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# An Iterative Method for Solving a Class of Fractional Functional Differential Equations with "Maxima"

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**Abstract:** In the present work, we deal with nonlinear fractional differential equations with "maxima" and deviating arguments. The nonlinear part of the problem under consideration depends on the maximum values of the unknown function taken in time-dependent intervals. Proceeding by an iterative approach, we obtain the existence and uniqueness of the solution, in a context that does not fit within the framework of fixed point theory methods for the self-mappings, frequently used in the study of such problems. An example illustrating our main result is also given.

**Keywords:** functional differential equations; fractional calculus; iterative procedures

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## 1. Introduction

One of the most interesting kinds of nonlinear functional differential equations is the case when the nonlinear part depends on the maximum values of the unknown function. These equations, called functional differential equations with "maxima", arise in many technological processes. For instance, in the automatic control theory of various technical systems, it occurs that the law of regulation depends on the maximal deviation of the regulated quantity (see [1,2]). Such problems are often modeled by differential equations that contain the maximum values of the unknown function (see [3–5]). Recently, ordinary differential equations with "maxima" have received wide attention and have been investigated in diverse directions (see, for example, [4,6–11] and the references therein). As far as we know, in the fractional case, these equations are not yet sufficiently discussed in the existing literature, and thus form a natural subject for further investigation. Motivated by the previous fact and inspired by [11], in this work, we focus on the existence and uniqueness of the solution for similar systems in a fractional context, and in more general terms. We consider the following nonlinear fractional differential equation with "maxima" and deviating arguments:

$$ {}^{C}D^{\alpha}u\left(t\right)=f\left(t,\max_{\sigma\in\left[a\left(t\right),b\left(t\right)\right]}u\left(\sigma\right),u\left(t-\tau_{1}\left(t\right)\right),...,u\left(t-\tau_{N}\left(t\right)\right)\right),\quad t>0, \tag{1} $$

with the initial condition function

$$ u\left(t\right)=\phi\left(t\right),\quad t\leq 0, \tag{2} $$

where ${}^{C}D^{\alpha}$ denotes the Caputo fractional derivative operator of order $\alpha\in[0,1]$, $N$ is a positive integer, $a$, $b$ and $\tau_i$ (with $1\leq i\leq N$) are real continuous functions defined on $\mathbb{R}_{+}=[0,+\infty]$ subject to conditions that will be specified later, $\phi:[-\infty,0]\longrightarrow\mathbb{R}$ is a continuous function such that $\phi\left(0\right)=\phi_{0}>0$, and $f:\mathbb{R}_{+}\times\mathbb{R}^{1+N}\longrightarrow\mathbb{R}$ is a nonlinear continuous function.

Our aim is to give sufficient assumptions leading to an iterative process that converges to the unique continuous solutions of Equations (1) and (2). These being under weaker conditions compared to the usual contractions (see Remark 3), and in a setting for which the standard process of Picard's iterations fails to be well defined.

It should be pointed out here, that the maximums in Equation (1) are taken on time-dependent intervals and not on a fixed one as is the case of the example given in [11].

Moreover, the Equation (1) will be supposedly of mixed type, namely with both retarded and advanced deviations $\tau_i$, while, in [11], only the delays are considered. It is also important to note that, in the Lipschitz condition of the nonlinear function $f$, we take into account the direction of maximums too, which is not the case of the corresponding assumption in [11].

Due to all of these generalizations, our work attempts to extend the application of [11] (Theorem 3) to the fractional case by a constructive approach.

To our knowledge, the studies devoted to the question of the existence and uniqueness of the solutions for fractional differential equations are based on different variants from the fixed point theory for self-mappings, or on the upper and lower solutions method (see, e.g., [12–18] and the references therein). We emphasize here that our result answers this question for a class of problems of the forms Equations (1) and (2), even when the previous versions of the theory fail to do so directly. That is, when the integral operator associated with Equations (1) and (2) is allowed to be a non self-mapping (see Remark 1).

The rest of the paper is organized as follows. In the next section, we introduce some basic definitions from the fractional calculus as well as preliminary lemmas. In Section 3, under some sufficient conditions allowing the integral operator associated with Equations (1) and (2) to be non self, we prove an existence–uniqueness result by means of an iterative process. The applicability of our theoretical result is illustrated in Section 4.

## 2. Preliminaries

We start by recalling the definitions of the Riemann–Liouville fractional integrals and the Caputo fractional derivatives on the half real axis. For further details on the historical account and essential properties about the fractional calculus, we refer to [19–22].

**Definition 1.** *The Riemann–Liouville fractional integral of a function $u : \mathbb{R}_+ \longrightarrow \mathbb{R}$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I^{\alpha} u\left(t\right) := \frac{1}{\Gamma\left(\alpha\right)} \int_0^t \left(t-s\right)^{\alpha-1} u\left(s\right) ds, \quad t > 0,$$

*where $\Gamma\left(.\right)$ is the Gamma function, provided that the right side is pointwise defined on $[0, \infty]$.*

In the following definition, $n$ denotes the positive integer such that $n-1 < \alpha \leq n$ and $d^n/dt^n$ is the classical derivative operator of order $n$. For simplicity, we set $du/dt = u'\left(t\right)$.

**Definition 2.** *The Caputo fractional derivative of a function $u : \mathbb{R}_+ \longrightarrow \mathbb{R}$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$^{C}D^{\alpha} u\left(t\right) := I^{n-\alpha} \frac{d^n}{dt^n} u\left(t\right) := \frac{1}{\Gamma\left(n-\alpha\right)} \int_0^t \left(t-s\right)^{n-\alpha-1} \frac{d^n}{ds^n} u\left(s\right) ds, \quad t > 0,$$

*provided that the right-hand side exists pointwise on $[0, \infty]$.*

In particular, when $0 < \alpha < 1$,

$$^{C}D^{\alpha} u\left(t\right) := I^{1-\alpha} u'\left(t\right) := \frac{1}{\Gamma\left(1-\alpha\right)} \int_0^t \frac{u'\left(s\right)}{\left(t-s\right)^{\alpha}} ds, \quad t > 0. \tag{3}$$

Let us denote by $\mathcal{C}(\mathbb{R})$ the set of all real continuous functions on $\mathbb{R}$. Applying the Riemann–Liouville fractional integral operator $I^{\alpha}$ of order $\alpha$ to both sides of Equation (1) and using its properties (see [19,21]), together with the initial condition Equation (2), we easily get the following lemma.

**Lemma 1.** *If $f, a, b$ and $\tau_i$ (with $1 \le i \le N$) are continuous functions, then $u \in \{v \in \mathcal{C}(\mathbb{R}) \text{ s.t. } v(t) = \phi(t) \text{ for } t \le 0\}$ is a solution of Equations (1) and (2) if and only if $u(t) = Fu(t)$, where*

$$Fu(t) = \phi_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, \max_{\sigma \in [a(s), b(s)]} u(\sigma), u(s - \tau_1(s)), ..., u(s - \tau_N(s))\right) ds, \ t > 0, \tag{4}$$

$$Fu(t) = \phi(t), \quad t \le 0. \tag{5}$$

**Proof.** Let $u \in \mathcal{C}(\mathbb{R})$. The functions $a$ and $b$ are continuous, so, according to the Remark in [7] (page 8) , see also [4] (Remark 3.1.1, page 62), $\max_{\sigma \in [a(t), b(t)]} u(\sigma)$ is continuous too. Moreover, since $\tau_i$ are continuous, then $f\left(t, \max_{\sigma \in [a(t), b(t)]} u(\sigma), u(t - \tau_1(t)), ..., u(t - \tau_N(t))\right)$ as a composition of continuous functions, it is also continuous. Now, we are able to follow the usual approach to show this type of result (see [15,19,21,23,24]). Note first that the Caputo fractional derivative of order $\alpha \in [0, 1]$ can be expressed by means of the Riemann–Liouville fractional derivative denoted by $D^{\alpha}$, as follows (see [21] (2.4.4) or [19] (Definition 3.2)):

$$^{C}D^{\alpha} u(t) = D^{\alpha}[u(t) - u(0)] := \frac{d}{dt} I^{1-\alpha}[u(t) - u(0)]. \tag{6}$$

Let now $u \in \{v \in \mathcal{C}(\mathbb{R}) \text{ s.t. } v(t) = \phi(t) \text{ for } t \le 0\}$ be a solution of Equations (1) and (2). Thus, in view of the first equality in Equation (6), Equation (1) can be rewritten as

$$D^{\alpha}[u(t) - u(0)] = f\left(t, \max_{\sigma \in [a(t), b(t)]} u(\sigma), u(t - \tau_1(t)), ..., u(t - \tau_N(t))\right), \quad t > 0. \tag{7}$$

Since the right-hand side of Equation (7) is continuous, then according to the definition of the Riemann–Liouville fractional derivative given by the second equality in Equation (6), we have

$$I^{1-\alpha}[u(t) - u(0)] \in \mathcal{C}^1(\mathbb{R}_+). \tag{8}$$

Thus, using [21] (Lemma 2.9, (d) with $\gamma = 0$), we have

$$I^{\alpha} D^{\alpha}[u(t) - u(0)] = [u(t) - u(0)] - \frac{1}{\Gamma(\alpha)} I^{1-\alpha} U(0)\, t^{\alpha-1}, \tag{9}$$

where

$$U(t) := [u(t) - u(0)]. \tag{10}$$

Since $U$ is continuous, for every $T > 0$, there exists $L > 0$ such that $|U(t)| \le L :$ for all $t \in [0, T]$. Thus, the following inequality holds true for every $t > 0$, sufficiently small

$$\left| I^{1-\alpha} U(t) \right| \le \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{|U(s)|}{(t-s)^{\alpha}} ds \le \frac{L}{\Gamma(2-\alpha)} t^{1-\alpha}.$$

Hence, the fact that $1 - \alpha > 0$, together with the continuity of $I^{1-\alpha}U$ resulting from Equation (8), imply that $I^{1-\alpha}U(0) = 0$. Consequently, Equation (9) becomes

$$I^{\alpha} D^{\alpha} [u(t) - u(0)] = [u(t) - u(0)]. \quad (11)$$

Now, returning to Equation (7), applying the Riemann–Liouville fractional integral to both sides, and then using Equation (11) together with Equation (2), we obtain Equation (4).

Suppose now that $u \in \{v \in \mathcal{C}(\mathbb{R}) \text{ s.t. } v(t) = \phi(t) \text{ for } t \le 0\}$ is a solution of Equations (4) and (5). Then, in view of Definition 1, we can rewrite Equation (4) as

$$u(t) = \phi_0 + I^{\alpha} f\left(t, \max_{\sigma \in [a(t), b(t)]} u(\sigma), u(t - \tau_1(t)), ..., u(t - \tau_N(t))\right).$$

Since $u$ is continuous, then the right-hand side above is continuous too. By applying the Caputo fractional derivative operator ${}^{C}D^{\alpha}$ to both sides, then using its linearity (see [19] (Theorem 3.16)), as well as the fact that the derivative of a constant (in the sense of Caputo) is equal to zero [21] (Property 2.16), together with [21] (Lemma 2.21), we get Equation (1). □

In the present work, the state space will be regarded as a complete Hausdorff locally convex space. For further details on these spaces, we refer to [25]. In the sequel of this paper, we make use of the following lemma, which can be found in [26] ([Lemma 2).

**Lemma 2.** *Let $X$ be a complete Hausdorff locally convex space, $\mathbf{E}$ a closed subset of $X$ and $u, v \in X$. If $u \in \mathbf{E}$ and $v \notin \mathbf{E}$, then there exists $\beta \in [0,1]$ such that $w_\beta := (1-\beta)u + \beta v \in \partial\mathbf{E}$, where $\partial\mathbf{E}$ denotes the boundary of $\mathbf{E}$. Furthermore, if $u \notin \partial\mathbf{E}$, then $\beta \in ]0,1]$.*

## 3. The Main Results

In this section, we not only prove the existence–uniqueness result for Equations (1) and (2), but we also give this solution as a limit of an iterative process.

First, let us set the following hypotheses:

$(H_1)$ $\forall t \ge 0 : 0 \le a_* \le a(t) \le b(t) \le b^*$, with $a_* := \inf_{t \in [0,+\infty[} a(t)$, and $b^* := \sup_{t \in [0,+\infty[} b(t)$. Furthermore, for all $t \in [0, b^*]$, we assume that $a(t) = a_*$ and $b(t) = b^*$. In other words, the functions $a$ and $b$ are constant on the interval $[0, b^*]$.

$(H_2)$ $\exists \tau > 0$, such that, for $i = 1, ..., N : \tau_i(t) > t - \tau$, $\forall t > 0$.

$(H_3)$ For $i = 1, ..., N, \exists t_i > 0 : \tau_i(t) \ge t$, $\forall t \in [0, t_i]$, and $\tau_i(t) < t$, $\forall t \in [t_i, +\infty]$.

$(H_4)$ There exist positive constants $l_1$ and $l_2$, such that $f$ satisfies the Lipschitz condition

$$|f(t, \xi, x_1, ..., x_N) - f(t, \eta, y_1, ..., y_N)| \le l_1 |\xi - \eta| + l_2 \sum_{i=1}^{N} |x_i - y_i|.$$

$(H_5)$ There exists a positive constant $M > \phi_0$ such that

$$\frac{1}{\Gamma(\alpha)\alpha} f(t, M, x_1, ..., x_N) \le \frac{M - \phi_0}{b^{*\alpha}}, \quad \forall (t, x_1, ..., x_N) \in [0, b^*] \times \mathbb{R}^N.$$

$(H_6)$ $f$ is a non negative function, and, moreover, $\exists h \in [\phi_0, M]$ such that $\forall t \in [0, b^*]$

$$\frac{1}{\Gamma(\alpha)\alpha} f(t, h, x_1, ..., x_N) > \frac{M - \phi_0}{|b^* - \max_{1 \le i \le N} t_i|^{\alpha}}, \forall (x_1, ..., x_N) \in \left([\phi_0, \phi_0 + \tfrac{h - \phi_0}{b^*} \tau]\right)^N.$$

Let $X = \mathcal{C}(\mathbb{R})$ be the locally convex sequentially complete Hausdorff space of all real valued continuous functions defined on $\mathbb{R}$, and $\{P_K : K \in \mathcal{K}\}$ be the saturated family of semi-norms, generating the topology of $X$, defined by

$$P_K(u) = \sup_{t \in K} \left\{ e^{-\lambda t} |u(t)| \right\}, \quad (12)$$

where $K$ runs over the set of all compact subsets of $\mathbb{R}$ denoted by $\mathcal{K}$, and $\lambda$ is a positive real number to be specified later.

We denote by $\mathbf{E}_{\phi,M}$, the subset of $X$ defined by

$$\mathbf{E}_{\phi,M} = \{u \in X : u(t) = \phi(t) \text{ for } t \leq 0, \text{ and } u(t) \leq M \text{ for } t \in [a_*, b^*]\},$$

where $a_*, b^*$ and $M$ are the constants given by $(H_1)$ and $(H_5)$. It can be easily seen that $\mathbf{E}_{\phi,M}$ is a closed subset of $X$ and its boundary is

$$\partial\mathbf{E}_{\phi,M} = \left\{u \in X : u(t) = \phi(t) \text{ for } t \leq 0 \text{ and } \max_{t\in[a_*,b^*]} u(t) = M\right\}.$$

Throughout the remaining of this paper, $F$ denotes the operator defined on $\mathbf{E}_{\phi,M}$ by Equations (4) and (5). Thus, according to Lemma 1, $F$ maps $\mathbf{E}_{\phi,M}$ into $X$ and the fixed points of $F$ are continuous solutions of problems Equations (1) and (2).

**Remark 1.** *It should be pointed out that under hypotheses $(H_1)$–$(H_3)$, $(H_6)$ with the additional condition $\max\limits_{1\leq i\leq N} t_i < b^*$, $F$ is a non-self mapping on $\mathbf{E}_{\phi,M}$. Indeed, as is noted in the proof of [11] (Theorem 3) , for any function $u \in \mathbf{E}_{\phi,M}$ defined by $u(t) = \phi_0 + (h - \phi_0)\, t/b^*$, where $t \in [0\ , b^*]$ and $h$ is the constant given by $(H_6)$, it can be easily seen that $Fu \notin \mathbf{E}_{\phi,M}$. This will be checked by the example of the last section.*

The introduction of a self-mapping of the index set in uniform spaces is motivated by applications in the theory of neutral functional differential equations [11,27,28]. Following this idea, let us define a map $j : \mathcal{K} \longrightarrow \mathcal{K}$ by

$$j(K) := \begin{cases} K, & \text{if } K_+ = \emptyset, \\ [0,\ \max\{K_m, \tau, b^*\}], & \text{if } K_+ \neq \emptyset, \end{cases} \tag{13}$$

where $K_+ := K \cap [0, +\infty]$, $K_m = \sup K$, $\tau$ and $b^*$ are the positive constants given in $(H_1)$–$(H_2)$. For $n \in \mathbb{N}^*$, $j^n(K)$ is the compact set defined inductively by $j^n(K) = j\left(j^{n-1}(K)\right)$ and $j^0(K) = K$.

**Remark 2.** *Note that, for every $K \in \mathcal{K}$ and every integer $n$ greater than 1, we have $j^n(K) = j(K)$.*

In the next proposition, we show that $F$ satisfies Equation (14), which is a weakened version of the usual contraction when $L_\lambda < 1$ (see Remark 3).

**Proposition 1.** *Under hypotheses $(H_1)$–$(H_4)$, the operator $F : \mathbf{E}_{\phi,M} \to X$ satisfies for each $u, v \in \mathbf{E}_{\phi,M}$ and every $K \in \mathcal{K}$*

$$P_K(Fx - Fy) \leq L_\lambda P_{j(K)}(x - y) \tag{14}$$

*with*

$$L_\lambda = \frac{l_1}{\lambda^\alpha \Gamma(\alpha)} \Gamma\left(\alpha^2\right)^{\frac{1}{1+\alpha}} \left(\frac{\alpha}{1+\alpha}\right)^{\frac{\alpha}{1+\alpha}} e^{\lambda b^*} + \frac{N l_2 e^{\lambda\tau}}{\lambda^\alpha}. \tag{15}$$

**Proof.** Note that it suffices to consider $K_+ \neq \emptyset$, since otherwise $P_K(Fu - Fv) = 0$. Letting $t \in K_+$, we obtain by means of hypotheses $(H_3)$ and $(H_4)$

$$|Fu(t) - Fv(t)| \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} l_1 \left| \max_{\sigma\in[a(s),b(s)]} u(\sigma) - \max_{\sigma\in[a(s),b(s)]} v(\sigma) \right| ds$$

$$+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \; + \; l_2 \sum_{i=1}^{N} |u(s-\tau_i(s)) - v(s-\tau_i(s))| ds$$

$$\le l_1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \max_{\sigma\in[a(s),b(s)]} |u(\sigma)-v(\sigma)| \, ds + \; l_2 \sum_{i=1}^{N} \int_0^{\min\{t_i,t\}} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left|\phi(r_i(s))-\phi(r_i(s))\right| ds$$

$$+ \; l_2 \sum_{i\in\{1,\dots,N: t_i\le t\}} \int_{t_i}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |u(r_i(s)) - v(r_i(s))| \, ds$$

$$= l_1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \max_{\sigma\in[a(s),b(s)]} |u(\sigma)-v(\sigma)| \, ds + \; l_2 \sum_{i\in\{1,\dots,N: t_i\le t\}} \int_{t_i}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |u(r_i(s))-v(r_i(s))| \, ds$$

$$\le l_1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda b(s)} \max_{\sigma\in[a(s),b(s)]} e^{-\lambda\sigma} |u(\sigma) - v(\sigma)| ds$$

$$+ \; l_2 \sum_{i\in\{1,\dots,N: t_i\le t\}} \int_{t_i}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda r_i(s)} e^{-\lambda r_i(s)} \, |u(r_i(s)) - v(r_i(s))| \, ds$$

$$\le l_1 \max_{\sigma\in[a_*,b^*]} e^{-\lambda\sigma} \left|u(\sigma)-v(\sigma)\right| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda b(s)} ds$$

$$+ \; l_2 \sum_{i\in\{1,\dots,N: t_i\le t\}} \int_{t_i}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda r_i(s)} e^{-\lambda r_i(s)} \, |u(r_i(s)) - v(r_i(s))| \, ds,$$

where $r_i(s) = s - \tau_i(s)$ . Note that, due to the definition Equation (13) and under hypothesis $(H_1)$, it is clear that, for every $K \in \mathcal{K}$ with $K_+ \neq \emptyset$, we have $[a_*, \; b^*] \subset j(K)$ and further $(H_2)$–$(H_3)$ lead to $r_i(s) \in j(K)$ when $t_i \le s \le t$. Hence,

$$|Fx(t) - Fy(t)| \le l_1 P_{j(K)}(u-v) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda b(s)} ds$$

$$+ \; l_2 \sum_{i\in\{1,\dots,N: t_i\le t\}} \int_{t_i}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda r_i(s)} \max_{\xi\in j(K)} e^{-\lambda\xi} \left|u(\xi)-v(\xi)\right| ds$$

$$= l_1 P_{j(K)}(u-v) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda b(s)} ds + \; l_2 P_{j(K)}(u-v) \sum_{i\in\{1,\dots,N: t_i\le t\}} \int_{t_i}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda r_i(s)} ds.$$

Now, multiplying the both sides of the above inequality by $e^{-\lambda t}$, then performing the change of variable $u = \lambda(t-s)$, we get

$$e^{-\lambda t} |Fu(t) - Fv(t)| \le l_1 \; P_{j(K)}(u-v) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda(t-b(s))} ds$$

$$+ \; l_2 \; P_{j(K)}(u-v) \sum_{i\in\{1,\dots,N: t_i\le t\}} \int_{t_i}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda(t-r_i(s))} ds$$

$$= \frac{l_1}{\lambda^\alpha \Gamma(\alpha)} \; p_{j(K)}(u-v) \int_0^{\lambda t} x^{\alpha-1} e^{-\lambda(t-b(t-\frac{x}{\lambda}))} dx$$

$$+\frac{l_2}{\lambda^\alpha}P_{j(K)}(u-v)\sum_{i\in\{1,\ldots,N:t_i\leq t\}}\int_0^{\lambda(t-t_i)}\frac{x^{\alpha-1}}{\Gamma(\alpha)}e^{-x}e^{-\lambda\tau_i\left(t-\frac{x}{\lambda}\right)}dx$$

$$\leq\frac{l_1}{\lambda^\alpha\Gamma(\alpha)}P_{j(K)}(u-v)\int_0^{\lambda t}x^{\alpha-1}e^{-x}e^{-\lambda\left(\left(t-\frac{x}{\lambda}\right)-b\left(t-\frac{x}{\lambda}\right)\right)}dx$$

$$+\frac{l_2}{\lambda^\alpha}P_{j(K)}(u-v)\sum_{i\in\{1,\ldots,N:t_i\leq t\}}\int_0^{\lambda(t-t_i)}\frac{x^{\alpha-1}}{\Gamma(\alpha)}e^{-x}e^{-\lambda\tau_i\left(t-\frac{x}{\lambda}\right)}dx.$$

Let $\mu := 1+\alpha$ and $\nu := 1+1/\alpha$. Taking into account $(H_3)$, Hölder's inequality gives

$$e^{-\lambda t}\left|Fu(t)-Fv(t)\right|\leq,$$

$$\left\{\frac{l_1}{\lambda^\alpha\Gamma(\alpha)}\left(\int_0^{\lambda t}x^{\mu(\alpha-1)}e^{-\mu x}dx\right)^{\frac{1}{\mu}}\left(\int_0^{\lambda t}e^{-\nu\lambda\left(\left(t-\frac{x}{\lambda}\right)-b\left(t-\frac{x}{\lambda}\right)\right)}dx\right)^{\frac{1}{\nu}}+\frac{Nl_2e^{\lambda\tau}}{\lambda^\alpha}\right\}P_{j(K)}(u-v)$$

$$=\left\{\frac{l_1}{\lambda^\alpha\Gamma(\alpha)}\left(\int_0^{\lambda t}x^{\mu(\alpha-1)}e^{-\mu x}dx\right)^{\frac{1}{\mu}}\left(\lambda\int_0^{t}e^{-\nu\lambda(s-b(s))}ds\right)^{\frac{1}{\nu}}+\frac{Nl_2e^{\lambda\tau}}{\lambda^\alpha}\right\}P_{j(K)}(u-v)$$

$$\leq\left\{\frac{l_1}{\lambda^\alpha\Gamma(\alpha)}\left(\int_0^{\lambda t}x^{\mu(\alpha-1)}e^{-\mu x}dx\right)^{\frac{1}{\mu}}\left(\lambda\int_0^{t}e^{-\nu\lambda(s-b^*)}ds\right)^{\frac{1}{\nu}}+\frac{Nl_2e^{\lambda\tau}}{\lambda^\alpha}\right\}P_{j(K)}(u-v)$$

$$\leq\left\{\frac{l_1}{\lambda^\alpha\Gamma(\alpha)}\Gamma\left(\alpha^2\right)^{\frac{1}{\mu}}\frac{1}{\nu}^{\frac{1}{\nu}}e^{\lambda b^*}+\frac{Nl_2e^{\lambda\tau}}{\lambda^\alpha}\right\}P_{j(K)}(u-v).$$

Thus, the result is obtained by taking the supremum on $K$. □

**Remark 3.** *Since $K_+\subset j(K)$, if $P_K(Fu-Fv)\leq L_\lambda P_K(u-v)$ is satisfied, then Equation (14) holds true. Therefore, due to the choice of $j$, in the present context, the usual contraction is a particular case of Equation (14) when $L_\lambda<1$.*

To reach our aim, we proceed by adapting the proof of [11], [Theorem 1] with some completeness, for the construction of an iterative process converging to the unique continuous solutions of Equations (1) and (2).

According to Remark 1, the standard process of Picard's iterations fails to be well defined. To overcome this fact, we make use of Lemma 2 to construct a sequence of elements of $\mathbf{E}_{\phi,M}$ as follows: starting from an arbitrary point $u_0\in\mathbf{E}_{\phi,M}$, we define the terms of a sequence $\{u_n\}_{n\in\mathbb{N}^*}$ in $\mathbf{E}_{\phi,M}$ iteratively as follows:

$$\begin{cases} u_n = Fu_{n-1}, \quad \text{if} \quad Fu_{n-1}\in\mathbf{E}_{\phi,M}, \\ \\ u_n=(1-\beta_n)u_{n-1}+\beta_nFu_{n-1}\in\partial\mathbf{E}_{\phi,M} \text{ with } \beta_n\in[0,1[\,,\text{ if } Fu_{n-1}\notin\mathbf{E}_{\phi,M}. \end{cases} \tag{16}$$

Note that the terms of the sequence $\{u_n\}_{n\in\mathbb{N}^*}$ belong to $\mathbf{A}\cup\mathbf{B}\subset\mathbf{E}_{\phi,M}$, with $\mathbf{B}\subset\partial\mathbf{E}_{\phi,M}$, where

$$\mathbf{A}:=\{u_i\in\{u_n\}_{n\in\mathbb{N}^*}\,:\,u_i=Fu_{i-1}\} \text{ and } \mathbf{B}:=\{u_i\in\{u_n\}_{n\in\mathbb{N}^*}\,:\,u_i\neq Fu_{i-1}\}.$$

Furthermore, if $u_n\in\mathbf{B}$, a straightforward computation leads to

$$P_K(u_{n-1}-u_n)+P_K(u_n-Fu_{n-1})=P_K(u_{n-1}-Fu_{n-1})\ \ \forall K\in\mathcal{K}. \tag{17}$$

**Proposition 2.** *Let $u_0 \in \mathbf{E}_{\phi,M}$, and $\{u_n\}_{n\in\mathbb{N}^*}$ be the sequence defined iteratively by Equation (16). Then, under hypotheses $(H_1)$–$(H_5)$, for each $K \in \mathcal{K}$ and every integer $m$ greater than or equal to 1, the following estimation holds true:*

$$\max\{P_K(u_{2m} - u_{2m+1}),\, P_K(u_{2m+1} - u_{2m+2})\} \le 2^{2m-1} L_\lambda^m C_K, \tag{18}$$

*where $L_\lambda$ is given by Equation (15) and*

$$C_K = \max\{P_{j(K)}(u_0 - u_1),\, P_{j(K)}(u_1 - u_2),\, P_{j(K)}(u_2 - u_3)\}.$$

**Proof.** If $\max_{t\in[a_*,b^*]} u\,(t) = M$, then, for $a_* \le t \le b^*$, hypothesis $(H_5)$ gives

$$\begin{aligned} Fu\,(t) &= \phi_0 + \frac{1}{\Gamma\,(\alpha)} \int_0^t (t-s)^{\alpha-1} f\,(s, M, u\,(s-\tau_1\,(s))\,, ..., u\,(s-\tau_N\,(s)))\, ds \\ &\le \phi_0 + \tfrac{\alpha(M-\phi_0)}{b^{*\alpha}} \int_0^t (t-s)^{\alpha-1} ds \le M, \end{aligned}$$

which means that $F(\partial\mathbf{E}_{\phi,M}) \subset \mathbf{E}_{\phi,M}$. Consequently, two consecutive terms of the sequence $\{u_n\}_{n\in\mathbb{N}^*}$ can not belong to $\mathbf{B}$ (recall that $\mathbf{B} \subset \partial\mathbf{E}_{\phi,M}$). Thus, it suffices to consider the three cases below.

Case 1. $u_n,\, u_{n+1} \in \mathbf{A}$. From Equation (14), we have

$$P_K(u_n - u_{n+1}) = P_K(Fu_{n-1} - Fu_n) \le L_\lambda \cdot P_{j(K)}(u_{n-1} - u_n).$$

Case 2. $u_n \in \mathbf{A}$, $u_{n+1} \in \mathbf{B}$. From the condition Equation (14) together with Equation (17) (for $u_{n+1}$ instead of $u_n$), we get

$$\begin{aligned} P_K(u_n - u_{n+1}) &= P_K(u_n - Fu_n) - P_K(u_{n+1} - Fu_n) \\ &\le P_K(Fu_{n-1} - Fu_n) \le L_\lambda \cdot P_{j(K)}(u_{n-1} - u_n). \end{aligned}$$

Case 3. $u_n \in \mathbf{B}$, $u_{n+1} \in \mathbf{A}$. Then, $\exists \beta_n \in [0,1] : u_n = (1-\beta_n)u_{n-1} + \beta_n F u_{n-1}$, which implies that

$$P_K(u_n - u_{n+1}) \le \max\{P_K(u_{n-1} - u_{n+1}),\, P_K(Fu_{n-1} - u_{n+1})\}.$$

Thus, by Equation (14), for every integer number $n \ge 2$, we obtain either

$$P_K(u_n - u_{n+1}) \le L_\lambda \cdot p_{j(K)}(u_{n-1} - u_n), \text{ or } \; P_K(u_n - u_{n+1}) \le L_\lambda \cdot P_{j(K)}(u_{n-2} - u_n).$$

Moreover,

$$\begin{aligned} P_{j(K)}(u_{n-2} - u_n) &\le P_{j(K)}(u_{n-2} - u_{n-1}) + P_{j(K)}(u_{n-1} - u_n) \\ &\le 2\max\{P_{j(K)}(u_{n-2} - u_{n-1}),\, P_{j(K)}(u_{n-1} - u_n)\}. \end{aligned}$$

In summary, the following inequality is true in all cases

$$P_K(u_n - u_{n+1}) \le 2L_\lambda \max\{P_{j(K)}(u_{n-2} - u_{n-1}),\, P_{j(K)}(u_{n-1} - u_n)\}. \tag{19}$$

We now prove Equation (18) by induction. Using Equation (19), we have either

$$P_K(u_2 - u_3) \le 2L_\lambda \cdot P_{j(K)}(u_0 - u_1) \le 2L_\lambda \cdot C_K,$$

or

$$P_K(u_2 - u_3) \le 2L_\lambda \cdot P_{j(K)}(u_1 - u_2) \le 2L_\lambda \cdot C_K,$$

and similarly we obtain

$$P_K(u_3 - u_4) \le 2L_\lambda \cdot C_K.$$

Consequently, Equation (18) is satisfied for $m = 1$. Assume now that Equation (18) holds true for some $m > 1$. Using Equation (19), we get either

$$P_K(u_{2m+2} - u_{2m+3}) \le 2L_\lambda \cdot P_{j(K)}(u_{2m} - u_{2m+1}),$$

or

$$P_K(u_{2m+2} - u_{2m+3}) \le 2L_\lambda \cdot P_{j(K)}(u_{2m+1} - u_{2m+2}).$$

Thus, the fact that $C_{j(K)} = C_K$, which follows from Remark 2, leads to

$$P_K(u_{2m+2} - u_{2m+3}) \le 2L_\lambda 2^{2m-1} {L_\lambda}^m C_{j(K)} = 2^{2m} {L_\lambda}^{m+1} C_K \le 2^{2(m+1)-1} {L_\lambda}^{m+1} C_K.$$

In the same way, we get

$$P_K(u_{2m+3} - u_{2m+4}) \le 2^{2(m+1)-1} {L_\lambda}^{m+1} C_K,$$

which means that Equation (18) holds for $m + 1$, and this completes the proof. □

We are now ready to prove our main result.

**Theorem 1.** *Let $u_0 \in \mathbf{E}_{\phi,M}$, then under hypotheses $(H_1)$–$(H_6)$ with*

$$\max_{1 \le i \le N} t_i < b^*, \tag{20}$$

*the sequence $\{u_n\}_{n\in\mathbb{N}^*}$ defined iteratively by Equation (16), converges in $\mathbf{E}_{\phi,M}$ to the unique continuous solution of Equations (1) and (2) provided that*

$$\left\{ \frac{l_1}{\Gamma(\alpha)} \Gamma\left(\alpha^2\right)^{\frac{1}{1+\alpha}} \left(\frac{\alpha}{1+\alpha}\right)^{\frac{\alpha}{1+\alpha}} + Nl_2 \right\} e \max\{\tau, b^*\}^\alpha < \frac{1}{4}. \tag{21}$$

**Proof.** Let us put $\lambda = 1/\max\{\tau, b^*\}$ in Equation (12). Thus, according to Proposition 1, for every $K \in \mathcal{K}$ and $u, v \in \mathbf{E}_{\phi,M}$, Equation (14) holds true with $L_\lambda < 1/4$. Therefore, for an arbitrary fixed $K \in \mathcal{K}$, and, for each $\varepsilon > 0$, there exists a positive integer $s$ satisfying

$$\sum_{m=s}^{\infty} 2^{2m} L_\lambda^m < \frac{\varepsilon}{C_K}. \tag{22}$$

Hence, for $n \ge 2s$, $q \ge 1$ and a sufficiently large $l$, we get, by means of Equations (18) and (22),

$$\begin{aligned} P_K(u_n - u_{n+q}) &\le P_K(u_n - u_{n+1}) + P_K(u_{n+1} - u_{n+2}) + \cdots + P_K(u_{n+q-1} - u_{n+q}) \\ &\le \sum_{m=s}^{l} \{P_K(u_{2m} - u_{2m+1}) + P_K(u_{2m+1} - u_{2m+2})\} \\ &\le \sum_{m=s}^{l} 2^{2m} L_\lambda^m \cdot C_K \le C_K \cdot \sum_{m=s}^{\infty} 2^{2m} L_\lambda^m < \varepsilon. \end{aligned}$$

Consequently, $\{u_n\}_{n\in\mathbb{N}^*}$ is a Cauchy sequence in the closed subset $\mathbf{E}_{\phi,M}$ of the complete locally convex space $X$, and so it converges to a point $u \in \mathbf{E}_{\phi,M}$. Let $\{u_{n_k}\}_{k\ge1}$ be a sub-sequence of $\{u_n\}_{n\ge1}$ in $\mathbf{A}$, which is $u_{n_k+1} = Fu_{n_k}$ for every positive integer $k$. Then, for each compact $K \in \mathcal{K}$, we have

$$\begin{aligned} P_K(u - Fu) &\leq P_K(u - u_{n_k}) + P_K(u_{n_k} - Fu) = P_K(u - u_{n_k}) + P_K(Fu_{n_k-1} - Fu) \\ &\leq P_K(u - u_{n_k}) + L_\lambda \cdot P_{j(K)}(u_{n_k-1} - u) \xrightarrow[k\to+\infty]{} 0. \end{aligned}$$

Therefore, $u = Fu$ and so, according to Lemma 1, $u$ is a solution of Equations (1) and (2). For the uniqueness, assume that there exists another solution $v \in \mathbf{E}_{\phi,M}$ such that $u \neq v$. Since $X$ is Hausdorff, then $P_{K_0}(u - v) \neq 0$ for some compact $K_0 \in \mathcal{K}$. Using Equation (14) and Remark 2, we get for every positive integer $n$

$$\begin{aligned} 0 < P_{K_0}(u - v) &= P_{K_0}(Fu - Fv) \leq L_\lambda \cdot P_{j(K_0)}(u - v) = L_\lambda \cdot P_{j(K_0)}(Fu - Fv) \\ &\leq L_\lambda^2 \cdot P_{j(K_0)}(u - v) \leq \ldots \leq L_\lambda^n P_{j(K_0)}(u - v), \end{aligned}$$

which contradicts the fact that $L_\lambda < \frac{1}{4}$. This completes the proof. □

## 4. Example

The following example illustrates the applicability of our theoretical result. Let us consider the following equation

$$^{C}D^{0.5}u\,(t) = \frac{1.132}{2.45 + 10^{-2}\left|\max\limits_{t\in[10^{-1},2]} u(t)\right| + 10^{-4}\left|u\left(\frac{0.994}{1+t}t - 0.004\right)\right|}, \quad t > 0, \tag{23}$$

subject to the initial condition function

$$u\,(t) = 2t^2 + 1.168, \quad t \leq 0, \tag{24}$$

Problems in Equations (23) and (24) are identified to Equations (1) and (2) with $\alpha = 0.5$, $N = 1$, $a(t) = 10^{-1}$, $b(t) = 2$, $\tau_1\,(t) = t - \dfrac{0.994}{1+t}t + 0.004$, $\phi\,(t) = 2t^2 + 1.168$ and

$$f\,(t, \xi, \eta) = \frac{1.132}{2.45 + 10^{-2}\,|\xi| + 10^{-4}\,|\eta|}.$$

It can be easily seen that hypotheses $(H_1)$–$(H_4)$ are satisfied with $a_* = 10^{-1}$, $b^* = 2$, $\tau = 0.994$, $t_1 = 4 \times 10^{-3}/(1 - 10^{-2})$, $l_1 = 1132 \times 10^{-5}$ and $l_2 = 1132 \times 10^{-7}$.

In addition, there exists $M = 1.9$ ($M > \phi_0 = 1.168$), such that, for every $\eta \in \mathbb{R}$, we have

$$f\,(t, M, \eta) \leq \frac{1.132}{2.45 + 10^{-2} \times M} \simeq 0.45845216686918,$$

and

$$0.5\Gamma\,(0.5)\,\frac{M - \phi_0}{b^{*0.5}} \simeq 0.458712974257473.$$

Thus, $(H_5)$ holds true. To check $(H_6)$, let $h = 1.168001$ ($\phi_0 < h < M$), and then we have for every $\eta \in [\phi_0, \phi_0 + \frac{h-\phi_0}{b^*}\tau[ = ]1.168, 1.168 + (9.97 \times 10^{-5})]$

$$f\,(t, h, \eta) > \frac{1.132}{2.45 + h \times 10^{-2} + \left(\phi_0 + \frac{h-\phi_0}{b^*}\tau\right) \times 10^{-4}} \simeq 0.461798645149363,$$

and

$$0.5\Gamma\,(0.5)\,\frac{M - \phi_0}{(b^* - t_1)^{0.5}} \simeq 0.459177023920154.$$

Since, moreover, $f$ is clearly non negative, hypothesis $(H_6)$ is satisfied too.

Furthermore, we have $t_1 < b^*$ and

$$\left\{ \frac{l_1}{\Gamma(0.5)} \Gamma(0.25)^{\frac{1}{1.5}} \left(\frac{0.5}{1.5}\right)^{\frac{0.5}{1.5}} + N l_2 \right\} e \max\{\tau, b^*\}^{0.5} \simeq 0.041231690678434,$$

which is all conditions of Theorem 1 being fulfilled. Now, let $u_0$ be the function defined by

$$u_0(t) = \begin{cases} 1.168 + (5 \times 10^{-7})t : & t \in [0, b^*], \\ 1.168001 : & t \geq b^*, \\ 2t^2 + 1.168 : & t \leq 0. \end{cases}$$

It is clear that $u_0 \in \mathbf{E}_{\phi,1.9}$, where

$$\mathbf{E}_{\phi,1.9} = \left\{ u \in \mathcal{C}(\mathbb{R}) : u(t) = 2t^2 + 1.168 \quad \text{for} \quad t \leq 0 \quad \text{and} \quad u(t) \leq 1.9 \quad \text{for } t \in \left[10^{-1}, 2\right] \right\}.$$

Note that, for $0 < t - \tau_1(t) < \tau$, we have

$$\phi_0 = 1.168 < \ u_0(t - \tau_1(t)) < 1.168 + (5 \times 10^{-7})\tau = \phi_0 + \frac{h - \phi_0}{b^*}\tau.$$

Then, hypothesis $(H_6)$ yields to

$$Fu_0(2) = Fu_0(b^*) = \phi_0 + \int_0^{b^*} \frac{(b^* - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, h, u_0(s - \tau_1(s)))\, ds$$

.

$$\geq \phi_0 + \int_{t_1}^{b^*} \frac{(b^* - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, h, u_0(s - \tau_1(s)))\, ds$$

$$> \phi_0 + \alpha \frac{M - \phi_0}{(b^* - t_1)^{\alpha}} \int_{t_1}^{b^*} (b^* - s)^{\alpha-1}\, ds = M = 1.9,$$

which means that $Fu_0 \notin \mathbf{E}_{\phi,1.9}$. Thus, in this framework, the iterative processes usually used in the self-mapping context can not be applied, while, according to Theorem 1, the process defined by Equation (16), converges in $\mathbf{E}_{\phi,1.9}$, to the unique continuous solutions of Equations (23) and (24). The first term is approximately given by

$$u_1(t) \simeq \begin{cases} 1.168 + (12115 \times 10^{-13})\, t + 0.5176017180\sqrt{t} : \ 0 < t \leq t_1, \\ 1.168 + (12115 \times 10^{-13})\, t + (509761847 \times 10^{-18})\sqrt{27225\, t - 110} + 0.5176017180\sqrt{t} : \\ \quad t_1 < t \leq b^*, \\ 1.168000002 + (509761847 \times 10^{-18})\sqrt{27225\, t - 110} + 0.517601718\sqrt{t} : \quad t \geq b^*, \\ 2t^2 + 1.168 : \quad t \leq 0. \end{cases}$$

For $t > 0$, the other terms can be computed using the following formulas:

$$u_2(t) = \phi_0 + \frac{1.132}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{2.45 + 10^{-2}|\max\limits_{t \in [10^{-1}, 2]} u_1(s)| + 10^{-4}|u_1(\frac{0.994}{1+s}s - 0.004)|} ds.$$

By successive iterations up to the order $n - 1$, the term $u_n$ is given by

$$u_n(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{2.45 + 10^{-2}|\max\limits_{t \in [10^{-1}, 2]} u_{n-1}(s)| + 10^{-4}|u_{n-1}(\frac{0.994}{1+s}s - 0.004)|} ds,$$

if the right-hand side belongs to $\mathbf{E}_{\phi,1.9}$. If not, we have

$$u_n(t) = (1-\beta_n)u_{n-1} + \beta_n\left(\phi_0 + \frac{1.132}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}}{2.45+10^{-2}|\max\limits_{t\in[10^{-1},2]} u_{n-1}(s)| + 10^{-4}|u_{n-1}(\frac{0.994}{1+s}s-0.004)|}ds\right),$$

with $\beta_n \in [0,1]$, such that the right-hand side belongs to $\partial\mathbf{E}_{\phi,1.9}$.

## 5. Conclusions

In this contribution, the investigated question concerns the existence and uniqueness of the solution for a class of nonlinear functional differential equations of fractional order. The considered problems in Equations (1) and (2) are distinguished by the fact that the nonlinear part depends on maximum values of the unknown function, which is not frequently discussed in the existing literature. These maximums are taken on time-dependent intervals and, moreover, the equation is of mixed type, i.e., with both retarded and advanced deviations. It should be noted that, if the hypotheses $(H_6)$ and Equation (20) are omitted, the operator $F$ can be a self mapping, and thus, by the usual contraction methods, it can be shown that the result of Theorem 1 remains valid with the bound in Equation (21) weakened to 1. When additional conditions are necessary to meet the physical or mechanical requirements of the phenomenon governed by Equations (1) and (2), we leave the previous usual framework of study. In this case, our main result of Theorem 1 shows that the condition in Equation (21) is sufficient for the existence and uniqueness of the solution.

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## References

1. Popov, E.P. *Automatic Regulation and Control*; Nauka: Moscow, Russia, 1966.
2. Popov, E.P. *The Dynamics of Automatic Control Systems*; Pergamon Press: Oxford, UK, 1962.
3. Agarwal, R.P.; Hristova, S. Strict stability in terms of tow measures for impulsive differential equations with supremum. *Appl. Anal.* **2012**, *91*, 1379–1392.
4. Bainov, D.D.; Hristova, S.G. *Differential Equations with Maxima*; Chapman and Hall/CRC:Boca Raton, FL, USA, 2011.
5. Stepanov, E. On solvability of some boundary value problems for differential equations with maxima. *Topol. Methods Nonlinear Anal.* **1996**, *8*, 315–326.
6. Agarwal, R.P.; Hristova, S. Quasilinearization for initial value problems involving differential equations with "maxima". *Math. Comput. Model.* **2012**, *55*, 2096–2105.
7. Angelov, V.G.; Bainov, D.D. On functional differential equations with Maximums. *Period. Math. Hung.* **1987**, *18*, 7–15.
8. Bohner, M.J.; Georgieva, A.T.; Hristova, S.G. Nonlinear differential equations with maxima: Parametric stability in terms of two measures. *Appl. Math. Inf. Sci.* **2013**, *7*, 41–48.
9. Golev, A.; Hristova, S.; Rahnev, A. An algorithm for approximate solving of differential equations with "maxima". *Comput. Math. Appl.* **2010**, *60*, 2771–2778.
10. Otrocol, D. Systems of functional differential equations with maxima of mixed type, Volume 191 of Graduate Texts in Mathematics. *Electron. J. Qual. Theory Differ. Equ.* **2014**, *5*, 1–9.
11. Tsachev, T.; Angelov, V.G. Fixed points of nonself-mappings and applications. *Nonlinear Anal. Theory Methods Appl.* **1993**, *21*, 1–12.

12. Araci, A.; Sen, E.; Acikgoz, M.; Srivastava, H.M. Existence and uniqueness of positive and nondecreasing solutions for a class of fractional boundary value problems involving the p-Laplacian operator. *Adv. Differ. Equ.* **2015**, *40*, doi:10.186/s13662-015-0375-0.
13. Jankowski, T. Fractional equations of Volterra type involving a Riemann-Liouville derivative. *Appl. Math. Lett.* **2013**, *26* , 344–350.
14. Mishra, L.N.; Srivastava, H.M.; Sen, M. Existence results for some nonlinear functional-integral equations in Banach algebra with applications. *Int. J. Anal. Appl.* **2016**, *11*, 1–10.
15. Nisse, L.; Bouaziz, A. Existence and stability of the solutions for systems of nonlinear fractional differential equations with deviating arguments. *Adv. Differ. Equ.* **2014**, *275*, doi:10.1186/1687-1847-2014-275.
16. Wang, G.; Agarwal, R.P.; Cabada, A. Existence results and the monotone iterative technique for systems of nonlinear fractional differential equations. *Appl. Math. Lett.* **2012**, *25*, 1019–1024.
17. Ahmad, B.; Ntouyas, S.K. Initial value problems of fractional order Hadamard-type functional differential equations. *Electron. J. Differ. Equ.* **2015**, *77*, 1–9.
18. Tisdell, C.C. Basic existence and a priori bound results for solutions to systems of boundary value problems for fractional differential equations. *Electron. J. Differ. Equ.* **2016**, *84*, 1–9.
19. Diethelm, K. *The Analysis of Fractional Differential Equations*; Springer: Berlin, Germany, **2004**.
20. Gorenflo, R.; Mainardi, F. Fractional calculus: Integral and differential equations of fractional order. In *Fractional Calculus in Continuum and Statistical Mechanics (Udine, 1996)*; CISM Courses and Lectures; Springer: Viena, Austria, 1997; pp. 223–276.
21. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
22. Mainardi, F. Fractional calculus: Some basic problems in Continuum and Statistical Mechanics. In *Fractal and Fractional Calculus in Continuum Mechanics (Udine, 1996)*; CIAM Courses and Lectures; Springer: Wien, Austria; New York, NY, USA, 1997; pp. 291–348.
23. Diethelm, K. Analysis of Fractional Differential Equations. *J. Math. Anal. Appl.* **2002**, *265*, 229–248.
24. Jalilian, Y.; Jalilian, R. Existence of solution for delay fractional differential equations. *Mediterr. J. Math.* **2013**, *10*, 1731–1747.
25. Koethe, G. *Topological Vector Spaces I*; Springer: Berlin, Germany, 1969.
26. Sehgal, V.M.; Singh, S.P. On a fixed point theorem of Krasnoselskii for locally convex spaces. *Pac. J. Math.* **1976**, *62*, 561–567.
27. Angelov, V.G. A Converse to a Contraction Mapping Theorem in Uniform Spaces. *Nonlinear Anal. Theory Methods Appl.* **1988**, *12*, 989–996.
28. Angelov, V.G. Fixed point theorems in uniform spaces and applications. *Czechoslov. Math. J.* **1987**, *37*, 19–33.