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# Best Proximity Point Results in $b$-Metric Space and Application to Nonlinear Fractional Differential Equation

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**Abstract:** Based on the concepts of contractive conditions due to Suzuki (Suzuki, T., A generalized Banach contraction principle that characterizes metric completeness, Proceedings of the American Mathematical Society, 2008, 136, 1861–1869) and Jleli (Jleli, M., Samet, B., A new generalization of the Banach contraction principle, J. Inequal. Appl., 2014, 2014, 38), our aim is to combine the aforementioned concepts in more general way for set valued and single valued mappings and to prove the existence of best proximity point results in the context of $b$-metric spaces. Endowing the concept of graph with $b$-metric space, we present some best proximity point results. Some concrete examples are presented to illustrate the obtained results. Moreover, we prove the existence of the solution of nonlinear fractional differential equation involving Caputo derivative. Presented results not only unify but also generalize several existing results on the topic in the corresponding literature.

**Keywords:** Suzuki-contraction; $b$-metric space; Caputo derivative

## 1. Introduction and Preliminaries

Metric fixed point theory progressed a lot after the classical result due to Banach [1], known as the Banach contraction principle and it states that "Every contractive self mapping on a complete metric space has a unique fixed point". Due to its importance, several researchers have obtained many interesting generalizations of Banach's principle (see [2–10] and the references therein). Later on, Nadler [11] extended the Banach contraction principle to the context of set valued contraction.

**Theorem 1.** *[11] Every multivalued mapping $T : X \to CB(X)$, where $(X, d)$ a complete metric space, satisfying*

$$H(Tx, Ty) \leq kd(x, y)$$

*for all $x, y \in X$, where $k \in [0, 1)$ has at least one fixed point.*

In 2009, Suzuki [12] proved the following result in compact metric spaces.

**Theorem 2.** *[12] Let $(X, d)$ be a compact metric space and $T : X \to X$ be a mapping. Assume that, for all $x, y \in X$ with $x \neq y$,*

$$\frac{1}{2}d(x, Tx) < d(x, y) \Rightarrow d(Tx, Ty) < d(x, y),$$

*then $T$ has a unique fixed point in $X$.*

Recently, Jleli et al. [13] introduced the class $\Theta$ of all functions $\theta : (0,\infty) \to (1,\infty)$ satisfying the following conditions:

($\theta_1$) $\theta$ is non-decreasing;
($\theta_2$) for each sequence $\{t_n\} \subseteq (0,\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;
($\theta_3$) there exists $r \in (0,1)$ and $l \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$,

and proved the following result:

**Theorem 3.** *[13] Let $(X,d)$ be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exist $\theta \in \Theta$ and $k \in (0,1)$ such that*

$$x, y \in X,\ d(Tx,Ty) \neq 0 \ \Rightarrow \ \theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k.$$

*then $T$ has a unique fixed point.*

Observe that Banach contraction is a $\theta$-contraction for $\theta(t) = e^t$. So Theorem 3 is a generalization of the Banach contraction principle [1].

Liu et al. [14] proved some fixed point results for $\theta$-type contraction and $\theta$-type Suzuki contraction in complete metric spaces. Hancer et al. [15] introduced the notion of multi-valued $\theta$-contraction mapping as follows:

Let $(X,d)$ be a metric space and $T : X \to CB(X)$ a multivalued mapping. Then $T$ is said to be multi-valued $\theta$-contraction if there exists $\theta \in \Theta$ and $0 < k < 1$ such that

$$\theta(H(Tx,Ty)) \leq [\theta(d(x,y))]^k \tag{1}$$

for any $x, y \in X$ provided that $H(Tx,Ty) > 0$, where $CB(X)$ is a collection of all nonempty closed and bounded subsets of $X$.

Bakhtin [2] initiated the study of a generalized metric space named as $b$-metric space and presented a version of Banach contraction principle [1] in the context of $b$-metric spaces. Subsequently, several researchers studied fixed point theory for single-valued and set-valued mappings in $b$-metric spaces (see [2,3,5,6,16–18] and references therein).

**Definition 1.** *[2] Let $X$ be a nonempty set, and let $k \geq 1$ be a given real number. A functional $d_b : X \times X \to [0,\infty)$ is said to be a $b$-metric if for all $x, y, z \in X$, following conditions are satisfied:*

1. $d_b(x,y) = 0 \Leftrightarrow x = y$;
2. $d_b(x,y) = d_b(y,x)$;
3. $d_b(x,y) \leq k(d_b(x,z) + d_b(z,y))$.

*The pair $(X, d_b)$ is called $b$-metric space.*

**Example 1.** *[3] The space $L_p (0 < p < 1)$ for all real function $x(t), t \in [0,1]$ such that $\int_0^1 |x(t)|^p dt < \infty$, is $b$-metric space if we take*

$$d_b(x,y) = \left( \int_0^1 |x(t) - y(t)|^p dt \right)^{\frac{1}{p}}.$$

On the other hand, let $A$ and $B$ be two nonempty subsets of a metric space $(X,d)$ and $T : A \to CB(B)$. A point $x^* \in A$ is called a best proximity point of $T$ if

$$D(x^*, Tx^*) \ = \ \inf\{d(x^*,y) : y \in Tx^*\} = \mathrm{dist}(A,B),$$

where
$$\operatorname{dist}(A,B) = \inf\{d(a,b) : a \in A, b \in B\}.$$

If $A \cap B \neq \phi$, then $x^*$ is a fixed point of $T$. If $A \cap B = \phi$, then $D(x,Tx) > 0$ for all $x \in A$ and $T$ has no fixed point.

Consider the following optimization problem:
$$\min\{D(x,Tx) : x \in A\}. \tag{2}$$

It is then important to study necessary conditions so that the above minimization problem has at least one solution.

Since
$$d(A,B) \leq D(x,Tx) \tag{3}$$
for all $x \in A$. Hence the optimal solution to the problem
$$\min\{D(x,Tx) : x \in A\} \tag{4}$$
for which the value $d(A,B)$ is attained is indeed a best proximity point of multivalued mapping $T$.

In the sequel, we denote $(X, d_b)$ a $b$-metric space, $C(X)$, CB(X) and $K(X)$ by the families of all nonempty closed subsets, closed and bounded subsets and compact subsets of $(X, d_b)$. For any $A, B \in C(X)$ and $x \in X$, define
$$\begin{aligned} A_0 &= \{a \in A : \text{there exists some } b \in B \text{ such that } d_b(a,b) = D(A,B)\} \\ B_0 &= \{b \in B : \text{there exists some } a \in A \text{ such that } d_b(a,b) = D(A,B)\} \\ \delta(A,B) &= \sup\{D(a,B) : a \in A\} \\ H(A,B) &= \max\{\delta(A,B), \delta(B,A)\}. \end{aligned}$$

The function $H$ is called the Pompeiu-Hausdorff $b$-metric.

**Definition 2.** *[19] Let $(A,B)$ be a pair of nonempty subsets of a b-metric space $(X, d_b)$ with $A_0 \neq \emptyset$. Then the pair $(A,B)$ is said to have the weak P-property if and only if for any $x_1, x_2 \in A$ and $y_1, y_2 \in B$,*
$$\left.\begin{aligned} d_b(x_1,y_1) &= D(A,B) \\ d_b(x_2,y_2) &= D(A,B) \end{aligned}\right\} \quad \text{implies} \quad d_b(x_1,x_2) \leq d_b(y_1,y_2).$$

**Definition 3.** *[20] Let $T : A \to B$ and $\alpha : A \times A \to [0,\infty)$. We say that $T$ is $\alpha$-proximal admissible if*
$$\left.\begin{aligned} \alpha(x_1,x_2) &\geq 1 \\ d(u_1,Tx_1) &= D(A,B) \\ d(u_2,Tx_2) &= D(A,B) \end{aligned}\right\} \quad \text{implies} \quad \alpha(u_1,u_2) \geq 1,$$
*for all $x_1, x_2, u_1, u_2 \in A$.*

The aim of this paper is to define multivalued Suzuki type ($\alpha$, $\theta$)-contraction and prove the existence of best proximity point results in the setting of $b$-metric spaces. Moreover, we obtain best proximity point results in $b$-metric spaces endowed with a graph through our main results. Examples are given to prove the validity of our results. Moreover, we show the existence of solution of nonlinear fractional differential equation.

## 2. Existence Results for Multivalued Mappings

We first define the notions of continuity of non-self multivalued mapping and continuity of the underlying $b$-metric.

**Definition 4.** *Let $(X, d_b)$ be a b-metric space and $A, B$ be two nonempty subsets of $X$. A function $T : A \to CB(B)$ is called continuous if for all sequences $x_n$ and $y_n$ of elements from $A$ and $B$ respectively and $x \in A$, $y \in B$ such that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ and $y_{n+1} \in T(x_n)$ for every $n \in \mathbb{N}$, we have $y \in T(x)$.*

**Definition 5.** *Let $(X, d_b)$ be a b-metric space. The b-metric $d_b$ is called sequentially continuous if for every $A, B \in CB(B)$, every $x \in A$, $y \in B$ and every sequence $x_n$ in $A$, $y_n$ in $B$ such that $x_n \to x$, $y_n \to y$, we have $d_b(x_n, y_n) \to d_b(x, y)$.*

**Definition 6.** *Let $(X, d_b)$ be a b-metric space with constant $k \geq 1$, $A$ and $B$ be nonempty subsets of $X$. A mapping $T : A \to CB(B)$ is called multivalued (MV) Suzuki type $(\alpha, \theta)$-contraction if there exist a function $\alpha : A \times A \to [0, \infty)$, $\theta \in \Theta$ and $s \in (0, 1)$ such that*

$$\frac{1}{k}D(x, Tx) - D(A, B) \leq \alpha(x, y)d_b(x, y)$$

*implies that*

$$\theta(H(Tx, Ty)) \leq [\theta(M(x, y))]^s, \tag{5}$$

*where $M(x, y) = \max\{d_b(x, y), D(x, Tx), D(y, Ty)\}$ for all $x, y \in A$.*

**Example 2.** *Let $X = \mathbb{R}$ with a b-metric $d_b = |x - y|^2$ for all $x, y \in X$. Let $A = [2, 3]$ and $B = [0, 1]$, then $D(A, B) = 1$, define $T : A \to CB(B)$ by*

$$Tx = \begin{cases} [0, \frac{x}{4}] & \text{if } x \in (2, 3) \\ \{\frac{x}{10}\} & \text{if } x \in \{2, 3\}, \end{cases}$$

*$\alpha : A \times A \to [0, \infty)$ by*

$$\alpha(x, y) = 1 \ \text{ if } x,y \in [2, 3]$$

*and $\theta : (0, \infty) \to (1, \infty)$ by*

$$\theta(t) = e^{\sqrt{te^t}}$$

*for all $t > 0$. It is easy to see that $\theta \in \Theta$. Now for all $x, y \in A$*

$$\frac{1}{k}D(x, Tx) - D(A, B) = \alpha(x, y)d_b(x, y)$$

*and*

$$\begin{aligned} \theta(H(Tx, Ty)) &= \theta\left(\frac{|x-y|^2}{16}\right) \\ &= e^{\sqrt{\frac{|x-y|^2}{16}e^{\frac{|x-y|^2}{16}}}} \\ &\leq e^{\frac{1}{2}\sqrt{|x-y|^2 e^{|x-y|^2}}} \\ &= e^{\frac{1}{2}\sqrt{d_b(x,y)e^{d_b(x,y)}}} \\ &\leq e^{\frac{1}{2}\sqrt{M(x,y)e^{M(x,y)}}} \\ &= [\theta(M(x, y))]^{\frac{1}{2}}. \end{aligned}$$

*Hence, $T$ is MV Suzuki type $(\alpha, \theta)$-contraction.*

**Theorem 4.** *Let $A$ and $B$ be two nonempty closed subsets of a complete b-metric space $(X, d_b)$ such that $A_0$ is nonempty. Let $T : A \to K(B)$ be a MV Suzuki type $(\alpha, \theta)$-contraction. Suppose that the following conditions hold:*

*(i) for each $x \in A_0$, we have $Tx \subseteq B_0$ and the pair $(A, B)$ satisfies weak P-property;*
*(ii) there exist $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that*

$$d_b(x_1, y_1) = D(A, B) \ \text{and} \ \alpha(x_0, x_1) \geq 1;$$

*(iii) $T$ is $\alpha$-proximal admissible;*
*(iv) $d_b$ is sequentially continuous and $T$ is continuous.*

*Then $T$ has a best proximity point.*

**Proof.** By hypothesis $(ii)$, there exist $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that

$$d_b(x_1, y_1) = D(A, B) \ \text{and} \ \alpha(x_0, x_1) \geq 1. \tag{6}$$

If $y_1 \in Tx_1$, then we obtain

$$D(A, B) \leq D(x_1, Tx_1) \leq d_b(x_1, y_1) = D(A, B),$$

so $x_1$ is best proximity point of $T$ and the proof is complete.

Next, we suppose that $y_1 \notin Tx_1$. Since $y_1 \in Tx_0$, we have

$$D(x_0, Tx_0) \leq d_b(x_0, y_1) \leq k[d_b(x_0, x_1) + d(x_1, y_1)]. \tag{7}$$

Using (6) in (7), we have

$$\frac{1}{k} D(x_0, Tx_0) - D(A, B) \leq \alpha(x_0, x_1) d_b(x_0, x_1).$$

From (5), it follows that

$$\theta(H(Tx_0, Tx_1)) \leq [\theta(M(x_0, x_1))]^s, \tag{8}$$

where

$$M(x_0, x_1) = \max\{d_b(x_0, x_1), D(x_0, Tx_0), D(x_1, Tx_1)\}.$$

Since $Tx_0$ is compact, so we have

$$\begin{aligned} M(x_0, x_1) &= \max\{d_b(x_0, x_1), d_b(x_0, x_1), D(x_1, Tx_1)\} \\ &= \max\{d_b(x_0, x_1), D(x_1, Tx_1)\}. \end{aligned}$$

Suppose that $M(x_0, x_1) = D(x_1, Tx_1)$, then

$$\begin{aligned} 1 &< \theta(D(x_1, Tx_1)) \\ &\leq \theta(H(Tx_0, Tx_1)) \\ &\leq [\theta(M(x_0, x_1))]^s \\ &= [\theta(D(x_1, Tx_1)]^s, \end{aligned}$$

a contradiction. Therefore,

$$\theta(H(Tx_0, Tx_1)) \leq [\theta(d_b(x_0, x_1))]^s. \tag{9}$$

On the other hand, since $0 < D(y_1, Tx_1) \leq H(Tx_0, Tx_1)$ and from $(\theta_1)$, we obtain that

$$\theta(D(y_1, Tx_1) \quad \leq \quad \theta(H(Tx_0, Tx_1)) \leq [\theta(d_b(x_0, x_1))]^s$$

implies

$$\theta(D(y_1, Tx_1) \quad \leq \quad [\theta(d_b(x_0, x_1))]^s. \tag{10}$$

Since $Tx_1$ is compact, there exists $y_2 \in Tx_1$ such that $D(y_1, Tx_1) = d_b(y_1, y_2)$ and so

$$\theta(d_b(y_1, y_2)) \leq [\theta[d_b(x_0, x_1))]^s. \tag{11}$$

By hypothesis $(i)$, we have $Tx_1 \subseteq B_0$ and so there exists $x_2 \in A_0$ such that

$$d_b(x_2, y_2) = D(A, B). \tag{12}$$

Since $T$ is $\alpha$-proximal admissible, from (6) and (12), it follows that

$$\alpha(x_1, x_2) \geq 1. \tag{13}$$

Since $(A, B)$ satisfies weak P-property, we have

$$d_b(x_1, x_2) \leq d_b(y_1, y_2). \tag{14}$$

If $x_1 = x_2$, then $x_1$ is best proximity point of $T$ and proof is complete. From (11), (14) and $(\theta_1)$, we have

$$\theta(d_b(x_1, x_2)) \leq \theta(d_b(y_1, y_2)) \leq [\theta[d_b(x_0, x_1))]^s. \tag{15}$$

If $y_2 \in Tx_2$, then $x_2$ is best proximity point of $T$. Now suppose that $y_2 \notin Tx_2$, since $y_2 \in Tx_1$, then by similar arguments given above we have. Since $y_1 \in Tx_0$, we have

$$\theta(d_b(x_2, x_3)) \leq \theta(d_b(y_2, y_3)) \leq [\theta[d_b(x_1, x_2))]^s. \tag{16}$$

Thus, by induction, we can find two sequences $\{x_n\} \subseteq A_0$ and $\{y_n\} \subseteq B_0$ such that

(a) $\alpha(x_n, x_{n+1}) \geq 1$ with $x_n \neq x_{n+1}$;
(b) $y_n \in Tx_{n-1}$ and $y_n \notin Tx_n$;
(c) $d_b(x_n, y_n) = D(A, B)$ and

$$\theta(d_b(x_n, x_{n+1})) \leq \theta(d_b(y_n, y_{n+1})) \leq (\theta[d_b(x_{n-1}, x_n)])^s. \tag{17}$$

Now,

$$\begin{aligned} 1 < \theta(d_b(x_n, x_{n+1})) \quad &\leq \quad [\theta(d_b(x_{n-1}, x_n))]^s \leq [\theta(d_b(x_{n-2}, x_{n-1}))]^{s^2} \\ &\quad\;\; . \\ &\quad\;\; . \\ &\quad\;\; . \\ &\leq \quad [\theta[d_b(x_0, x_1))]^{s^n}, \end{aligned} \tag{18}$$

for all $n \in \mathbb{N} \cup \{0\}$. This shows that $\lim_{n\to\infty} \theta(d_b(x_n, x_{n+1})) = 1$ and $(\theta_2)$ gives

$$\lim_{n\to\infty} d_b(x_n, x_{n+1}) = 0. \tag{19}$$

As consequence, there exist $r \in [0,1)$ and $l \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(d_b(x_n,x_{n+1}))-1}{d_b(x_n,x_{n+1})^r} = l.$$

We distinguish two cases.

**Case-I:** If $0 < l < \infty$.

By definition of the limit, there exists some natural number $n_0$ such that

$$\lim_{n\to\infty} \frac{\theta(d_b(x_n,x_{n+1}))-1}{d_b(x_n,x_{n+1})^r} \geq \frac{l}{2} \text{ for all } n \geq n_0,$$

which yields

$$n[d_b(x_n,x_{n+1})]^r \leq \frac{2}{l} n[\theta(d_b(x_n,x_{n+1}))-1] \text{ for all } n \geq n_0.$$

**Case-II:** If $l = \infty$.

Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists some natural number $n_0$ such that

$$\lim_{n\to\infty} \frac{\theta(d_b(x_n,x_{n+1}))-1}{d_b(x_n,x_{n+1})^r} \geq B \text{ for all } n \geq n_0,$$

which yields

$$n[d_b(x_n,x_{n+1})]^r \leq \frac{1}{B} n[\theta(d_b(x_n,x_{n+1}))-1] \text{ for all } n \geq n_0.$$

As consequence, in all cases, there exist $A > 0$ and natural number $n_0$ such that

$$n[d_b(x_n,x_{n+1})]^r \leq An[\theta(d_b(x_n,x_{n+1}))-1] \text{ for all } n \geq n_0.$$

Using (18), we obtain

$$n[d_b(x_n,x_{n+1})]^r \leq An([\theta(d_b(x_0,x_1))]^{s^n}-1) \text{ for all } n \geq n_0.$$

Taking $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} n[d_b(x_n,x_{n+1})]^r = 0. \tag{20}$$

It follows from (20) that there exists $n_1 \in \mathbb{N}$ such that

$$n[d_b(x_n,x_{n+1})]^r \leq 1 \text{ for all } n > n_1.$$

This implies that

$$d_b(x_n,x_{n+1}) \leq \frac{1}{n^{\frac{1}{r}}} \text{ for all } n \geq n_1. \tag{21}$$

Now, for all $m = 1, 2, ..., n = n_1, n_1 + 1, ...$ and using (21), we have

$$\begin{aligned} d_b(x_n, x_{n+m}) &\leq kd_b(x_n, x_{n+1}) + kd_b(x_{n+1}, x_{n+m}) \\ &\leq kd_b(x_n, x_{n+1}) + k^2 d_b(x_{n+1}, x_{n+2}) + k^2 d_b(x_{n+2}, x_{n+m}) \\ & . \\ & . \\ & . \\ &\leq kd_b(x_n, x_{n+1}) + k^2 d_b(x_{n+1}, x_{n+2}) + ... + k^{m-2} d_b(x_{n+m-3}, x_{n+m-2}) \\ &\quad + k^{m-1} d_b(x_{n+m-2}, x_{n+m-1}) + k^m d_b(x_{n+m-1}, x_{m+s}) \\ &= \frac{1}{k^n}[k^{n+1} d_b(x_n, x_{n+1}) + k^{n+2} d_b(x_{n+1}, x_{n+2}) + ... \\ &\quad + k^{n+m-1} d_b(x_{n+m-2}, x_{n+m-1}) + k^{n+m} d_b(x_{n+m-1}, x_{m+s})] \\ &= \frac{1}{k^n} \sum_{i=n+1}^{n+m} k^i . d_b(x_i, x_{i+1}) \\ &< \frac{1}{k^n} \sum_{i=n+1}^{\infty} k^i . d_b(x_i, x_{i+1}) \leq \frac{1}{k^n} \sum_{i=n+1}^{\infty} \frac{k^i}{i^{\frac{1}{r}}}. \end{aligned}$$

Since $0 < r < 1$, $\sum\limits_{i=n+1}^{\infty} \frac{1}{i^{\frac{1}{r}}}$ converges. Therefore

$$\frac{1}{k^n} \sum_{i=n+1}^{\infty} \frac{k^i}{i^{\frac{1}{r}}} \to 0, \text{ as } n \to \infty,$$

which yields that $\{x_n\}$ is a Cauchy sequence in complete $b$-metric space $(X, d_b)$. From (17), it follows that

$$d_b(y_{n+1}, y_n) \leq d_b(x_{n-1}, x_n). \tag{22}$$

Similarly, we can show that $\{y_n\}$ is a Cauchy sequence in $B$. Since $A$ and $B$ are closed subsets of a complete $b$-metric space $(X, d_b)$, there exist $x^* \in A$ and $y^* \in B$ such that $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$, respectively. Since $d_b(x_n, y_n) \to D(A, B)$ for all $n \in \mathbb{N}$ and $d_b$ is sequentially continuous, we conclude that

$$\lim_{n \to \infty} d_b(x_n, y_n) = d_b(x^*, y^*) = D(A, B).$$

Since $T$ is continuous, we have $y^* \in Tx^*$. Furthermore,

$$D(A, B) \quad \leq \quad D(x^*, Tx^*) \leq d_b(x^*, y^*) = D(A, B)$$

implies

$$D(x^*, Tx^*) = D(A, B).$$

Therefore, $x^*$ is a best proximity point of $T$. This completes the proof. □

**Example 3.** *Let $X = [0, \infty) \times [0, \infty)$ be endowed with b-metric*

$$d_b(x, y) = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

*for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$ and $k = 2$. Let $A = \{\frac{1}{5}\} \times [0, \infty)$ and $B = \{0\} \times [0, \infty)$. Define $T : A \to K(B)$ by*

$$T\left(\frac{1}{5}, a\right) = \begin{cases} \{(0, \frac{x}{10}) \ : 0 \leq x \leq a\} & \text{if } a \leq 1, \\ \{(0, x^2) \ : 0 \leq x \leq a^2\} & \text{if } a > 1, \end{cases}$$

*and a function* $\alpha : A \times A \to [0, \infty)$ *as follows:*

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in \{(\frac{1}{5}, a)\ : 0 \leq a \leq 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

*Take* $\theta(t) = t + 1$ *for all* $t > 0$.

*Note that* $A_0 = A$, $B_0 = B$, $D(A,B) = \frac{1}{25}$ *and* $Tx \subseteq B_0$ *for all* $x \in A_0$ *and the pair* $(A, B)$ *satisfies weak P-property. Let* $x_0, x_1 \in \{(\frac{1}{5}, 0) : 0 \leq x \leq 1\}$. *Then we have*

$$Tx_0, Tx_1 \subseteq \left\{ \left(0, \frac{x}{10}\right) : 0 \leq x \leq 1 \right\}.$$

*Consider* $y_1 \in Tx_0$, $y_2 \in Tx_1$ *and* $u_1, u_2 \in A$ *such that* $d_b(u_1, y_1) = D(A, B)$, $d_b(u_2, y_2) = D(A, B)$. *Then we have* $u_1, u_2 \in \{(\frac{1}{5}, x) : 0 \leq x \leq \frac{1}{10}\}$. *Hence* $\alpha(u_1, u_2) = 1$ *implies that* $T$ *is an* $\alpha$*-proximal admissible.*

*For* $x_0 = (\frac{1}{5}, 1)$ *and* $y_1 = (0, \frac{1}{10}) \in Tx_0 \subseteq B_0$, *we have* $x_1 = (\frac{1}{5}, \frac{1}{10}) \in A_0$ *such that* $d_b(x_1, y_1) = D(A, B)$ *and* $\alpha(x_0, x_1) = 1$. *Furthermore,*

$$\begin{aligned} D(x_0, Tx_0) &= d_b\left(\left(\frac{1}{5}, 1\right), \left(0, \frac{1}{10}\right)\right) \\ &\leq 2\left[d_b\left(\left(\frac{1}{5}, 1\right), \left(\frac{1}{5}, \frac{1}{10}\right)\right) + d_b\left(\left(\frac{1}{5}, \frac{1}{10}\right), \left(0, \frac{1}{10}\right)\right)\right] \\ &= 2\left[d_b(x_0, x_1) + d_b(x_1, y_1)\right]. \end{aligned}$$

*Since* $d_b(x_1, y_1) = D(A, B)$ *and* $\alpha(x_0, x_1) \geq 1$, *we obtain*

$$\frac{1}{2} D(x_0, Tx_0) - D(A, B) \leq \alpha(x_0, x_1) d_b(x_0, x_1).$$

*Noting that* $Tx = \{(0, \frac{a}{10})\ : 0 \leq a \leq 1\}$ *and* $Ty = \{(0, \frac{b}{10})\ : 0 \leq b \leq \frac{1}{10}\}$, *so*

$$\begin{aligned} M(x_0, x_1) &= \max\{d_b(x_0, x_1), D(x_0, Tx_0), D(x_1, Tx_1)\} \\ &= \max\left\{\frac{81}{100}, \frac{85}{100}, \frac{481}{10000}\right\} \\ &= \frac{85}{100} \end{aligned}$$

*and* $H(Tx_0, Tx_1) = \frac{81}{10000}$. *Thus*

$$\theta(H(Tx_0, Tx_1)) = \theta\left(\frac{81}{10000}\right) = \frac{81}{10000} + 1 = \frac{1081}{10000} \tag{23}$$

*and*

$$[\theta(M(x_0, x_1))]^s = \left(\theta\left[\frac{85}{100}\right]\right)^{\frac{1}{2}} = \left(\frac{85}{100} + 1\right)^{\frac{1}{2}} = \frac{\sqrt{185}}{10}. \tag{24}$$

*From* (23) *and* (24), *we get that*

$$\theta(H(Tx_0, Tx_1)) \leq [\theta(M(x_0, x_1))]^s.$$

*Hence, T is MV Suzuki type (α, θ)-contraction. Furthermore, T is continuous and hypothesis (ii) of Theorem 4 is verified. Indeed, for* $x_0 = (\frac{1}{5}, 1)$, $x_1 = (\frac{1}{5}, 0)$ *and* $y_1 = (0, 0)$, *we obtain*

$$d_b(x_1, y_1) = d_b((\frac{1}{5}, 0), (0, 0)) = \frac{1}{25} = D(A, B) \text{ and } \alpha(x_0, x_1) = 1.$$

*Hence all the hypothesis of Theorem 4 are verified. Therefore, T has a best proximity point, which is* $(\frac{1}{5}, 0)$.

In the next result, we replace the continuity of the mapping $T$ by the following property:

If $\{x_n\}$ is a sequence in $A$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $\alpha(x_{n_m}, x) \geq 1$ for all $m \geq 1$. If the above condition is satisfied then we say that the set $A$ satisfies $\alpha$-subsequential property.

**Theorem 5.** *Let $A$ and $B$ be two nonempty closed subsets of a complete b-metric space $(X, d_b)$ such that $A_0$ is nonempty. Let $T : A \to K(B)$ be a MV Suzuki type $(\alpha, \theta)$-contraction such that conditions $(i)$–$(iii)$ of Theorem 4 are satisfied together with sequentially continuity of $d_b$. Then $T$ has a best proximity point in $A$ provided that $A$ satisfies $\alpha$-subsequential property.*

**Proof.** From the proof of Theorem 4, we obtain two sequences $\{x_n\}$ in $A_0$ and $\{y_n\}$ in $B_0$ such that

(a) $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \neq x_{n+1}$;
(b) $y_n \in Tx_{n-1}$ and $y_n \notin Tx_n$;
(c) $d_b(x_n, y_n) = D(A, B)$ and

$$\theta(d_b(x_n, x_{n+1})) \leq \theta(d_b(y_n, y_{n+1})) \leq [\theta[d_b(x_{n-1}, x_n))]^s. \tag{25}$$

Also, there exist $x^* \in A$, $y^* \in B$ such that $x_n \to x^*$, $y_n \to y^*$ as $n \to \infty$, respectively, and $d_b(x^*, y^*) = D(A, B)$.

Now, we show that $x^*$ is a best proximity point of $T$. If there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $Tx_{n_m} = Tx^*$ for all $m \geq 1$, then we obtain

$$D(A, B) \leq D(x_{n_m+1}, Tx_{n_m}) \leq d_b(x_{n_m+1}, y_{n_m+1}) = D(A, B),$$

which yields that

$$D(A, B) \leq D(x_{n_m+1}, Tx^*) \leq D(A, B),$$

for all $m \geq 1$. Letting $m \to \infty$, we obtain

$$D(A, B) \leq D(x^*, Tx^*) \leq D(A, B).$$

Hence $x^*$ is a best proximity point of $T$. So, without loss of generality, we may assume that $Tx_n \neq Tx^*$ for all $n \in \mathbb{N}$. By $\alpha$-subsequential property, there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $\alpha(x_{n_m}, x^*) \geq 1$ for all $m \geq 1$. From the hypothesis $(ii)$, we obtain $y_{n_k+1} \in Tx_{n_m}$ such that

$$\begin{aligned} D(x_{n_m}, Tx_{n_m}) &\leq d_b(x_{n_m}, y_{n_m+1}) \\ &\leq k[d_b(x_{n_m}, x_{n_m+1}) + d_b(x_{n_m+1}, y_{n_m+1})]. \end{aligned}$$

Since $d_b(x_{n_m+1}, y_{n_m+1}) = D(A, B)$ and $\alpha(x_{n_m}, x^*) \geq 1$, we obtain

$$\frac{1}{k} D(x_{n_m}, Tx_{n_m}) - D(A, B) \leq d_b(x_{n_m}, x_{n_m+1}) \leq \alpha(x_{n_m}, x^*) d_b(x_{n_m}, x_{n_m+1}).$$

From (5), we have

$$\theta(H(T_{n_m}, Tx^*)) \leq [\theta(M(x_{n_m}, x^*))]^s.$$

Thus

$$\theta(H(T_{n_m}, Tx^*)) \leq [\theta(M(x_{n_m}, x^*))]^s = [\theta(d_b(x_{n_m}, x^*))]^s < \theta(d_b(x_{n_m}, x^*)).$$

From $(\theta_1)$, we obtain

$$H(Tx_{n_m}, Tx^*) \leq d_b(x_{n_m}, x^*).$$

On the other hand

$$\begin{aligned} D(y^*, Tx^*) &\leq k[d_b(y^*, y_{n_m+1}) + D(y_{n_m+1}, Tx^*)] \\ &\leq k[d_b(y^*, y_{n_m+1}) + H(Tx_{n_m}, Tx^*)] \\ &\leq k[d_b(y^*, y_{n_m+1}) + d_b(x_{n_m}, x^*)]. \end{aligned}$$

Letting $m \to \infty$, we obtain $D(y^*, Tx^*) = 0$. Hence, we have

$$D(A, B) \leq D(x^*, Tx^*) \leq d_b(x^*, y^*) = D(A, B).$$

Therefore, $x^*$ is a best proximity point of $T$. □

Following results are direct consequences of Theorems 4 and 5:

**Corollary 1.** *Let $A$ and $B$ be two nonempty closed subsets of a complete b-metric space $(X, d_b)$ such that $A_0$ is nonempty and $d_b$ is sequentially continuous. Let $T : A \to K(B)$ be multivalued contraction. Suppose that the following conditions hold:*

*(i) for each $x \in A_0$, we have $Tx \subseteq B_0$ and the pair $(A, B)$ satisfies weak P-property;*
*(ii) there exist $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that*

$$d_b(x_1, y_1) = D(A, B) \ \text{ and } \ \alpha(x_0, x_1) \geq 1;$$

*(iii) $T$ is $\alpha$-proximal admissible;*
*(iv) there exist $\theta \in \Theta$ and $s \in (0, 1)$ such that*

$$\frac{1}{k} D(x, Tx) - D(A, B) \leq \alpha(x, y) d_b(x, y)$$

*implies that*

$$\theta(H(Tx, Ty)) \leq [\theta[d_b(x, y))]^s.$$

*(iv) $T$ is continuous or $A$ satisfied $\alpha$-subsequential property.*

*Then $T$ has a best proximity point.*

**Proof.** If we take $M(x, y) = d_b(x, y)$ in Theorem 4 (Theorem 5), we get the desire result. □

*Existence Results for Single Valued Mappings*

**Definition 7.** *Let $(X, d_b)$ be a b-metric space with constant $k \geq 1$, $A$ and $B$ be nonempty subsets of $X$. A mapping $T : A \to B$ is called Suzuki type $(\alpha, \theta)$-contraction if there exist functions $\alpha : A \times A \to [0, \infty)$, $\theta \in \Theta$ and $s \in (0, 1)$ such that*

$$\frac{1}{k} d(x, Tx) - d(A, B) \leq \alpha(x, y) d_b(x, y)$$

*implies that*

$$\theta(d(Tx, Ty)) \leq [\theta(M(x, y))]^s,$$

*where* $M(x,y) = \max\{d_b(x,y), d_b(x,Tx), d_b(y,Ty)\}$ *for all* $x, y \in A$.

**Theorem 6.** *Let $A$ and $B$ be two nonempty closed subsets of a complete b-metric space $(X, d_b)$ such that $A_0$ is nonempty and $d_b$ is sequentially continuous. Let $T : A \to B$ be Suzuki type $(\alpha, \theta)$-contraction. Suppose that the following conditions hold:*

*(i)* *$T(A_0) \subseteq B_0$ and the pair $(A, B)$ satisfies weak P-property;*
*(ii)* *there exist $x_0, x_1 \in A_0$ such that*

$$d_b(x_1, Tx_0) = d_b(A, B) \text{ and } \alpha(x_0, x_1) \geq 1;$$

*(iii)* *$T$ is $\alpha$-proximal admissible;*
*(iv)* *$T$ is continuous or $A$ satisfies $\alpha$-subsequential property.*

*Then $T$ has a best proximity point.*

Taking $A = B = X$ in Theorem 6, with an extra condition as follows:

If $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. If the above condition is satisfied then we say $A$ has $\alpha$-sequential property.

**Theorem 7.** *Let $(X, d_b)$ be a complete b-metric space and $T : X \to X$ be a Suzuki type $(\alpha, \theta)$-contraction. Suppose that the following conditions hold:*

*(i)* *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
*(ii)* *$T$ is $\alpha$ admissible;*
*(iii)* *$T$ is continuous or $A$ has $\alpha$-sequential property*

*Then $T$ has a fixed point.*

**Proof.** The proof is similar to that of Theorem 6. □

## 3. Existence Results in *b*-Metric Space Endowed with Graph

Jachymski [21] was the first who has presented an analogue of Banach contraction principle for mappings on a metric space endowed with a graph. Dinevari [22] took initiative to extend the Nadler's theorem on the lines of Jachymski [21].

In this section, we give the existence of best proximity point theorems in $b$-metric space endowed with graph. The following notions will be used in the sequel:

**Definition 8.** *Let $(X, d_b)$ be a b-metric space.*

1. *The set $\Delta = \{(x,x) : x \in X\} \subseteq X \times X$ is known as diagonal of the Cartesian product.*
2. *In a graph $G_b$, the set $V(G_b)$ of its vertices coincides with $X$ and the set $E(G_b)$ of its edges contains all loops, i.e., $\Delta \subseteq E(G_b)$.*
3. *The graph $G_b$ has no parallel edges and so we can identify $G_b$ with the pair $(V(G_b), E(G_b))$.*
4. *The graph $G_b$ is a weighted graph by assigning to each edge the distance between its vertices.*

**Definition 9.** *Let $(X, d_b)$ be a b-metric space endowed with a graph $G_b$ and $A, B$ be two nonempty subsets of $X$. A function $T : A \to CB(B)$ is called $E(G_b)$-continuous if for all sequences $x_n$ and $y_n$ of elements from $A$ and $B$ respectively and $x \in A$, $y \in B$ such that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, $y_{n+1} \in T(x_n)$ and $(x_n, x_{n+1}), (y_n, y_{n+1}) \in E(G_b)$ for every $n \in \mathbb{N}$, we have $y \in T(x)$.*

**Definition 10.** *Let $(X, d_b)$ be a b-metric space endowed with a graph $G_b$. The b-metric $d_b$ is called $E(G_b)$-sequentially continuous if for every $A, B \in CB(B)$, every $x \in A$, $y \in B$ and every sequence $x_n$ in $A$, $y_n$ in $B$ such that $x_n \to x$, $y_n \to y$ and $(x_n, x_{n+1}), (y_n, y_{n+1}) \in E(G_b)$ we have $d_b(x_n, y_n) \to d_b(x, y)$.*

**Definition 11.** *Let $A$ and $B$ be nonempty subsets of a b-metric space $(X, d_b)$ endowed with a graph $G_b$. A mapping $T : A \to CB(B)$ is said to be $G_b$-proximal if*

$$\left.\begin{array}{rcl} (x_1, x_2) & \in & E(G_b) \\ d_b(u_1, y_1) & = & D(A, B) \\ d_b(u_2, y_2) & = & D(A, B) \end{array}\right\} \quad \textit{implies} \quad (u_1, u_2) \in E(G_b),$$

*for all $x_1, x_2, u_1, u_2 \in A$ and $y_1 \in Tx_1, y_2 \in Tx_2$.*

**Definition 12.** *Let $(X, d_b)$ be a b-metric space endowed with graph $G_b$, $A$ and $B$ be nonempty subsets of $X$. A mapping $T : A \to CB(B)$ is called MV Suzuki type $(\alpha, \theta_{G_b})$-contraction if there exist $\alpha : A \times A \to [0, \infty)$, $\theta \in \Theta$ and $s \in (0, 1)$ such that*

$$\frac{1}{k} D(x, Tx) - D(A, B) \leq d_b(x, y)$$

*implies that*

$$\theta(H(Tx, Ty)) \leq [\theta[M(x, y))]^s,$$

*where $M(x, y) = \max\{d_b(x, y), D(x, Tx), D(y, Ty)\}$ and $H(Tx, Ty) > 0$ for all $x, y \in A$ with $(x, y) \in E(G_b)$.*

**Theorem 8.** *Let $A$ and $B$ be two nonempty closed subsets of a b-metric space $(X, d_b)$ endowed with a graph $G_b$ such that $A_0$ is nonempty. Let $T : A \to K(B)$ be a MV Suzuki type $(\alpha, \theta_{G_b})$-contraction. Suppose that the following conditions hold:*

*(i)* $(X, d_b)$ *is an $E(G_b)$-complete b-metric space;*
*(ii)* *for each $x \in A_0$, we have $Tx \subseteq B_0$ and the pair $(A, B)$ satisfies weak P-property;*
*(iii)* *there exist $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that*

$$d_b(x_1, y_1) = D(A, B) \;\; \textit{and} \;\; (x_0, x_1) \in E(G_b);$$

*(iv)* *$d_b$ is $E(G_b)$-sequentially continuous;*
*(v)* *$T$ is $G_b$-proximal and $E(G_b)$-continuous.*

*Then $T$ has a best proximity point.*

**Proof.** Define $\alpha : A \times A \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } (x, y) \in E(G_b), \\ 0 & \text{otherwise.} \end{cases}$$

The conclusion follows from Theorem 4. □

Now to remove the condition of $E(G_b)$-continuous on $T$, we need following condition:

If $\{x_n\}$ is a sequence in $A$ such that $(x_n, x_{n+1}) \in E(G_b)$ for all $n \in \mathbb{N}$ and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $(x_{n_m}, x) \in E(G_b)$ for all $m \geq 1$. If the above condition is satisfied then we say that the set $A$ satisfied $\alpha_{G_b}$-subsequential property.

**Theorem 9.** *Let $A$ and $B$ be two nonempty closed subsets of a b-metric space $(X, d_b)$ endowed with a graph $G_b$ such that $A_0$ is nonempty. Let $T : A \to K(B)$ be a MV Suzuki type $(\alpha, \theta_{G_b})$-contraction. Suppose that the following conditions hold:*

*(i)* $(X, d_b)$ *is an* $E(G_b)$*-complete b-metric space;*
*(ii)* *for each* $x \in A_0$*, we have* $Tx \subseteq B_0$ *and the pair* $(A, B)$ *satisfies weak P-property;*
*(iii)* *there exist* $x_0, x_1 \in A_0$ *and* $y_1 \in Tx_0$ *such that*

$$d_b(x_1, y_1) = D(A, B) \;\; and \;\; (x_0, x_1) \in E(G_b);$$

*(iv)* $T$ *is* $G_b$*-proximal;*
*(v)* $d_b$ *is* $E(G_b)$*-sequentially continuous;*
*(vi)* $A$ *satisfied* $\alpha_{G_b}$*-subsequential property.*

*Then $T$ has a best proximity point.*

**Proof.** Define $\alpha : A \times A \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } (x, y) \in E(G_b), \\ 0 & \quad \text{otherwise.} \end{cases}$$

The conclusion follows from Theorem 5. □

## 4. Application to Fractional Calculus

First, we recall some notions (see [23]). For a continuous function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order $\beta$ is defined as

$$^{C}D^{\beta}(g(t)) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} g^{(n)}(s) ds \;\; (n-1 < \beta < n, n = [\beta]+1)$$

where $[\beta]$ denotes the integer part of real number $\beta$ and $\Gamma$ is gamma function.

In this section, we present an application of Theorem 7 to show the existence of the solution for nonlinear fractional differential equation:

$$^{C}D^{\beta}(x(t)) + f(t, x(t)) = 0 \;\; (0 \leq t \leq 1, \beta < 1) \tag{26}$$

via boundary conditions $x(0) = 0 = x(1)$, where $x \in C([0,1], \mathbb{R})$ and $C([0,1], \mathbb{R})$ is the set of all continuous functions from $[0,1]$ into $\mathbb{R}$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous function (see [24]). Recall Green function associated with the problem (26) is given by

$$G(t, s) = \begin{cases} (t(1-s))^{\alpha-1} - (t-s)^{\alpha-1} & \text{if } 0 \leq s \leq t \leq 1, \\ \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} & \text{if } 0 \leq t \leq s \leq 1. \end{cases}$$

First, let $X = C([0,1], \mathbb{R})$ be a $b$-metric space endowed with $b$-metric

$$d_b(x, y) = ||x||_{\infty,p} = \sup_{t \in [0,1]} |x(t) - y(t)|^p,$$

for all $x \in X$ with $k = 2^{p-1}$.

Now we prove the following existence theorem:

**Theorem 10.** *Suppose that*

(*i*) *there exist a function* $\mu : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $p, \tau > 1$ *such that*

$$\frac{1}{2^{p-1}} d_p(x, Tx) \leq d_p(x, y)$$

*implies that*

$$|f(t, a) - f(t, b)| \leq e^{\frac{-\tau}{p}} Q(a, b)$$

*for all* $t \in [0, 1]$ *and* $a, b \in \mathbb{R}$ *with* $\mu(a, b) \geq 0$, *where* $Q(a, b) = \max\{|a - b|, |a - Ta|, |b - Tb|\}$;

(*ii*) *There exists* $x_0 \in C([0, 1], \mathbb{R})$ *such that* $\mu(x_0(t). \int_0^1 Tx_0(t)dt) \geq 0$ *for all* $t \in [0, 1]$, *where* $T : C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$ *is defined by*

$$Tx(t) = \int_0^1 G(t, s) f(s, x(s)) ds;$$

(*iii*) *for each* $t \in [0, 1]$ *and* $x, y \in C([0, 1], \mathbb{R})$, $\mu(x(t), y(t)) \geq 0$ *implies* $\mu(Tx(t), Ty(t)) \geq 0$;

(*iv*) *for each* $t \in [0, 1]$, *if* $\{x_n\}$ *is a sequence in* $C([0, 1], \mathbb{R})$ *such that* $x_n \to x$ *in* $C([0, 1], \mathbb{R})$ *and* $\mu(x_n(t), x_{n+1}(t)) \geq 0$ *for all* $n \in \mathbb{N}$, *then* $\mu(x_n(t), x(t)) \geq 0$ *for all* $n \in \mathbb{N}$.

*Then, problem* (26) *has at least one solution.*

**Proof.** It is easy to see that $x \in X$ is a solution of (26) if and only if $x^* \in X$ is a solution of the equation $x(t) = \int_0^1 G(t, s) f(s, x(s)) ds$ for all $t \in [0, 1]$. Then the problem (26) is equivalent to finding $x^* \in X$ which is fixed point of $T$. From conditions (*i*) and (*ii*), for all distinct $x, y \in X$ such that $\mu(x(t), y(t)) \geq 0$ for all $t \in [0, 1]$, let $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned}
& |Tu(x) - Tv(x)|^p \\
= & \left| \int_0^1 G(t, s) f(s, x(s)) ds - \int_0^1 G(t, s) f(s, y(s)) ds \right|^p \\
\leq & \left| \int_0^1 G(t, s) [f(s, x(s)) - f(s, y(s))] ds \right|^p \\
\leq & \left( \int_0^1 |G(t, s) f(s, x(s)) - f(s, y(s))| ds \right)^p \\
\leq & \left[ \left( \int_0^1 G(t, s) ds \right)^{\frac{1}{q}} \left( \int_0^1 |f(s, x(s)) - f(s, y(s))|^p dx \right)^{\frac{1}{p}} \right]^p \\
\leq & \left[ \left( \int_0^1 G(t, s) ds \right)^{\frac{1}{q}} \left( \int_0^1 [e^{\frac{-\tau}{p}} Q(x(s), y(s))]^p dx \right)^{\frac{1}{p}} \right]^p \\
\leq & \left( \int_0^1 G(t, s) ds \right)^{\frac{p}{q}} \left( \int_0^1 [e^{\frac{-\tau}{p}} \max\{|x(s) - y(s)|^p, |x(s) - Tx(s)|^p, |y(s) - Ty(s)|^p\}]^p ds \right)^{\frac{p}{p}} \\
\leq & \left( \int_0^1 G(t, s) ds \right)^{p-1} \Bigg( \int_0^1 e^{-\tau} \max\{\sup_{t \in [0,1]} |x(s) - y(s)|^p, \\
& \sup_{t \in [0,1]} |x(s) - Tx(s)|^p, \sup_{t \in [0,1]} |y(s) - Ty(s)|^p\} ds \Bigg) \\
\leq & \left( \int_0^1 G(t, s) ds \right)^{p-1} e^{-\tau} \max\{d_b(x, y), d_b(x, Tx), d_b(y, Ty)\} \int_0^1 ds \\
\leq & e^{-\tau} M(x, y) \sup_{t \in [0,1]} \left( \int_0^1 G(t, s) ds \right)^{p-1} \\
\leq & e^{-\tau} M(x, y),
\end{aligned} \tag{27}$$

where

$$M(x, y) = \max\{d_b(x, y), d_b(x, Tx), d_b(y, Ty)\}.$$

Thus for each $x, y \in X$, with $\mu(x(t), y(t)) \geq 0$ for all $t \in [0,1]$ we have

$$d_b(Tx, Ty) = ||Tx - Ty||_{\infty,p} = \sup_{t\in[0,1]} |Tx(t) - Ty(t)|^p \leq e^{-\tau} M(x,y).$$

Let $\theta(t) = e^{\sqrt{t}} \in \Theta$, $t > 0$, we have

$$e^{\sqrt{d_b(Tx,Ty)}} \leq e^{\sqrt{e^{-\tau}M(x,y)}} = [e^{\sqrt{M(x,y)}}]^k, \ \forall \, x, y \in X,$$

where $k = \sqrt{e^{-\tau}}$. Since $\tau > 1$ then $k \in (0,1)$. Therefore, $T$ is Suzuki type $(\alpha, \theta)$-type contraction. Also define

$$\alpha(x,y) = \begin{cases} 1 & \text{if } \mu(x(t), y(t)) \geq 0, t \in [0,1], \\ -\infty & \text{otherwise.} \end{cases}$$

From $(ii)$ there exists $x_0 \in C[0,1]$ such that $\alpha(x_0, Tx_0) \geq 2^{p-1}$, for all $x, y \in C[0,1]$, we get that

$$\begin{aligned} \alpha(x,y) \geq 1 \quad &\Rightarrow \quad \mu(x(t), y(t)) \geq 0 \ \text{ for all } t \in [0,1] \\ &\Rightarrow \quad \mu(Tx(t), Ty(t)) \geq 0 \ \text{ for all } t \in [0,1] \\ &\Rightarrow \quad \alpha(Tx, Ty) \geq 1, \end{aligned}$$

hence $T$ is $\alpha$-admissible. Finally, from condition $(iv)$ in the hypothesis, condition $(iii)$ of Theorem 7 holds. Hence all the conditions of Theorem 7 are satisfied. Thus we conclude that there exists $x^* \in C[0,1]$ such that $Tx^* = x^*$ and so $x^*$ is a solution of the problem (26). This completes the proof. □

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