Properties of Functions Involving Struve Function

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Abstract: Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g_{p,b,c}(z) = z + \sum_{n=2}^{\infty} \frac{(-\frac{c}{4})^{n-1}}{(\frac{3}{2})_{n-1} (k)_{n-1}} z^n$ with $p, b, c \in \mathbb{C}$, $k = p + \frac{b+2}{2} \neq 0, -1, -2, \dots$ be two analytic functions in the unit disk $U = \{z : |z| < 1\}$. This paper gives conditions so that the function $T_{p,b,c}(z) = (f * g)(z)$, a function associated with the Struve function, is univalent, starlike, or convex in the unit disk.

Keywords: analytic; univalent; Struve function

MSC: 30C45

1. Introduction

In light of Louis de Brange using a special function, namely the generalized hypergeometric function, in proving the Bieberbach Conjecture, renewed interest was sparked among the mathematics community in special functions. Following this, many articles were presented in dealing with the geometric properties of different types of special functions including but not limited to generalized hypergeometric function, Gaussian, Kummer hypergeometric functions, Bessel functions, and, most recently, Struve functions [1–9]. Sufficient conditions on the parameters of these special functions were also determined by many authors for them to belong to a certain class of univalent functions [10–20].

Let $\mathcal{A}$ denote the class of analytic functions in the unit disk $U = \{z : |z| < 1\}$ of the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

and normalized by $f(0) = f'(0) - 1 = 0$. Denote $\mathcal{S}$ to be the subclass of $\mathcal{A}$ consisting of univalent functions. Subsequently, denote $\mathcal{S}^*$ and $\mathcal{C}$ to be subclasses of function $\mathcal{S}$ which are starlike and convex, respectively, in the unit disk U with the following definitions:

Definition 1. A set D in the plane is said to be starlike with respect to w_0 an interior point of D if each ray with initial point w_0 intersects the interior of D in a set that is either a line segment or a ray. If a function $f(z)$ maps U onto a domain that is starlike with respect to w_0 , then we say that $f(z)$ is starlike with respect to w_0 . In the special case that $w_0 = 0$, we say that $f(z)$ is a starlike function.

Definition 2. A set D in the plane is called convex if, for every pair of points w_1 and w_2 in the interior of D , the line segment joining w_1 and w_2 is also in the interior of D . If a function $f(z)$ maps U onto a convex domain, then $f(z)$ is called a convex function.

The analytical definition for the classes of starlike and convex functions are as follows, where $\Re$ denotes the real part of a complex function:

Proposition 1. Let $f \in \mathcal{S}$, then $f \in \mathcal{S}^*$ if and only if

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in U).$$

Proposition 2. Let $f \in \mathcal{A}$, then $f \in \mathcal{C}$ if and only if

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \quad (z \in U).$$

For more insights on these classes, refer to References [21,22].

According to the Alexander theorem [23], every function f is convex in the unit disk if and only if zf' is starlike in the unit disk, i.e., $f \in \mathcal{C} \Leftrightarrow zf' \in \mathcal{S}^*$.

Given any analytic functions, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, define the convolution function of f and g , denoted by $f * g$, as

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Trivially, $f * g$ is analytic and it is sometimes referred to as the Hadamard product of f and g in honour of J. Hadamard, where Hadamard used an alternative representation,

$$(f * g)(z) = \frac{1}{2\pi i} \int_{|\xi|=\rho} \frac{f\left(\frac{z}{\xi}\right)g(\xi)}{\xi} d\xi, \quad |z| < \xi < 1,$$

to illustrate the convolution.

Firstly, consider the following differential equation [24]:

$$z^2 w''(z) + zw'(z) + \left(z^2 - p^2\right)w(z) = \frac{4\left(\frac{z}{2}\right)^{p+1}}{\sqrt{\pi}\Gamma\left(p + \frac{1}{2}\right)}, \quad (2)$$

where $p \in \mathbb{R}$ and a particular solution for Equation (2) is

$$H_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma\left(n + \frac{3}{2}\right)\Gamma\left(p + n + \frac{3}{2}\right)} \left(\frac{z}{2}\right)^{2n+p+1} \quad (z \in \mathbb{C}).$$

The function $H_p(z)$ is known as the Struve function of order p . Next, consider the following differential equation which only differs in the coefficient of w :

$$z^2 w''(z) + zw'(z) - \left(z^2 + p^2\right)w(z) = \frac{4\left(\frac{z}{2}\right)^{p+1}}{\sqrt{\pi}\Gamma\left(p + \frac{1}{2}\right)}, \quad (3)$$

and a particular solution for Equation (3) is

$$\begin{aligned} L_p(z) &= -ie^{-\frac{ip\pi}{2}} H_p(iz) \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma\left(n + \frac{3}{2}\right)\Gamma\left(p + n + \frac{3}{2}\right)} \left(\frac{z}{2}\right)^{2n+p+1} \quad (z \in \mathbb{C}), \end{aligned}$$

where $L_p(z)$ is known as the modified Struve function of order p . Refer to Reference [25] for a more in-depth discussion on the Struve function. Now, consider the differential equation

$$z^2 w''(z) + bz w'(z) + [cz^2 - p^2 + (1-b)p] w(z) = \frac{4\left(\frac{z}{2}\right)^{p+1}}{\sqrt{\pi} \Gamma\left(p + \frac{b}{2}\right)}, \quad (4)$$

where $b, c, p \in \mathbb{C}$. Note that if $b = 1$ and $c = 1$, then Equation (4) reduces to Equation (2), and if $b = 1$ and $c = -1$, then Equation (4) reduces to Equation (3). A particular solution for Equation (4), denoted by $w_{p,b,c}(z)$, is

$$w_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{\Gamma(n + \frac{3}{2}) \Gamma(p + n + \frac{b+2}{2})} \left(\frac{z}{2}\right)^{2n+p+1} \quad (z \in \mathbb{C}).$$

Similarly to before, if $b = 1$ and $c = 1$, then $w_{p,1,1}(z) = H_p(z)$, and if $b = 1$ and $c = -1$, then $w_{p,1,-1}(z) = L_p(z)$. This generalization allows the study of $H_p(z)$ and $L_p(z)$ together. Thus, $w_{p,b,c}(z)$ is identified as the generalized Struve function of order p . Although the series representation of $w_{p,b,c}(z)$ is convergent everywhere in $\mathbb{C}$, the function is univalent generally in U [26]. Now, consider the function $u_{p,b,c}(z)$ defined as follows:

$$u_{p,b,c}(z) = 2^p \sqrt{\pi} \Gamma\left(p + \frac{b+2}{2}\right) z^{-\frac{p-1}{2}} w_{p,b,c}(\sqrt{z}).$$

Utilizing the Pochhammer symbol, $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} = \gamma(\gamma+1) \dots (\gamma+n-1)$, the following form of $u_{p,b,c}(z)$ can be written:

$$u_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{\left(-\frac{c}{4}\right)^n}{\left(\frac{3}{2}\right)_n (k)_n} z^n = b_0 + b_1 z + b_2 z^2 + \dots,$$

where $k = p + \frac{b+2}{2} \neq 0, -1, -2, \dots$ and

$$b_n = \frac{(-1)^n c^n \Gamma\left(\frac{3}{2}\right) \Gamma(k)}{4^n \Gamma\left(n + \frac{3}{2}\right) \Gamma(n+k)},$$

for $n \geq 0$. The function $u_{p,b,c}$ is analytic in $\mathbb{C}$, and satisfies the condition $u_{p,b,c}(0) = 1$, as well as the differential equation

$$4z^2 u''(z) + 2(2p + b + 3)zu'(z) + (cz + 2pb)u(z) = 2p + b.$$

For more discussion on generalized Struve function, refer to References [26–28].

The function $T_{p,b,c}(z)$ is a convolution of $f \in \mathcal{A}$ and $g_{p,b,c}(z) = zu_{p,b,c}(z)$, i.e.,

$$T_{p,b,c}(z) = (f * g_{p,b,c})(z) = z + \sum_{n=2}^{\infty} \frac{\left(-\frac{c}{4}\right)^{n-1}}{\left(\frac{3}{2}\right)_{n-1} (k)_{n-1}} a_n z^n \quad (z \in U), \quad (5)$$

where $p, b, c \in \mathbb{C}$ and $k = p + \frac{b+2}{2}$. In Reference [26], Orhan and Yagmur investigated the geometric properties for the function $g_{p,b,c}(z)$, and this prompts the motivation to seek similar properties for the function $T_{p,b,c}(z)$. The function $T_{p,b,c}$ was first introduced by Raza and Yagmur [29]. As such, this paper studies univalence, starlikeness, and convexity properties of the function $T_{p,b,c}$.

2. Preliminaries

The following preliminary results are needed to prove the results in the next section. These results can be found in References [30–33] respectively except for Lemma 3 which can be found in Reference [16].

Theorem 1. If $f \in \mathcal{S}$, then $|a_n| \leq n$ for $n \geq 2$.

Lemma 1. If $f \in \mathcal{A}$ satisfies the inequality

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < M \quad (z \in U),$$

where M is the solution of the equation $\cos(M) = M$, then $\Re\{f'(z)\} > 0$.

Lemma 2. If $f \in \mathcal{A}$ and $\Re\{f'(z)\} > 0$, then f is univalent.

Lemma 3. If $f \in \mathcal{A}$ and

$$\left| \frac{zf'(z)}{f(z)} - 1 \right|^{1-\beta} \left| \frac{zf''(z)}{f'(z)} \right|^\beta < (1-\alpha)^{1-2\beta} \left(1 - \frac{3\alpha}{2} + \alpha^2 \right)^\beta,$$

for some fixed $\alpha \in \left[0, \frac{1}{2}\right]$, $\beta \geq 0$ and for all $z \in U$, then $f \in \mathcal{S}^*(\alpha)$.

Proposition 3. Consider for $x \in \mathbb{R}$, the quartic function $P(x)$ of the form

$$P(x) = rx^4 + sx^3 + tx^2 + ux + v,$$

where $r, s, t, u, v \in \mathbb{R}$. Solutions of $P(x) = 0$ are given as follows:

$$\begin{aligned} x_{1,2} &= -\frac{s}{4r} - S \pm \frac{1}{2} \sqrt{-4S^2 - 2a + \frac{d}{S}}, \\ x_{3,4} &= -\frac{s}{4r} + S \pm \frac{1}{2} \sqrt{-4S^2 - 2a - \frac{d}{S}}, \end{aligned}$$

where a and d are the coefficients of the second and of the first degree, respectively, in the associated depressed quartic

$$\begin{aligned} a &= \frac{8rt - 3s^2}{8r^2}, \\ d &= \frac{s^3 - 4rst + 8r^2u}{8r^3}, \end{aligned}$$

and where

$$\begin{aligned} S &= \frac{1}{2} \sqrt{-\frac{2}{3}a + \frac{1}{3r} \left(Q + \frac{\Delta_0}{Q} \right)}, \\ Q &= \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}, \end{aligned}$$

with

$$\begin{aligned} \Delta_0 &= t^2 - 3su + 12rv, \\ \Delta_1 &= 2t^3 + 9stu + 27s^2v + 27r^2 - 72rtv. \end{aligned}$$

3. Results

Sufficient conditions for $T_{p,b,c}$ to be univalent, starlike, and convex are shown in the theorems below, respectively.

Theorem 2. Let $f \in \mathcal{A}$ and $T_{p,b,c}(z)$ be defined by Equation (5). If $p, b, c, \in \mathbb{C}$, $k_1 = p + \frac{b+2}{2}$ and

$$k_1 > \frac{7M+2+\beta+\sqrt{\frac{16M^3+(17\beta+100)M^2+(40\beta+80)M+12\beta+16-\beta^3}{\beta}}}{24M}|c| \\ \approx 1.098143352|c|,$$

where M is the solution the equation $\cos(M) = M$ and

$$\beta = \sqrt{\frac{1}{3}\left[\frac{49}{\alpha}M^3 + \left(\frac{28}{\alpha} + 17\right)M^2 + \left(\frac{4}{\alpha} + \alpha + 40\right)M + 12\right]} \\ \approx 3.469588501, \quad (6)$$

with

$$\alpha^3 = -343M^3 - 2886M^2 - 948M - 8 + 24\sqrt{3M}\sqrt{1029M^4 + 5113M^3 + 3138M^2 + 540M + 8} \\ \approx -28.2795216, \quad (7)$$

then $\Re\left\{T'_{p,b,c}(z)\right\} > 0$ for all $z \in U$.

Proof. Suppose $f \in \mathcal{A}$. Using $|z_1 + z_2| \leq |z_1| + |z_2|$, the inequalities $\left(\frac{3}{2}\right)_n \geq \frac{3}{2}n$ and $(k)_n \geq k^n$ ($n \in \mathbb{N}$) for $|z| < 1$,

$$\left|T'_{p,b,c}(z) - \frac{T_{p,b,c}(z)}{z}\right| = \left|\sum_{n=2}^{\infty} \frac{\left(-\frac{c}{4}\right)^{n-1}}{\left(\frac{3}{2}\right)_{n-1}(k)_{n-1}}(n-1)a_n z^{n-1}\right| \\ \leq \frac{2}{3} \sum_{n=2}^{\infty} n \left(\frac{|c|}{4k}\right)^{n-1} \quad (\text{as } |a_n| \leq n \text{ for } n \geq 2 \text{ by Theorem 1}) \\ = \frac{2}{3} \frac{|c|}{4k} \left[\frac{1}{1-\frac{|c|}{4k}} + \frac{1}{\left(1-\frac{|c|}{4k}\right)^2} \right] \\ = \frac{16k|c|-2|c|^2}{3(4k-|c|)^2}. \quad (8)$$

Obviously, the restriction on k is $k > \frac{|c|}{4}$. On the other hand, using $|z_1 - z_2| \geq ||z_1| - |z_2||$, the inequalities $\left(\frac{3}{2}\right)_n \geq \left(\frac{3}{2}\right)^n$ and $(k)_n \geq k^n$ ($n \in \mathbb{N}$) for $|z| < 1$,

$$\left|\frac{T_{p,b,c}(z)}{z}\right| = \left|1 + \sum_{n=2}^{\infty} \frac{\left(-\frac{c}{4}\right)^{n-1}}{\left(\frac{3}{2}\right)_{n-1}(k)_{n-1}}a_n z^{n-1}\right| \\ \geq 1 - \sum_{n=2}^{\infty} (n+1) \left(\frac{|c|}{6k}\right)^n \\ = 1 - \frac{|c|}{6k} \left[\frac{1}{1-\frac{|c|}{6k}} + \frac{1}{\left(1-\frac{|c|}{6k}\right)^2} \right] \\ = \frac{2(18k^2-12k|c|+|c|^2)}{(6k-|c|)^2}, \quad (9)$$

where the restriction on k is $k > \frac{|c|}{3} \left(1 + \frac{1}{\sqrt{2}}\right)$. Combining Equations (8) and (9) gives

$$\begin{aligned} \left| \frac{z T'_{p,b,c}(z)}{T_{p,b,c}(z)} - 1 \right| &= \left| T'_{p,b,c}(z) - \frac{T_{p,b,c}(z)}{z} \right| \left| \frac{z}{T_{p,b,c}(z)} \right| \\ &< \left(\frac{16k|c| - 2|c|^2}{3(4k - |c|)^2} \right) \left(\frac{(6k - |c|)^2}{2(18k^2 - 12k|c| + |c|^2)} \right) \\ &= \Psi(k, |c|). \end{aligned} \quad (10)$$

Next, to determine the values of k_1 such that it satisfies

$$\Psi(k_1, |c|) < M, \quad (11)$$

where M is the solution of the equation $\cos(M) = M$, the inequality can be written as

$$\begin{aligned} \Psi(k_1, |c|) &< M \\ \Rightarrow (8k_1|c| - |c|^2)(6k_1 - |c|)^2 &< 3M(4k_1 - |c|)^2(18k_1^2 - 12k_1|c| + |c|^2) \\ \Rightarrow 288k_1^3|c| - 132k_1^2|c|^2 + 20k_1|c|^3 + |c|^4 &< 3M(288k_1^4 - 336k_1^3|c| + 130k_1^2|c|^2 - 20k_1|c|^3 + |c|^4) \\ \Rightarrow F(k_1) = Mk_1^4 - \frac{7M+2}{6}k_1^3|c| + \frac{65M+22}{144}k_1^2|c|^2 - \frac{15M+5}{216}k_1|c|^3 + \frac{3M+1}{864}|c|^4 &> 0. \end{aligned} \quad (12)$$

Using Proposition 3 to find $F(x)$,

$$Mx^4 - \frac{7M+2}{6}x^3|c| + \frac{65M+22}{144}x^2|c|^2 - \frac{15M+5}{216}x|c|^3 + \frac{3M+1}{864}|c|^4 = 0.$$

Putting

$$\begin{aligned} a &= -\frac{17M^2 + 40M + 12}{288M^2}|c|^2, \\ d &= -\frac{4M^3 + 25M^2 + 20M + 4}{864M^3}|c|^4, \\ \Delta_0 &= \frac{49M^2 + 28M + 4}{20736}|c|^4, \\ \Delta_1 &= -\frac{343M^3 + 2886M^2 + 948M + 8}{1492992}|c|^6, \\ \Delta_1^2 - 4\Delta_0^3 &= \frac{1728M(1029M^4 + 5113M^3 + 3138M^2 + 540M + 8)}{1492992^2}|c|^{12}, \\ Q &= \frac{\alpha}{144}|c|^2 \text{ where } \alpha \text{ is (7),} \\ S &= \frac{\beta}{24M}|c| \text{ where } \beta \text{ is (6),} \end{aligned}$$

and since $\cos(M) = M$ gives $M \approx 0.7390851332$, the zeros of F are

$$\begin{cases} x_{1,2} \in \mathbb{C} & \text{as } -4S^2 - 2a + \frac{d}{S} < 0, \\ x_{3,4} \in \mathbb{R} & \text{as } -4S^2 - 2a - \frac{d}{S} > 0. \end{cases}$$

The real roots $x_{3,4}$ are given by

$$\begin{aligned} x_{3,4} &= -\frac{S}{4r} + S \pm \frac{1}{2} \sqrt{-4S - 2a - \frac{d}{S}} \\ &= \frac{7M+2+\beta \pm \sqrt{\frac{16M^3+(17\beta+100)M^2+(40\beta+80)M+12\beta+16-\beta^3}{\beta}}}{24M}|c|, \end{aligned}$$

which upon simplification, results in the following approximation:

$$\begin{aligned}x_3 &= \frac{7M+2+\beta-\sqrt{\frac{16M^3+(17\beta+100)M^2+(40\beta+80)M+12\beta+16-\beta^3}{\beta}}}{24M}|c| \\&\approx 0.1018970715|c|, \\x_4 &= \frac{7M+2+\beta+\sqrt{\frac{16M^3+(17\beta+100)M^2+(40\beta+80)M+12\beta+16-\beta^3}{\beta}}}{24M}|c| \\&\approx 1.09813352|c|.\end{aligned}$$

Thus, it can be concluded that $F(k_1) > 0$ for $k_1 < x_3$ or $k_1 > x_4$. Since $k > \frac{|c|}{3}\left(1 + \frac{1}{\sqrt{2}}\right)$, then

$$k > x_4 = \frac{7M+2+\beta+\sqrt{\frac{16M^3+(17\beta+100)M^2+(40\beta+80)M+12\beta+16-\beta^3}{\beta}}}{24M}|c|,$$

is the range of values of k_1 that satisfies Equation (11). As $T_{p,b,c}$ is an analytic function, then, by Lemma 1, $\Re\{T_{p,b,c}(z)\} > 0$ for all $z \in U$. $\square$

Remark 1. Obviously from Lemma 2, with c and k satisfying the constraints given in Theorem 2, the function $T_{p,b,c}$ is univalent in U .

Theorem 3. Let $f \in \mathcal{A}$ and $T_{p,b,c}(z)$ be defined by Equation (5). If $p, b, c, \in \mathbb{C}$, $k_2 = p + \frac{b+2}{2}$ and

$$k_2 > \frac{9 + \theta + \sqrt{\frac{-\theta^3 + 69\theta + 212}{\theta}}}{24}|c| \approx 0.970886809|c|,$$

where

$$\theta = \sqrt{23 + \sqrt[3]{-155 + 16\sqrt{91}} + \frac{9}{\sqrt[3]{-155 + 16\sqrt{91}}}} \approx 3.862149964, \quad (13)$$

then $T_{p,b,c}$ is starlike in U .

Proof. Suppose $f \in \mathcal{A}$. Similar to the previous result, the aim is to seek constraints on k_2 such that $T_{p,b,c}$ is starlike in U . Hence, replacing k_1 with k_2 and M with 1 in Equation (11) gives the following:

$$\begin{aligned}\Psi(k_2, |c|) &< 1 \\&\Rightarrow 4|c|^2 - 80k_2|c|^3 + 522k_2^2|c|^2 - 1296k_2^3|c| + 864k_2^4 > 0 \\&\Rightarrow G(k_2) = k_2^4 - \frac{3}{2}k_2^3|c| + \frac{29}{48}k_2^2|c|^2 - \frac{5}{54}k_2|c|^3 + \frac{1}{216}|c|^4 > 0.\end{aligned} \quad (14)$$

Once again, using Proposition 3 to find the zeros of $G(y)$,

$$y^4 - \frac{3}{2}y^3|c| + \frac{29}{48}y^2|c|^2 - \frac{5}{54}y|c|^3 + \frac{1}{216}|c|^4 = 0.$$

Putting

$$\begin{aligned}a &= -\frac{23}{96}|c|^2, \\d &= -\frac{56}{864}|c|^4, \\ \Delta_0 &= \frac{1}{256}|c|^4, \\ \Delta_1 &= -\frac{155}{55296}|c|^6, \\ \Delta_1^2 - 4\Delta_0^3 &= \frac{91}{11943936}|c|^{12}, \\ Q &= \sqrt[3]{\frac{-155 + 16\sqrt{91}}{48}}|c|^2, \\ S &= \frac{\theta}{24}|c| \text{ where } \theta \text{ is (13),}\end{aligned}$$

the zeros of G are

$$\begin{cases} y_{1,2} \in \mathbb{C} & \text{as } -4S^2 - 2a + \frac{d}{S} < 0, \\ y_{3,4} \in \mathbb{R} & \text{as } -4S^2 - 2a - \frac{d}{S} > 0. \end{cases}$$

The real roots $y_{3,4}$ are given by

$$y_{3,4} = -\frac{s}{4r} + S \pm \frac{1}{2}\sqrt{-4S - 2a - \frac{d}{S}} = \frac{9 + \theta \pm \sqrt{\frac{-\theta^3 + 69\theta + 212}{\theta}}}{24}|c|,$$

which, upon simplification, results in the following approximation:

$$\begin{aligned}y_3 &= \frac{9 + \theta - \sqrt{\frac{-\theta^3 + 69\theta + 212}{\theta}}}{24}|c| \approx 0.00958649|c|, \\ y_4 &= \frac{9 + \theta + \sqrt{\frac{-\theta^3 + 69\theta + 212}{\theta}}}{24}|c| \approx 0.970886809|c|.\end{aligned}$$

Thus, it can be concluded that $G(k_2) > 0$ for $k_2 < y_3$ or $k_2 > y_4$. Since $k > \frac{|c|}{3}\left(1 + \frac{1}{\sqrt{2}}\right)$, then

$$k_2 > y_4 = \frac{9 + \theta + \sqrt{\frac{-\theta^3 + 69\theta + 212}{\theta}}}{24}|c|,$$

is the range of values of k_2 that satisfies Equation (14) which, in turn, implies that $T_{p,b,c}$ is starlike in U by Lemma 3 when $\alpha = \beta = 0$. $\square$

Theorem 4. Let $f \in \mathcal{A}$ and $T_{p,b,c}(z)$ defined by Equation (5). If $p, b, c \in \mathbb{C}$, $k_3 = p + \frac{b+2}{2}$ and $k_3 > \frac{13+\sqrt{130}}{12}|c|$, then $T_{p,b,c}(z)$ is convex in U .

Proof. Suppose $f \in \mathcal{A}$. Using $|z_1 + z_2| \leq |z_1| + |z_2|$, the inequalities $(\frac{3}{2})_n > \frac{n(n+1)}{2}$ and $(k)_n \geq k^n$ ($n \in \mathbb{N}$) for $|z| < 1$,

$$\begin{aligned} |zT''_{p,b,c}(z)| &= \left| \sum_{n=2}^{\infty} \frac{(-\frac{c}{4})^{n-1}}{(\frac{3}{2})_{n-1}(k)_{n-1}} n(n-1)a_n z^{n-1} \right| \\ &\leq 2 \sum_{n=2}^{\infty} n \left(\frac{|c|}{4k} \right)^{n-1} \quad (\text{as } |a_n| \leq n \text{ for } n \geq 2 \text{ by Theorem 1}) \\ &= 2 \left(\frac{|c|}{4k} \right) \left[\frac{1}{1-\frac{|c|}{4k}} + \frac{1}{(1-\frac{|c|}{4k})^2} \right] \\ &= \frac{16k|c|-2|c|^2}{(4k-|c|)^2}. \end{aligned} \quad (15)$$

Obviously, the restriction on k is $k > \frac{|c|}{4}$. Using $|z_1 - z_2| \geq ||z_1| - |z_2||$, the inequalities $(\frac{3}{2})_n > \frac{3(n+1)}{4}$ and $(k)_n \geq k^n$ ($n \in \mathbb{N}$) for $|z| < 1$,

$$\begin{aligned} |T'_{p,b,c}(z)| &= \left| 1 + \sum_{n=2}^{\infty} \frac{(-\frac{c}{4})^{n-1}}{(\frac{3}{2})_n(k)_{n-1}} na_n z^{n-1} \right| \\ &\geq 1 - \frac{4}{3} \sum_{n=2}^{\infty} n \left(\frac{|c|}{4k} \right)^{n-1} \\ &= 1 - \frac{4}{3} \left(\frac{|c|}{4k} \right) \left[\frac{1}{1-\frac{|c|}{4k}} + \frac{1}{(1-\frac{|c|}{4k})^2} \right] \\ &= \frac{48k^2 - 56k|c| + 7|c|^2}{3(4k-|c|)^2}, \end{aligned} \quad (16)$$

where the restriction on k is $k > \frac{7+2\sqrt{7}}{12}|c|$. Comparing Equations (16) and (17) gives the following:

$$\left| \frac{zT''_{p,b,c}(z)}{T'_{p,b,c}(z)} \right| = \left| zT''_{p,b,c}(z) \right| \left| \frac{1}{T'_{p,b,c}(z)} \right| < \frac{48k|c| - 6|c|^2}{48k^2 - 56k|c| + 7|c|^2}.$$

The next step is to determine the values of k_3 such that it satisfies the following:

$$\frac{48k|c| - 6|c|^2}{48k^2 - 56k|c| + 7|c|^2} < 1. \quad (17)$$

From Equation (17),

$$\begin{aligned} \frac{48k_3|c| - 6|c|^2}{48k_3^2 - 56k_3|c| + 7|c|^2} &< 1 \\ \Rightarrow 48k_3|c| - 6|c|^2 &< 48k_3^2 - 56k_3|c| + 7|c|^2 \\ \Rightarrow 48k_3^2 - 104k_3|c| + 13|c|^2 &> 0 \\ \Rightarrow k_3^2 - \frac{13}{6}k_3|c| + \frac{13}{48}|c|^2 &> 0 \\ \Rightarrow k_3 < \frac{13-\sqrt{130}}{12}|c| \approx 0.133187|c| \text{ OR } k_3 > \frac{13+\sqrt{130}}{12}|c| \approx 2.03345|c|. \end{aligned}$$

Since $k > \frac{7+2\sqrt{7}}{12}|c|$, then $k_3 > \frac{13+\sqrt{130}}{12}|c|$ is the range of values of k_3 such that it satisfies Equation (17). Since the range of values of k_3 satisfies the condition in Theorem 3, then $zT'_{p,b,c}$ is starlike in U which, in turn, implies that $T_{p,b,c}$ is convex in U by the Alexander theorem. $\square$

4. Conclusions

In summary, the bounds on k for function $T_{p,b,c}$ to be univalent, starlike, and convex were obtained. Specifically, for $k = k_1 > 1.098143352|c|$, $T_{p,b,c}$ is univalent; for $k = k_2 > 0.970886809|c|$, $T_{p,b,c}$ is starlike; and for $k = k_3 > 2.03345|c|$, $T_{p,b,c}$ is convex. The bounds obtained for convexity and starlikeness of the

function $T_{p,b,c}$ are in agreement since $k_2 < k_3$ and all convex functions are starlike. Secondly, the same goes for the function to be univalent and convex since $k_1 < k_3$ and all convex functions are univalent for $z \in U$. However, since the bounds obtained in k_1 and k_2 are not necessarily sharp, the relationship between k_1 and k_2 does not imply the outcome that all starlike functions are univalent in the unit disk. This is probably due to the fact that the approach in establishing Theorem 3 used Lemma 3 (a necessary condition) as opposed to using Proposition 1 (necessary and sufficient). This allows for further research to explore better methods, such as using Proposition 1, to achieve better bounds. Nevertheless, the results attained in this paper are in adherence with the results in Reference [26]. Conducting further research in the future will hopefully produce a sharper result.

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