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A Numerical Solution of Fractional Lienard's Equation by Using the Residual Power Series Method

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Abstract: In this paper, we investigate a numerical solution of Lienard's equation. The residual power series (RPS) method is implemented to find an approximate solution to this problem. The proposed method is a combination of the fractional Taylor series and the residual functions. Numerical and theoretical results are presented.

Keywords: Lienard's equation; Caputo derivative; Taylor series; residual power series (RPS)

1. Introduction

The ordinary Lienard's equation is given by:

$$y''(x) + f(y)y'(x) + g(y) = r(x). \tag{1}$$

Different choices of $f,\ g,$ and r will produce different models. For example, if $f(y)\,y'(x)$ is the damping force, $g(y)$ is the restoring force, and $r(x)$ is the external force, we get the damped pendulum equation. However, if we choose $f(y) = \epsilon(y^2 - 1),\ g(y) = y,$ and $r(x) = 0$, we get a nonlinear model of electronic oscillation, see [1,2].

Several researchers have studied the exact solution of special cases of Equation (1). For example, Feng [3] investigated the exact solution of:

$$y''(x) + a\,y(x) + b\,y^3(x) + c\,y^5(x) = 0. \tag{2}$$

He found that one of the solutions to Equation (2) is given by:

$$y(x) = \sqrt{-\frac{2a}{b}(1 + \tan(\sqrt{-a}x)} \tag{3}$$

when $\frac{b^2}{4} - 4\frac{a\,c}{3} = 0,\ b > 0,$ and $a < 0$. Several methods are used to investigate the solution of nonlinear equation such as the homotopy analysis method (HAM), Adomian decomposition method, and variational method [4–12]. Equation (2) was studied by many researchers, notably Kong [13], Matinfar et al. [14,15], and others. In this paper, we generalize Equation (2) to the fractional case. It is not an easy task to solve highly nonlinear differential equations of fractional order. Many of the researchers tried to solve nonlinear equations by using different techniques. For example, Liao studied an analytical method termed as the HAM [9–11] to examine nonlinear problems. Furthermore, the HAM is used by many researchers to solve various types of nonlinear problems such as fractional Black–Scholes equation [12], natural convective heat and mass transfer in a steady 2-D MHD fluid flow over a stretching vertical surface via porous media [16], micropolar flow in a porous channel in the presence of mass injection [17], etc. The standard classical analytic schemes require more computational time and computer memory. Some researchers combine analytical techniques with Laplace transform to study nonlinear problems such as a class of nonlinear differential equations [18], the nonlinear boundary value problem on a semi infinite domain [19], the fractional convection–diffusion equation [20], the fractional Keller–Segel model [21], fractional coupled Burgers equations [16], and the synthesis of FO-PID controllers [22]. In this paper, we consider the following class of fractional Lienard's equation of the form:

$$D^{2\alpha}y(x) + a\,y(x) + b\,y^3(x) + c\,y^5(x) = 0,\ x > 0,\ \frac{1}{2} < \alpha \leq 1, \tag{4}$$

subject to:

$$y(0) = y_0,\ D^{\alpha}y(0) = y_1 \tag{5}$$

where $a,\ b,\ c,\ y_0,$ and y_1 are constants. The derivative in Equation (4) is in the Caputo sense. The Caputo derivative is defined as follows; see [23,24].

Definition 1. *Let n be the smallest integer greater than or equal to α. The Caputo fractional derivative of order $\alpha > 0$ is defined as:*

$$D^{\alpha}y(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} y^{(n)}(t)dt, & n-1 < \alpha < n, \\ y^{(n)}(x), & \alpha = n \in \mathbb{N}. \end{cases}.$$

The power rule of the Caputo derivative is given as follows.

Theorem 1. *The Caputo fractional derivative of the power function is given by:*

$$D^{\alpha}x^{p} = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha}, & n-1 < \alpha < n,\ p > n-1,\ p \in \mathbb{R}, \\ 0, & n-1 < \alpha < n,\ p \leq n-1,\ p \in \mathbb{N}. \end{cases}$$

We implement the residual power series (RPS) method [13,25–27] to solve Equation (4). We start by the following definition and some theorems related to the RPS, [13,27].

Definition 2. *A power series expansion of the form:*

$$\sum_{m=0}^{\infty} c_m (x-x_0)^{m\alpha} = c_0 + c_1(x-x_0)^{\alpha} + c_2(x-x_0)^{2\alpha} + \ldots$$

where $0 \leq n-1 < \alpha \leq n,\ x \leq x_0,$ is called fractional power series FPS about $x = x_0$.

Theorem 2. *Suppose that f has a RPS representation at $x = x_0$ of the form:*

$$f(x) = \sum_{m=0}^{\infty} c_m (x-x_0)^{m\alpha}, \quad x_0 \leq x < x_0 + R,$$

where R is the radius of convergence. If $D^{m\alpha}f(x),\ m = 0,1,2,\ldots$ are continuous on (x_0, x_0+R), then $c_m = \frac{D^{m\alpha}f(x_0)}{\Gamma(1+m\alpha)}$.

2. The Residual Power Series Method for Fractional Lienard's Equation

Write the solution of Equation (4) as a fractional power series of the form:

$$y(x) = \sum_{n=0}^{\infty} y_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)}. \tag{6}$$

Using the initial conditions in Equation (5), we approximate $y(x)$ in Equation (6) by:

$$y_k(x) = y_0 + y_1 \frac{x^{\alpha}}{\Gamma(1+\alpha)} + \sum_{n=2}^{k} y_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)}, \quad k = 2, 3, \ldots \tag{7}$$

where $y_1(x) = y_0 + y_1 \frac{x^{\alpha}}{\Gamma(1+\alpha)}$ is considered as the first RPS approximate solution of $y(x)$. To find the values of the RPS-coefficients $y_n, n = 2, 3, 4, \ldots$, we solve the equation:

$$D^{(n-2)\alpha} Res_n(0) = 0, \; n = 2, 3, 4, \ldots \tag{8}$$

where $Res_k(x)$ is the kth residual function [13,27] and it is defined by:

$$Res_k(x) = D^{2\alpha} y_k(x) + a\, y_k(x) + b\, y_k^3(x) + c\, y_k^5(x). \tag{9}$$

To find y_2 in Equation (7), we substitute the second RPS approximate solution $y_2(x) = y_0 + y_1 \frac{x^{\alpha}}{\Gamma(1+\alpha)} + y_2 \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}$ into:

$$\begin{aligned} Res_2(x) &= D^{2\alpha} y_2(x) + a\, y_2(x) + b\, y_2^3(x) + c\, y_2^5(x) \\ &= y_2 + a \left(y_0 + y_1 \tfrac{x^{\alpha}}{\Gamma(1+\alpha)} + y_2\, \tfrac{x^{2\alpha}}{\Gamma(1+2\alpha)} \right) \\ &\quad + b \left(y_0 + y_1 \tfrac{x^{\alpha}}{\Gamma(1+\alpha)} + y_2\, \tfrac{x^{2\alpha}}{\Gamma(1+2\alpha)} \right)^3 \\ &\quad + c \left(y_0 + y_1 \tfrac{x^{\alpha}}{\Gamma(1+\alpha)} + y_2\, \tfrac{x^{2\alpha}}{\Gamma(1+2\alpha)} \right)^5 . \end{aligned} \tag{10}$$

Then, we solve $Res_2(0) = 0$ to get:

$$y_2 = -\left(a\, y_0 + b\, y_0^3 + c\, y_0^5 \right). \tag{11}$$

To find y_3 in Equation (7), we substitute the third RPS approximate solution $y_3(x) = y_0 + y_1 \frac{x^{\alpha}}{\Gamma(1+\alpha)} + y_2 \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + y_3 \frac{x^{3\alpha}}{\Gamma(1+3\alpha)}$ into:

$$\begin{aligned} Res_3(x) &= D^{2\alpha} y_3(x) + a\, y_3(x) + b\, y_3^3(x) + c\, y_3^5(x) \\ &= y_2 + y_3 \tfrac{x^{\alpha}}{\Gamma(1+\alpha)} + a \left(y_0 + y_1 \tfrac{x^{\alpha}}{\Gamma(1+\alpha)} + y_2\, \tfrac{x^{2\alpha}}{\Gamma(1+2\alpha)} + y_3 \tfrac{x^{3\alpha}}{\Gamma(1+3\alpha)} \right) \\ &\quad + b \left(y_0 + y_1 \tfrac{x^{\alpha}}{\Gamma(1+\alpha)} + y_2\, \tfrac{x^{2\alpha}}{\Gamma(1+2\alpha)} + y_3 \tfrac{x^{3\alpha}}{\Gamma(1+3\alpha)} \right)^3 \\ &\quad + c \left(y_0 + y_1 \tfrac{x^{\alpha}}{\Gamma(1+\alpha)} + y_2\, \tfrac{x^{2\alpha}}{\Gamma(1+2\alpha)} + y_3 \tfrac{x^{3\alpha}}{\Gamma(1+3\alpha)} \right)^5 . \end{aligned} \tag{12}$$

Then, we solve $D^{\alpha} Res_3(0) = 0$ to get:

$$y_3 = -\left(a\, y_1 + 3b\, y_0^2\, y_1 + 5c\, y_0^4\, y_1 \right). \tag{13}$$

In general, to find $y_k(x)$, we substitute the k^{th} RPS approximate solution $y_k(x)$ into:

$$Res_k(x) = D^{2\alpha} y_k(x) + a\, y_k(x) + b\, y_k^3(x) + c\, y_k^5(x). \tag{14}$$

Then, we solve $D_t^{(k-2)\alpha} Res_k(0) = 0$ to get:

$$y_k = -\begin{pmatrix} a\, y_{k-2} + b \sum_{i=0}^{k-2} \sum_{j=0}^{k-2-i} \frac{y_i\, y_j\, y_{k-2-i-j}\, \Gamma(1+(k-2)\alpha)}{\Gamma(1+i\alpha)\Gamma(1+j\alpha)\Gamma(1+(k-2-i-j)\alpha)} \\ + c \sum_{i=0}^{k-2} \sum_{j=0}^{k-2-i} \sum_{l=0}^{k-2-i-j} \sum_{m=0}^{k-2-i-j-l} \frac{y_i\, y_j y_l y_m\, y_{k-2-i-j-l-m}\, \Gamma(1+(k-2)\alpha)}{\Gamma(1+i\alpha)\Gamma(1+j\alpha)\Gamma(1+l\alpha)\Gamma(1+m\alpha)\Gamma(1+(k-2-i-j-l-m)\alpha)} \end{pmatrix}. \tag{15}$$

3. Convergence Analysis

In this section, we prove the convergence of the proposed method. We start by the following lemma.

Lemma 1. *The classical power series* $\sum_{n=0}^{\infty} y_n x^n$, $-\infty < x < \infty$, *has a radius of convergence* R *if and only if the fractional power series* $\sum_{n=0}^{\infty} y_n x^{\alpha n}$, $x \geq 0$, *has a radius of convergence* $R^{\frac{1}{\alpha}}$.

Proof. See [28]. □

Theorem 3. *The fractional power series:*

$$y(x) = \sum_{n=0}^{\infty} y_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \tag{16}$$

where the coefficients are defined in Equation (15) has a positive radius of convergence.

Proof. From Equation (15), one can see that:

$$\frac{|y_k|}{\Gamma(1+k\alpha)} \leq \left(\begin{array}{c} A\, y_{k-2} + B \sum_{i=0}^{k-2} \sum_{j=0}^{k-2-i} |y_i|\ |y_j|\ \left|y_{k-2-i-j}\right| \\ +C \sum_{i=0}^{k-2} \sum_{j=0}^{k-2-i} \sum_{l=0}^{k-2-i-j} \sum_{m=0}^{k-2-i-j-l} |y_i|\ |y_j|\ |y_l|\ |y_m|\ \left|y_{k-2-i-j-l-m}\right| \end{array} \right) \tag{17}$$

where:

$$\begin{aligned} A &= \frac{|a|}{\Gamma(1+k\alpha)}, \\ B &= \underset{\substack{0 \leq j \leq k-2-i \\ 0 \leq i \leq k-2 \\ k}}{Max} \left\{ \frac{\Gamma(1+(k-2)\alpha)}{\Gamma(1+i\alpha)\Gamma(1+j\alpha)\Gamma(1+(k-2-i-j)\alpha)\Gamma(1+k\alpha)} \right\} |b|, \\ C &= \underset{\substack{0 \leq m \leq k-2-i-j-l \\ 0 \leq l \leq k-2-i-j \\ 0 \leq j \leq k-2-i \\ 0 \leq i \leq k-2 \\ k}}{Max} \left\{ \frac{\Gamma(1+(k-2)\alpha)}{\rho_{i,j,k,l,m}} \right\} |c|, \\ \rho_{i,j,k,l,m} &= \Gamma(1+i\alpha)\Gamma(1+j\alpha)\Gamma(1+l\alpha)\Gamma(1+m\alpha)\Gamma(1+(k-2-i-j-l-m)\alpha)\Gamma(1+k\alpha). \end{aligned}$$

Let:

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \tag{18}$$

where $a_0 = |y_0|$, $a_1 = \frac{|y_1|}{\Gamma(1+\alpha)}$, and

$$\begin{array}{c} a_k = A\, a_{k-2} + B \sum_{i=0}^{k-2} \sum_{j=0}^{k-2-i} a_i\, a_j\, a_{k-2-i-j} \\ +C \sum_{i=0}^{k-2} \sum_{j=0}^{k-2-i} \sum_{l=0}^{k-2-i-j} \sum_{m=0i}^{k-2-i-j-l} a_i\, a_j a_l a_m\, a_{k-2-i-j-l-m} \end{array} \tag{19}$$

for $k = 2, 3, \ldots$ Then,

$$\begin{aligned} \omega &= f(x) = a_0 + a_1\, x + x^2 \sum_{k=0}^{\infty} a_{k+2} x^k \\ &= a_0 + a_1\, x + x^2 \left(\begin{array}{c} A \sum_{k=0}^{\infty} a_k x^k + B \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} \sum_{j=0}^{k-i} a_i\, a_j\, a_{k-i-j} \right) x^k \\ + C \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} \sum_{j=0}^{k-i} \sum_{l=0}^{k-i-j} \sum_{m=0i}^{k-i-j-l} a_i\, a_j a_l a_m\, a_{k-i-j-l-m} \right) x^k \end{array} \right). \end{aligned} \tag{20}$$

Let:

$$H(x, \omega) = \omega - a_0 - a_1\, x - x^2 \left(A\omega + B\omega^3 + C\omega^5 \right). \tag{21}$$

It is easy to see that $H(x, \omega)$ is an analytic function in the $(x, \omega)-$ plane and

$$H(0, a_0) = 0,\ H_\omega(0, a_0) = 1 \neq 0. \tag{22}$$

By implicit function theorem [29], $f(x)$ is analytic function in a neighborhood of the point $(0, a_0)$ of the $(x, \omega)-$ plane with a positive radius of convergence. Thus, the series in Equation (6) is convergent by Lemma 1. $\square$

4. Numerical Results

In this section, we present four examples to show the efficiency of the proposed approach. Comparison with the exact solution presented in Equation (3) is reported in Tables 1–4 for different choices of $a,\ b,$ and c with $\alpha = 1$. Let:

$$error(x) = \left| y_{exact}(x) - y_5(x) \right|,\ x \geq 0. \tag{23}$$

In these two examples, we use $k = 3$. Then,

$$\begin{aligned} y_2 &= -\left(a\, y_0 + b\, y_0^3 + c\, y_0^5 \right), \\ y_3 &= -\left(a\, y_1 + 3b\, y_0^2\, y_1 + 5c\, y_0^4\, y_1 \right). \end{aligned} \tag{24}$$

Example 1. *Consider the following class of fractional differential equation:*

$$D^{2\alpha} y(x) - y(x) + 3y^3(x) - \frac{27}{16} y^5(x) = 0,\ \frac{1}{2} < \alpha \leq 1, \tag{25}$$

subject to:

$$y(0) = y_0 = \sqrt{\frac{2}{3}},\ D^\alpha y(0) = y_1 = \frac{1}{\sqrt{6}}. \tag{26}$$

Then, the error, when $\alpha = 1$, is reported in Table 1. Figure 1 shows the effect of α on the solution for $\alpha = 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1$.

Table 1. Error when $\alpha = 1$.

x	Exact Solution	$y_3(x)$	$error(x)$
0.00	0.816497	0.816497	0
0.02	0.824622	0.824620	2.2×10^{-6}
0.04	0.832675	0.832658	1.7×10^{-5}
0.06	0.840663	0.840606	5.8×10^{-5}
0.08	0.848595	0.848460	1.4×10^{-4}
0.10	0.856479	0.856216	2.6×10^{-4}

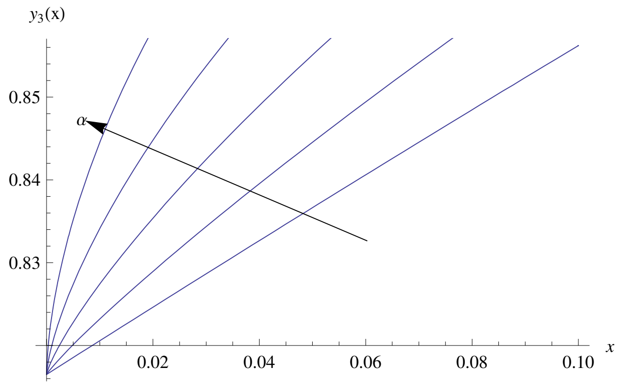

Figure 1. The approximate solution for $\alpha = 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1$.

Example 2. *Consider the following class of fractional differential equation:*

$$D^{2\alpha}y(x) - y(x) + 4y^3(x) - 3y^5(x) = 0,\ \frac{1}{2} < \alpha \leq 1, \tag{27}$$

subject to:

$$y(0) = y_0 = \sqrt{\frac{1}{2}},\ D^{\alpha}y(0) = y_1 = \frac{1}{\sqrt{8}}. \tag{28}$$

Then, the error, when $\alpha = 1$, is reported in Table 2. Figure 2 shows the effect of α on the solution for $\alpha = 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1$.

Table 2. Error when $\alpha = 1$.

x	Exact Solution	$y_3(x)$	$error(x)$
0.00	0.707107	0.707107	0
0.02	0.714142	0.714144	1.9×10^{-6}
0.04	0.721103	0.721118	1.5×10^{-5}
0.06	0.727986	0.728036	4.9×10^{-5}
0.08	0.734789	0.734905	1.1×10^{-4}
0.10	0.741508	0.741733	2.2×10^{-4}

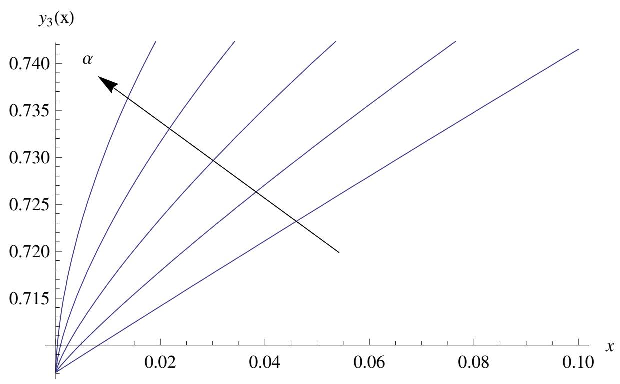

Figure 2. The approximate solution for $\alpha = 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1$.

Example 3. *Consider the following class of fractional differential equation:*

$$D^{2\alpha}y(x) - y(x) + 2y^3(x) - \frac{3}{4}y^5(x) = 0,\ \frac{1}{2} < \alpha \leq 1, \tag{29}$$

subject to:

$$y(0) = y_0 = 1,\ D^{\alpha}y(0) = y_1 = 1. \tag{30}$$

Then, the error, when $\alpha = 1$, is reported in Table 3. Figure 3 shows the effect of α on the solution for $\alpha = 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1$.

Table 3. Error when $\alpha = 1$.

x	**Exact Solution**	$y_3(x)$	$error(x)$
0.00	1.41421356	1.41421356	0
0.02	1.43407937	1.43407951	1.4×10^{-8}
0.04	1.45370489	1.45370501	1.2×10^{-6}
0.06	1.47313071	1.47313091	2.0×10^{-6}
0.08	1.49239558	1.49239579	2.1×10^{-6}
0.10	1.51153681	1.51153707	2.6×10^{-6}

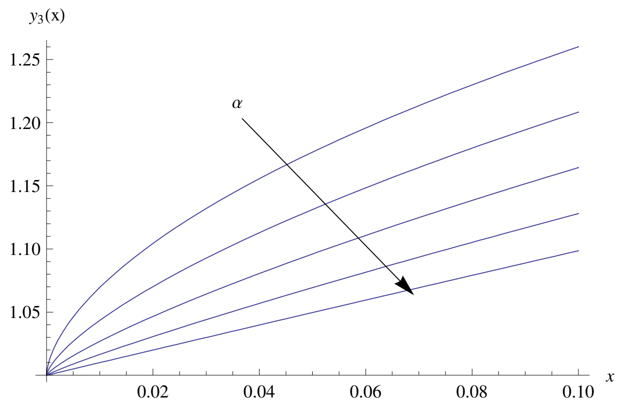

Figure 3. The approximate solution for $\alpha = 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1$.

Example 4. *Consider the following class of fractional differential equation:*

$$D^{2\alpha}y(x) - 4y(x) + 8y^3(x) - 3y^5(x) = 0,\ \frac{1}{2} < \alpha \leq 1, \tag{31}$$

subject to:

$$y(0) = y_0 = 1,\ D^{\alpha}y(0) = y_1 = 1. \tag{32}$$

Then, the error, when $\alpha = 1$, is reported in Table 4. Figure 4 shows the effect of α on the solution for $\alpha = 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1$.

Table 4. Error when $\alpha = 1$.

x	**Exact Solution**	$y_3(x)$	$error(x)$
0.00	1	1	0
0.02	1.01981437	1.01981441	4.0×10^{-8}
0.04	1.03931280	1.03931294	1.4×10^{-7}
0.06	1.05857420	1.05857444	2.4×10^{-7}
0.08	1.07767317	1.07767348	3.1×10^{-7}
0.10	1.09668137	1.09668171	3.4×10^{-7}

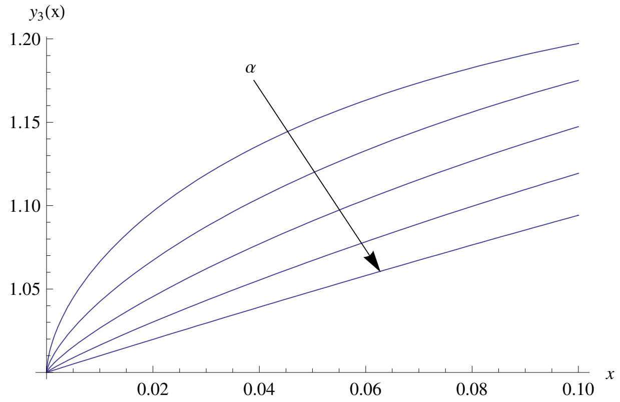

Figure 4. The approximate solution for $\alpha = 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1$.

5. Conclusions

In this paper, we have investigated the analytical solution of Lienard's equation based on the RPS method. Convergence of the proposed infinite series is presented. Four examples of our numerical results are presented. Comparison with the exact solution when $\alpha = 1$ is reported in Tables 1–4. From Tables 1–4, we see that the approximate solute is close to the exact solution with four terms only. From Figures 1 and 2, we see that as α is increasing, the approximate solution is increasing while from Figures 3 and 4, we see that as α is increasing, the approximate solution is decreasing. We see that this approach is cheap compared with other methods and we obtain accurate results using four terms only. A reasonably accurate solution can be achieved with only a few terms. Moreover, the proposed method can be applied to several nonlinear models in science and engineering.

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