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Common Fixed Point Results of ($\alpha - \psi$, φ)-Contractions for a Pair of Mappings and Applications

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Abstract: In this paper, we prove some fixed point theorems in a b-metric-like space setting using a new class of admissible mappings and types of $\alpha - k$ and (ψ, ϕ)-contractive conditions. Our results are supported by the application of finding solutions of integral equations and generalizing some well-known results of the literature.

Keywords: α_{s^p}-admissible pair; $(\alpha_{s^p} - \psi, \phi)$-contractive mapping; b-metric-like space; fixed point

MSC: 47H10, 54H25

1. Introduction and Preliminaries

Recently, several authors investigated fixed point theorems in generalized metric spaces, such as b-metric spaces, metric-like spaces, b-metric-like spaces, and so on, where "metric" d takes its values in more generalized conditions. The advantage of this approach is that they bring us much stronger applications. It is, among other things, shown by examples in the articles cited throughout this manuscript.

Presently, the study of (ψ, φ)-contractions using the concept of α-admissible mapping in b-metric-like spaces is the focus of many researchers. Later, many generalizations under $(\psi - \phi)$-, $\alpha - \psi$-, and $(\alpha - \psi - \phi)$-contractive conditions was provided in many works. For fixed point theorems related to these notions, see References [1–28].

In our work, following this direction, using the notion of α-admissible mapping, in the first part of the paper, we proved some fixed point theorems for contractions of rational types, by means of a function $\gamma : R^+ \times R^+ \to R^+$. In the second part, we introduce the notion of α_{s^p}-admissible pairs of mappings and also a general and much wider class of $(\alpha_{s^p} - \psi, \phi)$-contractive pairs of mappings where the framework was taken to be b-metric-like spaces. Various related fixed point theorems in the recent literature can be derived using our results.

Definition 1 ([2]). *Let M be a nonempty set and $s \geq 1$ be a given real number. A mapping $d : M \times M \to [0, \infty)$ is called a b-metric if for all $t, r, z \in M$, the following conditions are satisfied:*

$d(t, r) = 0$ *if and only if* $t = r$;

$d(t,r) = d(r,t)$;
$d(t,r) \leq s[d(t,z) + d(z,r)]$.

The pair (M,d) is called a b-metric space with parameter s.

The following space was introduced and studied for the first time in 1985 by Matthews [1] under the name "metric domains". In 2000, Hitzler and Seda [3] called these spaces "dislocated metric spaces". In 2012, Amini-Harandi [4] rediscovered dislocated metric spaces under the name "metric-like spaces".

Definition 2 ([4])**.** *Let M be a nonempty set. A mapping $\sigma : M \times M \to [0,\infty)$ is called metric-like if for all $t,r,z \in M$, the following conditions are satisfied:*

$\sigma(t,r) = 0$ *implies* $t = r$;
$\sigma(t,r) = \sigma(r,t)$;
$\sigma(t,r) \leq \sigma(t,z) + \sigma(z,r)$.

The pair (M,σ) is called a metric-like space.

Definition 3 ([5])**.** *Let M be a nonempty set and $s \geq 1$ be a given real number. A mapping $\sigma_b : M \times M \to [0,\infty)$ is called b-metric-like if for all $t,r,z \in M$, the following conditions are satisfied:*

$\sigma_b(t,r) = 0$ *implies* $t = r$;
$\sigma_b(t,r) = \sigma_b(r,t)$;
$\sigma_b(t,r) \leq s[\sigma_b(t,z) + \sigma_b(z,r)]$.

The pair (M,σ_b) is called a b-metric-like space.

In a b-metric-like space (M,σ_b), if $t,r \in M$ and $\sigma_b(t,r) = 0$, then $t = r$; however, the converse need not be true, and $\sigma_b(t,t)$ may be positive for $t \in M$.

Example 1 ([5])**.** *Let $M = R^+ \cup \{0\}$. Define the function $\sigma_b : M^2 \to [0,\infty)$ by $\sigma_b(t,r) = (t+r)^2$ for all $t,r \in M$. Then, (M,σ_b) is a b-metric-like space with parameter $s = 2$.*

Definition 4 ([5])**.** *Let (M,σ_b) be a b-metric-like space with parameter s, and let $\{t_n\}$ be any sequence in M and $t \in M$. Then, the following applies:*

(a) *The sequence $\{t_n\}$ is said to be convergent to t if $\lim_{n\to\infty} \sigma_b(t_n,t) = \sigma_b(t,t)$;*
(b) *The sequence $\{t_n\}$ is said to be a Cauchy sequence in (M,σ_b) if $\lim_{n,m\to\infty} \sigma_b(t_n,t_m)$ exists and is finite;*
(c) *The pair (M,σ_b) is called a complete b-metric-like space if, for every Cauchy sequence $\{t_n\}$ in M, there exists $t \in M$ such that $\lim_{n,m\to\infty} \sigma_b(t_n,t_m) = \lim_{n\to\infty} \sigma_b(t_n,t) = \sigma_b(t,t)$.*

Preposition 1 ([5])**.** *Let (M,σ_b) be a b-metric-like space with parameter s, and let $\{t_n\}$ be any sequence in M with $t \in M$ such that $\lim_{n\to\infty} \sigma_b(t_n,t) = 0$. Then, the following applies:*

(a) *t is unique;*
(b) *$s^{-1}\sigma_b(t,r) \leq \lim_{n\to\infty} \sigma_b(t_n,r) \leq s\sigma_b(t,r)$ for all $r \in M$.*

In 2012, Samet et al. [6] introduced the class of α-admissible mappings.

Definition 5. *Let M be a non-empty set, f a self-map on M, and $\alpha : M \times M \to R^+$ a given function. We say that f is an α-admissible mapping if $\alpha(t,r) \geq 1$ implies that $\alpha(ft,fr) \geq 1$ for all $t,r \in M$.*

Definition 6 ([7])**.** *Let (M,σ_b) be a b-metric-like space with parameter $s \geq 1$, and let $\alpha : M \times M \to R^+$ be a function, and arbitrary constants q,p such that $q \geq 1$ and $p \geq 2$. A self-mapping $f : M \to M$ is α_{qs^p}-admissible if $\alpha(t,r) \geq qs^p$ implies $\alpha(ft,fr) \geq qs^p$, for all $t,r \in M$.*

Examples 3.3 and 3.4 in Reference [20] illustrate Definition 6.

Recently, Aydi et al. [8] generalized Definition 5 to a pair of mappings.

Definition 7. *For a non-empty set M, let $f, g : M \to M$ and $\alpha : M \times M \to R^+$ be mappings. We say that (f, g) is an α-admissible pair if for all $t, r \in M$, we have*

$$\alpha(t, r) \geq 1 \Rightarrow \alpha(ft, gr) \geq 1 \text{ and } \alpha(gr, ft) \geq 1.$$

Examples 1.13 and 1.14 in Reference [8] illustrate Definition 7.

Lemma 1 ([8])**.** *Let (M, σ_b) be a b-metric-like space with parameter $s \geq 1$. If a given mapping $f : M \to M$ is continuous at $u \in M$, then, for all sequences $\{t_n\}$ in M convergent to u, we have that the sequence ft_n is convergent to the point fu, that is*

$$\lim_{n\to\infty} \sigma_b(ft_n, fu) = \sigma_b(fu, fu).$$

Lemma 2 ([5])**.** *Let (M, σ_b) be a b-metric-like space with parameter $s \geq 1$, and suppose that $\{t_n\}$ and $\{r_n\}$ are σ_b-convergent to t and r, respectively. Then we have*

$$\begin{aligned} s^{-2}\sigma_b(t, r) - s^{-1}\sigma_b(t, t) - \sigma_b(r, r) &\leq \liminf_{n\to\infty} \sigma_b(t_n, r_n) \\ \leq \limsup_{n\to\infty} \sigma_b(t_n, r_n) &\leq s\sigma_b(t, t) + s^2\sigma_b(r, r) + s^2\sigma_b(t, r). \end{aligned}$$

In particular, if $\sigma_b(t, r) = 0$, then we have $\lim_{n\to\infty} \sigma_b(t_n, r_n) = 0$.

Moreover, for each $z \in M$, we have

$$\begin{aligned} s^{-1}\sigma_b(t, z) - \sigma_b(t, t) &\leq \liminf_{n\to\infty} \sigma_b(t_n, z) \\ &\leq \limsup_{n\to\infty} \sigma_b(t_n, z) \leq s\sigma_b(t, z) + s\sigma_b(t, t). \end{aligned}$$

In particular, if $\sigma_b(t, t) = 0$, then

$$\begin{aligned} s^{-1}\sigma_b(t, z) &\leq \liminf_{n\to\infty} \sigma_b(t_n, z) \\ &\leq \limsup_{n\to\infty} \sigma_b(t_n, z) \leq s\sigma_b(t, z). \end{aligned}$$

The following result is useful.

Lemma 3 ([7])**.** *Let (M, σ_b) be a b-metric-like space with parameter $s \geq 1$. Then, the following applies:*

(a) *If $\sigma_b(t, r) = 0$, then $\sigma_b(t, t) = \sigma_b(r, r) = 0$;*
(b) *If $\{t_n\}$ is a sequence such that $\lim_{n\to\infty} \sigma_b(t_n, t_{n+1}) = 0$, then we have $\lim_{n\to\infty} \sigma_b(t_n, t_n) = \lim_{n\to\infty} \sigma_b(t_{n+1}, t_{n+1}) = 0$;*
(c) *If $t \neq r$, then $\sigma_b(t, r) > 0$.*

Lemma 4. *Let (M, σ_b) be complete b-metric-like space with parameter $s \geq 1$, and let $\{t_n\}$ be a sequence such that*

$$\lim_{n\to\infty} \sigma_b(t_n, t_{n+1}) = 0. \tag{1}$$

If for the sequence $\{t_n\}$ $\lim_{n,m\to\infty} \sigma_b(t_n, t_m) \neq 0$, then there exists $\varepsilon > 0$, and sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers with $n_k > m_k > k$, such that

$$\begin{aligned} \varepsilon \leq \sigma_b(t_{2n_k}, t_{2m_k}) \leq \varepsilon s,\ \varepsilon/s \leq \limsup_{k\to\infty} \sigma_b(t_{2m_k}, t_{2n_k-1}) \leq \varepsilon s, \\ \varepsilon/s^2 \leq \limsup_{k\to\infty} \sigma_b(t_{2n_k-1}, t_{2m_k+1}) \leq \varepsilon s^2, \end{aligned}$$

$$\varepsilon/s \leq \limsup_{k\to\infty} \sigma_b(t_{2m_k+1}, t_{2n_k}) \leq \varepsilon s^2.$$

Proof. Suppose that $\{t_{2n}\}$ is a sequence in (M, σ_b) satisfying (1) such that $\lim_{n,m\to\infty} \sigma_b(t_{2n}, t_{2m}) \neq 0$. Then, there exists $\varepsilon > 0$, and sequences $\{m(k)\}_{k=1}^{\infty}$ and $\{n(k)\}_{k=1}^{\infty}$ of positive integers with $n_k > m_k > k$, such that n_k is smallest index for which

$$n_k > m_k > k, \quad \sigma_b(t_{2n_k}, t_{2m_k}) \geq \varepsilon. \tag{2}$$

This means that

$$\sigma_b(t_{2n_k-2}, t_{2m_k}) < \varepsilon. \tag{3}$$

Consider

$$\begin{aligned} \varepsilon &\leq \sigma_b(t_{2n_k}, t_{2m_k}) \leq s\sigma_b(t_{2n_k}, t_{2n_k-2}) + s\sigma_b(t_{2n_k-2}, t_{2m_k}) \\ &\leq s^2\sigma_b(t_{2n_k}, t_{2n_k-1}) + s^2\sigma_b(t_{2n_k-1}, t_{2n_k-2}) + s\sigma_b(t_{2n_k-2}, t_{2m_k}). \end{aligned} \tag{4}$$

Hence, by (4), and (1)–(3), we have

$$\limsup_{k\to\infty} \sigma_b(t_{2n_k}, t_{2m_k}) \leq \varepsilon s. \tag{5}$$

Again, we consider

$$\varepsilon \leq \sigma_b(t_{2m_k}, t_{2n_k}) \leq s\sigma_b(t_{2m_k}, t_{2n_k-1}) + s\sigma_b(t_{2n_k-1}, t_{2n_k}). \tag{6}$$

Taking the limit superior in (6), we get

$$\limsup_{k\to\infty} \sigma_b(t_{2m_k}, t_{2n_k-1}) \geq \frac{\varepsilon}{s}.$$

Also,

$$\sigma_b(t_{2m_k}, t_{2n_k-1}) \leq s\sigma_b(t_{2m_k}, t_{2n_k-2}) + s\sigma_b(t_{2n_k-2}, t_{2n_k-1}). \tag{7}$$

By (7), and in view of (1) and (3), we get

$$\limsup_{k\to\infty} \sigma_b(t_{2m_k}, t_{2n_k-1}) \leq \varepsilon s. \tag{8}$$

As a result,

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} \sigma_b(t_{2m_k}, t_{2n_k-1}) \leq \varepsilon s.$$

Again, we consider

$$\begin{aligned} \varepsilon &\leq \sigma_b(t_{2n_k-1}, t_{2m_k+1}) \leq s\sigma_b(t_{2n_k-1}, t_{2n_k-2}) + s\sigma_b(t_{2n_k-2}, t_{2m_k+1}) \\ &\leq s\sigma_b(t_{2n_k-1}, t_{2n_k-2}) + s^2\sigma_b(t_{2n_k-2}, t_{2m_k}) + s^2\sigma_b(t_{2m_k}, t_{2m_k+1}). \end{aligned} \tag{9}$$

By (9), we get

$$\limsup_{k\to\infty} \sigma_b(t_{2n_k-1}, t_{2m_k+1}) \leq \varepsilon s^2. \tag{10}$$

Also,

$$\begin{aligned} \varepsilon &\leq \sigma_b(t_{2n_k}, t_{2m_k}) \leq s\sigma_b(t_{2n_k}, t_{2n_k-1}) + s\sigma_b(t_{2n_k-1}, t_{2m_k}) \\ &\leq s\sigma_b(t_{2n_k}, t_{2n_k-1}) + s^2\sigma_b(t_{2n_k-1}, t_{2m_k+1}) + s^2\sigma_b(t_{2m_k+1}, t_{2m_k}). \end{aligned}$$

Letting $k \to \infty$ and using (1), we get

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} \sigma_b(t_{2n_k-1}, t_{2m_k+1}). \tag{11}$$

From (5) and (6), we have

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} \sigma_b(t_{2n_k-1}, t_{2m_k+1}) \leq \varepsilon s^2.$$

Consider

$$\varepsilon \leq \sigma_b(t_{2n_k}, t_{2m_k}) \leq s\sigma_b(t_{2n_k}, t_{2m_k+1}) + s\sigma_b(t_{2m_k+1}, t_{2m_k}).$$

Letting $k \to \infty$ and by (1), we obtain

$$\frac{\varepsilon}{s} \leq \lim_{k\to\infty} \sup \sigma_b(t_{2m_k+1}, t_{2n_k}). \tag{12}$$

From

$$\sigma_b(t_{2m_k+1}, t_{2n_k}) \leq s\sigma_b(t_{2m_k+1}, t_{2m_k}) + s\sigma_b(t_{2m_k}, t_{2n_k}),$$

using (1) and (5), we get

$$\lim_{k\to\infty} \sup \sigma_b(t_{2m_k+1}, t_{2n_k}) \leq \varepsilon s^2,$$

and also

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} \sigma_b(t_{2m_k+1}, t_{2n_k}) \leq \varepsilon s^2.$$

□

Lemma 5. *Let $\{t_n\}$ be a sequence in a b-metric-like space (M, σ_b) with parameter $s \geq 1$, such that $\sigma_b(t_n, t_{n+1}) \leq \lambda\sigma_b(t_{n-1}, t_n)$ for all $n \in N$, for some λ, where $0 \leq \lambda < 1/s$. Then, the following applies:*

1. $\lim_{n\to\infty} \sigma_b(t_n, t_{n+1}) = 0$,
2. *$\{t_n\}$ is a Cauchy sequence in (M, σ_b) and $\lim_{n,m\to\infty} \sigma_b(t_n, t_m) = 0$.*

Proof. For the proof of the previous lemma, one can use the following clear inequalities:

$$\sigma_b(t_{n+1}, t_{n+2}) \leq \lambda\sigma_b(t_n, t_{n+1}) \leq \lambda^2\sigma_b(t_{n-1}, t_n) \leq \ldots \leq \lambda^{n+1}\sigma_b(t_0, t_1),$$

and

$$\sigma_b(t_m, t_n) \leq s\sigma_b(t_m, t_{m+1}) + s^2\sigma_b(t_{m+1}, t_{m+2}) + \ldots + s^{n-m-1}\sigma_b(t_{n-2}, t_{n-1}) + s^{n-m}\sigma_b(t_{n-1}, t_n),$$

where $m, n \in N$ and $n > m$. □

2. Main Results

We start the main section with generalization of Definitions 5 and 6, introducing α_{s^p}-admissible pairs of mappings and properties H_{s^p} and U_{s^p}.

Definition 8. *Let (f, g) be a pair of self-mappings in a b-metric-like space (M, σ_b) with parameter $s \geq 1$, and $\alpha : M \times M \to R^+$ be a given mapping, and some constant p with $p \geq 2$. We say that (f, g) is an α_{s^p}-admissible pair if $\alpha(t, r) \geq s^p$, implies $min\{\alpha(ft, gr), \alpha(gr, ft)\} \geq s^p$ for all $t, r \in M$.*

Remark 1.

- *By choosings = 1 and $g = f$, we derive further consequences of Definition 8.*

- *The function alpha is considered asymmetric.*

Example 2. *Let $M = R$ and $\alpha : M \times M \to R^+$ as $\alpha(t,r) = s^2 e^{tr}$ for all $t, r \in M$ and $s \geq 1$. Define the self-mappings f, g on M by $ft = t^2$ and $gt = t^4$. Then, (f,g) is an α_{s^p}-admissible pair, where $p = 2$.*

Example 3. *Let $M = R$ and constants $s \geq 1$, $p = 2$. Let $\alpha : M \times M \to R^+$ and $f, g : M \to M$ be defined by*

$$\alpha(t,r) = \begin{cases} s^2 & if\ t,r \in [0,1] \\ 0 & otherwise \end{cases}, \quad ft = t/3 \text{ and } gt = t^3.$$

Then, (f,g) is an α_{s^p}-admissible pair.

In the sequel, in a complete b-metric-like space (M, σ_b), we consider useful properties below.

(H_{s^p}): If $\{t_n\}$ is a sequencein M such that $t_n \to t \in M$ as $n \to \infty$ and $\alpha(t_n, t_{n+1}) \geq s^p$ and $\alpha(t_{n+1}, t_n) \geq s^p$, then there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $\alpha(t_{n_k}, t) \geq s^p$ and $\alpha(t, t_{n_k}) \geq s^p$ for all $k \in \mathrm{N}$.

(U_{s^p}): For all $t, r \in CF(f,g)$, we have $\alpha(t,r) \geq s^p$, where $CF(f,g)$ denotes the set of common fixed points of f and g (also $Fix(f)$ is the set of fixed points of f).

Now, we present some fixed point theorems for contractions of rational type in the setting of b-metric-like spaces. These theorems generalize some results appearing in References [9,10] and others in the literature.

According to Definition 3.1 in Reference [7], for $q = 1$, we obtain the following definition:

Definition 9. *Let (M, σ_b) be a complete b-metric-like space with parameter $s \geq 1$, and $f : M \to M$ and $\alpha : M \times M \to R^+$ be given mappings. We say that f is a generalized $\alpha_{s^p} - k$ rational contractive mapping (short $(\alpha_{s^p} - k, R)$ contraction) if there exists $\gamma : R^+ \times R^+ \to R^+$ as a continuous function with $\gamma(x,x) \leq 1$ and $\gamma(x,0) \leq 1$ for all $x \in R^+$, which satisfy the following condition:*

$$\alpha(t,r)\sigma_b(ft, fr) \leq kR(t,r), \tag{13}$$

for all $t, r \in M$ with $\alpha(t,r) \geq s^p$, where

$$R(t,r) = \max\{\sigma_b(t,r), \sigma_b(r, fr)\gamma(\sigma_b(t, ft), \sigma_b(t,r))\}.$$

Remark 2. *If in Definition 9, we take $\alpha(t,r) = s^p$ then we obtain a generalized $(s - k, R)$ contraction. Also other remarks can be taken for certain choices of the coefficients s and p.*

Theorem 1. *Let f be a continuous self-mapping in a complete b-metric-like space (M, σ_b) with coefficient $s \geq 1$, and $\alpha : M \times M \to R^+$ a given function. If the following conditions are satisfied:*

(i) *f is an α_{s^p}-admissible mapping;*
(ii) *f is an $(\alpha_{s^p} - k, R)$ contractive mapping;*
(iii) *there exists an $t_0 \in M$ such that $\alpha(t_0, ft_0) \geq s^p$.*

Then, f has a fixed point.

Proof. By hypothesis (iii), we have $t_0 \in M$ satisfying $\alpha(t_0, ft_0) \geq s^p$. With this $t_0 \in M$ as an initial point, we define an iterative sequence $\{t_n\}$ in M by $t_{n+1} = ft_n$ for all $n = 0, 1, 2, \ldots$. If $\sigma_b(t_n, t_{n+1}) = 0$ for some n, then $t_n = t_{n+1} = ft_n$ and t_n is a fixed point of f and the proof is done.

Hence, we assume that $\sigma_b(t_n, t_{n+1}) > 0$ (that is $t_n \neq t_{n+1}$) for all n.

From hypothesis (i), we get that

$$\alpha(t_0, t_1) = \alpha(t_0, ft_0) \geq s^p, \alpha(ft_0, ft_1) = \alpha(t_1, t_2) \geq s^p \text{ and } \alpha(ft_1, ft_2) = \alpha(t_2, t_3) \geq s^p.$$

On continuing this process, by induction, we get that

$$\alpha(t_n, t_{n+1}) \geq s^p \text{ for all } n.$$

Hence, applying Condition (13), we have

$$\begin{aligned} s^p \sigma_b(t_1, t_2) &= s^p \sigma_b(ft_0, ft_1) \\ &\leq \alpha(t_0, t_1)\sigma_b(ft_0, ft_1) \\ &\leq kR(t_0, t_1), \end{aligned} \tag{14}$$

where

$$\begin{aligned} R(t_0, t_1) &= \max\{\sigma_b(t_0, t_1), \sigma_b(t_1, ft_1)\gamma(\sigma_b(t_0, ft_0), \sigma_b(t_0, t_1))\} \\ &= \max\{\sigma_b(t_0, t_1), \sigma_b(t_1, t_2)\gamma(\sigma_b(t_0, t_1), \sigma_b(t_0, t_1))\} \\ &\leq \max\{\sigma_b(t_0, t_1), \sigma_b(t_1, t_2)\}. \end{aligned}$$

Now, if $\sigma_b(t_0, t_1) \leq \sigma_b(t_1, t_2)$, then $R(t_0, t_1) = \sigma_b(t_1, t_2)$,and from (14) we have

$$s^p \sigma_b(t_1, t_2) \leq k\sigma_b(t_1, t_2),$$

which is a contradiction. Therefore,

$$\max\{\sigma_b(t_0, t_1), \sigma_b(t_1, t_2)\} = \sigma_b(t_0, t_1), \tag{15}$$

and Inequality (14) implies that

$$\sigma_b(t_1, t_2) \leq \frac{k}{s^p}\sigma_b(t_0, t_1) = \lambda\sigma_b(t_0, t_1), \tag{16}$$

where $0 < \lambda = k/s^p < 1/s$.

In the same manner, one can show that

$$\sigma_b(t_2, t_3) \leq \frac{k}{s^p}\sigma_b(t_1, t_2) = \lambda\sigma_b(t_1, t_2).$$

Furthermore, in general, we have that

$$\sigma_b(t_n, t_{n+1}) \leq \lambda\sigma_b(t_{n-1}, t_n) \text{ for all } n \in N. \tag{17}$$

Then, in view of Lemma 4, we get

$$\lim_{n\to\infty} \sigma_b(t_n, t_{n+1}) = 0, \tag{18}$$

$\{t_n\}$ as a Cauchy sequence, and $\lim_{n,m\to\infty} \sigma_b(t_n, t_m) = 0$. Since M is complete, there exists $z \in M$ such that

$$0 = \lim_{n,m\to\infty} \sigma_b(t_n, t_m) = \lim_{n\to\infty} \sigma_b(t_n, z) = \sigma_b(z, z). \tag{19}$$

By using Lemma 1, we have $ft_n \to fz$, that is $\lim_{n\to\infty} \sigma_b(ft_n, fz) = \sigma_b(fz, fz)$.

On the other side, $\lim_{n\to\infty} \sigma_b(t_n, z) = 0 = \sigma_b(z, z)$; thus, by Preposition 1,

$$s^{-1}\sigma_b(z, fz) \leq \lim_{n\to\infty} \sigma_b(t_n, fz) \leq s\sigma_b(z, fz).$$

This implies that

$$s^{-1}\sigma_b(z, fz) \leq \sigma_b(fz, fz) \leq s\sigma_b(z, fz). \tag{20}$$

Since $p \geq 1$, in view of (19) and (20), and using (13), we have

$$\begin{aligned} s^p\sigma_b(z,fz) &\leq \alpha(z,z)\sigma_b(fz,fz) \leq kR(z,z) \\ &= k\max\{\sigma_b(z,z), \sigma_b(z,fz)\gamma(\sigma_b(z,fz),\sigma_b(z,z))\} \\ &= k\max\{0, \sigma_b(z,fz)\gamma(\sigma_b(z,fz),0)\} \\ &\leq k\sigma_b(z,fz). \end{aligned} \tag{21}$$

From (21), we get $\sigma_b(z,fz) = 0$, that is, $fz = z$ and z is a fixed point of f. □

Example 4. *Consider the set $M = [0,1]$ equipped with a b-metric-like $\sigma_b(t,r) = (t+r)^2$ for all $t,r \in M$. The pair (M,σ_b) is a complete b-metric-like space with coefficient $s = 2$. Define $f : M \to M$ and $\alpha : M \times M \to R^+$ by*

$$ft = \begin{cases} t/6 & if\, t \in [0,1] \\ 4 & if\, t > 1 \end{cases} \text{ and } \alpha(t,r) = \begin{cases} s^2 & t,r \in [0.1] \\ 0 & otherwise \end{cases}.$$

It is easy to show that conditions (i) and (iii) hold. With regards to (ii), for all $t,r \in [0,1]$, $s = 2$,we have

$$\begin{aligned} \alpha(t,r)\sigma_b(ft,fr) &= s^2\sigma_b(ft,fr) = 2^2\left(\tfrac{t}{6}+\tfrac{r}{6}\right)^2 = 4\tfrac{(t+r)^2}{36} = \tfrac{4}{36}(t+r)^2 \\ &= \tfrac{1}{9}\sigma_b(t,r) \leq k\sigma_b(t,r) \leq kR(t,r). \end{aligned}$$

Here, the conditions of Theorem 1 are verified and we see that $\{0,4\} \subset Fix(f)$.

Below, we present analogous theorems for Theorem 1 using properties H_{s^p} and U_{s^p}.

Theorem 2. *The conclusion of Theorem 1 remains true if the continuity property of the self-mapping f on (M,σ_b) is replaced by the property H_{s^p}.*

Proof. From arguments similar to the proof of Theorem 1, we obtain that the sequence $\{t_n\}$ defined by $t_{n+1} = ft_n$ for all $n \geq 0$ is a Cauchy sequence convergent to z, such that (18)–(20) hold. Since the condition H_{s^p} is satisfied, there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $\alpha(t_{n_k},z) \geq s^p$ for all $k \in \mathrm{N}$. Applying (13), with $t = t_{n_k}$ and $r = z$, we have

$$\begin{aligned} s^p\sigma_b(ft_n,fz) &\leq \alpha(t_n,z)\sigma_b(ft_n,fz) \leq kR(t_n,z) \\ &= k\max\{\sigma_b(t_n,z), \sigma_b(z,fz)\gamma(\sigma_b(t_n,ft_n),\sigma_b(t_n,z))\} \\ &= k\max\{\sigma_b(t_n,z), \sigma_b(z,fz)\gamma(\sigma_b(t_n,t_{n+1}),\sigma_b(t_n,z))\}. \end{aligned} \tag{22}$$

Taking the upper limit as $n \to \infty$ in (22), using Lemma 2, and (18), (19), and the property of γ, we obtain

$$s^{p-1}\sigma_b(z,fz) = s^p s^{-1}\sigma_b(z,fz) \leq k\sigma_b(z,fz). \tag{23}$$

From (23), $\sigma_b(fz,z) = 0$, which implies that $fz = z$. Hence, z is a fixed point of f. □

Theorem 3. *Adding condition U_{s^p} to the hypotheses of Theorem 1 (respective to Theorem 2), we obtain the uniqueness of fixed point of f.*

Proof. On the contrary we assume that $z,v \in Fix(f)$ with $z \neq v$. By the hypothesis U_{s^p}, $\alpha(z,v) \geq s^p$. We shall now prove that, if z is a fixed point of f, then $\sigma_b(z,z) = 0$. Applying (13), we have

$$\begin{aligned} s^p\sigma_b(z,z) &= s^p\sigma_b(fz,fz) \leq \alpha(z,z)\sigma_b(fz,fz) \\ &\leq kR(z,z) \\ &= k\max\{\sigma_b(z,z),\sigma_b(z,fz)\gamma(\sigma_b(z,fz),\sigma_b(z,z))\} \\ &= k\max\{\sigma_b(z,z),\sigma_b(z,z)\gamma(\sigma_b(z,z),\sigma_b(z,z))\} \\ &= k\sigma_b(z,z), \end{aligned}$$

which implies that $\sigma_b(z,z) = 0$.

Again, by the hypothesis U_{s^p} and applying (13), we have

$$\begin{aligned} s^p\sigma_b(z,v) &= s^p\sigma_b(fz,fv) \leq \alpha(z,v)\sigma_b(fz,fv) \\ &\leq kR(z,v) \\ &= k\max\{\sigma_b(z,v),\sigma_b(v,fv)\gamma(\sigma_b(z,fz),\sigma_b(z,v))\} \\ &= k\max\{\sigma_b(z,v),\sigma_b(v,v)\gamma(\sigma_b(z,z),\sigma_b(z,v))\} \\ &= k\sigma_b(z,v), \end{aligned}$$

which implies that $\sigma_b(z,v) = 0$, which is a contradiction. Hence, $z = v$. □

Some corollaries can be derived from above theorems, and to avoid repetition, we include all the properties H_{s^p} and U_{s^p}.

Corollary 1. *Let (M,σ_b) be a complete b-metric-like space with coefficient $s \geq 1$ and f a self-mapping on M satisfying*

$$s^p\sigma_b(ft,fr) \leq kR(t,r)$$

for all $t,r \in M$, where $0 \leq k < 1$. Then, f has a unique fixed point in M.

Proof. It suffices to take $\alpha(t,r) = s^p$ in Theorem 1. □

If in Theorem 1 we take $k = 1/s$ and $\alpha(t,r) = s^p$, then we obtain a weaker contractive condition below.

Corollary 2. *Let (M,σ_b) be a complete b-metric-like space with coefficient $s > 1$. Let f be a self-map on M satisfying*

$$s^{p+1}\sigma_b(ft,fr) \leq R(t,r)$$

for all $t,r \in M$. Then, f has a unique fixed point.

Definition 10. *Let (M,σ_b) be a b-metric-like space with coefficient $s \geq 1$. A self-mapping f on M is an α_{s^p}-Dass and Gupta contraction if it satisfies*

$$\alpha(t,r)\sigma_b(ft,fr) \leq \alpha\frac{\sigma_b(r,fr)[1+\sigma_b(t,ft)]}{1+\sigma_b(t,r)} + \beta\sigma_b(t,r)$$

for all $t,r \in M$, where $0 \leq \alpha + \beta < 1$.

Corollary 3. *Conclusions of Theorem 3 remain true if Condition (ii) is replaced byan α_{s^p}-Dass and Gupta contractive condition.*

Proof. Define $\gamma(x,y) = (1+x)/(1+y)$ for all $x,y \in R^+$. Then, the inequality of Definition 10 becomes

$$\begin{aligned}\alpha(t,r)\sigma_b(ft,fr) &\leq \alpha\tfrac{\sigma_b(r,fr)[1+\sigma_b(t,ft)]}{1+\sigma_b(t,r)} + \beta\sigma_b(t,r)\\ &= \alpha\sigma_b(r,fr)\gamma(\sigma_b(t,ft),\sigma_b(t,r)) + \beta\sigma_b(t,r)\\ &\leq (\alpha+\beta)\max\{\sigma_b(t,r),\sigma_b(r,fr)\gamma(\sigma_b(t,ft),\sigma_b(t,r))\}\\ &= k\max\{\sigma_b(t,r),\sigma_b(r,fr)\gamma(\sigma_b(t,ft),\sigma_b(t,r))\},\end{aligned}$$

where $k = \alpha + \beta < 1$, and the inequality is a special case of (13). □

Definition 11. *Let (M,σ_b) be a b-metric-like space with coefficient $s \geq 1$. A self-mapping f on M is an α_{s^p}-Jaggi contraction if it satisfies*

$$\alpha(t,r)\sigma_b(ft,fr) \leq \alpha\frac{\sigma_b(t,ft)\sigma_b(r,fr)}{\sigma_b(t,r)} + \beta\sigma_b(t,r)$$

for all $t,r \in M$ with $\sigma_b(t,r) > 0$, where $0 \leq \alpha + \beta < 1$.

Corollary 4. *If we replace Condition (ii) by an α_{s^p}-Jaggi contractive condition, then the conclusions of Theorem 1 (and respective to Theorems 2 and 3) remain true.*

Proof. Use $\gamma(x,y) = x/y$ for all $x,y \in R^+$ and $y \neq 0$ in the inequality of Definition 11. □

Remark 3.

(1) *Theorem 1 extends and generalizes Theorems 3.4 in Reference [9] and 3.13 in Reference [10].*
(2) *If we use different choices for the function γ (for example, $\gamma(x,y) = x/(x+y)$ with $x+y \neq 0$, $\gamma(x,y) = \sqrt{xy}/(1+y)$, $\gamma(x,y) = x/(1+y)$, ... for all $x,y \in R^+$), we obtain various corollaries.*
(3) *Similarly, we can get the corresponding conclusions in b-metric space.*
(4) *By taking $\alpha(t,r) = s^p$ in previous theorems, we obtain results for generalized $(s-k,R)$ contractions.*

Before proceeding further with the ongoing main theorem, we use the following denotations:

Ψ is the class of functions $\psi : R^+ \to R^+$ continuous and increasing;
Φ is the class of functions $\phi : R^+ \to R^+$ continuous and $\phi(x) < \psi(x)$ for every $x > 0$;
S is the class of functions $\beta : R^+ \to [0,1)$ satisfying the condition: $\beta(x_n) \to 1$ *as* $n \to \infty$ implies that $x_n \to 0$ *as* $n \to \infty$.

Let $f,g : M \to M$ be two self-mappings,

$$E(t,r) = \max\{\sigma_b(t,r),\sigma_b(t,ft),\sigma_b(r,gr),\frac{\sigma_b(t,gr)+\sigma_b(r,ft)}{4s}\} \tag{24}$$

for all $t,r \in M$.

Now, we introduce the definition of $(\alpha_{s^p} - \psi,\phi)$-contraction pairs of mappings.

Definition 12. *Let (f,g) be a pair of self-mappings in a b-metric-like space (M,σ_b) with coefficient $s \geq 1$. Also, suppose that $\alpha : M \times M \to [0,\infty)$ exists and some constant p such that $p \geq 2$. A pair (f,g) is called a generalized $(\alpha_{s^p} - \psi,\phi)$-contraction pair, if they satisfy*

$$\psi(\alpha(t,r)\sigma_b(ft,gr)) \leq \phi(E(t,r)) \tag{25}$$

for all $t,r \in M$ with $\alpha(t,r) \geq s^p$, where $\psi \in \Psi, \phi \in \Phi$ and $E(t,r)$ is defined by (24).

Remark 4.

(1) *If we take $g = f$, then we obtain the definition of $(\alpha_{s^p} - \psi,\phi)$-contractive mapping as in Reference [20].*

(2) *For s = 1, the definition reduces to an $\alpha - (\psi, \phi)$-contraction pair in a metric space.*
(3) *The above definition reduces to a (ψ, ϕ, s)-contraction pair for $\alpha(t, r) = s^p$.*

Theorem 4. *Let (f, g) be a pair of self-mappings in a complete b-metric-like space (M, σ_b) with coefficient $s \geq 1$. If (f, g) is a generalized $(\alpha_{s^p} - \psi, \phi)$-contraction pair, and the following conditions hold:*

(i) *(f, g) is an α_{s^p}-admissible pair;*
(ii) *There exists $t_0 \in M$ such that $\min\{\alpha(t_0, ft_0), \alpha(ft_0, t_0)\} \geq s^p$;*
(iii) *Property H_{s^p} is satisfied.*

Then, f and g have a common fixed point $u \in M$. Moreover, f and g have a unique common fixed point if property U_{s^p} is satisfied.

Proof. Since (f, g) is an α_{s^p}-admissible pair, then $t_0 \in M$ exists with $\alpha(t_0, ft_0) \geq s^p$ and $\alpha(ft_0, t_0) \geq s^p$. Take $t_1 = ft_0$ and $t_2 = gt_1$. By induction, we construct an iterative sequence $\{t_n\}$ in M, such that $t_{2n+1} = ft_{2n}$ and $t_{2n+2} = gt_{2n+1}$ for all $n \geq 0$. By Condition (ii), we have $\alpha(t_0, t_1) \geq s^p$ and $\alpha(t_1, t_0) \geq s^p$, and using (i), we obtain that

$$\alpha(t_1, t_2) = \alpha(ft_0, gt_1) \geq s^p \text{ and } \alpha(t_2, t_1) = \alpha(gt_1, ft_0) \geq s^p.$$

Also, we have

$$\alpha(t_3, t_2) = \alpha(ft_2, gt_1) \geq s^p \text{ and } \alpha(t_2, t_3) = \alpha(gt_1, ft_2) \geq s^p.$$

In general, by induction, we obtain

$$\alpha(t_n, t_{n+1}) \geq s^p \text{ and } \alpha(t_{n+1}, t_n) \geq s^p \text{ for all } n \geq 0. \tag{26}$$

If for some n, we have $\sigma_b(t_{2n+1}, t_{2n}) = 0$, then $\sigma_b(t_{2n+1}, t_{2n}) = 0$ gives $\sigma_b(t_{2n+1}, t_{2n+2}) = 0$. Indeed, by (24), we have

$$\begin{aligned}
E(t_{2n}, t_{2n+1}) &= \max\left\{ \begin{array}{l} \sigma_b(t_{2n}, t_{2n+1}), \sigma_b(t_{2n}, ft_{2n}), \sigma_b(t_{2n+1}, gt_{2n+1}), \\ \frac{\sigma_b(t_{2n+1}, ft_{2n}) + \sigma_b(t_{2n}, gt_{2n+1})}{4s} \end{array} \right\} \\
&= \max\left\{ \begin{array}{l} \sigma_b(t_{2n}, t_{2n+1}), \sigma_b(t_{2n}, t_{2n+1}), \sigma_b(t_{2n+1}, t_{2n+2}), \\ \frac{\sigma_b(t_{2n+1}, t_{2n+1}) + \sigma_b(t_{2n}, t_{2n+2})}{4s} \end{array} \right\} \\
&\leq \max\left\{ \begin{array}{l} \sigma_b(t_{2n}, t_{2n+1}), \sigma_b(t_{2n}, t_{2n+1}), \sigma_b(t_{2n+1}, t_{2n+2}), \\ \frac{2s\sigma_b(t_{2n}, t_{2n+1}) + s[\sigma_b(t_{2n}, t_{2n+1}) + \sigma_b(t_{2n+1}, t_{2n+2})]}{4s} \end{array} \right\} \\
&= \max\left\{ 0, 0, \sigma_b(t_{2n+1}, t_{2n+2}), \frac{\sigma_b(t_{2n+1}, t_{2n+2})}{4} \right\} = \sigma_b(t_{2n+1}, t_{2n+2}).
\end{aligned}$$

Using (25), we obtain

$$\begin{aligned}
\psi(\sigma_b(t_{2n+1}, t_{2n+2})) &\leq \psi(s^p \sigma_b(t_{2n+1}, t_{2n+2})) = \psi(s^p \sigma_b(ft_{2n}, gt_{2n+1})) \\
&\leq \psi(\alpha(t_{2n}, t_{2n+1}) \sigma_b(ft_{2n}, gt_{2n+1})) \\
&\leq \phi(E(t_{2n}, t_{2n+1})) = \phi(\sigma_b(t_{2n+1}, t_{2n+2})).
\end{aligned}$$

By property of ψ, ϕ, the previous inequality implies $\sigma_b(t_{2n+1}, t_{2n+2}) = 0$, that is, $t_{2n+1} = t_{2n+2}$. We deduce that $t_{2n} = t_{2n+1} = ft_{2n}$ and $t_{2n} = t_{2n+2} = gt_{2n+1} = gft_{2n} = gt_{2n}$. Hence, t_{2n} is a common fixed point of f and g, and the proof is completed. Now, we assume that $\sigma_b(t_n, t_{n+1}) > 0$ for all $n \geq 0$. By (26), applying Condition (25), we have

$$\begin{aligned}\psi(\sigma_b(t_{2n+1},t_{2n})) &\leq \psi(s^p\sigma_b(t_{2n+1},t_{2n})) = \psi(s^p\sigma_b(ft_{2n},gt_{2n-1}))\\ &\leq \psi(\alpha(t_{2n},t_{2n-1})\sigma_b(ft_{2n},gt_{2n-1}))\\ &\leq \phi(E(t_{2n},t_{2n-1})) < \psi(E(t_{2n},t_{2n-1})),\end{aligned} \tag{27}$$

where

$$\begin{aligned}E(t_{2n},t_{2n-1}) &= \max\left\{\begin{array}{l}\sigma_b(t_{2n},t_{2n-1}),\sigma_b(t_{2n},ft_{2n}),\sigma_b(t_{2n-1},gt_{2n-1}),\\ \frac{\sigma_b(t_{2n-1},ft_{2n})+\sigma_b(t_{2n},gt_{2n-1})}{4s}\end{array}\right\}\\ &= \max\left\{\begin{array}{l}\sigma_b(t_{2n},t_{2n-1}),\sigma_b(t_{2n},t_{2n+1}),\sigma_b(t_{2n-1},t_{2n}),\\ \frac{\sigma_b(t_{2n-1},t_{2n+1})+\sigma_b(t_{2n},t_{2n})}{4s}\end{array}\right\}\\ &\leq \max\left\{\begin{array}{l}\sigma_b(t_{2n},t_{2n-1}),\sigma_b(t_{2n},t_{2n+1}),\sigma_b(t_{2n-1},t_{2n}),\\ \frac{s[\sigma_b(t_{2n-1},t_{2n})+\sigma_b(t_{2n},t_{2n+1})]+2s\sigma_b(t_{2n},t_{2n+1})}{4s}\end{array}\right\}.\end{aligned} \tag{28}$$

If we suppose that $\sigma_b(t_{2n-1},t_{2n}) < \sigma_b(t_{2n},t_{2n+1})$ for some $n \in$, then, from (28), we get

$$E(t_{2n-1},t_{2n}) \leq \sigma_b(t_{2n},t_{2n+1}). \tag{29}$$

Again, by (27) and property of ψ, we get

$$\sigma_b(t_{2n+1},t_{2n}) \leq E(t_{2n},t_{2n-1}). \tag{30}$$

From (29) and (30), we have

$$E(t_{2n-1},t_{2n}) = \sigma_b(t_{2n},t_{2n+1}). \tag{31}$$

From (27), and using (31), we obtain

$$\begin{aligned}\psi(s^p\sigma_b(t_{2n+1},t_{2n})) &= \psi(s^p\sigma_b(ft_{2n},gt_{2n-1})) \leq \psi(\alpha(t_{2n},t_{2n-1})\sigma_b(ft_{2n},gt_{2n-1}))\\ &\leq \phi(E(t_{2n},t_{2n-1})) = \phi(\sigma_b(t_{2n+1},t_{2n})) < \psi(\sigma_b(t_{2n+1},t_{2n})).\end{aligned} \tag{32}$$

By property of ψ, Inequality (32) implies $\sigma_b(t_{2n},t_{2n+1}) \leq \sigma_b(t_{2n-1},t_{2n})$ for all $n \in N$.

Hence, the sequence of nonnegative numbers $\{\sigma_b(t_{2n+1},t_{2n})\}$ is non-increasing. Thus, it converges to a nonnegative number, say $\delta \geq 0$. That is, $\lim_{n\to\infty}\sigma_b(t_n,t_{n+1}) = \delta$, and also $\lim_{n\to\infty}\sigma_b(t_n,t_{n+1}) = \lim_{n\to\infty}E(t_{n-1},t_n) = \delta$. If $\delta > 0$, and we consider

$$\begin{aligned}&\psi(s^p\sigma_b(t_{2n+1},t_{2n})) = \psi(s^p\sigma_b(ft_{2n},gt_{2n-1}))\\ &\leq \psi(\alpha(t_{2n},t_{2n-1})\sigma_b(ft_{2n},gt_{2n-1})) \leq \phi(E(t_{2n},t_{2n-1})) = \phi(\sigma_b(t_{2n+1},t_{2n})),\end{aligned} \tag{33}$$

then, letting $n \to \infty$ in (33), we obtain $\psi(\delta) \leq \phi(\delta)$, which implies $\delta = 0$, that is,

$$\lim_{n\to\infty}\sigma_b(t_n,t_{n+1}) = \lim_{n\to\infty}E(t_{n-1},t_n) = 0. \tag{34}$$

Now, we prove that $\lim_{n,m\to\infty}\sigma_b(t_n,t_m) = 0$. It is sufficient to show that $\lim_{n,m\to\infty}\sigma_b(t_{2n},t_{2m}) = 0$. Suppose, on the contrary, that $\lim_{n,m\to\infty}\sigma_b(t_{2n},t_{2m}) \neq 0$. Then, using Lemma 4, we get that there exists $\varepsilon > 0$, and two subsequences $\{m_k\}$ and $\{n_k\}$ of positive integers, with $n_k > m_k > k$, such that

$$\begin{aligned}&\varepsilon \leq \limsup_{k\to\infty}\sigma_b(t_{2n_k},t_{2m_k}) \leq \varepsilon s,\ \tfrac{\varepsilon}{s} \leq \limsup_{k\to\infty}\sigma_b(t_{2m_k},t_{2n_k-1}) \leq \varepsilon s,\\ &\tfrac{\varepsilon}{s^2} \leq \limsup_{k\to\infty}\sigma_b(t_{2n_k-1},t_{2m_k+1}) \leq \varepsilon s^2,\ \tfrac{\varepsilon}{s} \leq \limsup_{k\to\infty}\sigma_b(t_{2m_k+1},t_{2n_k}) \leq \varepsilon s^2.\end{aligned} \tag{35}$$

Furthermore, $E(t,r)$ is

$$\begin{aligned} EI\big(t_{2m_k}, t_{2n_k-1}\big) &= \max\left\{ \begin{array}{l} \sigma_b(t_{2m_k}, t_{2n_k-1}), \sigma_b(t_{2m_k}, ft_{2m_k}), \sigma_b(t_{2n_k-1}, gt_{2n_k-1}), \\ \frac{\sigma_b(t_{2n_k-1}, ft_{2m_k}) + \sigma_b(t_{2m_k}, gt_{2n_k-1})}{4s} \end{array} \right\} \\ &= \max\left\{ \begin{array}{l} \sigma_b(t_{2m_k}, t_{2n_k-1}), \sigma_b(t_{2m_k}, t_{2m_k+1}), \sigma_b(t_{2n_k-1}, t_{2n_k}), \\ \frac{\sigma_b(t_{2n_k-1}, t_{2m_k+1}) + \sigma_b(t_{2m_k}, t_{2n_k})}{4s} \end{array} \right\}. \end{aligned} \tag{36}$$

Hence, by (34)–(36), we have

$$\begin{aligned} \limsup_{k\to\infty} E\big(t_{2m_k}, t_{2n_k-1}\big) &= \limsup_{k\to\infty} \max\left\{ \begin{array}{l} \sigma_b(t_{2m_k}, t_{2n_k-1}), \sigma_b(t_{2m_k}, t_{2m_k+1}), \sigma_b(t_{2n_k-1}, t_{2n_k}), \\ \frac{\sigma_b(t_{2n_k-1}, t_{2m_k+1}) + \sigma_b(t_{2m_k}, t_{2n_k})}{4s} \end{array} \right\} \\ &\le \max\left\{ \varepsilon s, 0, 0, \tfrac{\varepsilon s^2 + \varepsilon s}{4s} \right\} \le \varepsilon s. \end{aligned} \tag{37}$$

Since $\alpha\big(t_{2m_k}, t_{2n_k-1}\big) \ge s^p$ from (25), we have

$$\begin{aligned} &\psi\big(s^p \sigma_b(t_{2m_k+1}, t_{2n_k})\big) \le \psi\big(s^p \sigma_b(ft_{2m_k}, gt_{2n_k-1})\big) \\ &\le \psi\big(\alpha\big(t_{2m_k}, t_{2n_k-1}\big)\sigma_b(ft_{2m_k}, gt_{2n_k-1})\big) \le \phi\big(E\big(t_{2m_k}, t_{2n_k-1}\big)\big). \end{aligned} \tag{38}$$

Hence, by (35), (37), and (38), we obtain

$$\begin{aligned} \psi(\varepsilon s) &\le \psi\big(\varepsilon s^{p-1}\big) = \psi\big(s^p \tfrac{\varepsilon}{s}\big) \le \psi\left(\limsup_{k\to\infty} \sigma_b(t_{m_k}, t_{n_k}) \right) \\ &\le \phi\left(\limsup_{k\to\infty} \big(E\big(t_{m_k-1}, t_{n_k-1}\big)\big) \right) \le \phi(\varepsilon s), \end{aligned}$$

which implies that $\varepsilon = 0$, a contradiction with $\varepsilon > 0$. Thus, $\lim_{n,m\to\infty} \sigma_b(t_n, t_m) = 0$, that is, $\{t_n\}$ is a Cauchy sequence in M. By completeness of (M, σ_b), there exists $u \in M$ such that $\{t_n\}$ is convergent to u, that is, $\lim_{n\to\infty} \sigma_b(t_n, u) = \lim_{n\to\infty} \sigma_b(t_n, t_m) = \sigma_b(u, u) = 0$. By condition H_{s^p}, there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $\alpha\big(t_{n_k}, u\big) \ge s^p$ and $\alpha\big(u, t_{n_k}\big) \ge s^p$ for all $k \in \mathrm{N}$. Since $\alpha\Big(t_{2n(k)}, u\Big) \ge s^p$, applying (25), with $t = t_{2n_k}$ and $r = u$, we obtain

$$\begin{aligned} \psi\Big(s^p \sigma_b\Big(t_{2n(k)+1}, gu\Big)\Big) &= \psi\Big(s^p \sigma_b\Big(ft_{2n(k)}, gu\Big)\Big) \\ &\le \psi\Big(\alpha\Big(t_{2n(k)}, u\Big)\sigma_b\Big(ft_{2n(k)}, gu\Big)\Big) \\ &\le \phi\Big(E\Big(t_{2n(k)}, u\Big)\Big), \end{aligned} \tag{39}$$

where

$$\begin{aligned} E\Big(t_{2n(k)}, u\Big) &= \max\left\{ \begin{array}{l} \sigma_b\big(t_{2n_k}, u\big), \sigma_b\big(t_{2n_k}, ft_{2n_k}\big), \sigma_b(u, gu), \\ \frac{\sigma_b\big(t_{2n_k}, gu\big) + \sigma_b\big(u, ft_{2n_k}\big)}{4s} \end{array} \right\} \\ &= \max\left\{ \begin{array}{l} \sigma_b\big(t_{2n_k}, u\big), \sigma_b\big(t_{2n_k}, t_{2n_k+1}\big), \sigma_b(u, gu), \\ \frac{\sigma_b\big(t_{2n_k}, gu\big) + \sigma_b\Big(u, t_{2n_k+1}\Big)}{4s} \end{array} \right\}. \end{aligned} \tag{40}$$

By (40), Lemma 2, and (34), we obtain

$$\limsup_{n\to\infty} E\Big(t_{2n(k)}, u\Big) \le \max\left\{ 0, 0, \sigma_b(u, gu), \frac{s\sigma_b(u, gu)}{4s} \right\} = \sigma_b(u, gu). \tag{41}$$

Taking limit superior as $k \to \infty$ in (39), considering (41) and Lemma 2, we obtain

$$\begin{aligned}\psi\left(s^{p-1}\sigma_b(u,gu)\right) &= \psi\left(s^p s^{-1}\sigma_b(u,gu)\right) \leq \psi\left(s^p \limsup_{k\to\infty} \sigma_b\left(t_{n_k},gu\right)\right)\\ &\leq \phi\left(\limsup_{k\to\infty} E\left(t_{n_k},u\right)\right) \leq \phi(\sigma_b(u,gu)).\end{aligned} \tag{42}$$

From (42) we get $\sigma_b(u,gu) = 0$ and $gu = u$. Hence, u is a fixed point of g. Similarly, it can be proven that $\sigma_b(fu,u) = 0$ and u is a common fixed point of f and g.

Suppose that u and z are common fixed points of the pair (f,g) such that $u \neq z$. Then, by hypothesis U_{s^p} and applying (25), we have

$$\begin{aligned}\psi(s^p\sigma_b(u,u)) &\leq \psi(\alpha(u,u)\sigma_b(fu,gu))\\ &\leq \phi(E(u,u)) \leq \phi(\sigma_b(u,u)),\end{aligned} \tag{43}$$

where

$$E(u,u) = \max\left\{\sigma_b(u,u),\sigma_b(u,u),\sigma_b(u,u),\frac{\sigma_b(u,u)+\sigma_b(u,u)}{4s}\right\} = \sigma_b(u,u).$$

From Inequality (43), it follows that $\sigma_b(u,u) = 0$ (also $\sigma_b(z,z) = 0$).

Again, we have

$$\begin{aligned}\psi(s^p\sigma_b(u,z)) &\leq \psi(\alpha(u,z)\sigma_b(fu,gz))\\ &\leq \phi(E(u,z)) \leq \phi(\sigma_b(u,z)),\end{aligned}$$

where $E(u,z) = \sigma_b(u,z)$.

From the inequality above, follows $\sigma_b(u,z) = 0$. Thus, $u = z$, and the common fixed point is unique. □

Remark 5.

1. *If we take the mapping $g = f$ in Theorem 4, we obtain Theorem 3.13 of Zoto et al. in Reference [7].*
2. *By taking $\psi(t) = t$ and $p = 2$ in Theorem 4, we obtain Theorem 2.2 of Aydi et al. in Reference [8].*
3. *Theorem 4 generalizes and extends Theorem 2.7 in Reference [4], Theorem 2.7 in Reference [6], Theorems 3 and 4 in Reference [11], Theorems 2.9 and 2.16 in Reference [8], and Theorem 3.16 in Reference [12].*

Remark 6. *A variety of well-known contraction, can be derived by choosing the functions $\psi \in \Psi$ and $\phi \in \Phi$ suitably; for example, $\phi(x) = \psi(x) - \varphi(x)$, where $\varphi \in \Psi$; $\psi(x) = x$ and $\phi(x) = \beta(x)x$ where $\beta \in S$; $\psi(x) = x$; $\phi(x) = \lambda\psi(x)$.*

Corollary 5. *Let (f,g) be a pair of self-mappings in a b-metric-like space (M,σ_b) with coefficient $s \geq 1$, satisfying*

$$\psi(s^p\sigma_b(ft,gr)) \leq \phi(E(t,r))$$

for all $t,r \in M$, where $\psi \in \Psi$, $\varphi \in \Phi$, some $p \geq 2$, and $E(t,r)$ is defined by (24).

Then, f and g have a unique common fixed point $t \in M$.

Proof. It suffices to take $\alpha(t,r) = s^p$ in Theorem 4. □

3. Application

In this section, we provide an application for the existence of a solution of a system of integral equations. In particular, we apply Corollary 5 to show an existence theorem for a solution of a system of nonlinear integral equations given below.

$$\begin{aligned}t(h) &= \textstyle\int_0^h G_1(h,v,t(v))dv,\\ t(h) &= \textstyle\int_0^h G_2(h,v,t(v))dv.\end{aligned} \tag{44}$$

Let $M = C([0,H],R)$ be the set of real continuous functions defined on $[0,H]$ for $H > 0$.
A b-metric-like is given by

$$\sigma_b(t,r) = \max_{h\in[0,1]} (|t(h)| + |r(h)|)^q \text{ for all } t,r \in M.$$

It is noticed that (M,σ_b) is a complete b-metric-like space with parameter $s = 2^{q-1}$, where $q > 1$.
Take the self-mappings $f,g : M \to M$ by

$$\begin{aligned} ft(h) &= \textstyle\int_0^h G_1(h,v,t(v))dv, \\ gt(h) &= \textstyle\int_0^h G_2(h,v,t(v))dv. \end{aligned}$$

Then, the existence of a solution to (44) is equivalent to the existence of a common fixed point of f and g.

Theorem 5. *Consider the system of integral Equation (44), and suppose that the following applies:*

(a) $G_1, G_2 : [0,H] \times [0,H] \times R \to R^+$ *(that is* $G_1(h,v,t(v)) \geq 0, G_2(h,v,r(v)) \geq 0$*) are continuous;*
(b) *There exists a continuous function* $\mu : [0,H] \times [0,H] \to R$ *such that for all* $(h,v) \in [0,H]^2$ *and* $t,r \in M$, *(c) is satisfied;*
(c) $(|G_1(h,v,t(v))| + |G_2(h,v,r(v))|) \leq \mu(h,v)(|t(v)| + |r(v)|)$;
(d) *There exist* $p \geq 2$ *and* $L \in (0,1)$, *such that for all* $h \in [0,H]$ $\sup_{h\in[0,H]} \int_0^h \mu(h,v)dv \leq \sqrt[q]{\frac{L}{s^p}}$.

Then, the system of integral Equation (44) has a unique solution $t \in M$.

Proof. For $t,r \in M$ from Conditions (b) and (c), for all h, we have

$$\begin{aligned} \sigma_b(ft(h),gr(h)) &= (|ft(h)| + |gr(h)|)^q \\ &= \left(\left|\textstyle\int_0^h G_1(h,v,t(v))dv\right| + \left|\textstyle\int_0^h G_2(h,v,r(v))dv\right|\right)^q \\ &\leq \left(\textstyle\int_0^h |G_1(h,v,t(v))|dv + \int_0^h |G_2(h,v,r(v))|dv\right)^q \\ &= \left(\textstyle\int_0^h (|G_1(h,v,t(v))| + |G_2(h,v,r(v))|)dv\right)^q \\ &\leq \left(\textstyle\int_0^h \mu(h,v)(|t(v)| + |r(v)|)dv\right)^q \\ &\leq \left(\textstyle\int_0^h \mu(h,v)\left(((|t(v)| + |r(v)|)^q)^{\frac{1}{q}}\right)dr\right)^q \\ &\leq \left(\sigma_b^{\frac{1}{q}}(t(v),r(v))\textstyle\int_0^h \mu(h,v)dv\right)^q \\ &= \sigma_b(t(v),r(v))\left(\textstyle\int_0^h \mu(h,v)dv\right)^q \\ &\leq \sigma_b(t(v),r(v))\left(\sup_{h\in[0,H]} \textstyle\int_0^h \mu(h,v)dv\right)^q \\ &\leq \left(\left(\frac{L}{s^p}\right)^{\frac{1}{q}}\right)^q \sigma_b(t(v),r(v)) \\ &\leq \tfrac{L}{s^p} E(t,r). \end{aligned}$$

which, in turn, give $s^p\sigma_b(ft(h),gr(h)) \leq LE(t,r)$.

Taking $\psi(x) = x$, and $\phi(x) = Lx$ where $L \in (0,1)$, then all the assertions in Corollary 5 are satisfied; hence, applying Corollary 5, we get that the system of integral Equation (44) has a unique solution. □

4. Conclusions

This paper presents some common fixed point theorems for a pair of α_{s^p}-admissible mappings under (ψ, ϕ)-contractive type conditions. Our results extend, generalize, and improve many new and classical results in fixed point theory.

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