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On p-Common Best Proximity Point Results for $\mathcal{S}$-Weakly Contraction in Complete Metric Spaces

Chayut Kongban [1,2], Poom Kumam [1,2,*], Somayya Komal [1,2] and Kanokwan Sitthithakerngkiet [3]

1 KMUTT-Fixed Point Research Laboratory, Department of Mathematics, Room SCL 802 Fixed Point Laboratory, Science Laboratory Building, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand; chayut_kb@hotmail.com (C.K.); komal.musab@gmail.com (S.K.)

2 KMUTT-Fixed Point Theory and Applications Research Group (KMUTT-FPTA), Theoretical and Computational Science Center (TaCS), Science Laboratory Building, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand

3 Intelligent and Nonlinear Dynamic Innovations Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok (KMUTNB), Wongsawang, Bangsue, Bangkok 10800, Thailand; kanokwan.s@sci.kmutnb.ac.th

* Correspondence: poom.kum@kmutt.ac.th

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Abstract: In this work, we introduced new notions of a new contraction named $\mathcal{S}$-weakly contraction; after that, we obtained the p-common best proximity point results for different types of contractions in the setting of complete metric spaces by using weak P_p-property and proved the uniqueness of these points. Also, we presented some examples to prove the validity of our results.

Keywords: p-common best proximity point; weak P_p-property; $\mathcal{S}$-weakly contraction

MSC: 47H10; 54H25

1. Introduction

Banach Contraction Principle [1] is a very familiar theorem that helps out in the branch of fixed point theory to describe the tools for finding a solution to non-linear equations of the type $Ux = x$ if given mapping U is a self-mapping defined on any non-empty subset of metric space or any other relevant framework. If the given mapping U is non-self then it is possible that given mapping has no solution $Ux = x$. Then, in those cases we try to find those points for that non-self mapping U which give us a close solution to the equation $Ux = x$, with this idea we approach towards the best approximation problems and then we obtain the solution which is not optimal but is an approximate solution to the equation $Ux = x$. With the help of these approximate solutions, we attain a target to find the solution which is optimal because the error $d(x, Ux)$ is minimum and $d(x, Ux) = d(A, B)$ and that optimal approximate solution is called the best proximity point for given mapping which is non-self. To find out the best proximity point, it is necessary that we should have only one non-self mapping; with the help of that mapping, we can find a best proximity point, but whenever we have more than one non-self mappings in a problem and we have to find the optimal solution for those mappings defined on same subsets of any space, then that type of optimal solution is known as a common best proximity point for given mappings.

The basic purpose of this paper is to construct some new theorems with new notions and contractions; with the help of these new results, we will describe a common best proximity point for given mappings in metric spaces. Then, we will establish some examples for the justification of our results. The given results are more general than earlier ones.

2. Preliminaries and Mathematical Definition

In this section, let us recall some definitions, lemmas and theorems that will be used in what follows.

Definition 1. *[2] Let A and B be two nonempty subsets of a metric space (X, d). We define the sets*

$$A_0 = \{a \in A : \text{there exists some } b \in B \text{ such that } d(a, b) = d(A, B)\},$$

and

$$B_0 = \{b \in B : \text{there exists some } a \in A \text{ such that } d(a, b) = d(A, B)\},$$

where $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$ is the distance between the sets A and B.

Definition 2. *[3] Let (A, B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the weak P-property if and only if for any $x_1, x_2 \in A_0$ and $x_3, x_4 \in B_0$,*

$$\left.\begin{array}{ll} d(x_1, x_3) & = d(A, B) \\ d(x_2, x_4) & = d(A, B) \end{array}\right\} \Rightarrow d(x_1, x_2) \leq d(x_3, x_4).$$

Definition 3. *[4] Given a non-self mapping $f : A \to B$, then an element x^* is called a best proximity point of the mapping f if*

$$d(x^*, fx^*) = d(A, B),$$

and denote the set of all best proximity points of f by $BPP(f)$.

Definition 4. *[5] Let $f : A \to B$ and $g : A \to B$ be non-self mappings. An element x^* is called a common best proximity point of the mappings f and g if this condition is satisfied:*

$$d(x^*, fx^*) = d(A, B) = d(x^*, gx^*).$$

Lemma 1. *[4] Let $\{x_n\}$ be a sequence in X such that $d(x_{n+1}, x_n) \leq kd(x_n, x_{n-1})$ for all $n \in N$ and $0 \leq k < 1$. Then $\{x_n\}$ is a Cauchy sequence.*

Theorem 1. *[4] Let (A, B) be a pair of non-empty closed subsets of a complete metric space (X, d) and let $S : A \to B$ and $T : A \to B$ be the mappings such that A_0 is nonempty. Assume that the following conditions are satisfied:*

1. *The pair (A, B) has weak P-property;*
2. *$d(Sx, Ty) \leq kd(x, y)$ for $0 \leq k < 1$.*

Then there exists a unique common best proximity point x to the pair (S, T) that is $d(x, Sx) = d(x, Tx) = d(A, B)$.

Theorem 2. *[4] Let (A, B) be a pair of non-empty closed subsets of a complete metric space (X, d) and let $S : A \to B$ and $T : A \to B$ be the mappings such that A_0 is nonempty. Assume that the following conditions are satisfied:*

1. *The pair (A, B) has weak P-property;*
2. *S and T are continuous;*
3. *$d(Sx, Ty) \leq k[d(x, Sx) + d(y, Ty) - 2d(A, B)]$ for $0 \leq k < 1$.*

Then there exists a unique common best proximity point x to the pair (S, T) that is $d(x, Sx) = d(x, Tx) = d(A, B)$.

Theorem 3. *[6] A C-contraction defined on a complete metric space (X, d) has a unique fixed point that is if $T : X \to X$ satisfies*

$$d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)],$$

where $0 < \alpha < 1$ and $x, y \in X$, then T has a unique fixed point.

Next, we recall w-distance on a metric space (X, d) and give some facts by using w-distance function.

Definition 5. *[7] Let (X, d) be a metric space. Then a function $p : X \times X \to [0, \infty)$ is called w-distance on X if the following are satisfied:*

1. *$p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$;*
2. *for any $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semi continuous;*
3. *for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ implies $d(x, y) \leq \epsilon$.*

Note that the metric d is an example of w-distance.

Definition 6. *[7] Let (X, d) be a metric space. A set valued mapping $T : X \to X$ is called weakly contractive if there exists a w-distance p on X and $r \in [0, 1)$ such that for any $x_1, x_2 \in X$ and $y_1 \in Tx_1$ there is $y_2 \in Tx_2$ with $p(y_1, y_2) \leq rp(x_1, x_2)$.*

3. On p-Common Best Proximity Point Theorems for $\mathcal{S}$-Weakly Contractive Mappings

Before giving our main results, we first introduce some notations by considering the concept of the w_s-distance.

Definition 7. *Let (X, d) be a metric space. Then a function $p : X \times X \to [0, \infty)$ is called w_s-distance on X if the following are satisfied:*

1. *$p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$;*
2. *$p(x, y) \geq 0$, for any $x, y \in X$;*
3. *if $\{x_m\}$ and $\{y_m\}$ be any sequences in X such that $x_n \to x,\ y_n \to y$ as $n \to \infty$, then $p(x_n, y_n) \to p(x, y)$ as $n \to \infty$;*
4. *for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ implies $d(x, y) \leq \epsilon$.*

Note that the metric d is also an example of w_s-distance.

Definition 8. *Let (X, d) be a metric space and p be w_s-distance on X. Let A and B be two nonempty subsets of X, define*

$$A_{0,p} = \{a \in A : \text{there exists some } b \in B \text{ such that } p(a, b) = p(A, B)\}$$

and

$$B_{0,p} = \{b \in B : \text{there exists some } a \in A \text{ such that } p(a, b) = p(A, B)\},$$

where $p(A, B) = \inf\{p(a, b) : a \in A, b \in B\}$.

Definition 9. *Let (X, d) be a metric space and $A, B \subseteq X$. Let p be w_s-distance on X such that $A_{0,p} \neq \emptyset$. A set valued mapping $T : A \to B$ with $T(A_0, p) \subseteq B_{0,p}$ is called $\mathcal{S}$-weakly contractive or P_p-contractive if there exists a w_s-distance p on A and $r \in [0, 1)$ such that for any $x_1, x_2 \in A$ and $y_1 \in Tx_1$ in B there is $y_2 \in Tx_2$ in B with $p(y_1, y_2) \leq rp(x_1, x_2)$.*

Definition 10. *Let (A, B) be a part of nonempty subsets of a metric space (X, d) and p be w_s-distance on X with $A_{0,p} \neq \emptyset$. Then the pair (A, B) is said to have weak P_p-property if and only if for any $x_1, x_2 \in A_{0,p}$ and $y_1, y_2 \in B_{0,p}$*

$$\left.\begin{array}{ll} p(x_1, y_1) & = p(A, B) \\ p(x_2, y_2) & = p(A, B) \end{array}\right\} \Rightarrow p(x_1, x_2) \leq p(y_1, y_2).$$

Definition 11. *Let p be w_s-distance on a metric space (X,d) and $A,B \subseteq X$. Given two non-self mappings $f : A \to B$ and $g : A \to B$, then an element x^* is called p-common best proximity point of the mappings if*

$$p(x^*, fx^*) = p(A,B) = p(x^*, gx^*).$$

Lemma 2. *Let p be w_s-distance on a metric space (X,d) and $\{x_n\}$ be a sequence in X such that $p(x_{n+1},x_n) \leq kp(x_n,x_{n-1})$ for all $n \in N$ and $0 \leq k < 1$. Then $\{x_n\}$ is a Cauchy sequence.*

Proof. We have, $p(x_{n+1},x_n) \leq kp(x_n,x_{n-1}) \leq k^2 p(x_{n-1},x_{n-2}) \leq ... \leq k^n p(x_1,x_0)$.

Let $m > n \geq n_0$ for some $n_0 \in N$. Then

$$\begin{aligned} p(x_m,x_n) &\leq p(x_m,x_{m-1}) + p(x_{m-1},x_{m-2}) + ... + p(x_{n+1},x_n) \\ &\leq (k^{m-1} + k^{m-2} + ... + k^n)p(x_1,x_0) \\ &\leq (k^n + k^{n+1} + ...)p(x_1,x_0) \\ &= \frac{k^n}{1-k} d(x_1,x_0) \to 0, \text{ as } n \to \infty, \text{ and } 0 \leq \text{k} < 1. \end{aligned}$$

This implies $\{x_n\}$ is a Cauchy sequence. □

Theorem 4. *Let (X,d) be a metric space and A,B are nonempty closed subsets of X. Suppose that $T : A \to B$ and $U : A \to B$ are continuous set valued, S-weakly contractives or p_p-contractive mappings with (A,B) satisfies the weak P_p-property where p is the w_s-distance with $A_{0,p} \neq \emptyset$. If $T(A_{0,p}) \subseteq B_{0,p}$ and $U(A_{0,p}) \subseteq B_{0,p}$ then there exists a unique p-common best proximity point.*

Proof. Since T and U are $\mathcal{S}$-weakly-contractive mappings and $A_{0,p}$ is nonempty. Thus, we take $x_0 \in A_{0,p}$, there exists $x_1 \in A_{0,p}$ such that

$$p(x_1, Tx_0) = p(A,B). \tag{1}$$

and similarly

$$p(x_1, Ux_0) = p(A,B). \tag{2}$$

Again, since $T(A_{0,p}) \subseteq B_{0,p}$ and $U(A_{0,p}) \subseteq B_{0,p}$, there exists $x_2 \in A_{0,p}$ such that

$$p(x_2, Tx_1) = p(A,B). \tag{3}$$

Also,

$$p(x_2, Ux_1) = p(A,B). \tag{4}$$

Repeating this process, we get a sequence $\{x_n\}$ in $A_{0,p}$ satisfying

$$p(x_{n+1}, Tx_n) = p(A,B) = p(x_{n+1}, Ux_n),$$

for any $n \in \mathbb{N}$.

Since (A,B) has weak P_p-property, we have that

$$p(x_n, x_{n+1}) \leq p(Tx_{n-1}, Tx_n)$$

and

$$p(x_n, x_{n+1}) \leq p(Ux_{n-1}, Ux_n),$$

for any $n \in \mathbb{N}$.

Note that T and U are $\mathcal{S}$-weakly-contractive mappings and (A, B) has weak P_p-property, so for any $n \in \mathbb{N}$, we have that

$$\begin{aligned} p(x_n, x_{n+1}) &\leq p(Tx_{n-1}, Tx_n) \\ &\leq rp(x_{n-1}, x_n) \\ &< p(x_{n-1}, x_n), \end{aligned}$$

and also

$$\begin{aligned} p(x_n, x_{n+1}) &\leq p(Ux_{n-1}, Ux_n) \\ &\leq rp(x_{n-1}, x_n) \\ &< p(x_{n-1}, x_n), \end{aligned}$$

where $0 \leq r < 1$. Then we have

$$p(x_n, x_{n+1}) < p(x_{n-1}, x_n).$$

This implies that $\{p(x_n, x_{n+1})\}$ is strictly decreasing sequence of nonnegative real numbers. Then, we can suppose that there exists $n_0 \in \mathbb{N}$ such that $p(x_{n_0}, x_{n_0+1}) = 0$. In this case,

$$0 = p(x_{n_0}, x_{n_0+1}) = p(Tx_{n_0-1}, Tx_{n_0}) = p(Ux_{n_0-1}, Ux_{n_0}),$$

and consequently

$$Tx_{n_0-1} = Tx_{n_0},$$

and

$$Ux_{n_0-1} = Ux_{n_0},$$

Therefore,

$$p(A, B) = p(x_{n_0}, Tx_{n_0-1}) = p(x_{n_0}, Tx_{n_0}) = p(x_{n_0}, Ux_{n_0}).$$

Note that $x_{n_0} \in A_0$, $Ux_{n_0-1} \in B_0$, $Tx_{n_0-1} \in B_0$, and $x_{n_0} = Tx_{n_0-1}$, $x_{n_0} = Ux_{n_0-1}$, for any $n_0 \in \mathbb{N}$, so $A \bigcap B$ is nonempty, then $p(A, B) = 0$. Thus in this case, there exists p-common best proximity point, i.e., there exists unique x^* in A such that $p(x^*, Tx^*) = p(A, B) = p(x^*, Ux^*)$.

In the contrary case, suppose that $p(Tx_{n_0}, Tx_{n_0-1}) > 0$ and $p(Ux_{n_0}, Ux_{n_0-1}) > 0$ this implies that $p(x_n, x_{n+1}) > 0$ for any $n \in N$. Since $\{p(x_n, x_{n+1})\}$ is strictly decreasing sequence of nonnegative real numbers and hence there exists $k \geq 0$ such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = k.$$

We have to show that $k = 0$. Let $k \neq 0$ and $k > 0$, then from

$$p(x, y) = \lim_{n\to\infty} p(x_n, x_{n+1})$$

and

$$p(x, y) \leq \liminf_{n\to\infty} p(x, x_{n+1}) \leq 0,$$

we have

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0.$$

for any $n \in \mathbb{N}$. Which yields that

$$\lim_{n\to\infty} p(x_{n-1}, x_n) = 0.$$

Hence $k = 0$ and this contradicts our assumption that $k > 0$. Therefore,

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0.$$

Since $p(x_{n+1}, Tx_n) = p(A, B)$ for any $n \in \mathbb{N}$, for fixed $p, q \in \mathbb{N}$, we have

$$p(x_p, Tx_{p-1}) = p(x_q, Tx_{q-1}) = p(A, B)$$

and since (A, B) satisfies weak P_p-property, so

$$p(x_p, x_q) \leq p(Tx_{p-1}, Tx_{q-1})$$

and

$$p(x_p, x_q) \leq p(Ux_{p-1}, Ux_{q-1}).$$

By Lemma 2, we conclude that $\{x_n\}$ is a Cauchy sequence in A. Since $\{x_n\} \subseteq A$ and A is closed subset of a complete metric space (X, d). There is $x^* \in A$ such that $x_n \to x^*$ as $n \to \infty$. Since T and U are continuous, so we have

$$Tx_n \to Tx^* \text{ and } Ux_n \to Ux^* \quad \text{as } n \to \infty.$$

Then we conclude that

$$p(x_{n+1}, Tx_n) \to p(x^*, Tx^*) \text{ and } p(x_{n+1}, Ux_n) \to p(x^*, Ux^*) \quad \text{as } n \to \infty.$$

Taking into account that $\{p(x_{n+1}, Tx_n)\}$ and $\{p(x_{n+1}, Ux_n)\}$ are constant sequences with a value $p(A, B)$, we deduce

$$p(x^*, Tx^*) = p(A, B) = p(x^*, Ux^*),$$

i.e., x^* is p-common best proximity point of T.

Next, we will prove the uniqueness of a p-common best proximity point. Since p is a w-distance and also T and U are P_p-contractives then $p(Tx, Ty) \leq rp(x, y)$ for every $x, y \in A$ of X. We suppose that given mappings T and U have two distinct p-common best proximity points $x_0, x_1 \in A$, that is $p(x_0, Tx_0) = p(x_0, Ux_0) = p(A, B)$, and $p(x_1, Tx_1) = p(x_1, Ux_1) = p(A, B)$. Since T and U have P_p-property, then

$$\begin{aligned} p(x_0, x_1) &= p(Tx_0, Tx_1) \\ &\leq rp(x_0, x_1), \end{aligned}$$

and

$$\begin{aligned} p(x_0, x_1) &= p(Ux_0, Ux_1) \\ &\leq rp(x_0, x_1), \end{aligned}$$

which shows

$$p(x_0, y_0) \leq rp(x_0, y_0).$$

It contradicts our assumption and so we get $x_0 = y_0$. Therefore, there exists a unique p-common best proximity point for the pair (T, U). $\square$

4. Characterizations Related to p-Contractive Type Mappings

In this section, now we are in a position to show the results for different p-contractive type mappings.

Theorem 5. *Let (A, B) be a pair of non empty closed subsets of a complete metric space X and p be the w_s-distance on X. Let $S : A \to B$ and $T : A \to B$ such that $A_{0,p}$ is nonempty and $S, T(A_{0,p}) \subseteq B_{0,p}$. Assume that the following conditions are satisfied:*

1. *The pair (A, B) has weak P_p-property;*
2. *$p(Sx, Ty) \leq kp(x, y)$ for $0 \leq k < 1$.*

Then there exists a unique p-common best proximity point x to the pair (S, T) that is $p(x, Sx) = p(x, Tx) = p(A, B)$.

Proof. We consider $x_0 \in A_{0,p}$ as $A_{0,p}$ is non empty, since $Sx_0 \in S(A_{0,p}) \subseteq B_{0,p}$, then by definition of $A_{0,p}$ we can find $x_1 \in A_{0,p}$, such that $p(x_1, Sx_0) = p(A, B)$. Again $Tx_1 \in T(A_{0,p}) \subseteq B_{0,p}$, we find $x_2 \in A_{0,p}$ such that $p(x_2, Tx_1) = p(A, B)$. Since $x_2 \in A_{0,p}$ and $S(A_{0,p}) \subseteq B_{0,p}$, we have $x_3 \in A_{0,p}$ such that $p(x_3, Sx_2) = p(A, B)$. In this manner we can get $x_4 \in A_{0,p}$ such that $p(x_4, Tx_3) = p(A, B)$ as $T(A_{0,p}) \subseteq B_{0,p}$ and $Tx_3 \in B_{0,p}$. Repeating the process, we obtain a sequence $\{x_n\}$ in $A_{0,p}$ satisfying $p(x_{2n}, Tx_{2n-1}) = p(A, B)$, for all $n \in N$ and $p(x_{2n-1}, Sx_{2n-2}) = p(A, B)$, for all $n \in N$ Since (A, B) has weak P_p-property, we obtain that

$$p(x_{2n}, x_{2n-1}) \leq p(Tx_{2n-1}, Sx_{2n-2}) = p(Sx_{2n-2}, Tx_{2n-1})$$

for any $n \in N$ and

$$p(x_{2n+1}, x_{2n}) \leq p(Sx_{2n}, Tx_{2n-1}) = p(Tx_{2n-1}, Sx_{2n})$$

for any $n \in N$. Now $p(x_{2n+2}, x_{2n+1}) \leq p(Sx_{2n}, Tx_{2n+1}) \leq kp(x_{2n}, x_{2n+1})$. Again $p(x_{2n+1}, x_{2n}) \leq p(Sx_{2n}, Tx_{2n-1}) \leq kp(x_{2n}, x_{2n-1})$.

Hence, we get $p(x_{n+1}, x_n) \leq kp(x_n, x_{n-1})$ for all $n \in N$, where $0 \leq k < 1$. Then by Lemma 2, $\{x_n\}$ is a Cauchy sequence in A. As A is closed subset of a complete metric space so A is complete. Hence there exists $x \in A$ such that $x_n \to x$ as $n \to \infty$. Now we claim that $p(Sx_n, Sx) = 0$ and $p(Tx_m, Tx) = 0$ as $n, m \to \infty$. Note that

$$\begin{aligned} p(Sx_n, Sx) &\leq p(Sx_n, Tx_m) + p(Tx_m, Sx) \\ &\leq k[p(x_n, x_m) + p(x_m, x)] \\ &\to 0 \text{ as } n, m \to \infty. \end{aligned}$$

Similarly, one can show that $p(Tx_m, Tx) = 0$. Now as $n \to \infty$, we have

$$p(x_{2n-1}, Sx_{2n-2}) = p(A, B))p(x, Sx) = p(A, B)$$

and

$$p(x_{2n}, Tx_{2n-1}) = p(A, B)p(x, Tx) = p(A, B).$$

Therefore, $p(x, Sx) = p(x, Tx) = p(A, B)$ that is x is a p-common best proximity point for the pair of mappings (S, T). Now, we shall prove uniqueness of the p-common best proximity point to the pair of mappings (S, T). Let us consider another p-common best proximity point y for the pair of mappings (S, T) then

$$p(y, Sy) = p(y, Ty) = p(A, B).$$

Then by weak P_p-property,

$$p(x, Sx) = p(x, Tx) = p(A, B),$$

and

$$p(y, Sy) = p(y, Ty) = p(A, B)$$

imply

$$p(x, y) \leq p(Sx, Ty) \leq kp(x, y)$$

or

$$p(x, y) \leq p(Sx, Sy) \leq p(Sx, Ty) + p(Ty, Sy) \leq k[p(x, y) + p(y, y)] = kp(x, y)$$

or

$$p(x, y) \leq p(Tx, Ty) \leq p(Tx, Sy) + p(Sy, Ty) \leq k[p(x, y) + p(y, y)] = kp(x, y).$$

As $0 \leq k < 1$, in any of the above three cases, we conclude a contradiction. Hence there exists a unique p-common best proximity point to the pair (S, T) that is $p(x, Sx) = p(x, Tx) = p(A, B)$. □

Theorem 6. *Let (A, B) be a pair of non empty closed subsets of a complete metric space (X, d) and p be the w_s-distance on X. Let $S : A \to B$ and $T : A \to B$ such that $A_{0,p}$ is nonempty, $S(A_{0,p}) \subseteq B_{0,p}$, $T(A_{0,p}) \subseteq B_{0,p}$ and $B_{0,p}$ is closed. Assume that the following conditions are satisfied:*

1. *The pair (A, B) has weak P_p-property;*
2. *S and T are continuous;*
3. *$p(Sx, Ty) \leq \frac{k}{2}[p(x, Ty) + p(y, Sx) - 2p(A, B)]$ for $0 \leq k < 1$.*

Then there exists a unique p-common best proximity point x to the pair (S, T) that is $p(x, Sx) = p(x, Tx) = p(A, B)$.

Proof. Since $A_{0,p} \neq \emptyset$ and the pair (A, B) satisfies weak P_p-property, also $B_{0,p}$ is closed. We have $S(A_{0,p}) \subseteq B_{0,p}$ and $T(A_{0,p}) \subseteq B_{0,p}$. Let us define an operator $PA_{0,p} : S(\overline{A_{0,p}}) \to A_{0,p}$, by $PA_{0,p}y = \{x \in A_{0,p} : p(x, y) = p(A, B)\}$. Since the pair (A, B) has weak P_p-property, then

$$p(PA_{0,p}(Sx), Sx) = p(A, B)$$

and

$$p(PA_{0,p}(Sy), Sy) = p(A, B).$$

imply that

$$\begin{aligned}
p(PA_{0,p}(Sx)PA_{0,p}(Sy)) &\leq p(Sx, Sy) \\
&\leq \frac{k}{2}[p(x, Sy) + p(y, Sx) - 2p(A, B)] \\
&\leq \frac{k}{2}[p(x, PA_{0,p}(Sy)) + p(PA_{0,p}(Sy), Sy) + p(y, PA_{0,p}(Sx)) \\
&+ p(PA_{0,p}(Sx), Sx) - 2p(A, B)] \\
&\leq \frac{k}{2}[p(x, PA_{0,p}(Sy)) + p(y, PA_{0,p}(Sx))].
\end{aligned}$$

for any $x, y \in \overline{A_{0,p}}$ and $0 \leq k < 1$. This gives that $PA_{0,p}oS : \overline{A_{0,p}} \to \overline{A_{0,p}}$ is C-contractive mapping from complete metric subspace $\overline{A_{0,p}}$ into itself then by [6], we can see that $PA_{0,p}oS$ has a unique p-fixed point say x_1. That is $PA_{0,p}oSx_1 = x_1 \in A_{0,p}$, which implies that $p(x_1, Sx_1) = p(A, B)$. In the same fashion, we can take a mapping $PA_{0,p}oT : \overline{A_{0,p}} \to \overline{A_{0,p}}$ and also that $PA_{0,p}oS$ has a unique p-fixed point say x_2. That is $PA_{0,p}oTx_2 = x_2 \in A_{0,p}$, which implies that $p(x_2, Tx_2) = p(A, B)$.

Now, we will show that $x_1 = x_2$. Since (A, B) satisfies weak P_p-property, then $p(x_1, Sx_1) = p(A, B)$ and $p(x_2, Tx_2) = p(A, B)$ imply that

$$\begin{aligned} p(x_1, x_2) &\leq p(Sx_1, Tx_2) \\ &\leq \frac{k}{2}\{p(x_1, Tx_2) + p(x_2, Sx_1) - 2p(A, B)\} \\ &\leq \frac{k}{2}\{p(x_1, x_2) + p(x_2, Tx_2) + p(x_2, x_1) + p(x_1, Sx_1) - 2p(A, B)\} \\ &= \frac{k}{2}\{p(x_1, x_2)\} \\ &= kp(x_1, x_2), \end{aligned}$$

which shows that $x_1 = x_2 := x(say)$. Therefore

$$p(x, Sx) = p(x, Tx) = p(A, B).$$

That is x is a p-common best proximity point.

Next, we will prove the uniqueness of the p-common best proximity point. Let y be another p-common best proximity point for the pair of mappings (S, T). Then

$$p(x, Sx) = p(x, Tx) = p(A, B).$$

$$p(y, Sy) = p(y, Ty) = p(A, B).$$

Then by weak P_p-property, we have

$$\begin{aligned} p(x, y) &\leq p(Sx, Ty) \\ &\leq \frac{k}{2}\{p(x, Ty) + p(y, Sx) - 2p(A, B)\} \\ &\leq \frac{k}{2}\{p(x, y) + p(y, Ty) + p(y, x) + p(x, Sx) - 2p(A, B)\} \\ &= kp(x, y) \end{aligned}$$

or

$$\begin{aligned} p(x, y) &\leq p(Sx, Sy) \\ &\leq \{p(Sx, Ty) + p(Ty, Sy)\} \\ &\leq \frac{k}{2}\{p(x, Ty) + p(y, Sx) - 2p(A, B)\} + \frac{k}{2}\{p(y, Ty) + p(y, Sy) - 2p(A, B)\} \\ &\leq \frac{k}{2}\{p(x, y) + p(y, Ty) + p(y, x) + p(x, Sx) - 2p(A, B)\} \\ &\quad + \frac{k}{2}\{p(y, Ty) + p(y, Sy) - 2p(A, B)\} \\ &= kp(x, y) \end{aligned}$$

or

$$\begin{aligned} p(x,y) &\leq p(Tx,Ty) \\ &\leq \{p(Tx,Sy)+p(Sy,Ty)\} \\ &\leq \frac{k}{2}\{p(x,Sy)+p(y,Tx)-2p(A,B)\}+\frac{k}{2}\{p(y,Sy)+p(y,Ty)-2p(A,B)\} \\ &\leq \frac{k}{2}\{p(x,y)+p(y,Sy)+p(y,x)+p(x,Tx)-2p(A,B)\} \\ &= kp(x,y). \end{aligned}$$

As $0 \leq k < 1$, in any of the above three different situations we conclude that $x = y$. Hence there exists a unique p-common best proximity point x to the pair (S,T) that is

$$p(x,Sx) = p(x,Tx) = p(A,B).$$

□

Example 1. *Consider* $X = \mathbb{R}^2$, *with the with the* p*-distance defined as* $p((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$. *Let* $A = \{(x,1) : 0 \leq x < \infty\}$ *and* $B = \{(x,0) : 0 \leq x < \infty\}$. *Obviously,* $p(A,B) = 1$ *and* A, B *are nonempty subsets of* X, *take* $A_{0,p} = A$ *and* $B_{0,p} = B$.

We define $S : A \to B$ *as:*

$$S(x,1) = (\frac{x+1}{3},0),$$

where $(x,1) \in A$.

Let $T : A \to B$ *defined as:*

$$T(x,1) = (\frac{x+1}{4},0).$$

Then, we see that $S(\overline{A_{0,p}}) \subseteq B_{0,p}$ *and* $T(\overline{A_{0,p}}) \subseteq B_{0,p}$. *Also, the pair* (A,B) *has weak* P_p*-property as:*

$$p((x_1,1),(y_1,1)) = \sqrt{(1-0^2)+(x_1-y_1)^2} = p(A,B) = 1,$$

and

$$p((x_2,1),(y_2,1)) = \sqrt{(1-0^2)+(x_2-y_2)^2} = p(A,B) = 1,$$

then one can easily obtain $x_1 = y_1$ *and* $x_2 = y_2$, *hence* $p((x_1,1),(x_2,1)) = |x_1-x_2| = |y_1-y_2| \leq p((y_1,0),(y_2,0))$. *Furthermore,* $p((0,1),(0,2)) = 1 = p(A,B)$ *and* $p((0,1),(0,0)) = 1 = p(A,B)$, *implies that* $p((0,1),(0,0)) = 1 = p(A,B)$. *Thus, the given pair* (A,B) *satisfies the weak* P_p*-property but not* P_p*-property.*

Next, for any different x, y, *let us suppose two elements* $(x_1,1),(x_2,1) \in A$,

$$\begin{aligned} p(S(x_1,1),(x_2,1)) &= p((\frac{x_1+1}{3},0),(\frac{x_2+1}{4},0)) \\ &= \frac{x}{3}-\frac{y}{4}+\frac{1}{12} \\ &\leq k|x-y| \\ &\leq kp((x_1,1),(x_2,1)) \end{aligned}$$

for any $k \in [0,1)$. *If* $x_1 = x_2$ *then surely this satisfied. So every condition of the Theorem 4 is satisfied thus one can find the unique* p*-common best proximity point for given pair of mappings* (S,T). *Hence, that* p*-common best proximity point is* $(0,1) \in A$.

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References

1. Fan, K. Extensions of two fixed point theorems of F. E. Browder. *Math. Z.* **1969**, *112*, 234–240. [CrossRef]
2. Basha, S.S. Best proximity point theorems generalizing the contraction principle. *Nonlinear Anal.* **2011**, *74*, 5844–5850. [CrossRef]
3. Zhang, J.; Su, Y.; Chang, Q. A note on a best proximity point theorem for Geraghty contractions. *Fixed Point Theory Appl.* **2013**, *2013*, 99. [CrossRef]
4. Mondal, S.; Dey, L.K. Some common best proximity point theorems in a complete metric space. *Afr. Mat.* **2017**, *28*, 85–97. [CrossRef]
5. Basha, S.S. Common best proximity points: Global minimization of multi-objective functions. *J. Glob. Optim.* **2012**, *54*, 367–373. [CrossRef]
6. Chatterjea, S.K. Fixed point theorems *C. R. Acad. Bulgare Sci.* **1972**, *25*, 727–730. [CrossRef]
7. Chaipunya, P.; Sintunavarat, W.; Kumam, P. On $\mathcal{P}$-contractions in ordered metric spaces. *Fixed Point Theory Appl.* **2012**, *2012*, 219. [CrossRef]