda, E.H. Fuzzy dual graphs. *J. Al-Nahrain Univ.* **2009**, *12*, 168–171.
27. Yager, R.R. On the theory of bags. *Int. J. Gen. Syst.* **1986**, *13*, 23–37. [CrossRef]
28. Nirmala, G.; Dhanabal, K. Interval-valued fuzzy planar graphs. *Int. J. Mach. Learn. Cybern.* **2016**, *7*, 653–664.
29. Alshehri, N.; Akram, M. Intuitionistic fuzzy Planar Graphs. Discret. *Dyn. Nat. Soc.* **2014**, *2014*, 397823. [CrossRef]
30. Akram, M.; Samanata, S.; Pal, M. Application of bipolar fuzzy sets in planar graph. *Int. J. Appl. Comput. Math.* **2017**, *3*, 773–785. [CrossRef]
31. Peng, X.; Yang, Y. Some results for Pythagorean fuzzy sets. *Int. J. Intell. Syst.* **2015**, *30*, 1133–1160. [CrossRef]
32. Ren, P.; Xu, Z.; Gou, X. Pythagorean fuzzy TODIM approach to multi-criteria decision making. *Appl. Soft Comput.* **2016**, *42*, 246–259. [CrossRef]
33. Rangasamy, P.; Palaniappan, N. Some operations on intuitionistic fuzzy sets of second type. *Notes Intuit. Fuzzy Sets* **2003**, *10*, 1–19.
34. Peng, X.; Selvachandran, G. Pythagorean fuzzy set: state of the art and future directions. *Artif. Intell. Rev.* **2017**, 1–55. [CrossRef]
35. Garg, H. Some methods for strategic decision-making problems with immediate probabilities in Pythagorean fuzzy environment. *Int. J. Intell. Syst.* **2018**, *33*, 687–712. [CrossRef]
36. Garg, H. Linguistic Pythagorean fuzzy sets and its applications in multi attribute decision making process. *Int. J. Intell. Syst.* **2018**, *33*, 1234–1263. [CrossRef]
37. Zadeh, L.A. Similarity relations and fuzzy orderings. *Inf. Sci.* **1971**, *3*, 177–200. [CrossRef]