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Bounds of Riemann-Liouville Fractional Integrals in General Form via Convex Functions and Their Applications

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Abstract: In this article, we establish bounds of sum of the left and right sided Riemann Liouville (RL) fractional integrals and related inequalities in general form. A new and novel approach is followed to obtain these results for general Riemann Liouville (RL) fractional integrals. Monotonicity and convexity of functions are used with some usual and straight forward inequalities. The presented results are also have connection with some known and already published results. Applications and motivations of presented results are briefly discussed.

Keywords: convex functions; fractional integrals; bounds

1. Introduction

The aim of this paper is to study several fractional integral operators in Fractional Calculus via convexity. The subject of Fractional Calculus is basically the generalization of classical calculus, which consists of the study of differentiation and integration of non-integer order. Historically it is as old as calculus. Work on it started from 1695, when Leibniz thought about derivative of fractional order and L' Hospital emphasize on half order derivative. A paradox of that age changed into reality due to the great interest of many mathematicians and physicist like Euler, Liouville, Laplace, Riemann, Grunwald, Letnikov, Sonin, Laurent and many others. A final formulation of first fractional integral operator is the Riemann-Liouville fractional integral operators due to a finishing work by Sonin [1], Letnikov [2] and then Laurent [3]. Now a days a variety of fractional integral operators are under discussion and many generalized fractional integral operators also take a part in generalizing the theory of fractional calculus [4–8].

Our focus is to use convexity property of functions as well as of absolute values of their derivatives in the establishment of bounds of Riemann-Liouville fractional integrals in general form defined in Definition 1. We start with the definition of convex function.

A real valued function $f : I \to \mathbb{R}$ is called a convex function on interval I if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

is valid for all $t \in [0,1]$, $x, y \in I$.

If inequality (1) is reversed, then f is called a concave function. Convex functions are very useful because of their interesting properties: A convex and finite function on a closed interval is bounded, it also satisfies Lipschitzian condition on any closed interval $[a,b] \subset I^{\circ}$. Therefore a convex function is absolutely continuous on $[a,b] \subset I^{\circ}$ and continuous on I°, see [9].

In the next, definition of well known Riemann-Liouville (RL) fractional integrals is given.

Let $f \in L[a,b]$. Then the left and right sided Riemann Liouville (RL) fractional integrals of order $\mu > 0$ with $a \geq 0$ are defined as

$$I^{\mu}_{a+}f(z) = \frac{1}{\Gamma(\mu)} \int_a^z (z-u)^{\mu-1} f(u)du, \quad z > a \tag{2}$$

and

$$I^{\mu}_{b-}f(z) = \frac{1}{\Gamma(\mu)} \int_z^b (u-z)^{\mu-1} f(u)du, \quad z < b \tag{3}$$

where $\Gamma(.)$ denotes Gamma function.

In [10], Farid gave some bounds and related modulus inequalities for Riemann-Liouville fractional integrals via convex functions. The presented results produce the results of this paper, in particular by setting $h(z) = z$. Further objective of this paper is to find bounds of the sum of the left-sided and right-sided (RL) fractional integrals and related inequalities in general form which provide in particular results for fractional integrals defined by Katugampola et al. in [11], Jarad et al. in [12] and Khan et al. in [13]. Also some connections with results of [14] are mentioned as application point of view.

In Section 2, bounds of sum of the left and right sided (RL) fractional integrals in general form via convex functions have been established. Some related inequalities via convexity and monotonicity of used functions have also been proved. The presented results may be useful in the study of fractional integral operators. Also these results provide the bounds of (RL) fractional integrals which are published in [10] and some results of [14]. In Section 3, we discuss the applications and motivations of the established results of Section 2.

In the following general form of the Riemann-Liouville (RL) fractional integrals is given [15]. This general form of fractional integrals contains several fractional integral operators in particular which are independently defined by many authors (see, Remark 1). This is given as follows.

Definition 1. *Let $f : [a,b] \to \mathbb{R}$ be an integrable function. Also let h be a differentiable, increasing and positive function defined on $(a,b]$ such that h' is continuous on (a,b). Then the left and right sided (RL) fractional integrals of a function f with respect to another function h on $[a,b]$ of order $\mu > 0$ are defined as*

$$I^{\mu,h}_{a^+}f(z) = \frac{1}{\Gamma(\mu)} \int_a^z (h(z)-h(u))^{\mu-1} h'(u) f(u)du, \; z > a \tag{4}$$

and

$$I^{\mu,h}_{b_-}f(z) = \frac{1}{\Gamma(\mu)} \int_z^b (h(u)-h(z))^{\mu-1} h'(u) f(u)du, \; z < b. \tag{5}$$

Remark 1. *Fractional integral operators defined in (4) and (5) are the generalizations of the following fractional integral operators:*

1. if we take $h(z) = z$, it reduces to the left-sided and right-sided Riemann-Liouville fractional integrals,
2. if we take $h(z) = \frac{z^{\rho}}{\rho}$, $\rho > 0$ it reduces to the left-sided and right-sided Katugampola fractional integrals given in [11],
3. if we take $h(z) = \frac{z^{\rho}}{\rho}$, $\rho > 0$ it reduces to the left-sided and right-sided Katugampola fractional integrals given in [11],
4. if we take $h(z) = \frac{z^{\tau+s}}{\tau+s}$, it reduces to the left-sided and right-sided generalized conformable fractional integrals defined by Khan et al. in [13].

2. Bounds of Riemann-Liouville (RL) Fractional Integrals in General Form

First we give the following fractional integral inequality which provides the bound of sum of the left and the right sided (RL) fractional integrals in general form. To obtain this result we use the convexity and monotonicity of used functions.

Theorem 1. *Suppose that $f,h : [a,b] \longrightarrow \mathbb{R}$ be the functions such that f be a positive and convex, h be differentiable and strictly increasing with $h' \in L[a,b]$. Then for $\mu,\nu \geq 1, \xi \in [a,b]$, we have*

$$\begin{aligned}
&\Gamma(\mu) I_{a^+}^{\mu,h} f(\xi) + \Gamma(\nu) I_{b^-}^{\nu,h} f(\xi) \\
&\leq \frac{((h(\xi)-h(a))^{\mu-1}}{\xi-a}\left[(\xi-a)(f(\xi)h(\xi)-f(a)h(a)) - (f(\xi)-f(a))\int_a^\xi h(t)dt\right] \\
&+ \frac{((h(b)-h(\xi))^{\nu-1}}{b-\xi}\left[(b-\xi)(f(b)h(b)-f(\xi)h(\xi)) - (f(b)-f(\xi))\int_\xi^b h(t)dt\right].
\end{aligned} \tag{6}$$

Proof. Since the function h is differentiable and strictly increasing therefore $(h(\xi)-h(t))^{\mu-1} \leq (h(\xi)-h(a))^{\mu-1}$, where as $\xi \in [a,b]$ and $t \in [a,\xi]$, $\mu \geq 1$, and $h'(t) > 0$. Hence the following inequality holds true

$$h'(t)(h(\xi)-h(t))^{\mu-1} \leq h'(t)(h(\xi)-h(a))^{\mu-1}. \tag{7}$$

From convexity of f, it can be obtained

$$f(t) \leq \frac{\xi-t}{\xi-a} f(a) + \frac{t-a}{\xi-a} f(\xi). \tag{8}$$

From (7) and (8), one can has

$$\begin{aligned}
&\int_a^\xi (h(\xi)-h(t))^{\mu-1} f(t) h'(t) dt \\
&\leq \frac{(h(\xi)-h(a))^{\mu-1}}{\xi-a}\left[f(a)\int_a^\xi (\xi-t)h'(t)dt + f(\xi)\int_a^\xi (t-a)h'(t)dt\right].
\end{aligned}$$

By using (4) of Definition 1 we get

$$\begin{aligned}
&\Gamma(\mu) I_{a^+}^{\mu,h} f(\xi) \\
&\leq \frac{((h(\xi)-h(a))^{\mu-1}}{\xi-a}\left[(\xi-a)(f(\xi)h(\xi)-f(a)h(a)) - (f(\xi)-f(a))\int_a^\xi h(t)dt\right].
\end{aligned} \tag{9}$$

Now for $\xi \in [a,b]$, $t \in [\xi,b]$ and $\nu \geq 1$, the following inequality holds true

$$h'(t)(h(t)-h(\xi))^{\nu-1} \leq h'(t)(h(b)-h(\xi))^{\nu-1}. \tag{10}$$

From convexity of f, it can be obtained

$$f(t) \leq \frac{t-\xi}{b-\xi} f(b) + \frac{b-t}{b-\xi} f(\xi). \tag{11}$$

Following the same way as we have done for (7) and (8) one can get from (10) and (11) the following inequality

$$\begin{aligned}
&\Gamma(\nu) I_{b^-}^{\nu,h} f(\xi) \\
&\leq \frac{((h(b)-h(\xi))^{\nu-1}}{b-\xi}\left[(b-\xi)(f(b)h(b)-f(\xi)h(\xi)) - (f(b)-f(\xi))\int_\xi^b h(t)dt\right].
\end{aligned} \tag{12}$$

From inequalities (9) and (12), we get (6) which is required. □

Corollary 1. *If we take $\mu = \nu$ in (6), then we get the following inequality for (RL) fractional integrals in general form*

$$\begin{aligned}
&\Gamma(\mu) I_{a^+}^{\mu,h} f(\xi) + \Gamma(\mu) I_{b^-}^{\mu,h} f(\xi) \qquad (13)\\
&\leq \frac{((h(\xi)-h(a))^{\mu-1}}{\xi - a}\left[(\xi-a)(f(\xi)h(\xi)-f(a)h(a)) - (f(\xi)-f(a))\int_a^\xi h(t)dt\right]\\
&+ \frac{((h(b)-h(\xi))^{\mu-1}}{b-\xi}\left[(b-\xi)(f(b)h(b)-f(\xi)h(\xi)) - (f(b)-f(\xi))\int_\xi^b h(t)dt\right].
\end{aligned}$$

Remark 2. *By setting $h(\xi) = \xi$ in (6), we get fractional integral inequality for (RL) fractional integrals ([10], Theorem 2).*

In the next theorem we present a generalization of already publish result.

Theorem 2. *Suppose that $f, h : [a,b] \longrightarrow \mathbb{R}$ be the functions such that f be differentiable and $|f'|$ is convex, h be also differentiable and strictly increasing with $h' \in L[a,b]$. Then for $\mu, \nu > 0, \xi \in [a,b]$ we have*

$$\begin{aligned}
&\left|\Gamma(\mu+1) I_{a^+}^{\mu,h} f(\xi) + \Gamma(\nu+1) I_{b^-}^{\nu,h} f(\xi) - ((h(\xi)-h(a))^{\mu} f(a) + (h(b)-h(\xi))^{\nu} f(b))\right| \qquad (14)\\
&\leq \frac{(h(\xi)-h(a))^{\mu}(\xi-a)|f'(a)| + (h(b)-h(\xi))^{\nu}(b-\xi)|f'(b)|}{2}\\
&+ |f'(\xi)|\frac{((h(\xi)-h(a))^{\mu}(\xi-a) + (h(b)-h(\xi))^{\nu}(b-\xi))}{2}.
\end{aligned}$$

Proof. From convexity of $|f'|$, it can be obtained

$$|f'(t)| \leq \frac{\xi - t}{\xi - a}|f'(a)| + \frac{t-a}{\xi - a}|f'(\xi)|. \qquad (15)$$

From (15), we have

$$f'(t) \leq \frac{\xi - t}{\xi - a}|f'(a)| + \frac{t-a}{\xi - a}|f'(\xi)|. \qquad (16)$$

Since the function h is differentiable and strictly increasing therefore we have the following inequality

$$(h(\xi) - h(t))^{\mu} \leq (h(\xi) - h(a))^{\mu}, \qquad (17)$$

where as $\xi \in [a,b]$ and $t \in [a,\xi]$, $\mu > 0$.

From (16) and (17), one can has

$$(h(\xi)-h(t))^{\mu} f'(t) \leq \frac{(h(\xi)-h(a))^{\mu}}{\xi - a}\left((\xi - t)|f'(a)| + (t-a)|f'(\xi)|\right).$$

Integrating over $[a,\xi]$, we have

$$\begin{aligned}
&\int_a^\xi (h(\xi)-h(t))^{\mu} f'(t)dt \qquad (18)\\
&\leq \frac{(h(\xi)-h(a))^{\mu}}{\xi - a}\left[|f'(a)|\int_a^\xi (\xi - t)dt + |f'(\xi)|\int_a^\xi (t-a)dt\right]\\
&= (h(\xi)-h(a))^{\mu}(\xi - a)\left[\frac{|f'(a)| + |f'(\xi)|}{2}\right],
\end{aligned}$$

and

$$\int_a^\xi (h(\xi)-h(t))^\mu f'(t)dt = f(t)(h(\xi)-h(t))^\mu|_a^\xi + \mu\int_a^\xi (h(\xi)-h(t))^{\mu-1} f(t)h'(t)dt$$
$$= -f(a)(h(\xi)-h(a))^\mu + \Gamma(\mu+1) I_{a^+}^{\mu,h} f(\xi).$$

Therefore (18) takes the form

$$\Gamma(\mu+1) I_{a^+}^{\mu,h} f(\xi) - f(a)(h(\xi)-h(a))^\mu \le (h(\xi)-h(a))^\mu(\xi-a)\left[\frac{|f'(a)|+|f'(\xi)|}{2}\right]. \tag{19}$$

Also from (15), one can has

$$f'(t) \ge -\left(\frac{\xi-t}{\xi-a}|f'(a)| + \frac{t-a}{\xi-a}|f'(\xi)|\right). \tag{20}$$

Following the same procedure as we did for (16), we also have

$$f(a)(h(\xi)-h(a))^\mu - \Gamma(\mu+1) I_{a^+}^{\mu,h} f(\xi) \le (h(\xi)-h(a))^\mu(\xi-a)\left[\frac{|f'(a)|+|f'(\xi)|}{2}\right]. \tag{21}$$

From (19) and (21), we get

$$\left|\Gamma(\mu+1) I_{a^+}^{\mu,h} f(\xi) - f(a)(h(\xi)-h(a))^\mu\right| \le (h(\xi)-h(a))^\mu(\xi-a)\left[\frac{|f'(a)|+|f'(\xi)|}{2}\right]. \tag{22}$$

From convexity of $|f'|$, it can be obtained

$$|f'(t)| \le \frac{t-\xi}{b-\xi}|f'(b)| + \frac{b-t}{b-\xi}|f'(\xi)|. \tag{23}$$

Now for $\xi \in [a,b]$ and $t \in [\xi,b]$ and $\nu > 0$, the following inequality holds true

$$(h(t)-h(\xi))^\nu \le (h(b)-h(\xi))^\nu. \tag{24}$$

Following the same way as we have done for (16), (17) and (20) one can get from (23) and (24) the following inequality

$$\left|\Gamma(\nu+1) I_{b^-}^{\nu,h} f(\xi) - f(b)(h(b)-h(\xi))^\nu\right| \le (h(b)-h(\xi))^\nu(b-\xi)\left[\frac{|f'(b)|+|f'(\xi)|}{2}\right]. \tag{25}$$

From inequalities (22) and (25) via triangular inequality, we get (14) which is required. □

Special cases are discussed in the following.

Corollary 2. *If we take $\mu = \nu$ in (14), then we get the following inequality for (RL) fractional integrals in general form*

$$\left|\Gamma(\mu+1)(I_{a^+}^{\mu,h} f(\xi) + I_{b^-}^{\mu,h} f(\xi)) - ((h(\xi)-h(a))^\mu f(a) + (h(b)-h(\xi))^\mu f(b))\right|$$
$$\le \frac{(h(\xi)-h(a))^\mu(\xi-a)|f'(a)| + (h(b)-h(\xi))^\mu(b-\xi)|f'(b)|}{2}$$
$$+ |f'(\xi)|\frac{((h(\xi)-h(a))^\mu(\xi-a) + (h(b)-h(\xi))^\mu(b-\xi))}{2}.$$

Remark 3. *By setting $h(\xi) = \xi$ in (14), we get the fractional integral inequality for (RL) fractional integrals ([10], Theorem 1).*

The following lemma is important to prove the next result.

Lemma 1. *Suppose that $f : [a, b] \longrightarrow \mathbb{R}$ is a convex function which is symmetric about $\frac{a+b}{2}$ [10]. Then we have*

$$f\left(\frac{a+b}{2}\right) \leq f(\xi) \quad \xi \in [a, b]. \tag{26}$$

Theorem 3. *Suppose that $f, h : [a, b] \longrightarrow \mathbb{R}$ be the functions such that f be positive convex and symmetric about $\frac{a+b}{2}$, h be differentiable and strictly increasing with $h' \in L[a, b]$. Then for $\mu, \nu > 0$, we have*

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right)\left[\Gamma(\mu+1) I_{a^+}^{\mu,h} h(b) - \Gamma(\nu+1) I_{b^-}^{\nu,h} h(a)\right. \\
& \left.-(h(b)-h(a))^{\mu} h(a) + (h(b)-h(a))^{\nu} h(b)\right] \\
& \leq \Gamma(\mu+1) I_{a^+}^{\mu+1,h} f(b) + \Gamma(\nu+1) I_{b^-}^{\nu+1,h} f(a) \\
& \leq \frac{((h(b)-h(a))^{\nu} + (h(b)-h(a))^{\mu})}{b-a} \\
& \times\left[(b-a)(f(b)h(b) - f(a)h(a)) - (f(b)-f(a)) \int_a^b h(\xi) d\xi\right].
\end{aligned} \tag{27}$$

Proof. Since the function h is differentiable and strictly increasing therefore $(h(\xi) - h(a))^{\nu} \leq (h(b) - h(a))^{\nu}$, where as $\xi \in [a, b]$, $\nu > 0$, and $h'(\xi) > 0$. Hence the following inequality holds true

$$h'(\xi)(h(\xi) - h(a))^{\nu} \leq h'(\xi)(h(b) - h(a))^{\nu}. \tag{28}$$

From convexity of f, it can be obtained

$$f(\xi) \leq \frac{\xi - a}{b-a} f(b) + \frac{b-\xi}{b-a} f(a). \tag{29}$$

From (28) and (29), one can has

$$\begin{aligned}
& \int_a^b (h(\xi) - h(a))^{\nu} f(\xi) h'(\xi) d\xi \\
& \leq \frac{(h(b)-h(a))^{\nu}}{b-a}\left[f(b) \int_a^b (\xi - a) h'(\xi) d\xi + f(a) \int_a^b (b-\xi) h'(\xi) d\xi\right].
\end{aligned}$$

By using (5) of Definition 1 we get

$$\begin{aligned}
& \Gamma(\nu+1) I_{b^-}^{\nu+1,h} f(a) \\
& \leq \frac{(h(b)-h(a))^{\nu}}{b-a}\left[(b-a)(f(b)h(b) - f(a)h(a)) - (f(b)-f(a)) \int_a^b h(\xi) d\xi\right].
\end{aligned} \tag{30}$$

Now for $\xi \in [a, b]$, $t \in [\xi, b]$ and $\mu > 0$, the following inequality holds true

$$h'(\xi)(h(b) - h(\xi))^{\mu} \leq h'(\xi)(h(b) - h(a))^{\mu}. \tag{31}$$

Following the same way as we have done for (28) and (29) one can get from (29) and (31) the following inequality

$$\begin{aligned}
& \Gamma(\mu+1) I_{a^+}^{\mu+1,h} f(b) \\
& \leq \frac{(h(b)-h(a))^{\mu}}{b-a}\left[(b-a)(f(b)h(b) - f(a)h(a)) - (f(b)-f(a)) \int_a^b h(\xi) d\xi\right].
\end{aligned} \tag{32}$$

From (30) and (32), we get

$$\begin{aligned}
&\Gamma(\nu+1)I_{b^-}^{\nu+1,h}f(a)+\Gamma(\mu+1)I_{a^+}^{\mu+1,h}f(b) \\
&\leq \frac{((h(b)-h(a))^{\nu}+(h(b)-h(a))^{\mu})}{b-a} \\
&\times\left[(b-a)(f(b)h(b)-f(a)h(a))-(f(b)-f(a))\int_a^b h(\xi)d\xi\right].
\end{aligned} \tag{33}$$

Using Lemma 1 and multiplying (26) with $(h(\xi)-h(a))^{\nu}h'(\xi)$ and integrating over $[a,b]$ we get

$$f\left(\frac{a+b}{2}\right)\int_a^b (h(\xi)-h(a))^{\nu}h'(\xi)d\xi \leq \int_a^b (h(\xi)-h(a))^{\nu}h'(\xi)f(\xi)d\xi. \tag{34}$$

By using (5) of Definition 1 we get

$$f\left(\frac{a+b}{2}\right)\left[(h(b)-h(a))^{\nu}h(b)-\Gamma(\nu+1)I_{b^-}^{\nu,h}h(a)\right] \leq \Gamma(\nu+1)I_{b^-}^{\nu+1,h}f(a). \tag{35}$$

Similarly, using Lemma 1 and multiplying (26) with $(h(b)-h(\xi))^{\mu}h'(\xi)$, then integrating over $[a,b]$, we have

$$f\left(\frac{a+b}{2}\right)\left[\Gamma(\mu+1)I_{a^+}^{\mu,h}h(b)-(h(b)-h(a))^{\mu}h(a)\right] \leq \Gamma(\mu+1)I_{a^+}^{\mu+1,h}f(b). \tag{36}$$

From (35) and (36), we get

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)\left[\Gamma(\mu+1)I_{a^+}^{\mu,h}h(b)-\Gamma(\nu+1)I_{b^-}^{\nu,h}h(a)\right. \\
&\left.-(h(b)-h(a))^{\mu}h(a)+(h(b)-h(a))^{\nu}h(b)\right] \\
&\leq \Gamma(\mu+1)I_{a^+}^{\mu+1,h}f(b)+\Gamma(\nu+1)I_{b^-}^{\nu+1,h}f(a).
\end{aligned} \tag{37}$$

From inequalities (33) and (37), we get (27) which is required. □

Corollary 3. *If we take $\mu=\nu$ in (27), then we get the following inequality for (RL) fractional integrals in general form*

$$\begin{aligned}
&\left[\Gamma(\mu+1)I_{a^+}^{\mu,h}h(b)-\Gamma(\mu+1)I_{b^-}^{\mu,h}h(a)-(h(b)-h(a))^{\mu+1}\right] \\
&\leq \Gamma(\mu+1)I_{a^+}^{\mu+1,h}f(b)+\Gamma(\mu+1)I_{b^-}^{\mu+1,h}f(a) \\
&\leq \frac{2((h(b)-h(a))^{\mu}}{b-a}\left[(b-a)(f(b)h(b)-f(a)h(a))-(f(b)-f(a))\int_a^b h(\xi)d\xi\right].
\end{aligned}$$

Remark 4. *By setting $h(\xi)=\xi$ in (27), we get the fractional integral inequality for (RL) fractional integrals ([10], Theorem 3).*

3. Applications

In this section, we are interested in applying some of the results proved in the previous section. In particular, bounds of general Riemann-Liouville fractional integrals are obtained which contain bounds of all fractional integrals comprises in Remark 1. At the end, we give a refined bound of the Hadamard inequality. First, we apply Theorem 1 and get the following result.

Theorem 4. *Suppose that assumptions of Theorem 1 are valid, then we have*

$$\begin{aligned}
&\Gamma(\mu)I_{a^+}^{\mu,h}f(b)+\Gamma(\nu)I_{b^-}^{\nu,h}f(a) \\
&\leq \left(\frac{(h(b)-h(a))^{\mu-1}+(h(b)-h(a))^{\nu-1}}{b-a}\right) \\
&\times \left((b-a)(f(b)h(b)-f(a)h(a))-(f(b)-f(a))\int_a^b h(t)dt\right).
\end{aligned} \tag{38}$$

Proof. If we take $x=a$ and $x=b$ in (6), then adding resulting inequalities, we get (38). □

Corollary 4. *If we take $\mu=\nu$ in (38), then we get the following inequality for (RL) fractional integrals in general form*

$$\begin{aligned}
&\Gamma(\mu)(I_{a^+}^{\mu,h}f(b)+I_{b^-}^{\mu,h}f(a)) \\
&\leq \frac{2(h(b)-h(a))^{\mu-1}}{b-a}\left((b-a)(f(b)h(b)-f(a)h(a))-(f(b)-f(a))\int_a^b h(t)dt\right).
\end{aligned} \tag{39}$$

Corollary 5. *If we take $\mu=1$ and $h(\xi)=\xi$ in (39), then we get the following right Hadamard inequality*

$$\frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{40}$$

Next the applications of Theorem 2 to have been presented.

Theorem 5. *Suppose that assumptions of Theorem 2 are valid, then we have*

$$\begin{aligned}
&\Bigg|\Gamma(\mu+1)I_{a^+}^{\mu,h}f\left(\frac{a+b}{2}\right)+\Gamma(\nu+1)I_{b^-}^{\nu,h}f\left(\frac{a+b}{2}\right) \\
&-\left(\left(h\left(\frac{a+b}{2}\right)-h(a)\right)^{\mu}f(a)+\left(h(b)-h\left(\frac{a+b}{2}\right)\right)^{\nu}f(b)\right)\Bigg| \\
&\leq \frac{\left(\frac{b-a}{2}\right)\left(\left(h\left(\frac{a+b}{2}\right)-h(a)\right)^{\mu}|f'(a)|+\left(h(b)-h\left(\frac{a+b}{2}\right)\right)^{\nu}|f'(b)|\right)}{2} \\
&+\left|f'\left(\frac{a+b}{2}\right)\right|\frac{\left(\frac{b-a}{2}\right)\left(\left(h\left(\frac{a+b}{2}\right)-h(a)\right)^{\mu}+(h(b)-h\left(\frac{a+b}{2}\right))^{\nu}\right)}{2}.
\end{aligned} \tag{41}$$

Proof. If we take $\xi=\frac{a+b}{2}$ in (14), then we get (41). □

Corollary 6. *If we take $\mu=\nu$ in (41), then we get the following inequality for (RL) fractional integrals in general form*

$$\begin{aligned}
&\Bigg|\Gamma(\mu+1)\left(I_{a^+}^{\mu,h}f\left(\frac{a+b}{2}\right)+I_{b^-}^{\mu,h}f\left(\frac{a+b}{2}\right)\right) \\
&-\left(\left(h\left(\frac{a+b}{2}\right)-h(a)\right)^{\mu}f(a)+\left(h(b)-h\left(\frac{a+b}{2}\right)\right)^{\nu}f(b)\right)\Bigg| \\
&\leq \frac{\left(\frac{b-a}{2}\right)\left(\left(h\left(\frac{a+b}{2}\right)-h(a)\right)^{\mu}|f'(a)|+\left(h(b)-h\left(\frac{a+b}{2}\right)\right)^{\nu}|f'(b)|\right)}{2} \\
&+\left|f'\left(\frac{a+b}{2}\right)\right|\frac{\left(\frac{b-a}{2}\right)\left(\left(h\left(\frac{a+b}{2}\right)-h(a)\right)^{\mu}+(h(b)-h\left(\frac{a+b}{2}\right))^{\nu}\right)}{2}.
\end{aligned} \tag{42}$$

Corollary 7. *If we take $\mu = 1$ and $h(\xi) = \xi$ in (42), then we get the following inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{f(a)+f(b)}{2} \right| \le \frac{b-a}{8} \left[|f'(a)| + |f'(b)| + 2 \left| f'\left(\frac{a+b}{2}\right) \right| \right]. \tag{43}$$

Remark 5. *If $f'\left(\frac{a+b}{2}\right) = 0$, then (43) gives [14], Theorem 2.2. If $f'(\xi) \le 0$, then (43) provide a refinement of [14], Theorem 2.2.*

By applying Theorem 3 similar relations can be established we leave it for the reader.

4. Conclusions

The aim of this work was to investigate fractional integral inequalities for the generalized and compact form of Riemann-Liouville fractional integrals via convex functions. These inequalities consist on the bounds of sum of left-sided and right-sided fractional integrals and inequalities for function having its first derivative in absolute value convex. Also fractional inequalities of Hadamard type for a symmetric and convex function are proved. All these results hold in particular for fractional integrals comprises in Remark 1. The method adopted to produce fractional inequalities is new and straightforward. It can be followed to develop further results for other classes of functions related to convex function using convenient fractional integral operators. These results may be useful in the theory of fractional calculus.

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