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# A Reliable Method for Solving Fractional Sturm–Liouville Problems

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**Abstract:** In this paper, a reliable method for solving fractional Sturm–Liouville problem based on the operational matrix method is presented. Some of our numerical examples are presented.

Keywords: caputo derivative; operational matrix; fractional eigenvalue problem

#### 1. Introduction

The Sturm–Liouville theory plays an important role for the development of spectral methods and the theory of self-adjoint operators [1]. Several applications on SLPs are studied as boundary-value problems [2]. The Sturm–Liouville eigenvalue problem has played an important role in modeling many physical problems. The theory of the problem is well developed and many results have been obtained concerning the eigenvalues and corresponding eigenfunctions. It should be noted that, since finding analytical solutions for this problem is an extremely difficult task, several numerical algorithms have been developed to seek approximate solutions. However, to date, mostly integer-order differential operators in SLPs have been used, and such operators do not include any fractional differential operators. Fractional calculus is a theory which unifies and generalizes the notions of integer-order differentiation and integration to any real order [3–5].

Recently, the fractional Sturm–Liouville problems were formulated in [6,7]. Authors in these papers considered several types of the fractional Sturm–Liouville equations and they investigated the eigenvalues and eigenfunctions properties of the fractional Sturm–Liouville operators.

Djrbashian [8] studied the existence of a solution to the fractional boundary value problem. In [9], authors discussed the aforementioned relation between eigenvalues and zeros of Mittag–Leffler function. In [10], they used the Homotopy Analysis method while, in [11], they used the fractional differential transform method to compute the eigenvalues. In [12], researchers used the Fourier series to solve this problem while, in [13,14], they used the method of Haar wavelet operational matrix. In [15–19], researchers extended the scope of some spectral properties of fractional Sturm–Liouville problems. Variational methods and Inverse Laplace transform method were applied in [20,21], respectively. Recently, in [22], authors constructed numerical schemes using radial basis functions while, in [23], they used Galerkin finite element method for such problems. Greenberg and Marletta [24,25] developed their own code using Theta Matrices (SLEUTH). In [26], researchers implemented the iterated variation method.

In this article, we present a numerical technique for solving class of FSLPs of the form

$$D^{\gamma}[f(t)D^{\gamma}y(t)] + \mu g(t)y(t) = h(t), \ 0 \le t \le 1, \frac{1}{2} < \gamma \le 1$$
(1)

subject to

$$c_0 y(0) + c_1 D^{\gamma} y(0) = 0, \ c_0^2 + c_1^2 > 0,$$
 (2)

$$c_2 y(1) + c_3 D^{\gamma} y(1) = 0, c_2^2 + c_3^2 > 0,$$
 (3)

where  $c_0, c_1, c_2$  and  $c_3$  are constants such that det  $\begin{pmatrix} c_0 & c_1 \\ c_2 & c_3 \end{pmatrix} \neq 0$ , f(t), g(t), h(t) are continuous functions with f(t), g(t) > 0 for all  $t \in [0, 1]$ , and  $D^{\gamma}$  is the Caputo derivative.

Next, we present some results related to the Caputo fractional derivative, as well as the definition of the fractional-order functions.

**Definition 1.** The Rimann–Liouville fractional integral operator  $I^{\gamma}$  of order  $\gamma > 0$  on  $L_1[0,1]$  is given by

$$\begin{split} I^{\alpha}y(t) &= \frac{1}{\Gamma(\gamma)}\int\limits_{0}^{t}\frac{y(s)}{(t-s)^{1-\gamma}}ds,\\ I^{0}y(t) &= y(t), \end{split}$$

where  $\Gamma(\gamma)$  is the Euler Gamma function (see [5,27]).

For any  $\gamma$ ,  $\zeta \ge 0$ , and  $\zeta > -1$ ,  $I^{\alpha}$  exists for any  $t \in [0, 1]$  and

$$I^{\gamma}t^{\zeta} = \frac{\Gamma(\zeta+1)}{\Gamma(\zeta+\gamma+1)}t^{\zeta+\gamma}.$$
(4)

**Definition 2.** The Caputo fractional derivative of order  $\gamma$  is defined by

$$D^{\gamma}y(t) = I^{n-\gamma}D^{l}y(t) = \frac{1}{\Gamma(l-\gamma)}\int_{0}^{t}\frac{y^{(l)}(s)}{(t-s)^{\gamma-l+1}}ds,$$

provided that the integral exists, where  $l = [\gamma] + 1$ ,  $[\gamma]$  is the integer part of the positive real number  $\gamma$ , t > 0.

For  $y \in L_1[0, 1]$  and  $\gamma \ge 0$ :

$$I^{\gamma}D^{\gamma}y(t) = y(t) - \sum_{r=0}^{l-1} y^{(r)}(0^{+})\frac{t^{r}}{r!},$$
(5)

Let  $\Delta_n$  be defined by

$$\Delta_n = Span\{1, t^{\gamma}, t^{2\gamma}, ..., t^{n\gamma}\}.$$

The inner product on the set  $\Delta_n$  is given by

$$(f(t), g(t)) = {}^{1}_{0}f(t)g(t)dt.$$

**Theorem 1.** *The sequence of functions defined as follows are orthogonal:* 

$$y_i(t) = (t^{\gamma} - a_i)y_{i-1}(t) - b_i y_{i-2}(t), \ i = 2, 3, \dots$$
(6)

with  $y_0(t) = 1$ ,  $y_1(t) = t^{\gamma} - a_1$ , and

$$a_{i} = \frac{(t^{\gamma}y_{i-1}(t), g(t))}{(y_{i-1}(t), y_{i-1}(t))}, \ b_{i} = \frac{(t^{\gamma}y_{i-1}(t), y_{i-2}(t))}{(y_{i-2}(t), y_{i-2}(t))}.$$

**Proof.** For i = 1,

$$\begin{aligned} (y_1(t), y_0(t)) &= (t^{\gamma} - a_1, y_0(t)) \\ &= (t^{\alpha}, 1) - \frac{(t^{\alpha}, 1)}{(1, 1)} (1, 1) = 0. \end{aligned}$$

Assume the result of the theorem is true for i > 1. Then, for any  $j \in \{0, 1, ..., i - 2\}$ , we have

$$\begin{aligned} \left(y_{i+1}(t), y_j(t)\right) &= \left((t^{\gamma} - a_{i+1})y_i(t) - b_{i+1}y_{i-1}(t), y_j(t)\right) \\ &= \left(t^{\gamma}y_i(t), y_j(t)\right) - a_{i+1}\left(y_i(t), y_j(t)\right) - b_{i+1}\left(y_{i-1}(t), y_j(t)\right) \\ &= \left(t^{\gamma}y_i(t), y_j(t)\right) \\ &= \left(y_i(t), t^{\gamma}y_j(t)\right) \\ &= \left(y_i(t), t^{\gamma}y_j(t) + a_{j+1}y_j(t) + b_{j+1}y_{j-1}(t)\right) \\ &= \left(y_i(t), y_{j+1}(t) + a_{j+1}\left(y_i(t), y_j(t)\right) + b_{j+1}\left(y_i(t), y_{j-1}(t)\right) \\ &= 0. \end{aligned}$$

For j = i - 1,

$$\begin{aligned} (y_{i+1}(t), y_{i-1}(t)) &= & ((t^{\gamma} - a_{i+1})y_i(t) - b_{i+1}y_{i-1}(t), y_{i-1}(t)) \\ &= & (t^{\gamma}y_i(t), y_{i-1}(t)) - a_{i+1}(y_i(t), y_{i-1}(t)) - b_{i+1}(y_{i-1}(t), y_{i-1}(t)) \\ &= & (t^{\gamma}y_i(t), y_{i-1}(t)) - b_{i+1}(y_{i-1}(t), y_{i-1}(t)) \\ &= & (t^{\gamma}y_i(t), y_{i-1}(t)) - \frac{(t^{\gamma}y_i(t), y_{i-1}(t))}{(y_{i-1}(t), y_{i-1}(t))} (y_{i-1}(t), y_{i-1}(t)) \\ &= & 0. \end{aligned}$$

## 2. Operational Matrices of Fractional Integration

A set of *l* Block Pulse Functions (BPFs) in the interval [0, 1) are given by  $\{b_0(t), b_1(t), ..., b_{l-1}(t)\}$  such that

$$b_i(t) = \left\{ \begin{array}{cc} 1, & \frac{i}{l} \le t < \frac{i+1}{l} \\ 0, & otherwise \end{array} \right\}$$
(7)

for i = 0, 1, ..., l - 1. The following are some of the BPFs properties

$$b_i(t) \ b_j(t) = \left\{ \begin{array}{cc} b_i(t), & i=j\\ 0, & i\neq j \end{array} \right\}$$
(8)

and

$${}_{0}^{1}b_{i}(t)b_{j}(t)dt = \left\{ \begin{array}{cc} \frac{1}{l}, & i=j\\ 0, & i\neq j \end{array} \right\}.$$

$$\tag{9}$$

If  $y \in L_2[0, 1]$ , then

$$y(t) = Y_{l-1}^T B_{l-1}(t)$$
(10)

where

$$Y_{l-1} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{l-1} \end{bmatrix}, B_{l-1}(t) = \begin{bmatrix} b_0(t) \\ b_1(t) \\ \vdots \\ b_{l-1}(t) \end{bmatrix},$$

and

$$y_i = l_{\frac{i}{l}}^{\frac{i+1}{l}} y(t) dt, \ i = 0, 1, \dots l - 1.$$
(11)

**Theorem 2.** Let  $I^{\gamma}$  be the Rimann–Liouville functional operator. Then,

$$I^{\gamma} B_{l-1}(t) = P_l^{\gamma} B_{l-1}(t)$$
(12)

where

$$P_l^{\gamma} = \frac{1}{l^{\gamma}} \frac{1}{\Gamma(\gamma+2)} \begin{bmatrix} 1 & \varepsilon_1 & \varepsilon_2 & \dots & \varepsilon_{l-1} \\ 0 & 1 & \varepsilon_1 & \dots & \varepsilon_{l-2} \\ 0 & 0 & 1 & \dots & \varepsilon_{l-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and  $\varepsilon_r = (r+1)^{\gamma+1} - 2r^{\gamma+1} + (r-1)^{\gamma+1}$ , r = 1: l-1.

**Proof.** For each i = 0, 1, ..., l - 1, we can write  $I^{\gamma}b_i$  as

$$I^{\gamma}b_i =_{j=0}^{l-1} c_{ij} b_j(t).$$

Multiply both sides by  $b_r(t)$ , for  $0 \le r \le l - 1$ , then integrate both sides to get

$$c_{ir} = r_{\frac{r}{l}}^{\frac{r+1}{l}} I^{\gamma} b_{i}(t) dt = \frac{r}{\Gamma(\gamma)} \frac{r+1}{\frac{r}{l}} \int_{0}^{t} \frac{b_{i}(t)}{(t-t)^{1-\gamma}} dt dt.$$
  
= 
$$\begin{cases} 0, & i > r \ge 0\\ 1, & i = r\\ (r+1)^{\gamma+1} - 2r^{\gamma+1} + (r-1)^{\gamma+1} & i < r \le l-1 \end{cases}$$

For more details, see [28,29].  $\Box$ 

**Theorem 3.** Let 
$$Y_{M-1}(t) = \begin{bmatrix} y_0(t) \\ y_1(t) \\ \vdots \\ y_{M-1}(t) \end{bmatrix}$$
. Then, there exists an  $M \times l$  matrix  $Q^{\gamma}$  such that
$$Y_{M-1}(t) = Q_{M \times l}^{\gamma} B_{l-1}(t)$$
(15)

where

$$(Q_{M\times l}^{\gamma})_{i,k} = l_k^{\frac{k+1}{l}} y_i(t) dt$$

for i = 0 : M - 1 and k = 0 : l - 1.

**Proof.** It is easy to see that  $y_i(t) \in L_2[0,1)$ , for each i = 0 : M - 1. Using Equations (10) and (11), we get

$$Y_{M-1}(t) = Q_{M \times l}^{\gamma} B_{l-1}(t)$$

where

$$(Q_{M\times l}^{\gamma})_{i,k} = l_{\frac{k}{l}}^{\frac{k+1}{l}} y_i(t) dt.$$

for i = 0: M - 1 and k = 0: l - 1 which ends the proof. From now on, let M = l.  $\Box$ 

(13)

**Theorem 4.** If  $0 < \gamma < 1$ , then  $Q_{l \times l}^{\gamma}$  is nonsingular matrix.

**Proof.** Theorem 3 implies that

$$Y_{l-1}(t)Y_{l-1}(t)^{T} = Q_{l\times l}^{\gamma}B_{l-1}(t)B_{l-1}(t)^{T}Q_{l\times l}^{\gamma T}.$$

Integrate both sides with respect to t on (0,1) to get

$${}_{0}^{1}Y_{l-1}(t)Y_{l-1}(t)^{T}dt = Q_{l\times l}^{\gamma} \left( {}_{0}^{1}B_{l-1}(t)B_{l-1}(t)^{T}dt \right) Q_{l\times l}^{\gamma T}.$$

Theorem 1 and Equation (9) yield

$$D_1 = Q_{l \times l}^{\gamma} D_2 Q_{l \times l}^{\gamma T} \tag{14}$$

where

$$D_{1} = \begin{bmatrix} \frac{1}{0}y_{0}(t)y_{0}(t)dt & 0 & \cdots & 0 \\ 0 & \frac{1}{0}y_{1}(t)y_{1}(t)dt & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{0}y_{l-1}(t)y_{l-1}(t)dt \end{bmatrix}$$

and

$$D_2 = \frac{1}{\gamma} \begin{bmatrix} \frac{1}{l\gamma} & 0 & \cdots & 0\\ 0 & \frac{2\gamma - 1}{l\gamma} & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & \frac{l^{\gamma} - (l-1)^{\gamma}}{l\gamma} \end{bmatrix}$$

Then,  $det(D_1) > 0$  and  $det(D_2) > 0$ . Equation (14) gives

$$\left(\det(Q_{l\times l}^{\gamma})\right)^2 = \frac{\det(D_1)}{\det(D_2)} > 0.$$

Thus,  $Q_{l \times l}^{\gamma}$  is nonsingular.  $\Box$ 

**Operational Matrix of Fractional Integration** 

If  $y \in C^{1}[0, 1]$ , then

$$y(t) =_{k=0}^{\infty} u_k y_k(t)$$

where

$$u_k = \frac{\frac{1}{0}u(t)y_k(t)dt}{\frac{1}{0}f_k(t)y_k(t)dt}.$$

Approximate the function y(t) by

$$U_{l-1}(t) =_{k=0}^{l-1} u y_k(t) = U^T Y_{l-1}(t),$$
(15)

where

$$U = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{l-1} \end{bmatrix} \text{ and } Y_{l-1}(t) = \begin{bmatrix} y_0(t) \\ y_1(t) \\ \vdots \\ y_{l-1}(t) \end{bmatrix}.$$
(16)

**Theorem 5.**  $I^{\gamma}Y_{l-1}(t) = H_l^{\gamma}Y_{l-1}(t)$ where

$$H_l^{\gamma} = Q_{l \times l}^{\gamma} P_l^{\gamma} (Q_{l \times l}^{\gamma})^{-1}$$

**Proof.** Let  $H_l^{\gamma}$  be given by

$$I^{\gamma}Y_{l-1}(t) = H_l^{\gamma}Y_{l-1}(t).$$
(17)

From Equations (13) and (17), we get

$$I^{\gamma}Y_{l-1}(t) = H_{l}^{\gamma}Y_{l-1}(t) = H_{l}^{\gamma}Q_{l\times l}^{\gamma}B_{l-1}(t)$$
(18)

and

$$I^{\gamma}Y_{l-1}(t) = I^{\gamma}Q_{l\times l}^{\gamma}B_{l-1}(t)$$

$$= Q_{l\times l}^{\gamma}I^{\gamma}B_{l-1}(t)$$

$$= Q_{l\times l}^{\gamma}P_{l}^{\gamma}B_{l-1}(t).$$
(19)

$$H_l^{\gamma} Q_{l \times l}^{\gamma} B_{l-1}(t) = Q_{l \times l}^{\gamma} P_l^{\gamma} B_{l-1}(t).$$
<sup>(20)</sup>

Therefore,

$$H_l^{\gamma} = Q_{l \times l}^{\gamma} P_l^{\gamma} (Q_{l \times l}^{\gamma})^{-1}$$

## 3. Method of Solution

Using Equations (10) and (13), we get

$$D^{\gamma}[f(t)D^{\gamma}y(t)] = U^{T}Y_{l-1}(t) = U^{T}Q_{l\times l}^{\gamma}B_{l-1}(t).$$

Thus,

$$f(t)D^{\gamma}y(t) - f(0)\omega = I^{\gamma}U^{T}Y_{l-1}(t)$$

where  $\omega = D^{\gamma} y(0)$ . Theorem 5 and Equations (10) and (13) imply that

$$D^{\gamma}y(t) = \frac{1}{f(t)} \left( U^{T}I^{\gamma}Y_{l-1}(t) + f(0)\omega \right)$$
  
=  $\frac{1}{f(t)} \left( U^{T}H_{l}^{\gamma}Y_{l-1}(t) + f(0)\omega \right)$   
=  $U^{T}H_{l}^{\gamma}Q_{l\times l}^{\gamma} \frac{B_{ml-1}(t)}{f(t)} + \frac{f(0)\omega}{f(t)}$   
=  $U^{T}H_{l}^{\gamma}Q_{l\times l}^{\gamma} \begin{bmatrix} b_{0}(t)/f(t) \\ b_{1}(t)/f(t) \\ \vdots \\ b_{l-1}(t)/f(t) \end{bmatrix} + f(0)\omega \begin{bmatrix} 1/f(t) \\ 1/f(t) \\ \vdots \\ 1/f(t) \end{bmatrix}$ 

Hence,

$$D^{\gamma}y(t) = \left(U^{T}H_{l}^{\gamma}Q_{l\times l}^{\gamma}PF_{1} + f(0)\varpi F_{2}\right)B_{l-1}(t)$$

Thus,

$$y(t) = \left( U^T H_l^{\gamma} Q_{l \times l}^{\gamma} F_1 + f(0) \omega F_2 \right) I^{\gamma} B_{l-1}(t) + \psi$$

where  $\psi = y(0)$ . Therefore,

$$y(t) = \left( U^T H_l^{\gamma} Q_{l \times l}^{\gamma} F_1 + f(0) \varpi F_2 \right) Y_l^{\gamma} B_{l-1}(t) + \psi.$$
(21)

Hence,

$$U^{T}Q_{l\times l}^{\gamma}B_{m-1}(t) + \mu g(t)\left(\left(U^{T}H_{l}^{\gamma}Q_{l\times l}^{\gamma}F_{1} + f(0)\varpi F_{2}\right)FY_{l}^{\gamma}B_{l-1}(t) + \psi\right) = h(t)$$

or

$$U^{T}(Q_{l\times l}^{\gamma} + \mu q(t)H_{l}^{\gamma}Q_{l\times l}^{\gamma}F_{1}Y_{l}^{\gamma})B_{l-1}(t) = h(t) - \mu g(t)f(0)\varpi F_{2}Y_{l}^{\gamma}B_{l-1}(t) - \mu \psi g(t).$$
(22)

Using the boundary conditions in Equations (2) and (3), we get the following cases

• if  $c_0 = 0$ ,  $\omega = 0$ ,  $c_1 \neq 0$ ,  $c_2 \neq 0$ , and

$$\psi = -c_2 U^T H_l^{\gamma} Q_{l \times m}^{\gamma} F_1 Y_l^{\gamma} B_{l-1}(1) - \frac{c_3}{c_2} U^T H_l^{\gamma} Q_{l \times l}^{\gamma} P F_1 B_{l-1}(1)$$

• if  $c_0 \neq 0$ ,  $\psi = -\frac{c_1}{c_0} \omega$  and

$$\varpi = \frac{-c_2 U^T H_l^{\gamma} Q_{l \times l}^{\gamma} F_1 Y_l^{\gamma} B_{l-1}(1) - c_3 U^T H_l^{\gamma} Q_{l \times l}^{\gamma} Y_1 B_{l-1}(1)}{f(0) F_2 Y_l^{\gamma} B_{l-1}(1) - \frac{c_1 c_2}{c_0} + c_3 f(0) Y_2 B_{l-1}(1)}.$$

Thus,

$$U^{T}\left(\begin{array}{c}Q_{l\times l}^{\gamma}+\mu g(t)H_{l}^{\gamma}Q_{l\times l}^{\gamma}F_{1}Y_{l}^{\gamma}B_{l-1}(t)+\\\left(\mu g(t)f(0)F_{2}Y_{l}^{\gamma}B_{l-1}(t)-\frac{c_{1}}{c_{0}}\mu g(t)\right)\left(\frac{-c_{2}H_{l}^{\gamma}Q_{l\times l}^{\gamma}F_{1}Y_{l}^{\gamma}B_{l-1}(1)-c_{3}H_{l}^{\gamma}Q_{l\times l}^{\gamma}F_{1}B_{l-1}(1)}{f(0)F_{2}F_{l}^{\gamma}B_{l-1}(1)-\frac{c_{1}c_{2}}{c_{0}}+c_{3}f(0)F_{2}B_{l-1}(1)}\right)\right)=h(t).$$
 (23)

We use the collocation points

$$t_i = \frac{r+1}{l+1}, i = 0: l-1.$$

Substitute these values into Equation (23) and take the transpose of both sides to get a system of linear equations in terms of U of the form

$$G(\mu)U = R. \tag{24}$$

To have a nonzero solution to the system in Equation (24),  $G(\mu)$  must be nonsingular. Thus,

$$\det(G(\mu)) = 0. \tag{25}$$

Therefore, we find the eigenvalues from Equation (25) and we find the corresponding eigenfunctions from Equation (21).

### 4. Numerical Results

We present two examples for l = 16. In this paper, we focus only on the eigenvalues.

Example 1. Consider

$$D^{\gamma}[D^{\gamma}y(t)] + \mu y(t) = 0, \ t \in [0,1], \gamma \in (0.5,1],$$
  
 $y(0) = 0, y(1) = 0.$ 

Using the procedure described in the previous section, the generated eigenvalues are reported in Table 1.

$\gamma = 0.75$	$\gamma=$ 0.95	$\gamma = 0.99$
8.7825905605957	8.2711826449023	9.6635258705797
14.0844503539395	58.990163159836	38.044080578817
	96.673652759078	84.971438095925
	148.295350243613	150.13372170205
	199.571402506686	233.59863572826
	277.107135647923	335.09777723091
	295.450149615306	454.76440655382
		590.93089519416

**Table 1.** Eigenvalues for different values of  $\gamma$ .

For  $\gamma = 1$ , the exact eigenvalues are well-known and they are given by

$$\mu_n = n^2 \pi^2$$
,  $n = 1, 2, 3, ...$ 

It is worth mentioning that the eigenvalues of the problem in this example approach to  $n^2 \pi^2$  when  $\gamma$  approaches to 1. Let

$$\delta_{i,j} = \begin{vmatrix} 1 \\ 0 \end{vmatrix} y_i(t) \ y_j(t) \ dt \end{vmatrix}$$

For  $\gamma = 0.75$ ,  $\delta_{1,2} = 5.7 \times 10^{-16}$ . Sample of these values for  $\gamma = 0.95$  are given as

$$\delta_{1,2} = 5.7 \times 10^{-16}, \ \delta_{4,6} = 2.6 \times 10^{-16}, \ \delta_{1,6} = 8.3 \times 10^{-16}.$$

Similarly, for  $\gamma = 0.99$ ,

$$\delta_{1,2} = 3.1 \times 10^{-16}, \ \delta_{4,6} = 4.2 \times 10^{-16}, \ \delta_{1,7} = 2.0 \times 10^{-16}.$$

This means the orthogonality relation holds. We notice that the eigenvalues satisfy the increasing property.

#### Example 2. Consider

$$D^{\alpha}[D^{\alpha}y(t)] + \lambda(1+t^{\alpha})y(t) = 0, \ t \in [0,1], \gamma \in (0.5,1],$$
$$u(0) = 0, u(1) = 0.$$

Using the procedure described in the previous section, the generated eigenvalues are reported in Table 2.

$\gamma = 0.501$	$\gamma=$ 0.75	$\gamma=0.95$
3.74496847027704 5.59359607314814 25.47511927569108 151.8458499360075	4.90596508821842 9.95423834570763 14.2468657217155 25.8797084072818 124.475138197374	5.82711926402061 21.8630977993855 100.868795211963 234.225682145921 439.200912754629 721.009344587213 984.124781340994

**Table 2.** Eigenvalues for different values of  $\gamma$ .

Let

$$\delta_{i,j} = \left| {}_0^1 y_i(t) \ y_j(t) \ g(t) dt \right|.$$

For  $\gamma = 0.502$ ,  $\delta_{1,2} = 3.3 \times 10^{-16}$  and  $\delta_{2,4} = 4.9 \times 10^{-16}$ . Samples of these values for  $\gamma = 0.75$  are given as

$$\delta_{1,2} = 2.2 \times 10^{-16}, \ \delta_{4,5} = 4.1 \times 10^{-16}, \ \delta_{1,5} = 6.9 \times 10^{-16}.$$

Similarly, for  $\gamma = 0.95$ ,

$$\delta_{1,2} = 1.2 \times 10^{-16}, \ \delta_{4,6} = 2.1 \times 10^{-16}, \ \delta_{1,7} = 4.6 \times 10^{-16}.$$

This means the orthogonality relation holds. We notice that the eigenvalues satisfy the property

$$\mu_1 \leq \mu_2 \leq \dots$$

#### 5. Conclusions

In this article, a reliable method for solving fractional Sturm–Liouville problem based on the operational matrix method is presented. Two of our numerical examples are presented. From the previous discussion, we notice the following.

• From previous section, we can find the eigenvalues with the following property

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots < \lambda_n < \dots$$

• From previous section, the orthogonality property

$${}^{1}_{0}y_{i}(t) y_{i}(t) q(t) = 0, i \neq j$$

holds.

• The proposed method can be generalized to other applications in Physics and Engineering.

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