mathematics

MDPI

Article

Nonlocal q-Symmetric Integral Boundary Value Problem for Sequential q-Symmetric Integrodifference Equations

Rujira Ouncharoen [1], Nichaphat Patanarapeelert [2,*] and Thanin Sitthiwirattham [3]

[1] Center of Excellence in Mathematics and Applied Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand; rujira.o@cmu.ac.th
[2] Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand
[3] Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok 10700, Thailand; thanin_sit@dusit.ac.th
* Correspondence: nichaphat.p@sci.kmutnb.ac.th

Received: 11 September 2018; Accepted: 22 October 2018; Published: 25 October 2018

Abstract: In this paper, we prove the sufficient conditions for the existence results of a solution of a nonlocal q-symmetric integral boundary value problem for a sequential q-symmetric integrodifference equation by using the Banach's contraction mapping principle and Krasnoselskii's fixed point theorem. Some examples are also presented to illustrate our results.

Keywords: q-symmetric difference; q-symmetric integral; q-symmetric integrodifference equation; existence

(2010) Mathematics Subject Classifications: 39A05; 39A13

1. Introduction

Quantum difference operators dealing with sets of nondifferentiable functions have been extensively studied as they can be used as a tool to understand complex physical systems. There are several kinds of difference operators. The q-difference operator was first introduced by Jackson [1] and was studied in intensive work especially by Carmichael [2], Mason [3], Adams [4] and Trjitzinsky [5]. The studies of quantum problems involving q-calculus have been presented. The recent works related to q-calculus theories can be found in [6–8] and the references cited therein.

The q-symmetric difference operators are a useful tool in several fields, especially in quantum mechanics [9]. However, there are few research works [10–13] involving the development of q-symmetric difference operators.

In 2012, A.M.C. Brito da Cruz and N. Martins [10] studied the q-deformed theory, in which the standard q-symmetric integral must be generalized to the basic integral defined.

Recently, Sun, Jin and Hou [12] introduced basic concepts of fractional q-symmetric integral and derivative operator. Moreover, Sun and Hou [13] introduced basic concepts of fractional q-symmetric calculus on a time scale.

In particular, the boundary value problem for q-symmetric difference equations has not been studied. The results mentioned are the motivation for this research. In this paper, we devote our attention to the estabished existence results for a nonlocal q-symmetric integral boundary value for sequential q-symmetric integrodifference equation of the form

$$\tilde{D}_q\tilde{D}_p u(t) = F\Big(t, u(t), (S_\theta u)(t), (Z_\omega u)(t)\Big), \;\; t \in I_\chi^T$$

$$u(0) = \lambda u(T), \tag{1}$$
$$\int_0^{\eta} g(s)u(s)\,\tilde{d}_r s = 0,\;\; \eta \in I_\chi^T - \{0,T\},$$

where $I_\chi^T := \{\chi^k T : k \in \mathbb{N}\} \cup \{0,T\}$, $p,q,r,\omega,\theta \in (0,1)$, $p = \frac{p_1}{p_2}$, $q = \frac{q_1}{q_2}$, $r = \frac{r_1}{r_2}$, $\omega = \frac{\omega_1}{\omega_2}$, $\theta = \frac{\theta_1}{\theta_2}$ and $\chi = \frac{1}{\mathrm{LCM}\left(p_2,q_2,r_2,\omega_2,\theta_2\right)}$ are the simplest form of proper fractions; $g \in C\big(I_\chi^T, \mathbb{R}^+\big)$ and $F \in C\big(I_\chi^T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\big)$ are given functions; and for $\varphi, \phi \in C\big(I_\chi^T \times I_\chi^T, [0,\infty)\big)$, define

$$(S_\theta u)\,(t) := \int_0^t \varphi(s,t)u(s)\tilde{d}_\theta s \;\;\text{and}\;\; (Z_\omega u)\,(t) := \int_0^t \phi(s,t)u(s)\tilde{d}_\omega s.$$

This paper is organized as follows. In Section 2, we provide basic definitions, properties of q-symmetric difference operator and lemmas used in this paper. In Section 3, the existence results of problem (1) will be proved by employing Banach's contraction mapping principle and Krasnoselskii's fixed point theorem. In Section 4, we give some examples to illustrate our results.

2. Preliminaries

We introduce some basic definitions and properties of q-symmetric difference calculus as follows.

Definition 1. *For $q \in (0,1)$, the q-symmetric difference of function $f : \mathbb{R} \to \mathbb{R}$ is defined by*

$$\tilde{D}_q f(t) = \frac{f(qt) - f(q^{-1}t)}{(q - q^{-1})t} \;\; \textit{and} \;\; \tilde{D}_q f(0) = f'(t).$$

The higher order q-symmetric derivatives of f is defined by

$$\tilde{D}_q^n f(t) = \tilde{D}_q D_q^{n-1} f(t), n \in \mathbb{N}.$$

We note that $\tilde{D}_q^0 f(t) = f(t)$.

Next, if f is a function defined on the interval I, q-symmetric integral is defined by

$$\begin{aligned} \int_a^b f(s)\tilde{d}_q s \;\; &:= \;\; \int_0^b f(s)\tilde{d}_q s - \int_0^a f(s)\tilde{d}_q s \\ \text{and} \;\; \int_0^x f(s)\tilde{d}_q s \;\; &:= \;\; x(1-q^2)\sum_{k=0}^{\infty} q^{2k} f\left(xq^{2k+1}\right) \end{aligned}$$

where the above infinite series is convergent.

We next discuss the following lemmas used to simplify our calculations.

Lemma 1. *Let $0 < q < 1$, and $f : I \to \mathbb{R}$ be continuous at 0. Then,*

$$\int_0^t \int_0^r x(s)\,\tilde{d}_q s\,\tilde{d}_q r = \int_0^t \int_{q^2 s}^{qt} x\,(qs)\;\tilde{d}_q r\,\tilde{d}_q s.$$

Proof. Using the definition of symmetric q-integral, we have

$$\begin{aligned} &\int_0^t \int_0^r x(s)\,\tilde{d}_q s\,\tilde{d}_q r \\ =\;\; &\int_0^t \left[r(1-q^2)\sum_{k=0}^{\infty} q^{2k}\,x\left(rq^{2k+1}\right)\right]\tilde{d}_q r = \sum_{k=0}^{\infty} q^{2k}\left[\int_0^t r(1-q^2)\,x\left(rq^{2k+1}\right)\tilde{d}_q r\right] \\ =\;\; &\sum_{k=0}^{\infty} q^{2k} t(1-q^2)\sum_{h=0}^{\infty} q^{2h}\left[tq^{2h+1}(1-q^2)\right]x\Big(\left(tq^{2h+1}\right)q^{2k+1}\Big) \end{aligned}$$

$$
\begin{aligned}
&= qt^2(1-q^2)^2 \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} q^{2k+4h}\, x\left(tq^{2k+2h+2}\right) \\
&= qt^2(1-q^2)^2 \sum_{h=0}^{\infty} \left[q^{4h}x\left(tq^{2h+2}\right) + q^{4h+2}x\left(tq^{2h+4}\right) + q^{4h+4}x\left(tq^{2h+6}\right) + \ldots\right] \\
&= qt^2(1-q^2)^2 \Big\{ \left[x(tq^2) + q^2x\left(tq^4\right) + q^4x\left(tq^6\right) + \ldots\right] \\
&\qquad + \left[q^4x(tq^4) + q^6x\left(tq^6\right) + q^8x\left(tq^8\right) + \ldots\right] \\
&\qquad + \left[q^8x(tq^6) + q^{10}x\left(tq^8\right) + q^{12}x\left(tq^{10}\right) + \ldots\right] + \ldots \Big\} \\
&= qt^2(1-q^2)^2 \Big\{x(tq^2) + q^2(1+q^2)\,x\left(tq^4\right) + q^4(1+q^2+q^4)\,x\left(tq^6\right) + \ldots\Big\} \\
&= qt^2(1-q^2)^2 \sum_{k=0}^{\infty} q^{2k}[k+1]_{q^2}\, x\left(tq^{2k+2}\right) \\
&= q\sum_{k=0}^{\infty} q^{2k}t^2(1-q^2)\left[1-q^{2(k+1)}\right] x\left(tq^{2k+2}\right) \\
&= \int_0^t \left[qt-q^2s\right] x\,(qs)\ \tilde{d}_q s \\
&= \int_0^t \left[\int_0^{qt} x\,(qs)\ \tilde{d}_q r - \int_0^{q^2s} x\,(qs)\ \tilde{d}_q r\right] \tilde{d}_q s \\
&= \int_0^t \int_{q^2s}^{qt} x\,(qs)\ \tilde{d}_q r\, \tilde{d}_q s.
\end{aligned}
$$

□

Lemma 2. *Let $0<q<1$. Then,*

$$\int_0^t \tilde{d}_q s = t \quad \text{and} \quad \int_0^t \int_{q^2s}^{qt} \tilde{d}_q r \tilde{d}_q s = \frac{qt^2}{1+q^2}.$$

Proof. Using the definition of symmetric q-integral, we have

$$
\begin{aligned}
\int_0^t \tilde{d}_q s &= t(1-q^2)\sum_{k=0}^{\infty} q^{2k} = t(1-q^2)\left[\frac{1}{1-q^2}\right] = t, \\
\int_0^t \int_{q^2s}^{qt} \tilde{d}_q r \tilde{d}_q s &= \int_0^t \left[qt-q^2s\right] \tilde{d}_q s \\
&= t(1-q^2)\sum_{k=0}^{\infty} q^{2k}\left[qt-q^{2k+3}t\right] \\
&= qt^2(1-q^2)\left[\frac{1}{1-q^2}-\frac{q^2}{1-q^4}\right] = \frac{qt^2}{1+q^2}.
\end{aligned}
$$

□

To study the solution of the boundary value problem (1), we first consider the solution of a linear variant of the boundary value problem (1) as follows.

Lemma 3. *Let $p,q \in (0,1)$, $p=\frac{p_1}{p_2}$, $q=\frac{q_1}{q_2}$, $r=\frac{r_1}{r_2}$ and $\kappa=\frac{1}{LCM(p_2,q_2,r_2)}$ are the simplest form of proper fractions; $\lambda\in\mathbb{R}$; $g\in C(I_\kappa^T,\mathbb{R}^+)$ and $h\in C(I_\kappa^T,\mathbb{R})$ are given functions. The solution for the problem*

$$\tilde{D}_q\tilde{D}_p u(t) = h(t), \quad t\in I_\kappa^T \tag{2}$$

$$u(0) = \lambda u(T), \tag{3}$$

$$\int_0^\eta g(s)u(s)\,\tilde{d}_r s = 0, \;\; \eta \in I_\kappa^T - \{0, T\}, \tag{4}$$

is in the form

$$\begin{aligned} u(t) = \;& \frac{1}{\Lambda}\left[\int_0^\eta sg(s)\,\tilde{d}_q s \cdot \lambda \int_0^T \int_0^x h(s)\,\tilde{d}_q s\,\tilde{d}_p x - \lambda T \int_0^\eta \int_0^y \int_0^x g(y)h(s)\,\tilde{d}_q s\,\tilde{d}_p x\,\tilde{d}_r y\right] \\ & + \frac{t}{\Lambda}\left[\int_0^\eta g(s)\,\tilde{d}_q s \cdot \lambda \int_0^T \int_0^x h(s)\,\tilde{d}_q s\,\tilde{d}_p x - (1-\lambda) \int_0^\eta \int_0^y \int_0^x g(y)h(s)\,\tilde{d}_q s\,\tilde{d}_p x\,\tilde{d}_r y\right] \\ & + \int_0^t \int_0^x h(x)\,\tilde{d}_q s\,\tilde{d}_p x, \end{aligned} \tag{5}$$

where $\Lambda := \lambda T \int_0^\eta g(s)\,\tilde{d}_q s + (1-\lambda) \int_0^\eta sg(s)\,\tilde{d}_q s$.

Proof. We first take the q-symmetric integral for (2) to obtain

$$\tilde{D}_p u(t) = C_1 + \int_0^t h(s)\,\tilde{d}_q s. \tag{6}$$

Next, taking the p-symmetric integral for (6), we have

$$u(t) = C_2 + C_1 t + \int_0^t \int_0^x h(s)\,\tilde{d}_q s\,\tilde{d}_p x. \tag{7}$$

To find C_1 and C_2, we first take the r-symmetric integral for $g(t)u(t)$. We find that

$$\int_0^t g(s)u(s)\,\tilde{d}_r s = C_2 \int_0^t g(s)\,\tilde{d}_r s + C_1 \int_0^t sg(s)\,\tilde{d}_r s + \int_0^t \int_0^y \int_0^x g(y)h(s)\,\tilde{d}_q s\,\tilde{d}_p x\,\tilde{d}_r y. \tag{8}$$

We apply condition (3) to (7). Then, we have

$$C_1 \lambda T - (1-\lambda)C_2 = -\lambda \int_0^T \int_0^x h(x)\,\tilde{d}_q s\,\tilde{d}_p x. \tag{9}$$

We next apply condition (4) to (8). Then we have

$$C_1 \int_0^\eta sg(s)\,\tilde{d}_q s + C_2 \int_0^\eta g(s)\,\tilde{d}_q s = -\int_0^\eta \int_0^y \int_0^x g(y)h(s)\,\tilde{d}_q s\,\tilde{d}_p x\,\tilde{d}_r y. \tag{10}$$

Constants C_1 and C_2 are obtained by solving the system of Equations (9) and (10) as follows

$$C_1 = -\frac{1}{\Lambda}\left[\int_0^\eta g(s)\,\tilde{d}_q s \cdot \lambda \int_0^T \int_0^x h(x)\,\tilde{d}_q s\,\tilde{d}_p x - (1-\lambda) \int_0^\eta \int_0^y \int_0^x g(y)h(s)\,\tilde{d}_q s\,\tilde{d}_p x\,\tilde{d}_r y\right],$$

$$C_2 = \frac{1}{\Lambda}\left[\int_0^\eta sg(s)\,\tilde{d}_q s \cdot \lambda \int_0^T \int_0^x h(x)\,\tilde{d}_q s\,\tilde{d}_p x - \lambda T \int_0^\eta \int_0^y \int_0^x g(y)h(s)\,\tilde{d}_q s\,\tilde{d}_p x\,\tilde{d}_r y\right].$$

Employing these results in (7), we get the solution (5). □

3. Main Results

To study the existence and uniqueness of solution of (1), we transform the boundary value problem (1) into a fixed point problem. Let $\mathcal{C} = C(I_\chi^T, \mathbb{R})$ denote the Banach space of all functions u. The norm is defined by $\|u\| = \sup_{t\in I_\chi^T} |u(t)|$. The operator $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is defined by

$$\begin{aligned}(\mathcal{T}u)(t) = {} & \frac{1}{\Lambda}\bigg[\int_0^\eta sg(s)\,\tilde{d}_q s \cdot \lambda \int_0^T \int_0^x F\Big(x, u(x), (S_\theta u)(x), (Z_\omega u)(x)\Big)\,\tilde{d}_q s\,\tilde{d}_p x \\ & - \lambda T \int_0^\eta \int_0^y \int_0^x g(y)\,F\Big(s, u(s), (S_\theta u)(s), (Z_\omega u)(s)\Big)\,\tilde{d}_q s\,\tilde{d}_p x\,\tilde{d}_r y\bigg] \\ & + \frac{t}{\Lambda}\bigg[\int_0^\eta g(s)\,\tilde{d}_q s \cdot \lambda \int_0^T \int_0^x F\Big(x, u(x), (S_\theta u)(x), (Z_\omega u)(x)\Big)\,\tilde{d}_q s\,\tilde{d}_p x \\ & - (1-\lambda) \int_0^\eta \int_0^y \int_0^x g(y)\,F\Big(s, u(s), (S_\theta u)(s), (Z_\omega u)(s)\Big)\,\tilde{d}_q s\,\tilde{d}_p x\,\tilde{d}_r y\bigg] \\ & + \int_0^t \int_0^x F\Big(x, u(x), (S_\theta u)(x), (Z_\omega u)(x)\Big)\,\tilde{d}_q s\,\tilde{d}_p x, \end{aligned} \tag{11}$$

where $I_\chi^T := \{\chi^k T : k \in \mathbb{N}\} \cup \{0, T\}$, $p, q \in (0,1)$, $p = \frac{p_1}{p_2}$, $q = \frac{q_1}{q_2}$, $r = \frac{r_1}{r_2}$, $\omega = \frac{\omega_1}{\omega_2}$, $\theta = \frac{\theta_1}{\theta_2}$ are the simplest form of proper fractions and $\chi = \frac{1}{\mathrm{LCM}\left(p_2, q_2, r_2, \omega_2, \theta_2\right)}$; $g \in C\big(I_\chi^T, \mathbb{R}^+\big)$ and $F \in C\big(I_\chi^T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\big)$; and for $\varphi, \phi \in C\big(I_\chi^T \times I_\chi^T, [0,\infty)\big)$, define

$$S_\theta u(t) := \int_0^t \varphi(s,t) u(s)\tilde{d}_\theta s \quad \text{and} \quad Z_\omega u(t) := \int_0^t \phi(s,t) u(s)\tilde{d}_\omega s.$$

We note that problem (1) has solutions if and only if the operator $\mathcal{T}$ has fixed points.

To present our results, we establish the following theorem based on Banach's fixed point theorem.

Theorem 1. *Assume $F \in C\big(I_\chi^T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\big)$, $g \in C\big(I_\chi^T, \mathbb{R}^+\big)$ and $\varphi, \phi \in C\big(I_\chi^T \times I_\chi^T, [0,\infty)\big)$. Let $\varphi_0 := \sup_{(t,s)\in I_\chi^T \times I_\chi^T} \{\varphi(t,s)\}$ and $\phi_0 := \sup_{(t,s)\in I_\chi^T \times I_\chi^T} \{\phi(t,s)\}$. Assume F and g satisfy the following conditions:*

(H_1) *there exist positive constants L_1, L_2, L_3 such that*

$$\begin{aligned} & \Big|F\Big(t, u, S_\theta u, Z_\omega u\Big) - F\Big(t, v, S_\theta v, Z_\omega v\Big)\Big| \\ \leq {} & L_1|u - v| + L_2\big|S_\theta u - S_\theta v\big| + L_3\big|Z_\theta u - Z_\theta v\big|, \end{aligned}$$

for all $t \in I_\chi^T$ and $u, v, S_\theta u, Z_\omega u \in \mathbb{R}$,

(H_2) $0 < g(t) < G$ *for all $t \in I_\chi^T$.*

If

$$\Xi = \big[L_1 + T\,(L_2\varphi_0 + L_3\phi_0)\big]\left\{\Theta + \frac{pT^2}{1+p^2}\right\} < 1, \tag{12}$$

where

$$\Theta := \frac{1}{\hat{\Lambda}}\left[\frac{p\eta^2 TG}{1+p^2}\left(\frac{q}{1+q^2}\lambda T + \frac{r}{(1+r^2)(1+r^4)}\lambda\eta\right)\right.$$

$$+\frac{p\eta TG}{\hat{\Lambda}(1+p^2)}\left(\lambda T^2+\frac{r}{(1+r^2)(1+r^4)}(1-\lambda)\eta^2\right)\Bigg], \tag{13}$$

where $\hat{\Lambda}$ *is given by (15), then the boundary value problem (1) has a unique solution.*

Proof. For any $u, v \in \mathcal{C}$ and for each $t \in I_\chi^T$, we have

$$\begin{aligned}
&\left|F\left(t,u,S_\theta u,Z_\omega u\right)-F\left(t,v,S_\theta v,Z_\omega v\right)\right| \\
\leq\ & L_1|u-v|+L_2|S_\theta u-S_\theta v|+L_3|Z_\omega u-Z_\omega v| \\
=\ & L_1|u-v|+L_2\int_0^t \varphi(s,t)|u(s)-v(s)|\tilde{d}_\theta s+L_3\int_0^t \phi(s,t)|u(s)-v(s)|\tilde{d}_\omega s \\
\leq\ & \|u-v\|\{L_1+T\,(L_2\varphi_0+L_3\phi_0)\,\},
\end{aligned} \tag{14}$$

and

$$|\Lambda|\geq \lambda Tg\int_0^\eta \tilde{d}_q s+|1-\lambda|g\int_0^\eta s\,\tilde{d}_q s=\lambda Tg\eta+\frac{qg\eta^2}{1+q^2}|1-\lambda| =: \hat{\Lambda}. \tag{15}$$

From (14) and (15), we have

$$\begin{aligned}
&\left|(\mathcal{T}u)(t)-(\mathcal{T}v)(t)\right| \\
\leq\ & \tfrac{\|u-v\|}{\hat{\Lambda}}\{L_1+T\,(L_2\varphi_0+L_3\phi_0)\,\}\times \\
&\Bigg\{\Bigg[G\lambda\textstyle\int_0^\eta s\,\tilde{d}_q s\cdot\int_0^T\int_0^x \tilde{d}_q s\,\tilde{d}_p x+\lambda TG\int_0^\eta\int_0^y\int_0^x \tilde{d}_q s\,\tilde{d}_p x\,\tilde{d}_r y\Bigg] \\
&+\tfrac{T}{\hat{\Lambda}}\Bigg[G\lambda\textstyle\int_0^\eta \tilde{d}_q s\cdot\int_0^T\int_0^x \tilde{d}_q s\,\tilde{d}_p x+(1-\lambda)G\int_0^\eta\int_0^y\int_0^x \tilde{d}_q s\,\tilde{d}_p x\,\tilde{d}_r y\Bigg]+\int_0^T\int_0^x \tilde{d}_q s\,\tilde{d}_p x\Bigg\} \\
\leq\ & \|u-v\|\left[L_1+T\,(L_2\varphi_0+L_3\phi_0)\,\right]\Bigg\{\tfrac{1}{\hat{\Lambda}}\Bigg[\tfrac{p\eta^2TG}{1+p^2}\left(\tfrac{q}{1+q^2}\lambda T+\tfrac{r}{(1+r^2)(1+r^4)}\lambda\eta\right) \\
&+\tfrac{p\eta TG}{\hat{\Lambda}(1+p^2)}\left(\lambda T^2+\tfrac{r}{(1+r^2)(1+r^4)}(1-\lambda)\eta^2\right)\Bigg]+\tfrac{pT^2}{1+p^2}\Bigg\} \\
=\ & \|u-v\|\left[L_1+T\,(L_2\varphi_0+L_3\phi_0)\,\right]\left\{\Theta+\tfrac{pT^2}{1+p^2}\right\}.
\end{aligned} \tag{16}$$

As $\Xi < 1$, $\mathcal{T}$ is a contraction. Therefore, the proof is done based on Banach's contraction mapping principle. □

Furthermore, we prove the existence of a solution to the boundary value problem (1) by using the Krasnoselskii's fixed point theorem.

Theorem 2. [14] *Let K be a bounded closed convex and nonempty subset of a Banach space X. Let A, B be operators such that:*

(i) *$Ax + By \in K$ whenever $x, y \in K$,*
(ii) *A is compact and continuous,*
(iii) *B is a contraction mapping.*

Then there exists $z \in K$ such that $z = Az + Bz$.

Theorem 3. *Assume that (H_1) and (H_2) hold. In addition we suppose that:*

(H_3) $\left|F\left(t,u,S_\theta u,Z_\omega u\right)\right| \leq \mu(t)$, *for all* $\left(t,u,S_\theta u,Z_\omega u\right) \in I_\chi^T \times \mathbb{R}\times\mathbb{R}\times\mathbb{R}$, *with* $\mu \in C(I_\chi^T, \mathbb{R}^+)$.

If

$$\Theta < 1, \tag{17}$$

where Θ *is given by* (13)*, then the boundary value problem* (1) *has at least one solution on* I_χ^T.

Proof. We let $\sup_{t\in I_\chi^T} |\mu(t)| = \|\mu\|$ and choose a constant

$$R \geq \|\mu\| \left\{ \Theta + \frac{pT^2}{1+p^2} \right\}, \tag{18}$$

where $B_R = \{u \in \mathcal{C} : \|u\| \leq R\}$.

Based on the results of Lemma 3, we define the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ on the ball B_R as

$$\begin{aligned}
(\mathcal{T}_1 u)(t) &= \int_0^t \int_0^x F\Big(x, u(x), (S_\theta u)(x), (Z_\omega u)(x)\Big)\, \tilde{d}_q s\, \tilde{d}_p x, \\
(\mathcal{T}_2 u)(t) &= \frac{1}{\Lambda} \Bigg[\int_0^\eta s g(s)\, \tilde{d}_q s \cdot \lambda \int_0^T \int_0^x F\Big(x, u(x), (S_\theta u)(x), (Z_\omega u)(x)\Big)\, \tilde{d}_q s\, \tilde{d}_p x \\
&\quad - \lambda T \int_0^\eta \int_0^y \int_0^x g(y)\, F\Big(s, u(s), (S_\theta u)(s), (Z_\omega u)(s)\Big)\, \tilde{d}_q s\, \tilde{d}_p x\, \tilde{d}_r y \Bigg] \\
&\quad + \frac{t}{\Lambda} \Bigg[\int_0^\eta g(s)\, \tilde{d}_q s \cdot \lambda \int_0^T \int_0^x F\Big(x, u(x), (S_\theta u)(x), (Z_\omega u)(x)\Big)\, \tilde{d}_q s\, \tilde{d}_p x \\
&\quad - (1-\lambda) \int_0^\eta \int_0^y \int_0^x g(y)\, F\Big(s, u(s), (S_\theta u)(s), (Z_\omega u)(s)\Big)\, \tilde{d}_q s\, \tilde{d}_p x\, \tilde{d}_r y \Bigg].
\end{aligned}$$

We proceed similarly to Theorem 1 for $u, v \in B_R$. Then we have

$$\|\mathcal{T}_1 x + \mathcal{T}_2 y\| \leq \|\mu\| \left\{ \Theta + \frac{pT^2}{1+p^2} \right\} \leq R,$$

which implies that $\mathcal{T}_1 x + \mathcal{T}_2 y \in B_R$. Using (17), we find that $\mathcal{T}_2$ is a contraction mapping.

From the continuity of F and the assumption (H_3), we see that an operator $\mathcal{T}_1$ is continuous and uniformly bounded on B_R. For $t_1, t_2 \in I_\chi^T$ with $t_1 \leq t_2$ and $u \in B_R$, we have

$$\begin{aligned}
|\mathcal{T}_1 x(t_2) - \mathcal{T}_1 x(t_1)| &= \Bigg| \int_0^{t_2} \int_0^x F\Big(x, u(x), (S_\theta u)(x), (Z_\omega u)(x)\Big)\, \tilde{d}_q s\, \tilde{d}_p x \\
&\quad - \int_0^{t_1} \int_0^x F\Big(x, u(x), (S_\theta u)(x), (Z_\omega u)(x)\Big)\, \tilde{d}_q s\, \tilde{d}_p x \Bigg| \\
&\leq \frac{p}{1+p^2} \|F\| |t_2^2 - t_1^2|.
\end{aligned}$$

We observe that the above inequality tends to zero when $t_2 - t_1 \to 0$. Therefore $\mathcal{T}_1$ is relatively compact on B_R. Hence, we can conclude by the Arzelá-Ascoli Theorem that $\mathcal{T}_1$ is compact on B_R. We find that the assumptions of Theorem 2 are satisfied implying that the boundary value problem (1) has at least one solution on I_χ^T. The proof is complete. □

4. Examples

In this section, we provide some examples to illustrate our main results. Consider the following boundary value problem of sequential q-symetric difference equations

$$\begin{cases} \tilde{D}_{\frac{1}{2}}\tilde{D}_{\frac{3}{4}}u(t) = \dfrac{e^{-1-sin^2(\pi t)}|u(t)|}{(100+t)^2(1+|u(t)|)} + \dfrac{1}{100e^2}\left(S_{\frac{1}{3}}u\right)(t) + \dfrac{1}{100\pi^2}\left(Z_{\frac{2}{5}}u\right)(t), \quad t \in I^{10}_{\frac{1}{60}}, \\ u\,(0) = \dfrac{1}{3}u\,(10)\,, \quad \displaystyle\int_0^{\frac{1}{21600}} [100e + 20\cos^2 s]u(s)\,\tilde{d}_{\frac{2}{3}}s = 0, \end{cases} \tag{19}$$

where $\left(S_{\frac{1}{3}}u\right)(t) = \int_0^t \frac{e^{-|t-s|}u(s)}{(t+10)^2}\tilde{d}_{\frac{1}{3}}s$ and $\left(Z_{\frac{2}{5}}u\right)(t) = \int_0^t \frac{e^{-2|t-s|}u(s)}{(t+20)^2}\tilde{d}_{\frac{2}{5}}s$.

Set $p = \frac{3}{4}$, $q = \frac{1}{2}$, $r = \frac{2}{3}$, $\omega = \frac{2}{5}$, $\theta = \frac{1}{3}$, $T = 10$, $\chi = \frac{1}{L.C.M(4,2,3,5,3)} = \frac{1}{60}$, $\eta = \chi^5 T = \frac{1}{21600}$, $\lambda = \frac{1}{3}$, $\varphi(s,t) = \frac{e^{-|t-s|}}{(t+10)^2}$ and $\phi(s,t) = \frac{e^{-2|t-s|}}{(t+20)^2}$,

(I) If $g(t) = 100e + 20\cos^2 t$. We can show that $\varphi_0 = \frac{1}{100}$, $\phi_0 = \frac{1}{400}$ and

$$\begin{aligned} \left|F\left(t,u,S_{\frac{1}{3}}u,Z_{\frac{2}{3}}u\right) - F\left(t,v,S_{\frac{1}{3}}v,Z_{\frac{2}{3}}v\right)\right| \quad &\le \quad \frac{1}{100^2e}|u-v| + \frac{1}{100e^2}\left|S_{\frac{1}{3}}u - S_{\frac{1}{3}}v\right| \\ &\quad + \frac{1}{100\pi^2}\left|Z_{\frac{2}{5}}u - Z_{\frac{2}{5}}v\right|. \end{aligned}$$

So, (H_1) is satisfied with $L_1 = \frac{1}{100^2e}$, $L_2 = \frac{1}{100e^2}$ and $L_3 = \frac{1}{100\pi^2}$.

By (H_2), we have $g = 100e \le g(t) \le 100e + 20 = G$.

Clearly, $\hat{\Lambda} = 0.0419$ and $\Theta = 1231.332$.

Hence, we get

$$\left[L_1 + T\left(L_2\varphi_0 + L_3\phi_0\right)\right]\left\{\Theta + \frac{pT^2}{1+p^2}\right\} \;=\; 0.253 \;<\; 1.$$

Therefore, by Theorem 1, problem (19) has a unique solution on $I^{10}_{\frac{1}{60}}$. □

(II) If $g(t) = 100^3 e + 30\cos^2 t$. We can show that

$$\begin{aligned} \left|F\left(t,u,S_{\frac{1}{3}}u,Z_{\frac{2}{3}}u\right)\right| \quad &\le \quad \frac{e^{-1-sin^2\pi t}}{(100+t)^2} + \frac{e^{-t}}{100e^2}\cdot\frac{t}{(t+10)^2} + \frac{e^{-2t}}{100\pi^3}\cdot\frac{t}{(t+20)^2} \\ &< \quad \frac{1}{e(100+t)^2} + \frac{1}{100e^{2+t}} + \frac{e^{-2t}}{100\pi^3} \;=:\; \mu(t). \end{aligned}$$

So, (H_3) is satisfied and (H_1) is still satisfied.

By (H_2), we have $g = 100^3 e \le g(t) \le 100^3 e + 30 = G$.

Clearly, $\hat{\Lambda} = 419.489$.

Hence, we get

$$\Theta = 0.123 \;<\; 1.$$

Therefore, by Theorem 3, problem (19) has at least one solution on $I^{10}_{\frac{1}{60}}$. □

5. Conclusions

In this article, we consider a nonlocal q-symmetric integral boundary value problem for sequential q-symmetric difference-sum equation. We study the condition under which the problem has existence and a unique solution by using Banach's contraction mapping principle. Furthermore, we provide the condition for the case of at least one solution by using Krasnoselskii's fixed point theorem. A further extension of this article is the study of stability, behaviour under perturbation and possible applications in economics and engineering.

Author Contributions: The authors declare that they carried out all the work in this manuscript, and read and approved the final manuscript.

Funding: King Mongkut's University of Technology North Bangkok. Contract no. KMUTNB-ART-60-37.

Acknowledgments: The first author of this research was also supported by Chiang Mai University.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Jackson, F.H. On q-difference equations. *Am. J. Math.* **1910**, *32*, 305–314. [CrossRef]
2. Carmichael, R.D. The general theory of linear q-difference equations. *Am. J. Math.* **1912**, *34*, 147–168. [CrossRef]
3. Mason, T.E. On properties of the solutions of linear q-difference equations with entire function coefficients. *Am. J. Math.* **1915**, *37*, 439–444. [CrossRef]
4. Adams, C.R. On the linear ordinary q-difference equation. *Am. Math. Ser. II* **1928**, *30*, 195–205. [CrossRef]
5. Trjitzinsky, W.J. Analytic theory of linear q-differece equations. *Acta Math.* **1933**, *61*, 1–38. [CrossRef]
6. Kac, V.; Cheung, P. *Quantum Calculus*; Springer: New York, NY, USA, 2002.
7. Ernst, T. *A New Notation for q-Calculus and a New q-Taylor Formula*; U.U.D.M. Report 1999:25; Department of Mathematics, Uppsala University: Uppsala, Sweden, 1999; ISSN 1101-3591.
8. Floreanini, R.; Vinet, L. q-gamma and q-beta functions in quantum algebra representation theory. *J. Comput. Appl. Math.* **1996**, *68*, 57–68. [CrossRef]
9. Lavagno, A. Basic-deformed quantum mechanics. *Rep. Math. Phys.* **2009**, *64*, 79–91. [CrossRef]
10. Brito da Cruz, A.M.C.; Martins, N. The q-symmetric variational calculus. *Comput. Math. Appl.* **2012**, *64*, 2241–2250. [CrossRef]
11. Brito da Cruz, A.M.C.; Martins, N.; Torres, D.F.M. A symmetric quantum calculus. In *Differential and Difference Equations with Applications*; Springer: New York, NY, USA, 2013; Volume 47; pp. 359–373.
12. Sun, M.; Jin, Y.; Hou, C. Certain fractional q-symmetric integrals and q-symmetric derivatives and their application. *Adv. Differ. Equ.* **2016**, *2016*, 222. [CrossRef]
13. Sun, M.; Hou, C. Fractional q-symmetric calculus on a time scale. *Adv. Differ. Equ.* **2017**, *2017*, 166. [CrossRef]
14. Krasnoselskii, M.A. Two remarks on the method of successive approximations. *Uspekhi Mat. Nauk* **1955**, *10*, 123–127.