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On Domain of Nörlund Matrix

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Abstract: In 1978, the domain of the Nörlund matrix on the classical sequence spaces l_p and l_∞ was introduced by Wang, where $1 \leq p < \infty$. Tuğ and Başar studied the matrix domain of Nörlund mean on the sequence spaces f_0 and f in 2016. Additionally, Tuğ defined and investigated a new sequence space as the domain of the Nörlund matrix on the space of bounded variation sequences in 2017. In this article, we defined new space $bs(N^t)$ and $cs(N^t)$ and examined the domain of the Nörlund mean on the bs and cs, which are bounded and convergent series, respectively. We also examined their inclusion relations. We defined the norms over them and investigated whether these new spaces provide conditions of Banach space. Finally, we determined their α-, β-, γ-duals, and characterized their matrix transformations on this space and into this space.

Keywords: nörlund mean; nörlund transforms; difference matrix; α-, β-, γ-duals; matrix transformations

1. Introduction

1.1. Background

In the studies on the sequence space, creating a new sequence space and research on its properties have been important. Some researchers examined the algebraic properties of the sequence space while others investigated its place among other known spaces and its duals, and characterized the matrix transformations on this space.

One way to create a new sequence space in addition to standard sequence space is to use the domain of infinite matrices. In 1978, Ng-Lee [1] studied the domain of an infinite matrix. In the same year, Wang [2] constructed a new sequence space using an infinite matrix, unlike the infinite matrix used by Ng-Lee. These studies have been followed by many researchers such as Malkovsky [3], Altay, and Başar [4]. This topic was first studied in the 1970s but rather intensively after 2000.

1.2. Problem of Interest

The theory of infinite matrices was formulated by the book "Infinite Matrices and Sequence Spaces" written by Cooke [5]. After the publication of this book in 1950, many researchers have used infinite matrices over the years. In some of these studies, the domain of infinite matrices on a sequence space was investigated. One problem is that we do not know the properties of the domain of the Nörlund matrix, which is a trianglular infinite matrix on bs and cs. The domain of the Nörlund matrix is a new sequence space. We intend to address algebraic properties of this new space, to determine its place among other known spaces, to determine its duals, and to characterize the matrix transformations on this space and into this space. Our aim is to provide solutions to these problems.

One difficulty of this study is to determine whether the new space created by the infinite matrix is the contraction or the expansion or overlap of the original space. Also, we have a matrix mapping problem where we must determine the collection of infinite matrices for which the map is a sequence space into another sequence space. We intend to address the first problem by giving a few inclusion theorems, similar to previous studies. For the second problem, we provide two theorems and use the matrix transformation between the standard sequences spaces.

1.3. Literature Survey

Many authors have used infinite matrices for the calculation of any matrix domain up to now. For more information, see [6–26]. Ng and Lee [1] built sequence spaces using the domain of the Cesaro matrix of order one on the classical sequences l_p and l_∞ in 1978, where $1 \leq p < \infty$. In the same year, the spaces $l_\infty(N^t)$ and $l_p(N^t)$ which are the domain of the Nörlund matrix on the sequence space l_p and l_∞ were studied by Wang [2], with $1 \leq p < \infty$. Malkovsky [3] constructed the domain of the Riesiz matrix on sequence spaces c, c_0, and l_∞ in 1997. Altay and Başar [27] worked on the domain of Riesiz matrix on l_∞ in 2002. Malkovsky and Savaş [28] built some sequence spaces derived from the concept of weighted means. Aydın and Başar [29] introduced sequence spaces, a_0^r and a_c^r, that are derived from the domain of the A^r matrix which are stronger than the Cesaro method, C_1. Aydın and Başar [30] studied the forms $a_0^r(u,p)$ and $a_c^r(u,p)$. Aydın and Başar [31] introduced the spaces $a_0^r(\Delta)$ and $a_c^r(\Delta)$ of difference sequences. Aydın and Başar [32] also introduced the sequence space a_p^r of a non-absolute type of A^r matrix. Altay and Başar [33] investigated and introduced the domain of the Euler matrix on c and c_0. Sengönül and Başar [34] introduced and investigated the domain of the Cesaro matrix of order one on sequence spaces c and c_0. Also, $f_0(N^t)$ and $f(N^t)$ were defined by Tuğ and Başar [35], where f_0 and f were almost null and almost convergent sequence spaces, respectively. Yeşilkayagil and Başar [36] investigated the paranormed Nörlund sequence space of the non-absolute type. Yeşilkayagil and Başar [37] worked on the domain of the Nörlund matrix in some Maddox's spaces. Yaşar and Kayaduman [38] introduced and investigated sequence spaces $bs(\hat{F}(r,s))$ and $cs(\hat{F}(r,s))$ using the domain of the Generalized Fibonacci matrix on bs and cs. Furthermore, Mears [39,40] introduced some theorems and the inverse of the Nörlund matrix for the Nörlund mean.

1.4. Scope and Contribution

In this paper, we conduct studies on the sequence space such as topological properties, inclusion relations, base, duals, and matrix transformation. We provide certain tools to researchers by using the concept of sequence spaces directly or indirectly.

We will use a method similar to the ones used in previous studies to solve these problems. We see in the previous studies that the new sequence space produced from original space is a linear space. The same is true for the spaces we produced. At the same time, spaces produced are normed spaces and Banach spaces. In general, the spaces produced and original spaces were found to be isomorphic. The spaces produced in some studies were the expansion of the original space while the others involved some overlap. For example, the space produced in the study of Yaşar and Kayaduman [38] is an expansion, while in this study, the space is a contraction. In this study, alpha, beta, and gamma duals of the spaces produced are available. However, the spaces produced in some previous studies do not have all the duals.

In addition, we try to close the existing deficits in the field the domain of the Nörlund matrix on classical sequence spaces.

1.5. Organization of the Paper

This article consists of eight sections. In Section 1, general information about the working problem is given and the history and importance of the problem is emphasized. A literature survey and the scope and contribution of the study are also presented. In Section 2, a mathematical background of this study is given. In Section 3, two new sequence spaces are constructed using the domain of the Nörlund matrix on the bs and cs sequence spaces. These spaces are $bs(N^t)$ and $cs(N^t)$, where N^t is the Nörlund matrix according to $t = (t_k)$. The formulation of the N^t-transform function of any sequence space is obtained, and it is shown that they are linear spaces. Also, their norms are defined. We find that $bs(N^t) \cong bs$ and $cs(N^t) \cong cs$. In Section 4, $bs(N^t)$ and $cs(N^t)$ are proven to be Banach spaces. Their inclusion relations are given and they are compared to other spaces. It is found that the $cs(N^t)$ space has a Schauder base. The α-, β-, and γ-duals of these two spaces are calculated. Finally, the necessary conditions for matrix transformations on and into these spaces are provided. They are in the form of $(bs(N^t), \lambda)$, $(cs(N^t), \lambda)$, $(\mu, bs(N^t))$, and $(\mu, cs(N^t))$, where we denote the class of infinite matrices moved from sequences of μ space to sequences of λ space with (μ,λ). In Sections 5 and 6, results and discussion of the study are given, respectively. In Section 7, simple numerical examples were given in order to illustrate the findings of the paper. In the last section, a summary and the conclusions of the paper were reported.

2. Mathematical Background

The set of all real-valued sequences is indicated by w. By a sequence space, we understand that it is a linear subspace of w. The symbols l_∞, c, c_0, l_p, bs, cs, cs_0, bv, bv_0, and l_1 are called sequence spaces bounded, convergent to zero, convergent, absolutely p-summable, bounded series, convergent series, series converging to zero, bounded variation, and absolutely convergent series, respectively.

Now let's give descriptions of some sequence spaces.

$$l_\infty = \left\{ x = (x_k) \in w : \sup_{k\in\mathbb{N}} |x_k| < \infty \right\},$$

$$c = \left\{ x = (x_k) \in w : \lim_{k\to\infty} |x_k - l| = 0 \text{for some } l \in \mathbb{C} \right\},$$

$$c_0 = \left\{ x = (x_k) \in w : \lim_{k\to\infty} |x_k| = 0 \right\},$$

$$l_p = \left\{ x = (x_k) \in w : \sum_k |x_k|^p < \infty \right\}, (0 < p < \infty),$$

$$bs = \left\{ x = (x_k) \in w : \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n} x_k \right| < \infty \right\},$$

$$cs = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \left| \sum_{k=0}^{n} x_k - l \right| = 0 \text{ for some } l \in \mathbb{C} \right\},$$

$$cs_0 = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \left| \sum_{k=0}^{n} x_k \right| = 0 \right\},$$

$$bv = \left\{ x = (x_k) \in w : \sum_k |x_k - x_{k-1}| < \infty \right\},$$

$$bv_0 = bv \cap c_0$$

$$l_1 = \left\{ x = (x_k) \in w : \sum_k |x_k| < \infty \right\}, (0 < p < \infty),$$

We indicate the set of natural numbers including 0 by $\mathbb{N}$. The class of the non-empty and finite subsets of $\mathbb{N}$ is denoted by $\mathcal{F}$.

We will transfer the matrix transformation between sequence spaces. Let $A = (a_{nk})$ be an infinite matrix for every $n,k \in \mathbb{N}$, where a_{nk} is a real number. A is defined as a matrix transformation from X to Y if, for every $x = (x_k) \in X$, sequence $Ax = \{A_n(x)\}$ is an A-transform of x and in Y; where

$$A_n(x) = \sum_k a_{nk} x_k \text{ for each } n \in \mathbb{N}. \tag{1}$$

Here, the series converges for every $n \in \mathbb{N}$ in Equation (1).

In Equation (1), although the limit of the summation is are not written, it is from 0 to ∞, and we will use it for the rest of the article. The family of all the matrix transformations from X to Y is denoted by (X,Y).

Let λ and K be an infinite matrix and a sequence space, respectively. Then, the matrix domain, λ_K, which is a sequence space is defined by:

$$\lambda_K = \{t = (t_k) \in w : Kt \in \lambda\} \tag{2}$$

Let A and B be linear spaces over the same scalar field. A map f: $A \rightarrow B$ is called linear if:

$$f(ax_1 + bx_2) = af(x_1) + bf(x_2)$$

for all scalars a,b and all x_1, $x_2 \in A$. An isomorphism f: $A \rightarrow B$ is a bijective linear map. We say that A and B are isomorphic if there is an isomorphism f: $A \rightarrow B$.

A normed space is $(A, \|.\|)$ consisting of a linear space A and a norm $\|.\|$:$A \rightarrow \mathbb{R}$ such that $\|a\| = 0$; $\|\mu a\| = |\mu| \|a\|$ for each scalar μ and each $a \in A$; $\|a + b\| \leq \|a\| + \|b\|$ for each $a,b \in$ A.

A Banach space is $(A, \|.\|)$, a complete normed linear space, where completeness means that for every sequence (a_n) in A with $\|a_m - a_n\| \rightarrow 0$ $(m,n \rightarrow \infty)$, there exists $a \in A$ such that $\|a_n - a\| \rightarrow 0$ $(n \rightarrow \infty)$.

Let us define the Schauder basis of A normed space. Let a sequence $(a_k) \in A$. There exists only one sequence of scalars (v_k) such that $y = \sum_k \nu_k a_k$ and $\lim_{n\to\infty} \|y - \sum_{k=0}^{n} \nu_k a_k\| = 0$. Then, (a_k) is called a Schauder basis for A.

Let R be a sequence space. α-, β-, and γ-duals R^α, R^β, and R^γ of R are defined respectively, as:

$$R^\alpha = \{a = (a_k) \in w : ar = (a_k r_k) \in l_1 \text{ for all } r \in R\},$$

$$R^\beta = \{a = (a_k) \in w : ar = (a_k r_k) \in cs \text{ for all } r \in R\},$$

$$R^\gamma = \{a = (a_k) \in w : ar = (a_k r_k) \in bs \text{ for all } r \in R\}.$$

Let us give almost-convergent sequences space. This was first defined by Lorentz [41]. Let $a = (a_k) \in l_\infty$. Sequence a is almost convergent to limit α if and only if $\lim_{m\to\infty} \sum_{k=0}^{m} \frac{a_{n+k}}{m+1} = \alpha$ uniformly in n. By f-lim $a = \alpha$, we indicate sequence a is almost convergent to limit α. The sequence spaces f and f_0 are:

$$f_0 = \left\{a = (a_k) \in l_\infty : \lim_{m\to\infty} \sum_{k=0}^{m} \frac{a_{n+k}}{m+1} = 0 \; uniformly \; in \; n\right\},$$
$$f = \left\{a = (a_k) \in l_\infty : \exists \alpha \in \mathbb{C} \lim_{m\to\infty} \sum_{k=0}^{m} \frac{a_{n+k}}{m+1} = \alpha \; uniformly \; in \; n\right\}.$$

Lemma 1. *[35] Let δ and μ be a subspace of w. Then, $S = (s_{nk}) \in (\delta(N^t), \mu)$ if, and only if, $P = (p_{nk}) \in (\delta, \mu)$, where:*

$$p_{nk} = \sum_{j=k}^{\infty} (-1)^{j-k} D_{j-k} T_k s_{nj} \text{ for all } k, n \in \mathbb{N}. \tag{3}$$

Lemma 2. *[35] Let δ and μ be a subspace of w and let the infinite matrices be $S = (s_{nk})$ and $V = (v_{nk})$. If S and V are connected with the relation:*

$$v_{nk} = \sum_{j=0}^{n} \frac{t_{n-j}}{T_n} s_{nk} \text{ for all } k, n \in \mathbb{N}, \tag{4}$$

then, $S \in (\delta, \mu(N^t))$ if, and only if, $V \in (\delta, \mu)$.

Lemma 3. *[42] Let $S = (s_{nk})$ and $r = (r_k) \in w$ and the inverse matrix $F = (f_{nk})$ of the triangle matrix $G = (g_{nk})$ by,*

$$s_{nk} = \begin{cases} \sum\limits_{j=k}^{n} r_j f_{jk}, & 0 \le k \le n \\ 0, & k > n \end{cases}$$

for all k,$n \in \mathbb{N}$. In that case,

$$\delta_G^{\gamma} = \{r = (r_k) \in w : S \in (\mu, l_\infty)\},$$

$$\delta_G^{\beta} = \{r = (r_k) \in w : S \in (\mu, c)\}$$

such that μ is any sequence space.

Now, we take a non-negative real sequence (t_k) with $t_k > 0$ and $T_n = \sum\limits_{k=0}^{n} t_k$ for all $n \in \mathbb{N}$. The Nörlund mean according to $t = (t_k)$ is defined by the matrix $N^t = (a^t_{nk})$ as:

$$a^t_{nk} = \begin{cases} \frac{t_{n-k}}{T_n}, & 0 \le k \le n \\ 0, & k > n \end{cases} \text{ for all } k, n \in \mathbb{N}. \tag{5}$$

The inverse matrix $U^t = (u^t_{nk})$ of $N^t = (a^t_{nk})$ is defined as:

$$u^t_{nk} = \begin{cases} (-1)^{n-k} D_{n-k} T_k, & 0 \le k \le n \\ 0, & k > n \end{cases} \tag{6}$$

for all n,$k \in \mathbb{N}$, $t_0 = D_0 = 1$ and D_n for $n \in \{1,2,3, \dots\}$ and,

$$D_n = \begin{vmatrix} t_1 & 1 & 0 & 0 & . & . & 0 \\ t_2 & t_1 & 1 & 0 & . & . & 0 \\ t_3 & t_2 & t_1 & 1 & . & . & 0 \\ . & . & . & . & . & & . \\ . & . & . & . & & . & . \\ t_{n-1} & t_{n-2} & t_{n-3} & t_{n-4} & & & 1 \\ t_n & t_{n-1} & t_{n-2} & t_{n-3} & & & t_1 \end{vmatrix}.$$

3. Auxiliary Results

In this section, spaces $bs(N^t)$ and $cs(N^t)$ are defined. Also, some of their properties are found. Let us define the sets $bs(N^t)$ and $cs(N^t)$, whose $N^t = (a^t_{nk})$ transforms are in bs and cs.

$$bs(N^t) = \left\{ x = (x_k) \in w : \sup_{n \in \mathbb{N}} \left| \sum_{j=0}^{n} \sum_{k=0}^{j} \frac{t_{j-k}}{T_j} x_k \right| < \infty \right\},$$

$$cs(N^t) = \left\{ x = (x_k) \in w : \left(\sum_{j=0}^{n} \sum_{k=0}^{j} \frac{t_{j-k}}{T_j} x_k \right)_n \in c \right\}.$$

Here, it can be seen from Equation (2) that $bs(N^t) = (bs)_{N^t}$ and $cs(N^t) = (cs)_{N^t}$.
If $x = (x_n) \in w$ and $y = N^t x$, such that $y = (y_n)$, then the equality,

$$y_n = (N^t x)_n = \sum_{k=0}^{n} \frac{t_{n-k}}{T_n} x_k \text{ for all } n \in \mathbb{N} \tag{7}$$

is satisfied. In this situation, we can see that $x_n = (U^t y)_n$, that is,

$$x_n = \sum_{k=0}^{n} (-1)^{n-k} D_{n-k} T_k y_k \text{ for all } n \in \mathbb{N}. \tag{8}$$

Now, let us detail one of the basic theorems of our article.

Theorem 1. *The set of $bs(N^t)$ is a linear space.*

Proof. The proof is left to the reader because it is easy to see that it provides the linear space conditions. □

Theorem 2. *The set of $cs(N^t)$ is a linear space.*

Proof. The proof is left to the reader because it is easy to see that it provides the linear space conditions. □

Theorem 3. *$bs(N^t)$ is a normed space with:*

$$\|x\|_{bs(N^t)} = \sup_{n\in\mathbb{N}} \left| \sum_{j=0}^{n} \sum_{k=0}^{j} \frac{t_{j-k}}{T_j} x_k \right|. \tag{9}$$

Proof. The proof is left to the reader because it is easy to see that it provides the normed space conditions. □

Theorem 4. *$cs(N^t)$ is a normed space with the norm in Equation (9).*

Proof. The proof is left to the reader because it is easy to see that it provides the normed space conditions. □

Theorem 5. *$bs(N^t)$ and bs spaces are isomorphic as normed spaces.*

Proof. Let us take the transformation:

$$\begin{aligned} T : bs(N^t) &\to bs \\ x &\to y = Tx = N^t x. \end{aligned}$$

It is clear that T is both injective and linear.

Let $y = (y_n) \in bs$. By using Equations (6) and (7), we find,

$$\begin{aligned} \|x\|_{bs(N^t)} &= \sup_{n\in\mathbb{N}} \left| \sum_{j=0}^{n} \sum_{k=0}^{j} \frac{t_{j-k}}{T_j} x_k \right| \\ &= \sup_{n\in\mathbb{N}} \left| \sum_{j=0}^{n} \sum_{k=0}^{j} \frac{t_{j-k}}{T_j} \sum_{i=0}^{n} (-1)^{k-i} D_{k-i} T_i y_i \right| \\ &= \sup_{n\in\mathbb{N}} \left| \sum_{j=0}^{n} y_j \right| = \|y\|_{bs}. \end{aligned}$$

Hence, x is an element of $bs(N^t)$ and T is surjective. We see that T preserves the norm. Here, $bs(N^t)$ and bs are isometric. That is, $bs(N^t) \cong bs$. □

Theorem 6. *$cs(N^t)$ and cs spaces are isomorphic as normed spaces.*

Proof. The proof can be made similar to Theorem 5. □

Now, let $S = (s_{nk})$ be an infinite matrix and give the equations below:

$$\lim_k s_{nk} = 0 \text{ for each } n \in \mathbb{N}, \tag{10}$$

$$\sup_m \sum_k \left| \sum_{n=0}^{m} (s_{nk} - s_{n,k+1}) \right| < \infty, \tag{11}$$

$$\sum_n s_{nk} \text{convergent for each } k \in \mathbb{N} \tag{12}$$

$$\sup_n \sum_k |s_{nk} - s_{n,k+1}| < \infty, \tag{13}$$

$$\lim_n s_{nk} = \alpha_k \text{ for each } k \in \mathbb{N},\ \alpha_k \in \mathbb{C}, \tag{14}$$

$$\sup_{N,K\in\mathcal{F}} \left| \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} (s_{nk} - s_{n,k+1}) \right| < \infty, \tag{15}$$

$$\sup_{N,K\in\mathcal{F}} \left| \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} (s_{nk} - s_{n,k-1}) \right| < \infty, \tag{16}$$

$$\lim_n (s_{nk} - s_{n,k+1}) = \alpha \text{ for each } k \in \mathbb{N},\ \alpha \in \mathbb{C}, \tag{17}$$

$$\lim_n \sum_k |s_{nk} - s_{n,k+1}| = \sum_k \left| \lim_n (s_{nk} - s_{n,k+1}) \right|, \tag{18}$$

$$\sup_n \left| \lim_k s_{nk} \right| < \infty, \tag{19}$$

$$\lim_n \sum_k |s_{nk} - s_{n,k+1}| = 0 \text{ uniformly in } n, \tag{20}$$

$$\lim_m \sum_k \left| \sum_{n=0}^{m} (s_{nk} - s_{n,k+1}) \right| = 0, \tag{21}$$

$$\lim_m \sum_k \left| \sum_{n=0}^{m} (s_{nk} - s_{n,k+1}) \right| = \sum_k \left| \sum_n (s_{nk} - s_{n,k+1}) \right| = 0, \tag{22}$$

$$\sup_{N,K\in\mathcal{F}} \left| \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} [(s_{nk} - s_{n,k+1}) - (s_{n-1,k} - s_{n-1,k+1})] \right| < \infty, \tag{23}$$

$$\sup_{m\in\mathbb{N}}\left|\lim_k\sum_{n=0}^{m}s_{nk}\right|<\infty, \tag{24}$$

$$\exists\alpha_k\in\mathbb{C}\sum_n s_{nk}=\alpha_k \text{ for each } k\in\mathbb{N}, \tag{25}$$

$$\sup_{N,K\in\mathcal{F}}\left|\sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}}[(s_{nk}-s_{n-1,k})-(s_{n,k-1}-s_{n-1,k-1})]\right|<\infty. \tag{26}$$

$$\exists m_k\in\mathbb{C}f-\lim s_{nk}=m_k \text{ for each } k\in\mathbb{N}, \tag{27}$$

$$\exists m_k\in\mathbb{C}\lim_q\sum_k\frac{1}{q+1}\left|\sum_{i=0}^{q}\Delta\left[\sum_{j=0}^{n+i}(s_{jk}-m_k)\right]\right|=0 \text{ uniformly in } n, \tag{28}$$

$$\sup_{n\in\mathbb{N}}\sum_k\left|\Delta\left[\sum_{j=0}^{n}s_{jk}\right]\right|<\infty, \tag{29}$$

$$\exists m_k\in\mathbb{C}f-\lim\sum_{j=0}^{n}s_{jk}=m_k \text{ for each } k\in\mathbb{N}, \tag{30}$$

$$\sup_{n\in\mathbb{N}}\sum_k\left|\sum_{j=0}^{n}s_{jk}\right|<\infty, \tag{31}$$

$$\exists m_k\in\mathbb{C}\lim_{n\to\infty}\sum_n\sum_k s_{nk}=m_k \text{ for each } k\in\mathbb{N}, \tag{32}$$

$$\lim_n\sum_k\left|\Delta\left[\sum_{j=0}^{n}(s_{jk}-m_k)\right]\right|=0, \tag{33}$$

$$\sup_{n\in\mathbb{N}}\sum_k\left|\sum_{j=0}^{n}s_{jk}\right|^p<\infty,\ q=\frac{p}{p-1}, \tag{34}$$

$$\sup_{m,n\in\mathbb{N}}\left|\sum_{n=0}^{m}s_{nk}\right|<\infty, \tag{35}$$

$$\sup_{m,l\in\mathbb{N}}\left|\sum_{n=0}^{m}\sum_{k=l}^{\infty}s_{nk}\right|<\infty, \tag{36}$$

$$\sup_{m,l\in\mathbb{N}}\left|\sum_{n=0}^{m}\sum_{k=0}^{l}s_{nk}\right|<\infty, \tag{37}$$

$$\lim_m\sum_k\left|\sum_{n=m}^{\infty}s_{nk}\right|=0, \tag{38}$$

$$\sum_n\sum_k s_{nk}, \text{ convergent for each } k\in\mathbb{N} \tag{39}$$

$$\lim_{m\to\infty}\sum_{n=0}^{m}(s_{nk}-s_{n,k+1})=\alpha \text{ for each } k\in\mathbb{N},\ \alpha\in\mathbb{C}. \tag{40}$$

$$\sup_m\sum_k\left|\sum_{n=0}^{m}(s_{nk}-s_{n,k-1})\right|<\infty, \tag{41}$$

Now, we provide some matrix transformations which are taken from Stieglitz and Tietz [43] to use in the inclusion theorems.

Lemma 4. *Let $S=(s_{nk})$ be an infinite matrix. Then, the following statements hold.*

(1) $S = (s_{nk}) \in (bs, l_\infty)$ *if, and only if, Equations (10) and (13) hold.*
(2) $S = (s_{nk}) \in (cs.c)$ *if, and only if, Equations (13) and (14) hold.*
(5) $S = (s_{nk}) \in (bs, l_1)$ *if, and only if, Equations (10) and (15) hold.*
(6) $S = (s_{nk}) \in (cs, l_1)$ *if, and only if, Equation (16) holds.*
(7) $S = (s_{nk}) \in (bs, c)$ *if, and only if, Equations (10), (17), and (18) hold.*
(8) $S = (s_{nk}) \in (cs, l_\infty)$ *if, and only if, Equations (13) and (19) hold.*
(9) $S = (s_{nk}) \in (bs, c_0)$ *if, and only if, Equations (10) and (20) hold.*
(10) $S = (s_{nk}) \in (bs, cs_0)$ *if, and only if, Equations (10) and (21) hold.*
(11) $S = (s_{nk}) \in (bs, cs)$ *if, and only if, Equations (10) and (22) hold.*
(12) $S = (s_{nk}) \in (bs, bv)$ *if, and only if, Equations (10) and (23) hold.*
(13) $S = (s_{nk}) \in (bs, bs)$ *if, and only if, Equations (10) and (11) hold.*
(14) $S = (s_{nk}) \in (cs, cs)$ *if, and only if, Equations (10) and (41) hold.*
(15) $S = (s_{nk}) \in (bs, bv_0)$ *if, and only if, Equations (13), (20), and (23) hold.*
(16) $S = (s_{nk}) \in (cs, c_0)$ *if, and only if, Equation (13) holds and Equation (14) also holds with* $\alpha_k = 0$ *for all* $\mathrm{k} \in \mathbb{N}$.
(17) $S = (s_{nk}) \in (cs, bs)$ *if, and only if, Equations (11) and (24) hold.*
(18) $S = (s_{nk}) \in (cs, cs_0)$ *if, and only if, Equation (11) holds and Equation (25) also holds with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$.
(19) $S = (s_{nk}) \in (cs, bv)$ *if, and only if, Equation (26) holds.*
(20) $S = (s_{nk}) \in (cs, bv_0)$ *if, and only if, Equation (26) holds and Equation (14) also holds with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$.
(21) $S = (s_{nk}) \in (l_\infty, bs) = (c, bs) = (c_0, bs)$ *if, and only if, Equation (31) holds.*
(22) $S = (s_{nk}) \in (l_p, bs)$ *if, and only if, Equation (34) holds.*
(23) $S = (s_{nk}) \in (l_1, bs)$ *if, and only if, Equation (35) holds.*
(24) $S = (s_{nk}) \in (bv, bs)$ *if, and only if, Equation (36) holds.*
(25) $S = (s_{nk}) \in (bv_0, bs)$ *if, and only if, Equation (37) holds.*
(26) $S = (s_{nk}) \in (l_\infty, cs)$ *if, and only if, Equation (38) holds.*
(27) $S = (s_{nk}) \in (c, cs)$ *if, and only if, Equations (31), (32), and (39) hold.*
(28) $S = (s_{nk}) \in (cs_0, cs)$ *if, and only if, Equations (11) and (40) hold.*
(29) $S = (s_{nk}) \in (l_p, cs)$ *if, and only if, Equations (12) and (34) hold.*
(30) $S = (s_{nk}) \in (l_1, cs)$ *if, and only if, Equations (12) and (35) hold.*
(31) $S = (s_{nk}) \in (bv, cs)$ *if, and only if, Equations (12), (35) and (37) hold.*
(32) $S = (s_{nk}) \in (bv_0, cs)$ *if, and only if, Equations (12) and (37) hold.*

Lemma 5. *Let* $S = (s_{nk})$ *be an infinite matrix for all k,n* $\in \mathbb{N}$.

(1) $S = (s_{nk}) \in (f, cs)$ *if, and only if, Equations (25) and (31)–(33) hold (Başar [44]).*
(2) $S = (s_{nk}) \in (cs, f)$ *if, and only if, Equations (13) and (27) hold (Başar and Çolak [45]).*
(3) $S = (s_{nk}) \in (bs, f)$ *if, and only if, Equations (10), (13), (27) and (28) hold (Başar and Solak [46]).*
(4) $S = (s_{nk}) \in (bs, f)$ *if, and only if, Equations (10) and (28)–(30) hold (Başar and Solak [46]).*
(5) $S = (s_{nk}) \in (cs, fs)$ *if, and only if, Equations (29) and (30) hold (Başar and Çolak [45]).*

4. Main Results

Theorem 7. $bs(N^t)$ *is a Banach space, according to Equation (9).*

Proof. Clearly, the norm conditions are satisfied. Let us take the sequence $x^i = (x^i)_n$ as a Cauchy sequence in $bs(N^t)$ for all $i,n \in \mathbb{N}$. We find,

$$y^i{}_n = \sum_{k=0}^{n} \frac{t_{n-k}}{T_n} x^i{}_k \text{ for all } i,k \in \mathbb{N}$$

by using Equation (7). Since the sequence $x^i = (x^i)_n$ is a Cauchy sequence, $\forall \varepsilon > 0$ and there exists $n_0 \in \mathbb{N}$, such that:

$$\begin{aligned} \|x^i - x^m\|_{bs(N^t)} &= \sup_{n\in\mathbb{N}} \left| \sum_{j=0}^{n} \sum_{k=0}^{j} \frac{t_{j-k}}{T_j}(x_k^i - x_k^m) \right| \\ &= \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n} (y_k^i - y_k^m) \right| = \|y^i - y^m\| < \varepsilon \end{aligned}$$

for all $i,m > n_0$. $y_i \to y$ $(i \to \infty)$ such that $y \in bs$ exists because bs is complete. $bs(N^t)$ is also complete because $bs(N^t)$ and bs are isomorphic. Hence, $bs(N^t)$ is a Banach space. □

Theorem 8. *$cs(N^t)$ is a Banach space, according to Equation (9).*

Proof. Clearly, the norm conditions are satisfied. Let us take the sequence $x^i = (x^i)_n$ is a Cauchy sequence in $cs(N^t)$ for all $i,n \in \mathbb{N}$. We find:

$$y^i{}_n = \sum_{k=0}^{n} \frac{t_{n-k}}{T_n} x^i{}_k \text{ for all } i,k \in \mathbb{N}$$

by using Equation (7). Since the sequence $x^i = (x^i)_n$ is a Cauchy sequence, $\forall \varepsilon > 0$ and there exists $n_0 \in \mathbb{N}$, such that:

$$\begin{aligned} \|x^i - x^m\|_{cs(N^t)} &= \sup_{n\in\mathbb{N}} \left| \sum_{j=0}^{n} \sum_{k=0}^{j} \frac{t_{j-k}}{T_j}(x_k^i - x_k^m) \right| \\ &= \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n} (y_k^i - y_k^m) \right| = \|y^i - y^m\| < \varepsilon \end{aligned}$$

for all $i,m > n_0$. $y_i \to y$ $(i \to \infty)$ such that $y \in cs$ exists because cs is complete. $cs(N^t)$ is also complete because the $cs(N^t)$ and cs are isomorphic. Hence, $cs(N^t)$ is a Banach space. □

Theorem 9. *$cs(N^t) \subset bs(N^t)$ is valid.*

Proof. Let $x \in cs(N^t)$. If $y = N^t x \in cs$, then $\sum_k N^t x \in c$. Since $cs \subset l_\infty$, $\sum_k N^t x \in l_\infty$. Hence, $y = N^t x \in bs$. Therefore, $x \in bs(N^t)$. We obtain that $cs(N^t) \subset bs(N^t)$. □

Theorem 10. *bs and $bs(N^t)$ have an overlap, but neither of them contains the other.*

Proof. We prove that bs and $bs(N^t)$ are not disjointed.

(i) Let $x = (x_k) = (1, 0, 0, \ldots)$ and $t = (t_k) = (1, 0, 0, \ldots)$. It is clear that $x \in bs$. If we do the necessary calculations, we find $x \in bs(N^t)$. Thus, $x \in bs \cap bs(N^t)$.

(ii) Now, let us take $x = (x_k)$ and,

$$x_k = \sum_{j=0}^{k} (-1)^k D_{k-j} T_j \tag{42}$$

for all $k \in \mathbb{N}$. Then, we obtain:

$$(N^t x)_n = \sum_{k=0}^{n} \frac{t_{n-k}}{T_n} \sum_{j=0}^{k} (-1)^k D_{k-j} T_j = (-1)^n$$

for all $n \in \mathbb{N}$. Thus $(N^t x)_n \in bs$. That is, $x \in bs(N^t)$. However, $x \notin bs(N^t)$. Therefore, $bs(N^t) \backslash bs$ is not empty.

(iii) Let $x = (x_k) = (1, 0, 0, \ldots)$ and $t = (t_k) = (1, 1, 1, \ldots)$. It is clear that $x \in bs$. If we do the necessary calculations, we find that $x \notin bs(N^t)$. Hence, $x \in bs \backslash bs(N^t)$. □

Theorem 11. *$bs(N^t)$ and l_∞ have an overlap, but neither of them contains the other.*

Proof. We prove that $bs(N^t)$ and l_∞ are not disjointed.

(i) Let $x = (x_k) = \left\{(-1)^k\right\}$ for all $k \in \mathbb{N}$. It is clear that $\left\{(-1)^k\right\} \in l_\infty$. If we do the necessary calculations, we find $x = (x_k) = \left\{(-1)^k\right\} \in bs(N^t)$. Thus, $x \in bs(N^t) \cap l_\infty$.

(ii) Now, we take $x = (x_k) = (1, 1, 1, \ldots)$. $N^t x = x \notin bs$. Thus, $x \notin bs(N^t)$, but $x \in l_\infty$. Then, $x \in l_\infty \backslash bs(N^t)$.

(iii) On the other hand, if we take Equation (42), then $N^t x \in bs$. So, $x \in bs(N^t)$, but $x \notin l_\infty$. Thus, $x \in bs(N^t) \backslash l_\infty$.

This is the desired result. □

Theorem 12. *cs and $cs(N^t)$ have an overlap, but neither of them contains the other.*

Proof. We prove that cs and $cs(N^t)$ are not disjointed.

(i) If we use the example in the (*i*) of the proof of Theorem 10, then we find $x \in cs \cap cs(N^t)$.

(ii) Now, let $x = (x_k) = (1, -\sqrt{2}, \sqrt{3}, \ldots (-1)^k \sqrt{k+1}, \ldots)$ and $t = (t_k) = (1, 1, 1, \ldots)$ for all $k \in \mathbb{N}$. Then, we obtain that $x \in cs(N^t)$. However, $x \notin cs$. Therefore, $cs(N^t) \backslash cs$ is not empty.

(iii) If we use the example in the (iii) of the proof of Theorem 10, then we find $x \in cs \backslash cs(N^t)$. □

Theorem 13. *$cs(N^t)$ and c have an overlap, but neither of them contains the other.*

Proof. Let us prove that $cs(N^t)$ and c are not disjointed.

(i) If we use the example in the (i) of the proof of Theorem 10, then we find that there exists at least one point belonging to both $cs(N^t)$ and c.

(ii) If we use the example in the (ii) of the proof of Theorem 11, then we find $x \in c \backslash cs(N^t)$.

(iii) Let $x = (x_k) = (1, -1, 1, -1, \ldots)$ and $t = (t_k) = (1, 1, 0, 0, \ldots)$. Then, $N^t x = (1, 0, 0, \ldots) \in cs$. Therefore, $x \in cs(N^t)$, but $x \notin c$. Thus, $x \in cs(N^t) \backslash c$.

This is the desired result. □

Lemma 6. *Let $r = (r_n) \in w$ and let $U^t = (u^t_{nk})$ be the inverse matrix of N^t Nörlund matrix. The infinite matrix $C = (c_{nk})$ is defined by:*

$$c_{nk} = \begin{cases} r_n u^t_{nk}, & 0 \le k \le n \\ 0, & k > n \end{cases}$$

for all k,n $\in \mathbb{N}$, $\mu \in \{cs, bs\}$. In that case $r \in \left\{\mu(N^t)^\alpha\right\}$ if, and only if, $C \in (\mu, l_1)$.

Proof. Let $r = (r_n)$ and $x = (x_n)$ be an element of w for all $n \in \mathbb{N}$. Let $y = (y_n)$ be such that $y = N^t x$ is defined by Equation (7). In that case,

$$rx = r_n x_n = r_n (U^t y)_n = (Cy)_n = Cy$$

for all $n \in \mathbb{N}$. Therefore, we find using Equation (7) that $rx = (r_n x_n) \in l_1$ with $x = (x_n) \in \mu(N^t)$ if, and only if, $Cy \in l_1$ with $y \in \mu$. That is, $C \in (\mu, l_1)$. □

Let us give the Schauder basis of $cs(N^t)$.

Corollary 1. *Let us define sequences $b^{(k)} = \left\{ b_n^{(k)} \right\}_{n\in\mathbb{N}}$ in the $cs(N^t)$, such that:*

$$b_n^{(k)} = \begin{cases} (-1)^{n-k} D_{n-k} T_k, & n \geq k \\ 0, & n < k. \end{cases}$$

Then $\left\{ b_n^{(k)} \right\}_{n\in\mathbb{N}}$ is called a basis for $cs(N^t)$ and every $x \in cs(N^t)$ has only one representation $x = \sum_k y_k b^{(k)}$, such that $y_k = (N^t x)_k$.

In this section, we give the α-, β-, and γ-duals of the spaces $bs(N^t)$ and $cs(N^t)$ and the matrix transformations related to these spaces.

If we use Lemmas 3, 4, and 6 together, the following corollary is found.

Corollary 2. *Let us $B = (b_{nk})$ and $C = (c_{nk})$ such that:*

$$b_{nk} = \begin{cases} r_n u_{nk}^t, & 0 \leq k \leq n \\ 0, & k > n \end{cases} \quad \text{and } c_{nk} = \sum_{j=k}^{n} (-1)^{j-k} D_{j-k} T_k r_j.$$

If we take m_1, m_2, m_3, m_4, m_5, m_6, m_7, and m_8 as follows:

$$\begin{aligned}
m_1 &= \left\{ r = r_k \in w : \sup_{N,K\in\mathcal{F}} \left| \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} (b_{nk} - b_{n,k+1}) \right| < \infty \right\}, \\
m_2 &= \left\{ r = r_k \in w : \sup_{N,K\in\mathcal{F}} \left| \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} (b_{nk} - b_{n,k-1}) \right| < \infty \right\}, \\
m_3 &= \left\{ r = r_k \in w : \lim_k c_{nk} = 0 \right\}, \\
m_4 &= \left\{ r = r_k \in w : \exists \alpha \in \mathbb{C} \lim_n (c_{nk} - c_{n,k+1}) = \alpha \right\}, \\
m_5 &= \left\{ r = r_k \in w : \lim_n \sum_k |c_{nk} - c_{n,k+1}| = \sum_k \left| \lim_n (c_{nk} - c_{n,k+1}) \right| \right\}, \\
m_6 &= \left\{ r = r_k \in w : \exists \alpha \in \mathbb{C} \lim_n c_{nk} = \alpha \text{ for all } k \in \mathbb{N} \right\}, \\
m_7 &= \left\{ r = r_k \in w : \sup_{n\in\mathbb{N}} \sum_k |c_{nk} - c_{n,k+1}| < \infty \right\}, \\
m_8 &= \left\{ r = r_k \in w : \sup_{n\in\mathbb{N}} \left| \lim_k c_{nk} \right| < \infty \right\}.
\end{aligned}$$

Then, the following statements hold:

(1) $bs(N^t)^\alpha = m_1$
(2) $cs(N^t)^\alpha = m_2$
(5) $bs(N^t)^\beta = m_3 \cap m_4 \cap m_5$
(6) $cs(N^t)^\beta = m_6 \cap m_7$

(7) $bs(N^t)^\gamma = m_3 \cap m_7$
(8) $cs(N^t)^\gamma = m_7 \cap m_8$

Now, let us list the following conditions, where p_{nk} is taken from Equation (3);

$$\lim_k p_{nk} = 0 \text{ for each } n \in \mathbb{N}, \tag{43}$$

$$\sup_n \sum_k |p_{nk} - p_{n,k+1}| < \infty, \tag{44}$$

$$\exists m_k \in \mathbb{C} \lim_{n\to\infty} (p_{nk} - p_{n,k+1}) = m_k \text{ for all } k, n \in \mathbb{N}, \tag{45}$$

$$\lim_n \sum_k |p_{nk} - p_{n,k+1}| = \sum_k \left|\lim_n (p_{nk} - p_{n,k+1})\right|, \tag{46}$$

$$\sup_{m\in\mathbb{N}} \sum_k \left| \sum_{n=0}^{m} (p_{nk} - p_{n,k+1}) \right| < \infty, \tag{47}$$

$$\lim_m \sum_k \left| \sum_{n=0}^{m} (p_{nk} - p_{n,k+1}) \right| = \sum_k \left| \sum_n (p_{nk} - p_{n,k+1}) \right| = 0, \tag{48}$$

$$\lim_m \sum_k \left| \sum_{n=0}^{m} (p_{nk} - p_{n,k+1}) \right| = 0, \tag{49}$$

$$\sup_{N,K\in\mathcal{F}} \left| \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} (p_{nk} - p_{n,k+1}) \right| < \infty. \tag{50}$$

$$\sup_{N,K\in\mathcal{F}} \left| \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} [(p_{nk} - p_{n,k+1}) - (p_{n-1,k} - p_{n-1,k+1})] \right| < \infty, \tag{51}$$

$$\sup_n \left| \lim_k p_{nk} \right| < \infty, \tag{52}$$

$$\exists m_k \in \mathbb{C} \lim_{n\to\infty} p_{nk} = m_k \text{ for all } k \in \mathbb{N}, \tag{53}$$

$$\sup_{m\in\mathbb{N}} \left| \lim_k \sum_{n=0}^{m} p_{nk} \right| < \infty, \tag{54}$$

$$\sup_{m\in\mathbb{N}} \sum_k \left| \sum_{n=0}^{m} (p_{nk} - p_{n,k-1}) \right| < \infty, \tag{55}$$

$$\exists m_k \in \mathbb{C} \sum_n p_{nk} = m_k \text{ for each } k \in \mathbb{N}, \tag{56}$$

$$\sup_{N,K\in\mathcal{F}} \sum_{n\in\mathbb{N}} \left| \sum_{k\in\mathbb{N}} (p_{nk} - p_{n,k-1}) \right| < \infty, \tag{57}$$

$$\exists m_k \in \mathbb{C} f - \lim p_{nk} = m_k \text{ for each } k \in \mathbb{N}, \tag{58}$$

$$\sup_{N,K\in\mathcal{F}} \left| \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} [(p_{nk} - p_{n-1,k}) - (p_{n,k-1} - p_{n-1,k-1})] \right| < \infty, \tag{59}$$

$$m_k \in \mathbb{C} \lim_q \sum_k \frac{1}{q+1} \left| \sum_{i=0}^{q} \Delta \left[\sum_{j=0}^{n+i} (p_{jk} - m_k) \right] \right| = 0 \text{ uniformly in } n, \tag{60}$$

$$\sup_{n\in\mathbb{N}}\sum_{k}\left|\sum_{j=0}^{n} p_{jk}\right| < \infty, \tag{61}$$

$$\exists m_k \in \mathbb{C}\lim_{n}\sum_{n}\sum_{k} p_{nk} = m_k \text{ for each } k \in \mathbb{N}, \tag{62}$$

$$m_k \in \mathbb{C}\lim_{n}\sum_{k}\left|\Delta\left[\sum_{j=0}^{n}(p_{jk} - m_k)\right]\right| = 0, \tag{63}$$

$$\sup_{n\in\mathbb{N}}\sum_{k}\left|\Delta\left[\sum_{j=0}^{n} p_{jk}\right]\right| < \infty, \tag{64}$$

$$\exists m_k \in \mathbb{C}f - \lim\sum_{j=0}^{n} p_{jk} = m_k \text{ for each } k \in \mathbb{N}, \tag{65}$$

$$\lim_{n}\sum_{k}|p_{nk} - p_{n,k+1}| = 0, \tag{66}$$

$$\sum_{n} p_{nk} \text{ convergent for each } k \in \mathbb{N}. \tag{67}$$

Now we can give several conclusions of Lemmas 1,2,4, and 5.

Corollary 3. *Let $S = (s_{nk})$ be an infinite matrix for all k,n $\in \mathbb{N}$. Then,*

(1) $S = (s_{nk}) \in (bs(N^t), c_0)$ *if, and only if, Equations (43) and (66) hold.*
(2) $S = (s_{nk}) \in (bs(N^t), cs_0)$ *if, and only if, Equations (43) and (49) hold.*
(5) $S = (s_{nk}) \in (bs(N^t), c)$ *if, and only if, Equations (43), (45), and (46) hold.*
(6) $S = (s_{nk}) \in (bs(N^t), cs)$ *if, and only if, Equations (43) and (48) hold.*
(7) $S = (s_{nk}) \in (bs(N^t), l_\infty)$ *if, and only if, Equations (43) and (44) hold.*
(8) $S = (s_{nk}) \in (bs(N^t), bs)$ *if, and only if, Equations (43) and (47) hold.*
(9) $S = (s_{nk}) \in (bs(N^t), l_1)$ *if, and only if, Equations (43) and (50) hold.*
(10) $S = (s_{nk}) \in (bs(N^t), bv)$ *if, and only if, Equations (43) and (51) hold.*
(11) $S = (s_{nk}) \in (bs(N^t), bv_0)$ *if, and only if, Equations (44), (51), and (65).*

Corollary 4. *Let $S = (s_{nk})$ be an infinite matrix for all k,n $\in \mathbb{N}$. Then,*

(1) $S = (s_{nk}) \in (cs(N^t), c_0)$ *if, and only if, Equation (44) holds and Equation (53) also holds with $m_k = 0$ for all $k \in \mathbb{N}$.*
(2) $S = (s_{nk}) \in (cs(N^t), cs_0)$ *if, and only if, Equation (47) holds and Equation (56) also holds with $m_k = 0$ for all $k \in \mathbb{N}$.*
(5) $S = (s_{nk}) \in (cs(N^t), c)$ *if, and only if, Equations (44) and (53) hold.*
(6) $S = (s_{nk}) \in (cs(N^t), cs)$ *if, and only if, Equations (47) and (67) hold.*
(7) $S = (s_{nk}) \in (cs(N^t), l_\infty)$ *if, and only if, Equations (44) and (52) hold.*
(8) $S = (s_{nk}) \in (cs(N^t), bs)$ *if, and only if, Equations (47) and (54) hold.*
(9) $S = (s_{nk}) \in (cs(N^t), l_1)$ *if, and only if, Equation (57) holds.*
(10) $S = (s_{nk}) \in (cs(N^t), bv)$ *if, and only if, Equation (59) holds.*
(11) $S = (s_{nk}) \in (cs(N^t), bv_0)$ *if, and only if, Equation (59) holds and Equation (53) also holds with $m_k = 0$ for all $k \in \mathbb{N}$.*

Corollary 5. *Let $S = (s_{nk})$ be an infinite matrix for all k,n $\in \mathbb{N}$. Then,*

(1) $S = (s_{nk}) \in (bs(N^t), f)$ *if, and only if, Equations (43), (44), (58), and (60) hold.*

(2) $S = (s_{nk}) \in (cs(N^t), f)$ *if, and only if, Equations (44) and (58) hold.*
(5) $S = (s_{nk}) \in (f, cs(N^t))$ *if, and only if, Equations (56) and (61)–(63) hold with* v_{nk} *instead of* p_{nk}*, where* v_{nk} *is defined by Equation (4).*
(6) $S = (s_{nk}) \in (bs(N^t), fs)$ *if, and only if, Equations (43), (58), (64), and (65) hold.*
(7) $S = (s_{nk}) \in (cs(N^t), fs)$ *if, and only if, Equations (64) and (65) hold.*

Corollary 6. *Let* $S = (s_{nk})$ *be an infinite matrix for all* $k,n \in \mathbb{N}$*. Then,*

(1) $S = (s_{nk}) \in (l_\infty, bs(N^t)) = (c, bs(N^t)) = (c_0, bs(N^t))$ *if, and only if, Equation (31) holds with* v_{nk} *instead of* s_{nk}*, where* v_{nk} *is defined by Equation (4).*
(2) $S = (s_{nk}) \in (l_p, bs(N^t))$ *if, and only if, Equation (34) holds with* v_{nk} *instead of* s_{nk} *where* v_{nk}*, is defined by Equation (4).*
(5) $S = (s_{nk}) \in (l_1, bs(N^t))$ *if, and only if, Equation (35) holds with with* v_{nk} *instead of* s_{nk}*, where* v_{nk} *is defined by Equation (4).*
(6) $S = (s_{nk}) \in (bv, bs(N^t))$ *if, and only if, Equation (36) holds with with* v_{nk} *instead of* s_{nk}*, where* v_{nk} *is defined by Equation (4).*
(7) $S = (s_{nk}) \in (bv_0, bs(N^t))$ *if, and only if, Equation (37) holds with* v_{nk} *instead of* s_{nk}*, where* v_{nk} *is defined by Equation (4).*
(8) $S = (s_{nk}) \in (l_\infty, cs(N^t))$ *if, and only if, Equation (38) holds with* v_{nk} *instead of* s_{nk}*, where* v_{nk} *is defined by Equation (4).*
(9) $S = (s_{nk}) \in (c, cs(N^t))$ *if, and only if, Equations (12), (31), and (39) hold with* v_{nk} *instead of* s_{nk}*, where* v_{nk} *is defined by Equation (4).*
(10) $S = (s_{nk}) \in (cs_0, cs(N^t))$ *if and only if Equations (11) and (40) hold with* v_{nk} *instead of* s_{nk} *where* v_{nk} *is defined by Equation (4).*
(11) $S = (s_{nk}) \in (l_p, cs(N^t))$ *if, and only if, Equations (12) and (34) hold with* v_{nk} *instead of* s_{nk}*, where* v_{nk} *is defined by Equation (4).*
(12) $S = (s_{nk}) \in (l_1, cs(N^t))$ *if, and only if, Equations (12) and (35) hold with* v_{nk} *instead of* s_{nk}*, where* v_{nk} *is defined by Equation (4).*
(13) $S = (s_{nk}) \in (bv, cs(N^t))$ *if, and only if, Equations (12), (35) and (37) hold with* v_{nk} *instead of* s_{nk} *where* v_{nk} *is defined by Equation (4).*
(14) $S = (s_{nk}) \in (bv_0, cs(N^t))$ *if, and only if, Equations (12) and (37) hold with* v_{nk} *instead of* s_{nk}*, where* v_{nk} *is defined by Equation (4).*

5. Results

The present paper is concerned with the domain of the trianglular infinite matrix. The triangular matrix we use in this study is the Nörlund matrix. We introduced the sequence spaces $cs(N^t)$ and $bs(N^t)$ as the domain of the Nörlund matrix, where cs and bs are convergent and bounded series, respectively. We found that these spaces are linear spaces and they have the same norm,

$$\|x\| = \sup_{n \in \mathbb{N}} \left| \sum_{j=0}^{n} \sum_{k=0}^{j} \frac{t_{j-k}}{T_j} x_k \right|,$$

where $x \in bs(N^t)$ or $x \in cs(N^t)$. $cs(N^t)$ and $bs(N^t)$ are Banach spaces with that norm. Some inclusion theorems of them were given. It was found that $cs(N^t) \subset bs(N^t)$ holds. At the same time, bs, $bs(N^t)$; cs, $cs(N^t)$; $bs(N^t)$, l_∞; and $cs(N^t)$, c have an overlap, but neither of them contains the other. It was shown that the space $bs(N^t)$ has no Schauder basis, but the space $cs(N^t)$ has a Schauder basis. We detected that both spaces have the α-, β-, and γ-duals and calculated them. Finally, the necessary conditions for the matrix transformations on and into these spaces were given.

6. Discussion

The spaces $l_\infty(N^t)$ and $l_p(N^t)$ were studied by Wang [2] while $1 \le p < \infty$. $f_0(N^t)$ and $f(N^t)$ were studied by Tuğ and Başar [35], where f_0 and f are almost-null and almost-convergent sequence spaces, respectively. Tuğ and Başar [35] have not investigated whether the space was the expansion or the contraction or overlap of the original space. However, it is determined to be the overlap in our study. Tuğ [47] defined and investigated a new sequence space as the domain of the Nörlund matrix in the space of all the sequences of the bounded variation. In our study, we determined that it is an expansion.

We introduced new sequence spaces, $bs(N^t)$ and $cs(N^t)$, as the sets of all sequences whose $N^t = (a^t_{nk})$ transforms are in the sequence space, bs and cs,

$$bs(N^t) = \left\{ x = (x_k) \in w : \sup_{n\in\mathbb{N}} \left| \sum_{j=0}^{n} \sum_{k=0}^{j} \frac{t_{j-k}}{T_j} x_k \right| < \infty \right\},$$

$$cs(N^t) = \left\{ x = (x_k) \in w : \left(\sum_{j=0}^{n} \sum_{k=0}^{j} \frac{t_{j-k}}{T_j} x_k \right)_n \in c \right\}.$$

We realize that these spaces are linear and have normed spaces with the same norm and Banach spaces as the convenient norm. The pairs $bs(N^t)$, bs and $cs(N^t)$, cs are isomorphic as normed spaces. Also, $cs(N^t) \subset bs(N^t)$ holds. At the same time, bs, $bs(N^t)$; cs, $cs(N^t)$; $bs(N^t)$, l_∞; and $cs(N^t)$, c have an overlap, but neither of them contains the other. It was determined that they have α-, β-, and γ-duals. Finally, we found some matrix transformations related to these new spaces.

7. Illustrative Examples

Example 1. *Let $S = (s_{nk})$ be infinite unit matrix for all $k,n \in \mathbb{N}$ such that,*

$$s_{nk} = \begin{cases} 1, & k = n \\ 0, & k \neq n. \end{cases}$$

We show that $S = (s_{nk}) \in (bs(N^t), l_\infty)$. For this, let's look at the conditions of Equations (43) and (44).

i- *The Equation (43):* $\lim_{k\to\infty} p_{nk} = 0$ for each $n \in N$.

$$p_{nk} = \sum_{j=k}^{\infty} (-1)^{j-k} D_{j-k} T_k s_{nj} = \begin{cases} T_n, & k = n \\ 0, & k \neq n. \end{cases}$$

In that case $\lim_{k\to\infty} p_{nk} = 0$.

ii- *The Equation (44):* $\sup_n \sum_k |p_{nk} - p_{n,k+1}| < \infty$. *We find,*

$$p_{nk} - p_{n,k+1} = \begin{cases} T_n, & k \le n \le k+1 \\ 0, & k \neq n \text{ or } k+1 \neq n. \end{cases}$$

Hence, $\sup_n \sum_k |p_{nk} - p_{n,k+1}| = \sup_n 2T_n = 2\sup_n \sum_{k=0}^{n} t_k$.
Consequently, $S = (s_{nk}) \in (bs(N^t), l_\infty)$ for every $t = (t_k) \in bs$.

Also, there is no non-negative t = (t_k) *such that* $S = (s_{nk}) \in (bs(N^t), bs)$*. This is because, if Equation (47) is investigated, we find,*

$$\sup_{m\in\mathbb{N}} \sum_k \left| \sum_{n=0}^{m} (p_{nk} - p_{n,k+1}) \right| = T_0 + 2 \sup_{m\in\mathbb{N}} \sum_{k=1}^{m} \sum_{j=0}^{k} t_j.$$

Since $t = (t_k)$ *is non-negative, Equation (47) is not bounded.*

Example 2. *Let* $S = (s_{nk})$ *be an infinite unit matrix for all k,n* $\in \mathbb{N}$*, such as Example 1.*
We show that $S = (s_{nk}) \in (bs(N^t), bv)$*. For this, let's look at the conditions of Equations (43) and (51). We know that the condition Equation (43) holds. For Equation (51), if we calculate, then we find:*

$$\sup_{N,K\in\mathcal{F}} \left| \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} [(p_{nk} - p_{n,k+1}) - (p_{n-1,k} - p_{n-1,k+1})] \right| = 2 \sup_{N,K\in\mathcal{F}} \sum_{k=0}^{\infty} t_k.$$

This result is the same as the result of Example 1. Hence, $S = (s_{nk}) \in (bs(N^t), bv)$ *for every* $t = (t_k) \in bs$*.*

Example 3. *Let* $S = (s_{nk})$ *be an infinite unit matrix for all k,n* $\in \mathbb{N}$*, such as Example 1.*
We show that $S = (s_{nk}) \in (cs(N^t), l_1)$*. For this, let's look at the condition of Equation (57).*
If we calculate, then we find:

$$\sup_{N,K\in\mathcal{F}} \sum_{n\in\mathbb{N}} \left| \sum_{k\in\mathbb{N}} (p_{nk} - p_{n,k-1}) \right| = 0$$

This result shows that $S = (s_{nk}) \in (cs(N^t), l_1)$*.*

Example 4. *Let* $S = (s_{nk})$ *be an infinite unit matrix for all k,n* $\in \mathbb{N}$*, such as Example 1.*
We show that $S = (s_{nk}) \in (cs(N^t), bv)$*. For this, let's look at the condition of Equation (59).*
If we calculate, then we find

$$\sup_{N,K\in\mathcal{F}} \left| \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} [(p_{nk} - p_{n-1,k}) - (p_{n,k-1} - p_{n-1,k-1})] \right| = 0$$

This result shows that $S = (s_{nk}) \in (cs(N^t), bv)$*.*

8. Summary and Conclusions

In this article, two new sequence spaces are constructed using the domain of the Nörlund matrix on the bs and cs sequence spaces. These Spaces are $bs(N^t)$ and $cs(N^t)$, where N^t is the Nörlund matrix according to $t = (t_k)$. The formulation of the N^t-transform function of any sequence space is obtained, and it is shown that they are linear spaces. Also, their norms are defined. We found that $bs(N^t) \cong bs$ and $cs(N^t) \cong cs$. That is, the pairs $bs(N^t)$, bs and $cs(N^t)$, cs are isomorphic spaces. At the same time, they are proven to be Banach spaces. Their inclusion relations are given and they are compared to other spaces. It is determined that the $cs(N^t)$ space has a Schauder base. Also, the α-, β-, and γ- duals of these two spaces are calculated. Finally, the necessary conditions for the matrix transformations on and into these spaces are provided. They are in the form of $(bs(N^t), \lambda)$, $(cs(N^t), \lambda)$, $(\mu, bs(N^t))$, and $(\mu, cs(N^t))$, where we denote the class of infinite matrices moved from sequences of μ space to sequences of λ space with (μ, λ).

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