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Invariant Solutions for a Class of Perturbed Nonlinear Wave Equations

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Abstract: Approximate symmetries of a class of perturbed nonlinear wave equations are computed using two newly-developed methods. Invariant solutions associated with the approximate symmetries are constructed for both methods. Symmetries and solutions are compared through discussing the advantages and disadvantages of each method.

Keywords: approximate symmetry; approximate invariant solution; nonlinear wave equation

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1. Introduction

Approximate Lie symmetry is based on the utilization of the perturbation approach in finding symmetries of certain equations. Baikov, Gazizov and Ibragimov [1] proved an approximate Lie theorem enabling one to construct approximate symmetries of differential equations that are stable under small perturbations. Fushchich and Shtelen [2] and later Gazizov [3] introduced approximate symmetries of differential equations with small perturbations and showed that the these symmetries form an approximate Lie algebra. Since then, many authors have used the approximate Lie symmetries to study nonlinear partial differential equations (PDEs) with a small parameter; see, for instance, [4–7] and the references therein.

Pakdermirli, Yurusoy and Dolapci [8] provided a comparison between several methods that use approximate symmetries. Valenti [9] calculated the solution of a model describing dissipative media using the generator of the first-order approximate symmetries. Bokhari, Kara and Zaman [4] considered some nonlinear evolution equations with a small parameter and their symmetries. On the other hand, a refined invariant subspace method to determine subspaces of solutions to nonlinear wave equations was discussed in [10]. Zhi-Yong, Yu-Fu and Xue-Lin [11] performed classification and gave approximate solutions to a class of perturbed nonlinear wave equation employing the method originated from Fushchich and Shtelen. In [12], the authors introduced a new method to obtain the approximate symmetry of the nonlinear evolution equation from perturbations.

In this paper, we study the approximate symmetries of a class of perturbed nonlinear wave equations given by:

$$u_{tt} + \alpha u_t = (g(u)u_x)_x + (h(u)u_y)_y + \beta f(u). \tag{1}$$

Lie group theory provides a systematic way of finding exact solutions of differential equations. If the problem involves a small parameter, then an approximate solution instead of an exact solution can be sought. We employ two methods in which a combination of Lie symmetries and perturbation theory is used to find approximate Lie symmetries and invariant solutions.

Method I was introduced by Baikov, Gazizov and Ibragimov [1,13]. In this method, an approximate generator is calculated to obtain the solution. The Lie operator is expanded in a perturbation series other than perturbation for dependent variables as in the usual case. In other words, it is assumed that the perturbed differential equation is of the form:

$$F(z) = F_0(z) + \varepsilon F_1(z) = 0, \tag{2}$$

where $z = (x, u, u(1), ..., u(n))$, $F_0(z) = 0$ is the unperturbed equation and $F_1(z)$ is the perturbed term.

Theorem 1. *[14] Equation (2) is approximately invariant with the generator $X = X^0 + \varepsilon X^1$ if and only if:*

$$[XF]_{F\approx 0} = O(\varepsilon) \qquad or \qquad [X^0F^0 + \varepsilon(X^1F_0 + X^0F_1]_{F\approx 0} = O(\varepsilon),$$

in which X^0 is a generator of Lie symmetry of $F_0 = 0$ and X^1 is a generator of Lie symmetry of F_1.

The exact symmetry of the unperturbed equation $F_0(z) = 0$ denoted by X^0 can be obtained using the equation $X^0F_0(z)|_{F_0(z)=0} = 0$. Applying the auxiliary function:

$$H = \frac{1}{\varepsilon}X^0(F_0(z) + \varepsilon F_1(z))|_{F0+eF1=0},$$

we deduce the vector field X^1 from the relation:

$$X^1F_0(z)|_{F0=0} + H = 0. \tag{3}$$

After computing the approximate symmetries, the corresponding invariant solutions are constructed via the classical Lie symmetry method [14]. One may refer the reader for some cases of studying unperturbed and perturbed non-linear wave equations to Bokhari, Kara, Karim, Zaman [15] and Zhi-Yong, Yu-Fu and Xue-Lin [12]. Ahmed, Bokhari, Kara and Zaman [16] provided a classification of the symmetries of the unperturbed nonlinear $(2+1)$ dimensional wave equation with its respective commutator table.

Method II is due to Fushchich and Shtelen [2] and later followed by Euler et al. [17] and Euler and Euler [18]. In this method, the dependent variables are expanded in a perturbation series as is done in the usual perturbation analysis (see, e.g., [19,20]). The approximate symmetry of the original equation is defined to be the exact symmetry of the coupled equations.

Consider the general m-th order nonlinear evolution equation:

$$E = E(x, t, u, u_1, u_2, ..., u_m, u_t; \varepsilon) = 0, \tag{4}$$

where $u_t = \partial u/\partial t, u_k = \partial^k u/\partial x^k, 1 \leq k \leq m$, ε is a small parameter and E is a smooth function of the indicated variables. Expanding the dependent variable in the small parameter yields:

$$u = u_0 + \varepsilon u_1 + ..., \qquad 0 < \varepsilon < 1. \tag{5}$$

Inserting expansion Equation (5) into the original Equation (4) and separating at each order of the perturbed parameter, one has:

$$Order\ \varepsilon^0 : E_0 = 0,\ Order\ \varepsilon^1 : E_1 = 0, \tag{6}$$

and hence, the exact symmetry of system Equation (6) is the approximate symmetry of the original Equation (4).

The outline of this paper is as follows. In Section 2, we construct invariant solutions of a perturbed nonlinear $(1+1)$-dimensional wave equation. In Section 3, we consider Equation (1) with $\beta = 0$ and obtain exact and approximate symmetries of the equation using the approximate Lie symmetry Method I. Moreover approximate invariant solutions of the perturbed non-linear wave equation based on the Lie group method are constructed. In Section 4, we discuss Equation (1) with $\alpha = 0$ and compute approximate symmetries of the equation with a forcing term using both the approximate Lie symmetry methods. We compare these different methods and discuss the advantages of using one over the other. Moreover, approximate invariant solutions of the nonlinear wave equation with a forcing term based on the Lie group method are constructed.

2. Perturbed Nonlinear (1 + 1)-Dimension Wave Equation

Consider the perturbed nonlinear wave equation (see e.g., [21]):

$$F_0(z) + \varepsilon F_1(z) = u_{tt} - (u^2 u_x)_x + \varepsilon u_t = 0. \tag{7}$$

The approximate group generator of Equation (7) is of the form:

$$X = X^0 + \varepsilon X^1 \equiv (\tau_0 + \varepsilon\tau_1)\frac{\partial}{\partial t} + (\xi_0 + \varepsilon\xi_1)\frac{\partial}{\partial x} + (\eta_0 + \varepsilon\eta_1)\frac{\partial}{\partial u}, \tag{8}$$

where $\tau_j, \xi_j, \eta_j (j = 0, 1)$ are all unknown functions of $t, x,$ and u. The infinitesimal generator for the unperturbed equation is a vector field in the three-dimensional space (two independent variables and one dependent variable):

$$X^0 = \tau_0 \frac{\partial}{\partial t} + \xi_0 \frac{\partial}{\partial x} + \eta_0 \frac{\partial}{\partial u}. \tag{9}$$

The prolongation of the infinitesimal symmetry generator is given by:

$$X^{0^{(2)}} = X^0 + \eta_0^t \frac{\partial}{\partial u_t} + \eta_0^x \frac{\partial}{\partial u_x} + \eta_0^{tt} \frac{\partial}{\partial u_{tt}} + \eta_0^{xt} \frac{\partial}{\partial u_{xt}} + \eta_0^{xx} \frac{\partial}{\partial u_{xx}}. \tag{10}$$

The symmetry criterion of Equation (10) yields the relation:

$$X^{0^{(2)}}\left(u_{tt} - (u^2 u_x)_x\right)\Big|_{u_{tt} - (u^2 u_x)_x} = 0. \tag{11}$$

Comparing coefficients of $u_x, u_x^2, \ldots,$ we obtain the following system of determining equations.

$$\begin{gathered}
\xi_{0_u} = 0, \qquad \xi_{0_t} = 0, \qquad \tau_{0_u} = 0, \qquad \tau_{0_x} = 0, \qquad \eta_{0_{uu}} = 0, \\
2\eta_0 + 2u\eta_{0_u} - 4u\xi_{0_x} + 4u\tau_{0_t} = 0, \qquad 2u^2\eta_{0xu} - u\eta_{0xx} + 4u\eta_{0_x} = 0, \\
-2\eta_{0tu} + \tau_{0tt} = 0, \qquad 2u\eta_0 + 2u^2\eta_{0_x} + 2u^2\tau_{0t} = 0, \qquad u^2\eta_{0xx} - \eta_{0tt} = 0
\end{gathered}$$

Solving this system of PDEs, we obtain:

$$\xi_0 = \alpha_0 + \alpha_1 x, \qquad \tau_0 = \alpha_3 t + \alpha_1 t + \alpha_2, \qquad \eta_0 = -\alpha_3 u, \tag{12}$$

where $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are arbitrary constants. Thus,

$$X^0 = (\alpha_0 + \alpha_1 x)\frac{\partial}{\partial x} + (\alpha_3 t + \alpha_1 t + \alpha_2)\frac{\partial}{\partial t} - \alpha_3 u \frac{\partial}{\partial u} \tag{13}$$

To determine the auxiliary function H, we consider:

$$H = \frac{1}{\varepsilon}\left[X^{0^{(2)}}[f_0(z) + \varepsilon F_1(z)]\right]\Bigg|_{\{F_0(z) + \varepsilon F_1(z) = 0\}}, \tag{14}$$

or:

$$H = \tfrac{1}{\varepsilon}\left[X^{0^{(2)}}[u_{tt} - 2uu_x^2 - u^2 u_{xx} + \varepsilon u_t]\right]\Big|_{\{u_{tt} - 2uu_x^2 - u^2 u_{xx} + \varepsilon u_t = 0\}}, \tag{15}$$

where $X^{0^{(2)}}$ is the second prolongation of X^0. This implies that:

$$H = \frac{1}{\varepsilon}\left[\eta(-2u_x^2 - 2uu_{xx}) + \eta^x(-4uu_x^2) + \eta^t(\varepsilon) + \eta^{xx}(-u^2) + \eta^{tt}\right]\Big|_{\{u_{tt}-2uu_x^2-u^2u_{xx}+\varepsilon u_t=0\}}. \tag{16}$$

Hence,

$$\eta = -\alpha_3 u, \qquad \eta^x = -\alpha_3 u_x, \qquad \eta^t = \alpha_4 u_t - \alpha_3 u_t,$$
$$\eta^{tt} = \alpha_4 u_{tt} - 2\alpha_3 u_{tt}, \qquad \eta^{xx} = -\alpha_4 u_{xx} - 2\alpha_3 u_{xx}.$$

Substituting $\eta, \eta^x, \eta^t, \eta^{tt}, \eta^{xx}$ and $u_{tt} = 2uu_x^2 + u^2u_{xx} - \varepsilon u_t$ into Equation (16) gives:

$$H = \alpha_3 u_t. \tag{17}$$

The determining equation for deformations is written as:

$$X^{1^{(2)}}(u_{tt} - u^2u_{xx} - 2uu_x^2)|_{u_{tt}=u^2u_{xx}+2uu_x^2} + H = 0, \tag{18}$$

where $X^{1^{(2)}}$ denotes the second prolongation of the operator:

$$X^1 = \tau_1\frac{\partial}{\partial t} + \xi_1\frac{\partial}{\partial x} + \eta_1\frac{\partial}{\partial u}. \tag{19}$$

We obtain the following system of the determining equations for Equation (18):

$$\begin{aligned}
&\xi_{1_u} = 0, \qquad \xi_{1_t} = 0, \qquad \tau_{1_u} = 0, \qquad \tau_{1_x} = 0, \qquad \eta_{1_{uu}} = 0,\\
&2\eta_1 + 2u\eta_{1_u} - 4u\xi_{1_x} + 4u\tau_{1_t} = 0, \qquad 2u^2\eta_{1xu} - u\eta_{1xx} + 4u\eta_{1x} = 0,\\
&-2\eta_{1tu} + \tau_{1tt} - \alpha_3 = 0, \qquad 2u\eta_1 + 2u^2\eta_{1x} + 2u^2\tau_{1t} = 0, \qquad u^2\eta_{1xx} - \eta_{1tt} = 0
\end{aligned}$$

Solving the above system yields:

$$\tau_1 = \beta_1 + \beta_3 t + \frac{1}{6}\alpha_3 t^2, \qquad \xi_1 = \beta_2 + (\beta_3 + \beta_4)x, \qquad \eta_1 = (\beta_4 - \frac{1}{3}\alpha_3 u)u. \tag{20}$$

Substituting Equations (12) and (20) into Equation (8), we obtain the following approximate symmetries for Equation (7):

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \frac{\varepsilon}{6}(t^2\frac{\partial}{\partial t} - 2tu\frac{\partial}{\partial u}),$$
$$X_4 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \qquad X_5 = \varepsilon X_1, \qquad X_6 = \varepsilon X_2, \qquad X_7 = \varepsilon X_4, \qquad X_8 = \varepsilon X_3.$$

In Table 1, we show that the generators span an eight-dimensional approximate Lie algebra and, hence, generate an eight-parameter approximate transformation group.

Table 1. Approximate commutators of approximate symmetry of the perturbed non-linear wave equation.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	0	$X_1 + \frac{1}{3}(X_8 - X_7)$	0	0	0	0	X_5
X_2	0	0	X_2	X_2	0	0	X_6	X_6
X_3	0	0	0	0	$-X_5$	$-X_6$	0	0
X_4	0	0	0	0	0	$-X_6$	0	0
X_5	0	0	X_6	X_6	0	0	0	0
X_6	0	0	X_6	0	0	0	0	0
X_7	0	$-X_6$	0	0	0	0	0	0
X_8	$-X_5$	$-X_6$	0	0	0	0	0	0

Approximate Invariant Solution

Using the symmetry $X = X_3 - X_4$, we obtain:

$$X = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} + \frac{\varepsilon}{6}\left(t^2\frac{\partial}{\partial t} - 2tu\frac{\partial}{\partial u}\right). \tag{21}$$

The approximate invariants of Equation (21) can be written as:

$$E(t,x,u,\varepsilon) = E^0(t,x,u) + \varepsilon E^1(t,x,u) + O(\varepsilon),$$

which lead to the system:

$$\begin{aligned} &t\frac{\partial E^0}{\partial t} - u\frac{\partial E^0}{\partial u} = 0, \\ &t\frac{\partial E^1}{\partial t} - u\frac{\partial E^1}{\partial u} = -\frac{1}{6}\left(t^2\frac{\partial E^0}{\partial t} - 2tu\frac{\partial E^0}{\partial u}\right), \end{aligned} \tag{22}$$

Solving Equation (22) gives two functionally independent invariants:

$$E_1 = E_1^0(t,x,u) + \varepsilon E_1^1(t,x,u), \qquad E_2 = E_2^0(t,x,u) + \varepsilon E_2^1(t,x,u), \tag{23}$$

for generator Equation (21).

The first equation in Equation (22) has two functionally independent solutions,

$$E_1^0 = x, \qquad E_2^0 = tu.$$

Substituting $E_1^0 = x$ into the second equation in Equation (22) and taking its simplest solution $E_1^1 = 0$, we obtain one invariant in Equation (23),

$$E_1 = x. \tag{24}$$

Now, we substitute the solution $E_2^0 = tu$ of the first equation in Equation (22) into the second equation in Equation (22) and get a non-homogeneous linear equation:

$$t\frac{\partial E_1^1}{\partial t} - u\frac{\partial E_2^1}{\partial u} = \frac{1}{6}t^2u.$$

The corresponding characteristic equation are:

$$\frac{dt}{t} = -\frac{du}{u} = 6\frac{dE_2^1}{t^2u}$$

for which the first integral $tu = \lambda = const$. We obtain:

$$E_2^1 = \frac{1}{6}t^2u + c. \tag{25}$$

Assuming $t = 0$, we get the second invariant in Equation (23),

$$E_2 = tu + \frac{\varepsilon}{6}t^2u.$$

Note that invariants' Equations (24) and (25) are functionally independent. Letting $E_2 = \phi(E_1)$, i.e.,

$$\left(1 + \frac{\varepsilon t}{6}\right)tu = \phi(x)$$

and solving for tu in the first order of precision,

$$tu = \left(1 + \frac{\varepsilon t}{6}\right)^{-1} \phi(x) = \left(1 + \frac{\varepsilon t}{6}\right) \phi(x) + O(\varepsilon).$$

The approximately invariant solution is given by:

$$u = \left(\frac{1}{t} - \frac{\varepsilon}{6}\right) \phi(x). \tag{26}$$

From Equation (7), we obtain:

$$\frac{d\phi}{dx} = \pm\sqrt{1 + c\phi^{-4}}.$$

Setting $c = 0$, we have $\phi(x) = \pm x$, and:

$$u = \pm\left(\frac{x}{t} - \varepsilon\frac{x}{6}\right).$$

3. Perturbed Nonlinear (2 + 1)-Dimension Wave Equation

Consider the perturbed nonlinear wave equation:

$$u_{tt} + \varepsilon u_t = (g(u)u_x)_x + (h(u)u_y)_y, \tag{27}$$

where ε is a small parameter. Putting $g(u) = h(u) = u$ gives:

$$u_{tt} + \varepsilon u_t = (uu_x)_x + (uu_y)_y. \tag{28}$$

The first method is used to obtain a complete approximate symmetry classification of Equation (28) with the first order of precision $o(\varepsilon)$. The approximate group generator of Equation (28) is of the form:

$$\begin{aligned} X &= X_0 + \varepsilon X_1 \\ &= (\tau_0 + \varepsilon\tau_1)\frac{\partial}{\partial t} + (\xi_0 + \varepsilon\xi_1)\frac{\partial}{\partial x} + (\theta_0 + \varepsilon\theta_1)\frac{\partial}{\partial y} + (\eta_0 + \varepsilon\eta_1)\frac{\partial}{\partial u}, \end{aligned} \tag{29}$$

where τ_i, ξ_i, θ_i and $\eta_i,\ i = 0, 1$, are unknown functions of t, x, y and u.

3.1. Exact Symmetries

To find the exact symmetries, we solve the determining equation:

$$X_0^{(2)} F_0(z)\big|_{F_0(z)=0} = 0, \tag{30}$$

where $F_0(z) = u_{tt} - (uu_x)_x - (uu_y)_y$ is the unperturbed part of Equation (28) and $X_0^{(2)}$ is the second prolongation of the infinitesimal generator X_0 given by:

$$\begin{aligned} X_0^{(2)} = X_0 &+ \eta_0^t\frac{\partial}{\partial u_t} + \eta_0^x\frac{\partial}{\partial u_x} + \eta_0^y\frac{\partial}{\partial u_y} + \eta_0^{tt}\frac{\partial}{\partial u_{tt}} + \eta_0^{tx}\frac{\partial}{\partial u_{tx}} \\ &+ \eta_0^{ty}\frac{\partial}{\partial u_{ty}} + \eta_0^{xx}\frac{\partial}{\partial u_{xx}} + \eta_0^{xy}\frac{\partial}{\partial u_{xy}} + \eta_0^{yy}\frac{\partial}{\partial u_{yy}}. \end{aligned} \tag{31}$$

Equation (30) takes the form:

$$\begin{aligned}\big(\eta_0(-u_{xx}-u_{yy})+\eta_0^x(-2u_x)+\eta_0^y(-2u_y)+\eta_0^{xx}(-u)\\ +\eta_0^{yy}(-u)+\eta_0^{tt}\big)\big|_{u_{tt}=(uu_x)_x+(uu_y)_y}=0,\end{aligned} \tag{32}$$

where:

$$\begin{aligned}\eta_0^x &= D_x\eta_0-(u_xD_x\xi_0+u_yD_x\theta_0+u_tD_x\tau_0),\\ \eta_0^y &= D_y\eta_0-(u_xD_y\xi_0+u_yD_y\theta_0+u_tD_y\tau_0),\\ \eta_0^t &= D_t\eta_0-(u_xD_t\xi_0+u_yD_t\theta_0+u_tD_t\tau_0),\\ \eta_0^{xx} &= D_x\eta_0^x-u_{xx}D_x\xi_0-u_{xy}D_x\theta_0-u_{xt}D_x\tau_0,\\ \eta_0^{yy} &= D_y\eta_0^y-u_{xy}D_y\xi_0-u_{yy}D_y\theta_0-u_{yt}D_y\tau_0),\\ \eta_0^{tt} &= D_t\eta_0^t-u_{xt}D_t\xi_0-u_{yt}D_t\theta_0+u_{tt}D_t\tau_0).\end{aligned} \tag{33}$$

Here, D_x, D_y and D_t denote the total derivative operators with respect to x, y and t, respectively,

$$\begin{aligned}D_x &= \frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+u_{xy}\frac{\partial}{\partial u_y}+u_{xt}\frac{\partial}{\partial u_t}+\cdots+u_{xtt}\frac{\partial}{\partial u_{tt}},\\ D_y &= \frac{\partial}{\partial y}+u_y\frac{\partial}{\partial u}+u_{xy}\frac{\partial}{\partial u_x}+u_{yy}\frac{\partial}{\partial u_y}+u_{ty}\frac{\partial}{\partial u_t}+\cdots+u_{ytt}\frac{\partial}{\partial u_{tt}},\\ D_t &= \frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+u_{xt}\frac{\partial}{\partial u_x}+u_{yt}\frac{\partial}{\partial u_y}+u_{tt}\frac{\partial}{\partial u_t}+\cdots+u_{ttt}\frac{\partial}{\partial u_{tt}}.\end{aligned} \tag{34}$$

Equation (32) gives the following system of equations:

$$\begin{gathered}\xi_{0u}=0,\quad \xi_{0t}=0,\quad \tau_{0u}=0,\quad \tau_{0y}=0,\quad \tau_{0x}=0,\quad \theta_{0u}=0,\quad \theta_{0t}=0,\\ \eta_{0uu}=0,\quad -2\eta_{0x}-2u\eta_{0xu}+u\xi_{0xx}+u\xi_{0yy}=0,\quad -\eta_{0u}+2\xi_{0x}-2\tau_{0t}=0,\\ -2\eta_{0y}+u\theta_{0xx}+u\theta_{0yy}=0,\quad -2\eta_{0y}+2\theta_{0y}+\eta_{0u}-2\tau_{0t}=0,\quad -u\eta_{0xx}+\eta_{0tt}-u\eta_{0yy}=0,\\ 2\eta_{0ut}-\tau_{0tt}=0,\quad 2u\xi_{0x}-2u\tau_{0t}-\eta_0=0,\quad \theta_{0x}+\xi_{0y}=0,\quad -2u\tau_{0t}+2u\theta_{0y}-\eta_0=0\end{gathered}$$

Solving this system of PDEs, one has:

$$\xi_0=a_3x+a_1y+a_2,\qquad \theta_0=a_3y-a_1x+a_6,\qquad \tau_0=a_4t+a_5,\qquad \eta_0=2u(a_3-a_4), \tag{35}$$

where a_1, a_2, a_3, a_4, a_5 and a_6 are arbitrary constants. Hence, the infinitesimal generator for Equation (28) is:

$$\begin{aligned}X_0 =&(a_4t+a_5)\frac{\partial}{\partial t}+(a_3x+a_1y+a_2)\frac{\partial}{\partial x}\\ &+(a_3y-a_1x+a_6)\frac{\partial}{\partial y}+(2u(a_3-a_4))\frac{\partial}{\partial u}.\end{aligned} \tag{36}$$

3.2. Approximate Symmetries

The auxiliary function H is given by:

$$H=\frac{1}{\varepsilon}\left[X_0^{(k)}\big(F_0(z)+\varepsilon F_1(z)\big)\big|_{F_0(z)+\varepsilon F_1(z)=0}\right]. \tag{37}$$

Substituting the generator X_0 into Equation (36) and:

$$F_0(z) + \varepsilon F_1(z) = u_{tt} + \varepsilon u_t - (uu_x)_x - (uu_y)_y$$

into Equation (37), we obtain:

$$H = a_4 u_t. \tag{38}$$

Now, we calculate operator X_1 by solving the inhomogeneous determining equation:

$$X_1^{(2)} F_0(z)\big|_{F_0(z)} + H = 0,$$

which can be written as:

$$\left[X_1^{(2)}\left(u_{tt} - (uu_x)_x - (uu_y)_y\right)\big|_{u_{tt}=(uu_x)_x+(uu_y)_y}\right] + a_4 u_t = 0. \tag{39}$$

Equation (39) generates the following system of equations:

$$\begin{array}{llll}
\xi_{1u} = 0, \quad \xi_{1t} = 0, & \tau_{1u} = 0, \quad \tau_{1y} = 0, & \tau_{1x} = 0, & \theta_{1u} = 0, \\
\theta_{1t} = 0, \quad \eta_{1uu} = 0, & -2\eta_1 x - 2u\eta_{1xu} + u\xi_{1xx} + u\xi_{1yy} = 0, & & -\eta_{1u} + 2\xi_{1x} - 2\tau_{1t} = 0, \\
-2\eta_{1y} + u\theta_{1xx} + u\theta_{1yy} = 0, & -2\eta_{1y} + 2\theta_{1y} + \eta_{1u} - 2\tau_{1t} = 0, & & -u\eta_{1xx} + \eta_{1tt} - u\eta_{1yy} = 0, \\
2\eta_{1ut} - \tau_{1tt} + a_4 = 0, & 2u\eta_{1x} - 2u\tau_{1t} - \eta_1 = 0, & \theta_{1x} + \xi_{1y} = 0, & -2u\tau_{1t} + 2u\theta_{1y} - \eta_1 = 0.
\end{array}$$

Solving this system of PDEs, we obtain:

$$\begin{array}{rl}
\xi_1 = & b_3 x + b_1 y + b_2, \quad \theta_1 = b_3 y - b_1 x + b_6, \\
\tau_1 = & \frac{a_4 t^2}{10} + b_4 t + b_5, \quad \eta_1 = 2u\left(b_3 - \frac{1}{5}a_4 t - b_4\right),
\end{array} \tag{40}$$

where b_1, b_2, b_3, b_4, b_5 and b_6 are arbitrary constants. Thus, the approximate symmetries of Equation (28) are:

$$X_1 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2u\frac{\partial}{\partial u},$$

$$X_4 = t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} + \varepsilon\left(\frac{t^2}{10}\frac{\partial}{\partial t} - \frac{2}{5}tu\frac{\partial}{\partial u}\right),$$

$$X_5 = \frac{\partial}{\partial t}, \qquad X_6 = \frac{\partial}{\partial y}, \qquad X_7 = \varepsilon X_1, \qquad X_8 = \varepsilon X_2,$$

$$X_9 = \varepsilon X_3, \qquad X_{10} = \varepsilon X_5, \qquad X_{11} = \varepsilon X_6, \qquad X_{12} = \varepsilon X_4.$$

Remark 1.

$$X_{12} = \varepsilon\left(t\frac{\partial}{\partial t} - 2ut\frac{\partial}{\partial u}\right)$$

In Table 2, we show that the previous generators span a twelve-dimensional approximate Lie algebra and, hence, generate a twelve-parameter approximate transformations group.

Table 2. Approximate commutator table of approximate symmetries of the perturbed non-linear wave equation.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}
X_1	0	X_6	0	0	0	$-X_2$	0	X_{11}	0	0	0	0
X_2	$-X_6$	0	X_2	0	0	0	$-X_{11}$	0	X_8	0	0	0
X_3	0	$-X_2$	0	0	0	$-X_6$	0	$-X_8$	0	0	X_{11}	0
X_4	0	0	0	0	$-X_5 - \frac{2}{5}X_{12}$	0	0	0	0	$-X_{10}$	0	0
X_5	0	0	0	$X_5 + \frac{2}{5}X_{12}$	0	0	0	0	0	0	0	X_{10}
X_6	X_2	0	X_6	0	0	0	X_8	0	X_{11}	0	0	0
X_7	0	X_{11}	0	0	0	$-X_8$	0	0	0	0	0	0
X_8	$-X_{11}$	0	X_8	0	0	0	0	0	0	0	0	0
X_9	0	$-X_8$	0	0	0	$-X_{11}$	0	0	0	0	0	0
X_{10}	0	0	0	X_{10}	0	0	0	0	0	0	0	0
X_{11}	0	0	X_{11}	0	0	0	0	0	0	0	0	0
X_{12}	0	0	0	0	$-X_{10}$	0	0	0	0	0	0	0

3.3. Approximate Invariant Solutions

Reconsider Equation (28):

$$u_{tt} - \varepsilon u_t = (uu_x)_x + (uu_y)_y. \tag{41}$$

and the symmetry:

$$X_4 = t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} + \frac{\varepsilon}{10}\left(t^2\frac{\partial}{\partial t} - 4ut\frac{\partial}{\partial u}\right). \tag{42}$$

The approximate invariant for Equation (42) is of the form:

$$E(t,x,y,u,\varepsilon) = E^0(t,x,y,u) + \varepsilon E^1(t,x,y,u) + o(\varepsilon),$$

determined by the equation $X(E) = o(\varepsilon)$. Using the notation:

$$X = X^0 + \varepsilon X^1,$$

where:

$$X^0 = t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}, \qquad X^1 = \frac{1}{10}\left(t^2\frac{\partial}{\partial t} - 4tu\frac{\partial}{\partial u}\right);$$

for operator Equation (42), we write the determining equation $X(E) = o(\varepsilon)$ for the approximate invariants in the form:

$$X^0(E^0) + \varepsilon\left[X^0(E^1) + X^1(E^0)\right] = 0, \qquad X^0(E^0) = 0,$$
$$X^0(E^1) + X^1(E^0) = 0,$$

or:

$$\begin{aligned} &t\frac{\partial E^0}{\partial t} - 2u\frac{\partial E^0}{\partial u} = 0 \\ &t\frac{\partial E^1}{\partial t} - 2u\frac{\partial E^1}{\partial u} = -\frac{1}{10}\left(t^2\frac{\partial E^0}{\partial t} - 4tu\frac{\partial E^0}{\partial u}\right). \end{aligned} \tag{43}$$

Solving Equation (43) gives two functionally independent invariants:

$$E_1 = E_1^0(t,x,y,u) + \varepsilon E_1^1(t,x,y,u), \qquad E_2 = E_2^0(t,x,y,u) + \varepsilon E_2^1(t,x,y,u), \tag{44}$$

for generator Equation (42).

The first equation in Equation (43) has two functionally independent solutions:

$$E_1^0 = xy, \qquad E_2^0 = t^2 u.$$

Substituting $E_1^0 = xy$ into the second equation in Equation (43) and taking its simplest solution $E_1^1 = 0$, we obtain one invariant in Equation (44),

$$E_1 = xy \tag{45}$$

Note that the dependent variable u does not appear in Equation (45). Now, we substitute the solution $E_2^0 = t^2 u$ of the first equation in Equation (43) into the second equation in Equation (43) and obtain non-homogeneous linear equation:

$$t\frac{\partial {E_1}^1}{\partial t} - 2u\frac{\partial {E_2}^1}{\partial u} = \frac{1}{5}(t^3 u).$$

The corresponding characteristic equations are:

$$\frac{dt}{t} = -\frac{du}{2u} = 5\frac{dE_2^1}{t^2 u},$$

with the first integral $t^2 u = \lambda = const$. Therefore, the second equation:

$$\frac{dt}{t} = 5\frac{dE_2^1}{t^3 u}$$

gives:

$$ {E_2}^1 = \frac{1}{5}t^3 u + c. \tag{46}$$

Assuming that $c = 0$, we obtain the second invariant in Equation (44),

$$E_2 = t^2 u + \frac{\varepsilon}{5}t^3 u. \tag{47}$$

Note that E_1 and E_2 are functionally independent. Letting $E_2 = \psi(E1)$, i.e.,

$$\left(t^2 u + \frac{\varepsilon}{5}t^3 u\right) = \psi(xy)$$

and solving for $t^2 u$ in the first order of precision,

$$t^2 u = \left(1 + \frac{\varepsilon}{5}t\right)^{-1} \phi(xy) = \left(1 - \frac{\varepsilon}{5}t\right)\phi(xy) + o(\varepsilon),$$

yield the approximate invariant solution:

$$u(t, x, y) = \left(\frac{1}{t^2} - \frac{\varepsilon}{5t}\right)\phi(xy). \tag{48}$$

From Equation (28), we obtain:

$$\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{\partial^2 \phi}{\partial x^2}\cdot\phi + \left(\frac{\partial \phi}{\partial y}\right)^2 + \frac{\partial^2 \phi}{\partial y^2}\cdot\phi - 6\phi = 0. \tag{49}$$

Case I: Let $\psi(xy)$ be of the form $\phi(xy) = (xy)^\alpha$. From Equation (49), one obtains:

$$[2\alpha^2 - \alpha]\left(x^{\alpha-2}y^\alpha + x^\alpha y^{\alpha-2}\right) - 6 = 0$$

For $\alpha = 2$, we have:

$$u(t,x,y) = \left(\frac{1}{t^2} - \frac{\varepsilon}{5t}\right)(x^2y^2) \;\; s.t \;\; x^2 + y^2 = 1.$$

An approximate solution for this case is depicted in Figure 1.
Case II: Let $\phi(xy) = A(x)B(x)$. Equation (49) gives the following equation:

$$B\left(\frac{A'^2 + AA''}{A}\right) + A\left(\frac{B'^2 + BB''}{B}\right) - 6 = 0 \tag{50}$$

where:

$$A' = \frac{\partial A}{\partial x}, \qquad A'' = \frac{\partial^2 A}{\partial x^2}, \qquad B' = \frac{\partial B}{\partial y}, \qquad B'' = \frac{\partial^2 B}{\partial y^2}$$

Equation (50) leads to the following ordinary differential equations:

$$AA'' + A'^2 - d_1A^2 - c_1A = 0, \tag{51}$$

and:

$$BB'' + B'^2 - d_2B^2 - c_2B = 0, \tag{52}$$

where c_1, c_2, d_1, d_2 are constants.

Let $\gamma(x) = A^2(x)$. From Equations (51) and (52), we obtain:

$$(2\ln(d_1A(x) + c_1) - 1)\left(\frac{A^2(x)}{2d_1} - \frac{c_1^2}{2d_1^2}\right) + \frac{c_1}{d_1^2}A^2(x) = \frac{x^2}{2} + c_3x + c_4, \tag{53}$$

and:

$$(2\ln(d_2B(x) + c_2) - 1)\left(\frac{B^2(x)}{2d_2} - \frac{c_2^2}{2d_2^2}\right) + \frac{c_2}{d_2^2}B^2(x) = \frac{x^2}{2} + c_5x + c_6, \tag{54}$$

where c_3, c_4, c_5, c_6 are arbitrary constants. Therefore, a solution in this case is of the form:

$$u(t,x,y) = \left(\frac{1}{t^2} - \frac{\varepsilon}{5t}\right)A(x)B(x). \tag{55}$$

We plot an approximate solution for this case in Figure 2.

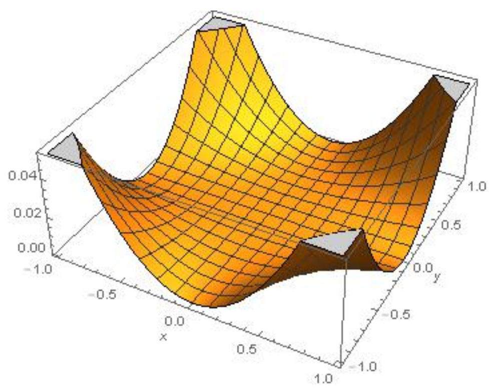

Figure 1. CaseI: approximate invariant solution of Equation (28) for $t = \pi$, $-1 \le x \le 1$, $-1 \le y \le 1$, $\varepsilon = 0.1$.

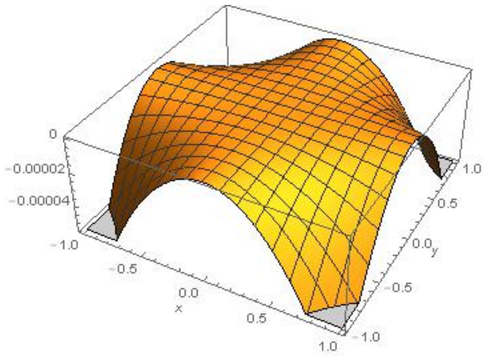

Figure 2. Case II:approximate invariant solution of Equation (28) for $t = 100$, $-1 \leq x \leq 1, -1 \leq y \leq 1$, $\varepsilon = 0.1$.

4. Nonlinear Wave Equation with a Forcing Term

In this section, we discuss the nonlinear (2 + 1)-dimensional wave equation with a forcing term:

$$u_{tt} - (uu_x)_x - (uu_y)_y = \varepsilon f(u). \tag{56}$$

4.1. Approximate Symmetries by Method I

Exact symmetries of the unperturbed part ($\varepsilon = 0$) of Equation (56) are given by:

$$\xi_0 = a_3 x + a_1 y + a_2, \qquad \theta_0 = a_3 y - a_1 x + a_6, \qquad \tau_0 = a_4 t + a_5, \qquad \eta_0 = 2u(a_3 - a_4), \tag{57}$$

where a_1, a_2, a_3, a_4, a_5 and a_6 are arbitrary constants.

Consider the auxiliary function:

$$H = \frac{1}{\varepsilon}\left[X^0(F_0(z) + \varepsilon F_1(z)\right) |_{F_0(z)+\varepsilon F_1(z)=0}, \tag{58}$$

where:

$$X^0 = \tau_0 \frac{\partial}{\partial t} + \xi_0 \frac{\partial}{\partial x} + \theta_0 \frac{\partial}{\partial y} + \eta_0 \frac{\partial}{\partial u}.$$

Using Equations (56) and (57), one obtains:

$$H = -2a_3 \left(uf'(u) + f(u)\right) + 2a_4 \left(uf'(u) - 2f(u)\right). \tag{59}$$

Now we calculate the operator X_1 with the condition that

$$X^1(F_0(z))\big|_{F_0(z)=0} + H = 0. \tag{60}$$

Condition Equation (60) can be written as:

$$\begin{aligned}\left[X_1^{(2)}(u_{tt} - (uu_x)_x - (uu_y)_y)\big|_{u_{tt}=(uu_x)_x+(uu_y)_y}\right] - 2c_3 \left(uf'(u) + f(u)\right) \\ +2c_4 \left(uf'(u) - 2f(u)\right) = 0,\end{aligned} \tag{61}$$

where $X_1^{(2)}$ is the second prolongation of X_1.

Equation (61) yields the following system of equations:

$$\begin{aligned}
&\xi_{1u} = 0, \quad \xi_{1t} = 0, \quad \tau_{1u} = 0, \quad \tau_{1y} = 0, \quad \tau_{1x} = 0,\\
&\theta_{1u} = 0, \quad \theta_{1t} = 0, \quad \eta_{1uu} = 0, \quad -2\eta_{1x} - 2u\eta_{1xu} + u\xi_{1xx} + u\xi_{1yy} = 0,\\
&-\eta_{1u} + 2\xi_{1x} - 2\tau_{1t} = 0 \quad -2\eta_{1y} + u\theta_{1xx} + u\theta_{1yy} = 0, \quad -2\eta_{1y} + 2\theta_{1y} + \eta_{1u} - 2\tau_{1t} = 0,\\
&-u\eta_{1xx} + \eta_{1tt} - u\eta_{1yy} - 2a_3(uf'(u) + f(u)) = 0, \quad 2\eta_{1ut} - \tau_{1tt} = 0,\\
&2u\xi_{1x} - 2u\tau_{1t} - \eta_1 = 0, \quad \theta_{1x} + \xi_{1y} = 0, \quad -2u\tau_{1t} + 2u\theta_{1y} - \eta_1 = 0,
\end{aligned}$$

Solving this system of PDEs, we obtain:

$$\begin{aligned}
\xi_1 = & \quad b_3x + b_1y + b_2, \qquad \theta_1 = b_3y - b_1x + b_6,\\
\tau_1 = & \quad b_4t + b_5, \qquad \eta_1 = 2u(b_3 - b_4),
\end{aligned} \tag{62}$$

where b_1, b_2, b_3, b_4, b_5 and b_6 are arbitrary constants.

Case I: $a_3 = 0$. The scaling operator:

$$X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2u\frac{\partial}{\partial u}$$

is not stable, and hence, Equation (56) does not inherit symmetries of its unperturbed part.

Case II: Solving the first order linear differential equation $uf'(u) + f(u) = 0$, we obtain $f(u) = k_1/u$, where k_1 is a constant. The approximate symmetry generator of Equation (56) is given by:

$$\begin{aligned}
X =& X_0 + \varepsilon X_1\\
=& [(a_3 + \varepsilon b_3)x + (a_1 + \varepsilon b_1)y + (a2 + \varepsilon b_2)]\frac{\partial}{\partial x}\\
&+ [(a_3 + \varepsilon b_3)y - (a_1 + \varepsilon b_1)x + (a_6 + \varepsilon b_6)]\frac{\partial}{\partial y}\\
&+ [(a_4 + \varepsilon b_4)t + (a_5 + \varepsilon b_5)]\frac{\partial}{\partial t} + [2u((a_3 + \varepsilon b_3) - (a_4 + \varepsilon b_4))]\frac{\partial}{\partial u}
\end{aligned} \tag{63}$$

These additional symmetries are actually the same as those obtained from the unperturbed equation that are considered as trivial symmetries. To summarize: in this case, Method I only gives trivial symmetries.

4.2. Approximate Symmetries by Method II

We expand the dependent variable to the first order of ε as follows:

$$u = v + \varepsilon w + o(\varepsilon), \qquad \varepsilon \to 0.$$

Taylor expansion of f in the first order of precision is given by:

$$f(u) = \frac{k_1}{v}\left(1 - \frac{\varepsilon w}{v} + o(\varepsilon)\right) = \frac{k_1}{v} - \frac{\varepsilon k_1 w}{v^2} + o(\varepsilon), \qquad \varepsilon \to 0.$$

Substituting the above expansion into Equation (56) and separating at each order of perturbation parameter, one may obtain:

$$\begin{aligned}
v_{tt} - {v_x}^2 - vv_{xx} - {v_y}^2 - vv_{yy} = 0,\\
w_{tt} - 2v_xw_x - vw_{xx} - wv_{xx} - 2v_yw_y - vw_{yy} - wv_{yy} = \frac{k_1}{v}.
\end{aligned} \tag{64}$$

Now, the infinitesimal generator for the problem is:

$$X = \tau(t,x,y,,v,w)\frac{\partial}{\partial t} + \xi(t,x,y,v,w)\frac{\partial}{\partial x} + \theta(t,x,y,v,w)\frac{\partial}{\partial y} + \phi(t,x,y,v,w)\frac{\partial}{\partial v} + \eta(t,x,y,v,w)\frac{\partial}{\partial w}. \tag{65}$$

Using standard Lie group analysis, we obtain the infinitesimals as follows:

$$\begin{aligned} &\tau = c_4 t + c_5, \quad \xi = c_1 x - c_3 y + c_6, \quad \theta = c_3 x + c_1 y + c_2, \\ &\phi = 2v(-c_4 + c_1), \quad \eta = -2w(c_1 - 2c_4), \end{aligned} \tag{66}$$

where c_1, c_2, c_3, c_4, c_5 and c_6 are arbitrary constants. Hence, we have the following symmetries:

$$\begin{aligned} &X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2v\frac{\partial}{\partial v} - 2w\frac{\partial}{\partial v}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \\ &X_4 = t\frac{\partial}{\partial t} - 2v\frac{\partial}{\partial v} + 4w\frac{\partial}{\partial w}, \quad X_5 = \frac{\partial}{\partial t}, \quad X_6 = \frac{\partial}{\partial x}. \end{aligned} \tag{67}$$

Table 3 shows that Equation (67) spans a sixth-dimensional Lie algebra.

Table 3. Commutators span six-dimensional Lie algebra.

	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	$\mathbf{X_6}$
X_1	0	$-X_2$	0	0	0	$-X_6$
X_2	X_2	0	$-X_6$	0	0	0
X_3	0	X_6	0	0	0	$-X_2$
X_4	0	0	0	0	$-X_5$	0
X_5	0	0	0	X_5	0	0
X_6	X_6	0	X_2	0	0	0

4.3. Approximate Invariant Solution

Using X_3 from Equation (67), we retrieve the following characteristic equations:

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{dv}{0} = \frac{dw}{0}. \tag{68}$$

The equations in Equation (68) yield $\alpha = x^2 + y^2$ and suggest that $w = w(\alpha)$, $v = v(\alpha)$. Derivatives of dependent variables v and w with respect to x and y are:

$$\begin{aligned} &v_t = v_\alpha \frac{\partial \alpha}{\partial t} = 0, \quad v_{tt} = 0 \\ &v_x = v - \alpha\frac{\partial \alpha}{\partial x} = 2xv_\alpha, \quad v_{xx} = 2v_\alpha + 2xv_{\alpha\alpha}\frac{\partial \alpha}{\partial x} = 2v_\alpha + 4x^2 v_{\alpha\alpha} \\ &v_y = v - \alpha\frac{\partial \alpha}{\partial y} = 2yv_\alpha, \quad v_{yy} = 2v_\alpha + 2yv_{\alpha\alpha}\frac{\partial \alpha}{\partial y} = 2v_\alpha + 4y^2 v_{\alpha\alpha} \\ &w_t = w_\alpha \frac{\partial \alpha}{\partial t} = 0, \quad w_{tt} = 0 \\ &w_x = w - \alpha\frac{\partial \alpha}{\partial x} = 2xw_\alpha, \quad w_{xx} = 2w_\alpha + 2xw_{\alpha\alpha}\frac{\partial \alpha}{\partial x} = 2w_\alpha + 4x^2 w_{\alpha\alpha} \\ &w_y = w - \alpha\frac{\partial \alpha}{\partial y} = 2yw_\alpha, \quad w_{yy} = 2w_\alpha + 2yw_{\alpha\alpha}\frac{\partial \alpha}{\partial y} = 2w_\alpha + 4y^2 w_{\alpha\alpha}. \end{aligned}$$

These equations lead to the following second order ordinary differential equations:

$$\begin{aligned} \alpha v_\alpha^2 + \alpha v v_{\alpha\alpha} + v v_\alpha &= 0, \\ 2\alpha v_\alpha w_\alpha + v w_\alpha + \alpha v w_{\alpha\alpha} + w v_\alpha + \alpha w v_{\alpha\alpha} &= \frac{k_1}{4v}. \end{aligned} \tag{69}$$

Substituting:

$$v(\alpha) = \frac{H(\alpha)^2}{2}$$

into the first equation of Equation (69), we obtain:

$$\frac{\partial H(\alpha)}{\partial \alpha} + \alpha \frac{\partial^2 H(\alpha)}{\partial \alpha^2} = 0.$$

We have $H(\alpha) = c_1 \ln \alpha + c_2$, where c_1 and c_2 are arbitrary constants of the integration. Thus, $v(\alpha) = \sqrt{c_2 + c_1 \ln \alpha}$. Put $c_2 = 0, c_1 = 1$. The second equation of Equation (69) is reduced to the following second-order ordinary differential equation:

$$(1 + \ln \alpha) w_\alpha + (\alpha \ln \alpha) w_{\alpha\alpha} - \frac{1}{4 \ln \alpha} w = \frac{k_1}{4}. \tag{70}$$

Observe that it is not straight forward to obtain a solution for Equation (70). However, we may obtain an asymptotic estimate of the solution of Equation (70) using the asymptotic expansions [22].

Definition 1. *The function $f(x) = O(g(x))$ as $x \to x_0$ if there exists a constant C such that $\lim_{x \to x_0} f/g = C$.*

In Equation (70), we have $(1 + \ln \alpha) = O(\alpha)$, $(\alpha \ln \alpha^2) = O(\alpha^2)$ and $\frac{1}{4 \ln \alpha} = O(1)$ as $\alpha \to \infty$. For large values of α, Equation (70) is asymptotically equivalent to the following equation:

$$\alpha^2 w_{\alpha\alpha} + \alpha w_\alpha + w = \frac{k_1}{4}. \tag{71}$$

The solution of the above non-homogeneous Cauchy–Euler equation is:

$$w(\alpha) = k_2 \sin(\ln \alpha) + k_3 \cos(\ln \alpha) + \frac{k_1}{4}$$

where k_2, k_3 are constants. Lastly, we re-cast the solution in original coordinates as:

$$\begin{aligned} u(t,x,y) &= v(t,x,y) + \varepsilon w(t,x,y) \\ &= \sqrt{\ln(x^2+y^2)} + \varepsilon \left(k_2 \sin(\ln(x^2+y^2)) + k_3 \cos(\ln(x^2+y^2)) + \frac{k_1}{4} \right). \end{aligned} \tag{72}$$

This is an approximate solution invariant under rotation in $x - y$, dilation in space and u coordinates. We depict an invariant solution for the unperturbed equation in Figure 3 and an approximate solution of the perturbed one in Figure 4.

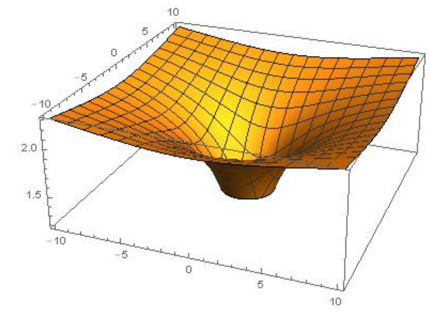

Figure 3. Invariant solution of the unperturbed equation of Equation (56) for $-10 \leq x \leq 10$, $-10 \leq y \leq 10$, $x^2 + y^2 \geq 1$.

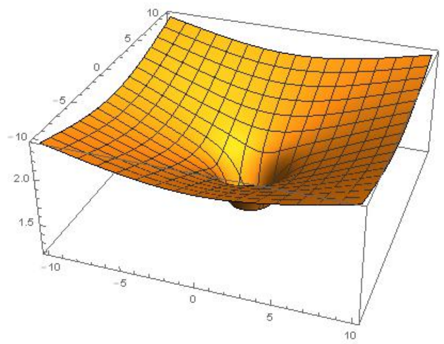

Figure 4. Approximate invariant solution of Equation (56) for $-10 \leq x \leq 10$, $-10 \leq y \leq 10$, $x^2 + y^2 \geq 1$, $k_1 = 4$, $k_2 = 1$, $k3 = 1$, $\varepsilon = 0.1$.

5. Concluding Remarks

In this work, we have studied a class of perturbed nonlinear wave equations via Lie symmetry analysis. Two methods have been employed to obtain approximate symmetries used to construct invariant solutions of the equations. There was a case where Method I gives only trivial solutions. We applied Method II to this case and obtained the invariant solutions of the equation. Many problems arising from physical or engineering situations may be dealt with by approximate Lie symmetry analysis. We plan to investigate modified and perturbed forms of Korteweg-de Vries (KdV) equations using this approach.

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Author Contributions: Waheed: working out symmetries and approximation. Zaman: proposing, guiding and leading. Saleh: working out plan, verifying, writing up and final form.

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