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A Constructive Method for Standard Borel Fixed Submodules with Given Extremal Betti Numbers

Marilena Crupi

Department of Mathematical and Computer Sciences, Physical Sciences and Earth Sciences, University of Messina, Viale Ferdinando Stagno d'Alcontres 31, 98166 Messina, Italy
* Correspondence: mcrupi@unime.it; Tel.: +39-090-676-5080

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Abstract: Let S be a polynomial ring in n variables over a field K of any characteristic. Given a strongly stable submodule M of a finitely generated graded free S-module F, we propose a method for constructing a standard Borel-fixed submodule $\widetilde{M}$ of F so that the extremal Betti numbers of M, values as well as positions, are preserved by passing from M to $\widetilde{M}$. As a result, we obtain a numerical characterization of all possible extremal Betti numbers of any standard Borel-fixed submodule of a finitely generated graded free S-module F.

Keywords: graded modules; monomial modules; minimal graded resolution; extremal Betti numbers

MSC: 13B25, 13D02, 16W50

1. Introduction

Let us consider the polynomial ring $S = K[x_1, \dots, x_n]$ as an $\mathbb{N}$-graded ring where $\deg x_i = 1$ $(i = 1, \dots, n)$, and let $F = \oplus_{i=1}^m Se_i$ $(m \geq 1)$ be a finitely generated graded free S-module with basis $e_1, \dots, e_m$, where $\deg(e_i) = f_i$ for $i = 1, \dots, m$, with $f_1 \leq f_2 \leq \cdots \leq f_m$. A graded submodule M of F is a strongly stable submodule if $M = \oplus_{i=1}^m I_i e_i$, and $I_i \subset S$ is a strongly stable ideal of S, for any $i = 1, \dots, m$ [1,2]. Strongly stable ideals play a fundamental role in commutative algebra, algebraic geometry and combinatorics. Indeed, their combinatorial properties make them useful for both theoretical and computational applications. In characteristic zero, the notion of strongly stable ideals coincides with the notion of Borel-fixed ideals (see, for instance, [3,4]). These are monomial ideals, which are fixed by the action of the Borel subgroup of triangular matrices of the linear group $\mathrm{Gl}(n)$, and correspond to the possible generic initial ideals by a well-known result by Galligo [5]. In [6], Pardue introduced the notion of the standard Borel-fixed submodule. A graded submodule M of F is a standard Borel-fixed submodule if $M = \oplus_{i=1}^m I_i e_i$, with $I_i \subset S$ $(i = 1, \dots, m)$ strongly stable ideals, and $(x_1, \dots, x_n)^{f_j - f_i} I_j \subseteq I_i$ for every $j > i$. If we compare such a definition with that of a strongly stable submodule, we notice that Pardue imposes the additional condition that the maximal ideal $(x_1, \dots, x_n)$ to an appropriate power multiplies I_j into I_i for every $j > i$. Such a condition is related to the fact that a standard Borel-fixed submodule is a Borel-fixed submodule [6], Definition 7, and the inclusion is a necessary and sufficient condition for a Borel-fixed submodule to be $B(F)$-fixed; $B(F)$ is the Borel subgroup of upper triangular matrices of the group $\mathrm{Gl}(F)$ of all graded S-module automorphisms of F [6]. On the other hand, it is worth being recalled that in characteristic zero, every Borel-fixed submodule is standard Borel-fixed [6]. In passing, we note that such a class of submodules arises naturally, as in the case where ideals are considered. Indeed, if K is a field of characteristic zero and M is a graded S-module, then the generic initial module $\mathrm{Gin}(M)$ is a standard Borel-fixed submodule with respect to the graded reverse lexicographic order [6].

Minimal graded free resolutions of modules over a polynomial ring are a classical and extremely interesting topic. Let M be a finitely generated graded S-module. A graded Betti number $\beta_{k,k+\ell}(M) \neq 0$ is called extremal if $\beta_{i,i+j}(M) = 0$ for all $i \geq k$, $j \geq \ell$, $(i,j) \neq (k,\ell)$. The pair (k,ℓ) is called a corner of M. Such special graded Betti numbers (nonzero top left corners in a block of zeroes in the Betti table) were introduced by Bayer, Charalambous and Popescu [7] as a refinement of the Castelnuovo–Mumford regularity. In characteristic zero, combinatorial characterizations of the possible extremal Betti numbers that a graded submodule of a finitely generated graded free S-module may achieve can be found in [1,8,9]. More precisely, the characterization regards classes of ideals of S in [8,9]; whereas it refers to classes of submodules of S^m $(m \geq 1)$ in [1].

Let $\mathrm{Corn}(M)$ be the set of all the corners of M. If $\mathrm{Corn}(M) = \{(k_1,\ell_1),\ldots,(k_r,\ell_r)\}$, with $n-1 \geq k_1 > k_2 > \cdots > k_r \geq 1$ and $2 \leq \ell_1 < \ell_2 < \cdots < \ell_r$, setting $a_i = \beta_{k_i,k_i+\ell_i}(M)$, for $i = 1,\ldots,r$, we call $b(M) = (a_1,\ldots,a_r)$ the corner values sequence of M.

In [10], we posed the following question.

Question 1. Suppose given a strongly stable submodule M of a finitely generated graded free S-module F with $\mathrm{Corn}(M) = \{(k_1,\ell_1),\ldots,(k_r,\ell_r)\}$ $(n-1 \geq k_1 > k_2 > \cdots > k_r \geq 1;\ 2 \leq \ell_1 < \ell_2 < \cdots < \ell_r)$ as the set of all its corners and $b(M) = (a_1,\ldots,a_r)$ as its corner values sequence, does there exist a standard Borel-fixed submodule $\widetilde{M}$ of F, such that $\mathrm{Corn}(\widetilde{M}) = \mathrm{Corn}(M)$ and $b(\widetilde{M}) = b(M)$?

Such a question was suggested by the fact that in [1], Theorem 4.6 , given two positive integers n, r, $1 \leq r \leq n-1$, r pairs of positive integers (k_1,ℓ_1), …, (k_r,ℓ_r) such that $n-1 \geq k_1 > k_2 > \cdots > k_r \geq 1$ and $2 \leq \ell_1 < \ell_2 < \cdots < \ell_r$ and r positive integers $a_1,\ldots,a_r$, the existence of a strongly stable submodule M of the finitely generated graded free S-module S^m, $m \geq 1$ such that $\beta_{k_1,k_1+\ell_1}(M) = a_1$, …, $\beta_{k_r,k_r+\ell_r}(M) = a_r$ are its extremal Betti numbers, has been proven. More precisely, a numerical characterization of all possible extremal Betti numbers of any strongly stable submodule of the finitely generated graded free S-module S^m, $m \geq 1$, has been given.

The strategy used in [10] has shown that the construction of the standard Borel-fixed submodule (general strongly stable submodule, in the sense of [10]) we are looking for often requires the increasing of the rank of the free S-module F given in the hypotheses. Indeed, the standard Borel-fixed submodule $\widetilde{M}$ obtained in [10] was a submodule of a free S-module $\widetilde{F}$, with $\mathrm{rank}\,\widetilde{F} \geq \mathrm{rank}\,F$.

Here, we succeed at overcoming this problem by implementing a procedure that swaps the monomial generators of the ideals appearing in the direct decomposition of M. As a result, both M and the standard Borel-fixed submodule obtained will be submodules of the same finitely generated graded free S-module F.

The paper is organized as follows. In Section 2, to keep the paper self-contained, some basic notions are recalled. In Section 3, we introduce and discuss the concepts of blocks and sub-blocks of a strongly stable ideal that will be crucial in the development of the paper. In Section 4, if M is a strongly stable submodule of a finitely generated graded free S-module F, the existence of a standard Borel-fixed submodule $\widetilde{M}$ of F, which preserves both the values and the positions of the extremal Betti numbers of M, is proven (Theorem 1); the underlying ideas behind the algorithm (Section 4.1) are discussed, and a straight description of the algorithm covering all exceptional cases is given (Section 4.2). Moreover, a not so short example (Example 6), suitably chosen to show that all the cases considered in Theorem 1 can really occur in a single case, is presented in detail. Finally, two further examples (Examples 7 and 8) illustrating how the procedure works are presented. Section 5 contains our conclusions and perspectives.

2. Preliminaries

Let us consider the polynomial ring $S = K[x_1,\ldots,x_n]$ as an $\mathbb{N}$-graded ring where each $\deg x_i = 1$, endowed with the lexicographic order $>_{\mathrm{lex}}$ induced by the ordering $x_1 >_{\mathrm{lex}} \cdots >_{\mathrm{lex}} x_n$. Let $F = \oplus_{i=1}^m Se_i$ $(m \geq 1)$ be a finitely generated graded free S-module with basis $e_1,\ldots,e_m$, where $\deg(e_i) = f_i$ for each $i = 1,\dots,m$, with $f_1 \le f_2 \le \cdots \le f_m$. The elements of the form $x^a e_i$, where $x^a = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$ for $a = (a_1,\dots,a_n) \in \mathbb{N}_0^n$, are called monomials of F, and $\deg(x^a e_i) = \deg(x^a) + \deg(e_i)$. In particular, if $F \simeq S^m$ and $e_i = (0,\dots,0,1,0,\dots,0)$, where one appears in the i-th place, we assume, as usual, $\deg(x^a e_i) = \deg(x^a)$, i.e., $\deg(e_i) = f_i = 0$. A monomial submodule M of F is a submodule generated by monomials, i.e., $M = \oplus_{i=1}^m I_i e_i$, where I_i are the monomial ideals of S generated by those monomials u of S such that $ue_i \in M$ [3]; if $m = 1$ and $f_1 = 0$, then a monomial submodule is a monomial ideal of S.

If I is a monomial ideal of S, we denote by $G(I)$ the unique minimal set of monomial generators of I and by $G(I)_\ell$ the set of monomials u of $G(I)$ such that $\deg u = \ell$; if $M = \oplus_{i=1}^m I_i e_i$ is a monomial submodule of F; we set:

$$\begin{aligned} G(M) &= \{ue_i \,:\, u \in G(I_i), i = 1,\dots,m\}, \\ G(M)_\ell &= \{ue_i \,:\, u \in G(I_i)_{\ell - f_i},\, i = 1,\dots,m\}. \end{aligned}$$

Finally, for a monomial $1 \neq u \in S$, let:

$$\operatorname{supp}(u) = \{i : x_i \text{ divides } u\},$$

and:

$$\mathrm{m}(u) = \max\{i : i \in \operatorname{supp}(u)\}.$$

Otherwise, if $u = 1$, we set $\mathrm{m}(u) = 0$.

Next definitions can be found in [11] and [2], respectively.

Definition 1. *A monomial ideal I of S is called stable if for all $u \in G(I)$, one has $(x_j u)/x_{\mathrm{m}(u)} \in I$ for all $j < \mathrm{m}(u)$. It is called strongly stable if for all $u \in G(I)$, one has $(x_j u)/x_i \in I$ for all $i \in \operatorname{supp}(u)$ and all $j < i$.*

Definition 2. *A graded submodule $M \subseteq F$ is a (strongly) stable submodule if $M = \oplus_{i=1}^m I_i e_i$ and $I_i \subset S$ is a (strongly) stable ideal of S, for any $i = 1,\dots,m$.*

For any finitely generated graded S-module M, there is a minimal graded free S-resolution [12]:

$$\mathbb{F}. : 0 \to F_s \to \cdots \to F_1 \to F_0 \to M \to 0,$$

where $F_i = \oplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}}$. The integers $\beta_{i,j} = \beta_{i,j}(M) = \dim_K \operatorname{Tor}_i(K, M)_j$ are called the graded Betti numbers of M.

Definition 3. *A graded Betti number $\beta_{k,k+\ell}(M) \neq 0$ is called extremal if $\beta_{i,i+j}(M) = 0$ for all $i \ge k$, $j \ge \ell$, $(i,j) \neq (k,\ell)$.*

Such a definition was introduced in [7].

The pair (k,ℓ) is called a corner of M (in degree ℓ). We denote by $\operatorname{Corn}(M)$ the set of all the corners of the module M, i.e.,

$$\operatorname{Corn}(M) = \{(k,\ell) \in \mathbb{N} \times \mathbb{N} : \beta_{k,k+\ell}(M) \text{ is an extremal Betti number of } M\}.$$

Remark 1. *In [1], the following definition was introduced.*
Let $(k_1,\dots,k_r)$ and $(\ell_1,\dots,\ell_r)$ be two sequences of positive integers such that $n-1 \ge k_1 > k_2 > \cdots > k_r \ge 1$ and $1 \le \ell_1 < \ell_2 < \cdots < \ell_r$. The set $\mathcal{C} = \{(k_1,\ell_1),\dots,(k_r,\ell_r)\}$ is called a corner sequence, and $\ell_1,\dots,\ell_r$

are called the corner degrees of $\mathcal{C}$. It is clear that if M is a finitely generated S-module, then $\mathrm{Corn}(M)$ *is a corner sequence.*

If $M = \oplus_{i=1}^m I_i e_i$ is a stable submodule of F, then we can use the Eliahou–Kervaire formula [11] for computing the graded Betti numbers of M:

$$\beta_{k,k+\ell}(M) = \beta_{k,k+\ell}(\oplus_{i=1}^m I_i e_i) = \sum_{i=1}^m \left[\sum_{u \in G(I_i)_{\ell - f_i}} \binom{\mathrm{m}(u)-1}{k} \right]. \tag{1}$$

From (1), one can deduce the following characterization of the extremal Betti numbers of a stable submodule [2,13].

Characterization 1. *Let $M = \oplus_{i=1}^m I_i e_i$ be a stable submodule of F. A graded Betti number $\beta_{k,k+\ell}(M)$ is extremal if and only if:*

$$k+1 = \max\{\mathrm{m}(u) : ue_i \in G(M)_\ell\,, i \in \{1,\ldots,m\}\},$$

and $\mathrm{m}(u) \le k$, for all $ue_i \in G(M)_j$ and all $j > \ell$.

As a consequence of the above result, one obtains that if $(k,\ell) \in \mathrm{Corn}(M)$, then:

$$\beta_{k,k+\ell}(M) = |\{ue_i \in G(M)_\ell \,:\, \mathrm{m}(u) = k+1,\, i \in \{1,\ldots,m\}\}|.$$

Definition 4. *A graded submodule M of F is a standard Borel-fixed submodule (SBF submodule, for short) if $M = \oplus_{i=1}^m I_i e_i$, with $I_i \subset S$ strongly stable ideal of S, for any $i = 1,\ldots,m$, and $(x_1,\ldots,x_n)^{f_j - f_i} I_j \subseteq I_i$ for every $j > i$.*

It is easy to verify that Definition 4 is equivalent to the following one (see [14] for the square-free case).

Definition 5. *A graded submodule M of F is a standard Borel-fixed submodule (SBF submodule) if $M = \oplus_{i=1}^m I_i e_i$, with strongly stable ideals $I_i \subset S$, for $i = 1,\ldots,m$, and $(x_1,\ldots,x_n)^{f_{j+1}-f_j}\, I_{j+1} \subseteq I_j$, for any $j = 1,\ldots,m-1$.*

We notice that if M is a graded submodule of the finitely generated graded free S-module S^m ($m \ge 1$), then M is an SBF submodule of S^m if and only if $M = \oplus_{i=1}^m I_i e_i$, with $I_i \subset S$ strongly stable ideal, for $i = 1,\ldots,r$, and $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_m$.

Example 1. *Let $S = K[x_1,x_2,x_3]$. The monomial submodule $M = (x_1^2,\, x_1x_2,\, x_1x_3,\, x_2x_3,\, x_2^4)\, e_1 \oplus (x_1^2,\, x_1x_2,\, x_2^4,\, x_2^3x_3)e_2$ of S^2 is an SBF submodule. On the contrary, the monomial submodule $N = (x_1^2, x_1x_2, x_1x_3)e_1 \oplus (x_1^2, x_1x_2, x_2^4)e_2$ is not an SBF submodule of S^2.*

For the reader's convenience, we recall some notations from [1]. Let $M = \oplus_{i=1}^m I_i e_i$ be a monomial submodule of the finitely generated graded free S-module $F = \oplus_{i=1}^m S e_i$; we denote by $\mathrm{Corn}_M(I_i e_i)$ the set of corners of $I_i e_i$ that are also corners of M. Moreover, if $\mathcal{D}(M)$ is the set of the ideals appearing in the direct decomposition of M, we define the following set of ideals of S:

$$\mathrm{Corn}(\mathcal{D}(M)) = \{I_i \in \mathcal{D}(M) : \mathrm{Corn}_M(I_i e_i) \ne \emptyset,\ \text{for } i = 1,\ldots,m\}.$$

We call each ideal of $\mathrm{Corn}(\mathcal{D}(M))$ a corner ideal of M. One can observe that if $(k,\ell) \in \mathrm{Corn}(M)$ and $\beta_{k,k+\ell-f_i}(I_i) \ne 0$, then $(k,\ell) \in \mathrm{Corn}_M(I_i e_i)$. Hence, we define the following m-tuple of non–negative integers:

$$C_{(k,\ell)} = (\beta_{k,k+\ell-f_1}(I_1), \ldots, \beta_{k,k+\ell-f_m}(I_m)).$$

We call such a sequence the (k,ℓ)-sequence of the module M. It is clear that $\beta_{k,k+\ell}(M) = \sum_{i=1}^{m} \beta_{k,k+\ell-f_i}(I_i)$. Therefore, if $\mathrm{Corn}(M) = \{(k_1,\ell_1),\ldots,(k_r,\ell_r)\}$ is the corner sequence of M, one associates with the module M a suitable $r \times m$ matrix whose i-th row is the (k_i,ℓ_i)-sequence of M, $1 \le i \le r$. We call such a matrix the corner matrix of M, and we denote it by C_M. The sum of the entries of the i-th row of C_M equals the value of the extremal Betti number $\beta_{k_i,k_i+\ell_i}(M)$, for $i = 1,\ldots,r$. Moreover, setting $a_i = \beta_{k_i,k_i+\ell_i}(M)$, for $i = 1,\ldots,r$, we call $b(M) = (a_1,\ldots,a_r)$ the corner values sequence of M.

3. Blocks and Sub-Blocks of a Monomial Ideal

If $T \subseteq S$, let us denote with $\mathrm{Mon}(T)$ ($\mathrm{Mon}_d(T)$, respectively) the set of all monomials (the set of all monomials of degree d, respectively) of T. Moreover, for a subset T of monomials of degree d of S, let $\max(T)$ ($\min(T)$, respectively) be the greatest monomial (smallest monomial, respectively) of T, with respect to the lexicographic ordering on S.

Definition 6. *A set T of monomials in S of degree d is said strongly stable if for all $u \in T$, $x_i u/x_j \in T$, for all $i < j$ and for all $j \in \mathrm{supp}(u)$.*

Remark 2. *One can observe that an ideal I is a strongly stable ideal if and only if $\mathrm{Mon}(I_d)$ is a strongly stable set in S for all d; I_d is the K-vector space of all homogeneous elements $f \in I$ of degree d.*

The next definitions are motivated by the above remark.

Let $T = \{u_1,\ldots,u_q\}$ be a strongly stable set of monomials of degree d. We can suppose, possibly after a permutation of the indices, that:

$$u_1 >_{\mathrm{lex}} u_2 >_{\mathrm{lex}} \cdots >_{\mathrm{lex}} u_q. \tag{2}$$

If u_i, u_j, $i < j$, are two monomials in (2), let us define the following subset of T:

$$[u_i,u_j] = \{w \in T : u_i \ge_{\mathrm{lex}} w \ge_{\mathrm{lex}} u_j\};$$

$[u_i,u_j]$ will be called a segment of T of initial element u_i and final element u_j; if $i = j$, we set $[u_i,u_j] = \{u_i\}$.

Example 2. *Let $S = K[x_1,x_2,x_3,x_4]$. Consider the strongly stable set $T = \{x_1^3,\ x_1^2x_2,\ x_1^2x_3,\ x_1x_2^2,\ x_1x_2x_3\}$ of monomials of degree 3 in S. Setting $B = \{x_1^2x_2,\ x_1^2x_3,\ x_1x_2^2\}$ and $\widetilde{B} = \{x_1^2x_2, x_1x_2^2, x_1x_2x_3\}$, one has $B = [x_1^2x_2, x_1x_2^2]$, whereas $\widetilde{B} \neq [x_1^2x_2, x_1x_2x_3]$.*

The previous definitions lead us to suitably represent strongly stable ideals of S. More in detail, let I be a strongly stable ideal; setting $I(\ell) = G(I)_\ell$, if I is generated in degrees $\ell_1 < \ell_2 < \cdots < \ell_r$, we write I as:

$$I = [I(\ell_1)|I(\ell_2)|\cdots|I(\ell_r)],$$

where $I(\ell_i)$ is called the ℓ_i-degree block of I. It is clear that every $I(\ell_i)$ $(1 \le i \le r)$ is a segment of the strongly stable set $\mathrm{Mon}(I_{\ell_i})$ (Remark 2).

For instance, if $I = (x_1^2,\ x_1x_2,\ x_1x_3,\ x_1x_4,\ x_1x_5,\ x_1x_6,\ x_2^3,\ x_2^2x_3,\ x_2^2x_4,\ x_2x_3^4,\ x_3^5)$ is a strongly stable ideal of $S = K[x_1,\ldots,x_6]$, then $I = [I(2)|I(3)|I(5)]$, with $I(2) = [x_1^2, x_1x_6]$, $I(3) = [x_2^3, x_2^2x_4]$ and $I(5) = [x_2x_3^4, x_3^5]$.

Definition 7. *Let $I(\ell)$ be a degree block of a strongly stable ideal I of S. A subset B of $I(\ell)$ is said to be an ℓ-degree sub-block of I if B is a segment of $\mathrm{Mon}(I_\ell)$.*

In the above example, if we consider the three-degree block $I(3) = [x_2^3, x_2^2x_4]$ of I, one can observe that $\{x_2^3, x_2^2x_4\}$ is not a three-degree sub-block of I; whereas $\{x_2^2x_3, x_2^2x_4\}$ and $\{x_2^3, x_2^2x_3\}$ are both three-degree sub-blocks of I.

Let M be a strongly stable submodule of S^m generated in degrees $\ell_1 < \ell_2 < \cdots < \ell_r$. Setting $I(\ell) := [\,]$, if $G(I)_\ell = \emptyset$, every ideal $I \in \mathcal{D}(M)$ can be written as follows:

$$I = [I(\ell_1)|I(\ell_2)|\cdots|I(\ell_r)],$$

with some blocks equal to $[\,]$. In other words, we may always assume that every ideal $I \in \mathcal{D}(M)$ has the same number of blocks.

Example 3. *Let $S = K[x_1, \dots, x_6]$. Consider the strongly stable submodule $M = \oplus_{i=1}^3 I_ie_i$ of S^3 generated in degrees $2, 3, 5$, with:*

$$\begin{aligned}
I_1 &= (x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_2^3, x_2^2x_3, x_2^2x_4, x_2x_3^2, x_2x_3x_4, x_2x_4^2, x_3^5),\\
I_2 &= (x_1^3, x_1^2x_2, x_1^2x_3, x_1^2x_4, x_1x_2^2, x_1x_2x_3, x_1x_2x_4, x_1x_3^2, x_1x_3x_4, x_1x_4^2, x_2^5, x_2^4x_3, x_2^3x_3^2),\\
I_3 &= (x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_2^2, x_2x_3, x_2x_4, x_2x_5, x_2x_6, x_3^5).
\end{aligned}$$

One has:

$I_1 = [I_1(2)|I_1(3)|I_1(5)]$, *with* $I_1(i) \neq [\,]$, *for* $i = 2, 3, 5$;
$I_2 = [I_2(2)|I_2(3)|I_2(5)]$, *with* $I_2(2) = [\,]$;
$I_3 = [I_3(2)|I_3(3)|I_3(5)]$, *with* $I_3(3) = [\,]$.

4. Construction of an SBF Submodule

In this section, if $M = \oplus_{i=1}^m I_ie_i$ is a strongly stable submodule of the finitely generated graded free S-module S^m ($m > 1$), we propose a method for constructing an SBF submodule $\widetilde{M}$ of S^m managing the monomial generators of the ideals $I_i \in \mathrm{Corn}(\mathcal{D}(M))$ ($i = 1, \dots, m$). Our method will return a new submodule of S^m with the same extremal Betti numbers as M.

We start this section with a definition and then prove a crucial result.

Definition 8. *Let I be a graded ideal of S, and $m_1, \dots, m_s$ monomials in I. The generators $m_1, \dots, m_s$ are called Borel generators of the ideal I if I is the smallest strongly stable ideal containing $m_1, \dots, m_s$.*

If $m_1, \dots, m_s$ are Borel generators of a graded ideal I, we write $I = \langle m_1, \dots, m_s \rangle$, and we call I a finitely generated Borel ideal. It is called a principal Borel ideal if there is a single Borel generator for I.

Given two positive integers k, ℓ, with $1 \le k < n$ and $\ell \ge 1$, let us introduce the following set of monomials:

$$A(k, \ell) = \{u \in \mathrm{Mon}_\ell(S) : \mathrm{m}(u) = k + 1\}.$$

Lemma 1. *Let $M = \oplus_{j=1}^r I_je_j$ be a strongly stable submodule of S^m, $m \ge 1$, with corner sequence $\mathrm{Corn}(M) = \{(k_1, \ell_1), \dots, (k_r, \ell_r)\}$ and corner values sequence $b(M) = (a_1, \dots, a_n)$. Then, there exists a strongly stable submodule $\widetilde{M}$ of S^m such that:*

(i) $\mathrm{Corn}(\widetilde{M}) = \mathrm{Corn}(M)$;
(ii) $b(\widetilde{M}) = b(M)$;
(iii) *every corner ideal of $\widetilde{M}$ is a finitely generated Borel ideal of S.*

Proof. From [1] Lemma 4.5, we may suppose that $\mathrm{Corn}_M(I_j) = \mathrm{Corn}(I_j)$, for every $I_j \in \mathrm{Corn}(\mathcal{D}(M))$. On the other hand, if:

$$C_M = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{r,1} & a_{r,2} & \cdots & a_{r,m} \end{pmatrix}$$

is the corner matrix of M, then $a_i = \beta_{k_i,k_i+\ell_i}(M) = \sum_{j=1}^m a_{i,j}$, with $a_{i,j} \neq 0$ whenever (k_i, ℓ_i) is a corner of I_j. Let $M_{i,j}$ be the set of all monomials in $A(k_i, \ell_i)$ that determine the corner (k_i, ℓ_i) of I_j $(i = 1, \ldots, r)$. One has that $|M_{i,j}| = a_{i,j}$, whenever $a_{i,j} \neq 0$. Let us denote by $\widetilde{M}_{i,j}$ the subset of $M_{i,j}$ with the following property: if $v, w \in \widetilde{M}_{i,j}$, $v >_{\mathrm{lex}} w$, one has $v \neq (x_i w)/x_j$, $j \in \mathrm{supp}(w)$, $i < j$; $\min(\widetilde{M}_{i,j})$ is the $a_{i,j}$-th monomial of degree ℓ_i with $\mathrm{m}(u) = k_i + 1$. Let us denote by $\langle \widetilde{M}_{1,j}, \ldots, \widetilde{M}_{r,j} \rangle$ the smallest strongly stable ideal containing all the monomials in $\cup_{k=1}^r \widetilde{M}_{k,j}$. The strongly stable submodule $\widetilde{M}$ of S^m obtained from M, replacing every ideal $I_j \in \mathrm{Corn}(\mathcal{D}(M))$ with the ideal $\langle \widetilde{M}_{1,j}, \ldots, \widetilde{M}_{r,j} \rangle$, and leaving unchanged the ideals of $\mathcal{D}(M) \setminus \mathrm{Corn}(\mathcal{D}(M))$, is a strongly stable submodule of S^m that preserves the extremal Betti numbers of M (values, as well as positions). □

4.1. The Underlying Idea behind the Algorithm

The basic idea that leads to the construction of the Algorithm 1 is suggested by the observation that every corner ideal I that appears in the decomposition of the given strongly stable submodule M of S^m generated in degrees $\ell_1 < \ell_2 < \cdots < \ell_r$ can be seen as a set of r blocks, i.e.,

$$I = [I(\ell_1)|I(\ell_2)|\cdots|I(\ell_r)],$$

with $I(\ell) = [\,]$, for some ℓ. Hence, one can try to suitably "interchange" the ℓ-degree blocks (or sub-blocks) of a corner ideal of M with the ones of the other corner ideals of M in order to obtain an SBF submodule.

We want to clarify such an idea by means of some examples. Let us consider some cases in which every ideal which appears in the direct decomposition of the given strongly stable submodule is a corner ideal.

At first, one can observe that sometimes, to achieve our purpose, a rearrangement of the ideals of $\mathcal{D}(M)$ is sufficient. For instance, in Example 3, we can quickly realize that the module $\widetilde{M} = I_3e_1 \oplus I_1e_2 \oplus I_2e_3$ is an SBF submodule that preserves the extremal Betti numbers (positions and values) of M.

Let T be a set of monomials of degree d of S. The following set of monomials of degree $d+1$ of S:

$$\mathrm{Shad}(T) = \{x_i u : u \in T, \quad i = 1, \ldots, n\}$$

is called the shadow of T. Moreover, let us define the i-th shadow recursively by $\mathrm{Shad}^i(T) = \mathrm{Shad}(\mathrm{Shad}^{i-1}(T))$, $\mathrm{Shad}^0(T) = T$.

Example 4. *Let $S = K[x_1, \ldots, x_6]$. Set $k_1 = 5$, $k_2 = 3$, $k_3 = 2$ and $\ell_1 = 2$, $\ell_2 = 3$, $\ell_3 = 5$ and $\mathcal{C} = \{(k_1, \ell_1), (k_2, \ell_2), (k_3, \ell_3)\}$. Consider the monomial submodule $M = \oplus_{i=1}^3 I_i e_i$ of S^3 (Table 1) generated in degrees* $2, 3, 5$ *with:*

Table 1. The non-standard Borel-fixed (SBF) submodule M.

I_1	$I_1(2) = \{x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6\}$ $I_1(3) = \{x_2^3, x_2^2x_3, x_2^2x_4\}$ $I_1(5) = \{x_2x_3^4\}$
I_2	$I_2(2) = []$ $I_2(3) = \{x_1^3, x_1^2x_2, x_1^2x_3, x_1^2x_4\}$ $I_2(5) = []$
I_3	$I_3(2) = \{x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5\}$ $I_3(3) = \{x_2^3, x_2^2x_3, x_2^2x_4, x_2x_3^2, x_2x_3x_4, x_2x_4^2\}$ $I_3(5) = []$

M is a strongly stable submodule with $\mathrm{Corn}(M) = \mathcal{C}$, $b(M) = (1, 5, 1)$ *and:*

$$C_M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 3 \\ 1 & 0 & 0 \end{pmatrix}.$$

The monomial module $\widetilde{M} = \oplus_{i=1}^{3} J_i e_i$ of S^3 (Table 2) generated in degrees $2, 3, 5$ *with:*

Table 2. SBF submodule $\widetilde{M}$.

J_1	$J_1(2) = I_1(2)$ $J_1(3) = I_3(3)$ $J_1(5) = I_3(5)$
J_2	$J_2(2) = I_3(2)$ $J_2(3) = I_1(3)$ $J_2(5) = I_1(5)$
J_3	$J_3(2) = I_2(2)$ $J_3(3) = I_2(3)$ $J_3(5) = I_2(5)$

is the SBF submodule we are looking for. Indeed, $\mathrm{Corn}(M) = \mathrm{Corn}(\widetilde{M})$, $b(M) = b(\widetilde{M})$ *and $J_1 \supseteq J_2 \supseteq J_3$. Moreover, its corner matrix is:*

$$C_{\widetilde{M}} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Note that $\widetilde{M}$ has been obtained from M only "interchanging" the blocks of the corner ideals of M.

It is worth being remarked that getting the desired SBF submodule can be more complicated, as the next example clearly shows.

Example 5. *Let $S = K[x_1, \ldots, x_7]$. Set $k_1 = 6$, $k_2 = 4$, $k_3 = 3$ and $\ell_1 = 2$, $\ell_2 = 3$, $\ell_3 = 5$ and $\mathcal{C} = \{(k_1, \ell_1), (k_2, \ell_2), (k_3, \ell_3)\}$. Consider the monomial submodule $M = \oplus_{i=1}^{4} I_i e_i$ of S^4 generated in degrees* $2, 3, 5$ *(Table 3) with:*

Table 3. The non-SBF submodule M.

I_1	$I_1(2) = \{x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7\}$ $I_1(3) = \{x_2^3, x_2^2x_3, x_2^2x_4, x_2^2x_5, x_2x_3^2, x_2x_3x_4, x_2x_3x_5\}$ $I_1(5) = \{x_2x_4^4, x_3^5, x_3^4x_4\}$
I_2	$I_2(2) = \{x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7, x_2^2, x_2x_3, x_2x_4, x_2x_5, x_2x_6, x_2x_7,$ $x_3^2, x_3x_4, x_3x_5, x_3x_6, x_3x_7\}$ $I_2(3) = []$ $I_2(5) = \{x_4^5\}$
I_3	$I_3(2) = \{x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7\}$ $I_3(3) = \{x_2^3, x_2^2x_3, x_2^2x_4, x_2^2x_5, x_2x_3^2, x_2x_3x_4, x_2x_3x_5, x_2x_4^2, x_2x_4x_5\}$ $I_3(5) = []$
I_4	$I_4(2) = \{x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7\}$ $I_4(3) = \{x_2^3, x_2^2x_3, x_2^2x_4, x_2^2x_5\}$ $I_4(5) = \{x_2x_3^4, x_2x_3^3x_4, x_2x_3^2x_4^2\}$

M is a strongly stable submodule with $\mathrm{Corn}(M) = \mathcal{C}$, $b(M) = (6, 6, 5)$ *and:*

$$C_M = \begin{pmatrix} 1 & 3 & 1 & 1 \\ 2 & 0 & 3 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix}.$$

The monomial module $\widetilde{M} = \oplus_{i=1}^4 J_ie_i$ (Table 4), where:

Table 4. The SBF submodule $\widetilde{M}$.

J_1	$J_1(2) = I_2(2)$ $J_1(3) = I_2(4)$ $J_1(5) = I_2(5)$
J_2	$J_2(2) = I_3(2)$ $J_2(3) = I_3(3)$ $J_2(5) = I_1(5) \setminus (\mathrm{Shad}^3(I_3(2)) \cup \mathrm{Shad}(I_3(3)) = \{x_3^5, x_3^4x_4\}$
J_3	$J_3(2) = I_1(2)$ $J_3(3) = I_1(3)$ $J_3(5) = I_1(5) \setminus (I_1(5) \setminus (\mathrm{Shad}^3(I_3(2)) \cup \mathrm{Shad}(I_3(3))) = \{x_2x_4^4\}$
J_4	$J_4(2) = I_4(2)$ $J_4(3) = I_4(3)$ $J_4(5) = I_4(5)$

is an SBF submodule of S^4 generated in degrees $2, 3, 5$ *with* $\mathrm{Corn}(M) = \mathrm{Corn}(\widetilde{M})$ *and* $b(M) = b(\widetilde{M})$. *Indeed, the corner matrix of $\widetilde{M}$ is:*

$$C_{\widetilde{M}} = \begin{pmatrix} 3 & 1 & 1 & 1 \\ 0 & 3 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}.$$

One can observe that to get the SBF module $\widetilde{M}$, some "exchanges" involving both "blocks and sub-blocks" of the ideals in M are required.

4.2. Description of the Algorithm

If I is a graded ideal of the polynomial ring S, we denote by $\alpha(I)$ the initial degree of I, i.e., the minimum t such that $I_t \neq 0$.

Moreover, for every integer $m \geq 1$, let $[m] = \{1, \dots, m\}$.

Algorithm 1. SBF Algorithm

INPUT: a strongly stable submodule M of S^m ($m > 1$) with corner sequence $\mathcal{C} = \{(k_1, \ell_1), \dots, (k_r, \ell_r)\}$ and corner values sequence $b(M) = (a_1, \dots, a_r)$.

OUTPUT: an SBF submodule $\widetilde{M}$ of S^m such that $\mathrm{Corn}(\widetilde{M}) = \mathcal{C}$ and $b(\widetilde{M}) = b(M)$.

(1) Let $M = \oplus_{i=1}^m I_i e_i$ be a strongly stable submodule of S^m with $\mathrm{Corn}(M) = \mathcal{C}$ and $b(M) = (a_1, \dots, a_r)$.

(2) Set $I_i = [I_i(\ell_1), I_i(\ell_2), \dots, I_i(\ell_r)]$, for $i = 1, \dots, m$, and assume:

(a) $\mathrm{Corn}(I_i) = \mathrm{Corn}_M(I_i)$, for all $I_i \in \mathcal{D}(M)$;

(b) each ideal $I_i \in \mathrm{Corn}(\mathcal{D}(M))$ is a finitely generated Borel ideal with $\mathrm{m}(\min(I_i(\ell_j))) = k_j + 1$, whenever $(k_j, \ell_j) \in \mathrm{Corn}(I_i)$.

(3) We distinguish two cases: $\mathrm{Corn}(\mathcal{D}(M)) = \mathcal{D}(M)$, and $\mathrm{Corn}(\mathcal{D}(M)) \subset \mathcal{D}(M)$.

Case 1. $\boxed{\text{Let } \mathrm{Corn}(\mathcal{D}(M)) = \mathcal{D}(M).}$

Step 1. Construction of $J_1 = [J_1(\ell_1), J_1(\ell_2), \dots, J_1(\ell_r)]$.

(i) $\boxed{J_1(\ell_1)}$ is given by the ℓ_1-degree blocks in M with the greatest number of monomials.

(ii) Let $t \in [m]$ such that $\boxed{J_1(\ell_1) = I_t(\ell_1)}$; let us consider the set:

$$S_{1,\ell_2} = \left\{ I_i(\ell_2) \setminus \mathrm{Shad}^{\ell_2 - \ell_1}(I_t(\ell_1)), i = 1, \dots, m \right\}.$$

- If $S_{1,\ell_2} = \emptyset$, we set $\boxed{J_1(\ell_2) = [\,]}$;
- if $S_{1,\ell_2} \neq \emptyset$, let $u_{1,\ell_2} \in \mathrm{Mon}(S)$ be the greatest monomial of degree ℓ_2 with $\mathrm{m}(u_{1,\ell_2}) \leq k_2 + 1$ not belonging to $\mathrm{Shad}^{\ell_2 - \ell_1}(I_t(\ell_1))$. If $\overline{I_s(\ell_2)} = I_s(\ell_2) \setminus \mathrm{Shad}^{\ell_2 - \ell_1}(I_t(\ell_1)) \in S_{1,\ell_2}$ ($s \in [m]$) is the set with the largest number of elements and such that $\max(\overline{I_s(\ell_2)}) = u_{1,\ell_2}$, let $\boxed{J_1(\ell_2) = \overline{I_s(\ell_2)}}$.

(iii) In order to construct the ℓ_3-degree generators of J_1, consider the set:

$$S_{1,\ell_3} = \left\{ I_i(\ell_3) \setminus \left(\cup_{i=1}^2 \mathrm{Shad}^{\ell_3 - \ell_i}(J_1(\ell_i)) \right), i = 1, \dots, m \right\}.$$

- If $S_{1,\ell_3} = \emptyset$, let $\boxed{J_1(\ell_3) = [\,]}$;
- if $S_{1,\ell_3} \neq \emptyset$, let u_{1,ℓ_3} be the greatest monomial of S of degree ℓ_3 with $\mathrm{m}(u_{1,\ell_3}) \leq k_3 + 1$ not belonging to $\cup_{i=1}^2 \mathrm{Shad}^{\ell_3 - \ell_i}(J_1(\ell_i))$. Setting $\overline{\overline{I_i(\ell_3)}} = I_i(\ell_3) \setminus \left(\cup_{i=1}^2 \mathrm{Shad}^{\ell_3 - \ell_i}(J_1(\ell_i)) \right)$ ($i \in [m]$), we consider the elements $\overline{\overline{I_i(\ell_3)}} \in S_{1,\ell_3}$ with the greatest cardinality and such that $\max(\overline{\overline{I_i(\ell_3)}}) = u_{1,\ell_3}$. If $\overline{\overline{I_q(\ell_3)}} \in S_{1,\ell_3}$ is such an element, for some $q \in [m]$, we set $\boxed{J_1(\ell_3) = \overline{\overline{I_q(\ell_3)}}}$.

(iv) Iterating such a method, one determines the ℓ_j-degree generators of J_1, for $j \in \{4, \dots, m\}$. More in detail, the ℓ_j-degree blocks $J_1(\ell_j)$ ($j \in \{4, \dots, m\}$) are determined by the set:

$$S_{1,\ell_j} = \left\{ I_i(\ell_j) \setminus \left(\cup_{p=1}^{j-1} \mathrm{Shad}^{\ell_j - \ell_i}(J_1(\ell_p)) \right), i = 1, \dots, m \right\}.$$

(v) Save the sets of monomials of degrees ℓ_i ($i = 2, \dots, r$) which can appear during the construction of J_1:

(I) $(I_s(\ell_2) \setminus \overline{I_s(\ell_2)}) \cup I_t(\ell_2)$ ($s \neq t$) in degree ℓ_2;

(II) $(I_q(\ell_3) \setminus \overline{\overline{I_q(\ell_3)}}) \cup I_s(\ell_3)$, ($s \neq q$) in degree ℓ_3;

(III) and so on.

(vi) The segments in (v), if not empty, will be involved in the computation of $G(\widetilde{M} \setminus \{J_1 e_1\})_{\ell_j}$ ($j = 2, \dots, r$).

Step 2. Construction of $J_2 = [J_2(\ell_1), J_2(\ell_2), \dots, J_2(\ell_r)]$. We manage the blocks and the sub-blocks not involved in the construction of J_1.

(i) Consider all the blocks $I_i(\ell_1)$, with $i \neq t$, where t is the integer defined in Step 1 (ii):

- if $I_i(\ell_1) = [\,]$ for all $i \in [m] \setminus \{t\}$, then we set $\boxed{J_2(\ell_1) = [\,]}$;
- otherwise, we consider among the blocks $I_i(\ell_1)$, with $i \neq t$, the ones which are maximal ℓ_1-degree blocks in M. If $I_a(\ell_1)$ is such a set, for some $a \in [m] \setminus \{t\}$, we choose $\boxed{J_2(\ell_1) = I_a(\ell_1)}$;

(ii) Let $J_2(\ell_1) = I_a(\ell_1)$. If $S_{2,\ell_2} = \{I_i(\ell_2) \setminus \mathrm{Shad}^{\ell_2 - \ell_1}(I_a(\ell_1)), i \in [m] \setminus \{s\}\}$, we check the set:

$$\widetilde{S}_{2,\ell_2} = S_{2,\ell_2} \cup \left\{ \left((I_s(\ell_2) \setminus \overline{I_s(\ell_2)}) \cup I_t(\ell_2) \right) \setminus \mathrm{Shad}^{\ell_2 - \ell_1}(I_a(\ell_1)) \right\},$$

where $I_s(\ell_2), \overline{I_s(\ell_2)}$ and $I_t(\ell_2)$ are the sets defined in the ℓ_2-degree case of Step 1.

- If $\widetilde{S}_{2,\ell_2} = \emptyset$, let $\boxed{J_2(\ell_2) = [\,]}$;
- if u_{2,ℓ_2} is the greatest monomial of S of degree ℓ_2 with $\mathrm{m}(u_{2,\ell_2}) \leq k_2 + 1$ not belonging to $\mathrm{Shad}^{\ell_2 - \ell_1}(I_a(\ell_2))$, we test all $X \in \widetilde{S}_{2,\ell_2}$ with the greatest cardinality, and such that $\max(X) = u_{2,\ell_2}$. If $\widetilde{X} \in \widetilde{S}_{2,\ell_2}$ is such an element, let $\boxed{J_2(\ell_2) = \widetilde{X}}$.

(iii) If $I_i(\ell_1) = [\,]$, for all $i \neq t$, then $\alpha(J_2) \geq \ell_2$, and we can construct $J_2(\ell_2)$ using the above arguments on $J_2(\ell_1) \neq [\,]$.

(iv) In order to get $\boxed{J_2(\ell_3)}$, setting $S_{2,\ell_3} = \{I_i(\ell_3) \setminus (\cup_{j=1}^{2} \mathrm{Shad}^{\ell_3 - \ell_j}(J_2(\ell_j))), i \in [m] \setminus \{q\}\}$, we consider the set:

$$\widetilde{S}_{2,\ell_3} = S_{2,\ell_3} \cup \left\{ \left((I_q(\ell_3) \setminus \overline{\overline{I_q(\ell_3)}}) \cup I_s(\ell_3) \right) \setminus \left(\cup_{j=1}^{2} \mathrm{Shad}^{\ell_3 - \ell_j}(J_2(\ell_j)) \right) \right\},$$

where $I_q(\ell_3)$, $\overline{\overline{I_q(\ell_3)}}$ and $I_s(\ell_3)$ are the sets defined in the ℓ_3-degree case of Step 1.

- If $\widetilde{S}_{2,\ell_3} = \emptyset$, let $\boxed{J_2(\ell_3) = [\,]}$;
- if $\widetilde{S}_{2,\ell_3} \neq \emptyset$, let $u_{2,\ell_3} \in \mathrm{Mon}(S)$ be the greatest monomial of degree ℓ_3 with $\mathrm{m}(u_{2,\ell_3}) \leq k_3 + 1$ not belonging to $\cup_{j=1}^{2} \mathrm{Shad}^{\ell_3 - \ell_j}(J_2(\ell_j))$. Hence, we test all $Y \in \widetilde{S}_{2,\ell_3}$ with the greatest cardinality, and such that $\max(Y) = u_{2,\ell_3}$. If $\widetilde{Y} \in \widetilde{S}_{2,\ell_3}$ is such an element, we set $\boxed{J_2(\ell_3) = \widetilde{Y}}$.

(v) Proceeding in this way, we obtain a strongly stable ideal J_2 of S, which is generated in at most $\ell_1 < \ell_2 < \cdots < \ell_r$ degrees and such that each block $J_2(\ell_j)$ ($j \in \{4 \ldots, r\}$) is determined either by the set:

$$S_{2,\ell_j} = \left\{ I_i(\ell_j) \setminus \left(\cup_{p=1}^{j-1} \mathrm{Shad}^{\ell_j - \ell_i}(J_2(\ell_p)) \right), i \in [m] \right\},$$

where the ℓ_j-degree blocks $I_i(\ell_j)$ have not been involved in the construction of J_1, or by a certain ℓ_j-degree sub-block arising in the construction of $G(J_1)_{\ell_j}$.

(vi) Save the sets of monomials of degrees ℓ_i ($i = 2, \ldots, r$) that can appear during the construction of J_2:

(I) $(I_b(\ell_2) \setminus \widetilde{X}) \cup I_a(\ell_2)$, if $\widetilde{X} \in S_{2,\ell_2}$; $((I_s(\ell_2) \setminus \overline{I_s(\ell_2)}) \cup I_t(\ell_2) \setminus \widetilde{X}) \cup I_a(\ell_2)$ ($b \neq s$), if $\widetilde{X} = ((I_s(\ell_2) \setminus \overline{I_s(\ell_2)}) \cup I_t(\ell_2)) \setminus \mathrm{Shad}^{\ell_2 - \ell_1}(I_a(\ell_1))$, in degree ℓ_2;

(II) $I_c(\ell_3) \setminus \widetilde{Y} \neq \emptyset$ ($c \neq q$) if $\widetilde{Y} \in S_{2,\ell_3}$, $((I_q(\ell_3) \setminus \overline{\overline{I_q(\ell_3)}}) \cup (I_s(\ell_3)) \setminus \widetilde{Y}$, if $\widetilde{Y} = ((I_q(\ell_3) \setminus \overline{\overline{I_q(\ell_3)}}) \cup I_s(\ell_3)) \setminus (\cup_{j=1}^{2} \mathrm{Shad}^{\ell_3 - \ell_j}(J_2(\ell_j)))$, in degree ℓ_3;

(III) and so on.

(vii) The segments in (vi), if not empty, will be involved in the computation of $G(\widetilde{M} \setminus \{J_1 e_1, J_2 e_2\})_{\ell_j}$ ($j = 4, \ldots, r$).

(viii) Repeating the same procedure as in Steps 1, and 2, the monomial submodule $\boxed{\widetilde{M} = \oplus_{i=1}^{m} J_i e_i}$ is an SBF submodule with $\mathrm{Corn}(\widetilde{M}) = \mathrm{Corn}(M)$ and $b(\widetilde{M}) = b(M)$.

Case 2. $\boxed{\text{Let } \mathrm{Corn}(\mathcal{D}(M)) = \{I_{j_1}, \ldots, I_{j_t}\} \subset \mathcal{D}(M).}$

(i) Set $\boxed{J_i = \langle x_{k_1+1}^{\ell_1 - 1} \rangle}$, for all $i = 1, \ldots, m - t$;

(ii) construct an SBF submodule $\boxed{M_2 = \oplus_{i=1}^{t} J_{m-t+i} e_{m-t+i}}$, with $\mathrm{Corn}(M_2) = \mathrm{Corn}(M)$ and $b(M_2) = b(M)$, by using the criterion described in Steps 1 and 2.

(iii) The submodule $\boxed{\widetilde{M} = M_1 \oplus M_2}$, with $M_1 = \oplus_{i=1}^{m-t} J_i e_i$, is an SBF submodule generated in degrees $\ell_1 - 1 < \ell_1 < \cdots < \ell_r$, which preserves the extremal Betti numbers (values and positions). Note that $\mathcal{D}(M_2) = \mathrm{Corn}(\mathcal{D}(\widetilde{M}))$.

The correctness of the SBF Algorithm is stated by the next theorem.

Theorem 1. *Let M be a strongly stable submodule of S^m ($m > 1$) with corner sequence $\mathcal{C} = \{(k_1, \ell_1), \ldots, (k_r, \ell_r)\}$ and corner values sequence $b(M) = (a_1, \ldots, a_r)$. Assume M is generated in degrees $\ell_1 < \ell_2 < \cdots < \ell_r$. Then, there exists an SBF submodule $\widetilde{M}$ of S^m such that:*

(i) $\mathrm{Corn}(\widetilde{M}) = \mathcal{C}$;
(ii) $b(\widetilde{M}) = b(M)$.

Proof. We construct an SBF submodule $\widetilde{M} = \oplus_{i=1}^{m} J_i e_i$ of S^m generated in at most the $r+1$ degrees $\ell_1 - 1 < \ell_1 < \ell_2 < \cdots < \ell_r$.

Let $M = \oplus_{i=1}^{m} I_i e_i$ be a strongly stable submodule of S^m with $\mathrm{Corn}(M) = \mathcal{C}$ and corner values sequence $b(M) = (a_1, \ldots, a_r)$. Set $I_i = [I_i(\ell_1), I_i(\ell_2), \ldots, I_i(\ell_r)]$, for $i = 1, \ldots, m$. From [1] Lemma 4.5, we may assume that $\mathrm{Corn}(I_i) = \mathrm{Corn}_M(I_i)$, for all $I_i \in \mathcal{D}(M)$; furthermore, by Lemma 1, we may suppose that each ideal $I_i \in \mathrm{Corn}(\mathcal{D}(M))$ is a finitely generated Borel ideal such that $\mathrm{m}(\min(I_i(\ell_j))) = k_j + 1$, whenever $(k_j, \ell_j) \in \mathrm{Corn}(I_i)$. We construct $\widetilde{M} = \oplus_{i=1}^m J_i e_i$ rearranging the blocks and the sub-blocks of the ideals $I_i \in \mathrm{Corn}(\mathcal{D}(M))$, for $i = 1, \ldots, m$.

We distinguish two cases: $\mathrm{Corn}(\mathcal{D}(M)) = \mathcal{D}(M)$; $\mathrm{Corn}(\mathcal{D}(M)) \subset \mathcal{D}(M)$.

First, we consider the case $\mathrm{Corn}(\mathcal{D}(M)) = \mathcal{D}(M)$.

Step 1. Construction of J_1.

Let us consider the ℓ_1-degree blocks in M with the greatest number of monomials. If $I_t(\ell_1)$ is such a block, we choose $J_1(\ell_1) = I_t(\ell_1)$. In order to construct $J_1(\ell_2)$, we proceed as follows. Consider the following set of monomials of degree ℓ_2:

$$S_{1,\ell_2} = \left\{ I_i(\ell_2) \setminus \mathrm{Shad}^{\ell_2-\ell_1}(I_t(\ell_1)), i = 1, \ldots, m \right\}.$$

If $S_{1,\ell_2} = \emptyset$, we set $J_1(\ell_2) = [\,]$.

Otherwise, if $S_{1,\ell_2} \neq \emptyset$, let $u_{1,\ell_2} \in \mathrm{Mon}(S)$ be the greatest monomial of degree ℓ_2 with $\mathrm{m}(u_{1,\ell_2}) \leq k_2 + 1$ not belonging to $\mathrm{Shad}^{\ell_2-\ell_1}(I_t(\ell_1))$. If $\overline{I_s(\ell_2)} = I_s(\ell_2) \setminus \mathrm{Shad}^{\ell_2-\ell_1}(I_t(\ell_1)) \in S_{1,\ell_2}$ $s \in [m]$, is a set with the largest number of elements, and such that $\max(\overline{I_s(\ell_2)}) = u_{1,\ell_2}$, we set $J_1(\ell_2) = \overline{I_s(\ell_2)}$.

If $s = t$, then $J_1(\ell_2) = \overline{I_t(\ell_2)}$ and $I_s(\ell_2) \setminus \overline{I_s(\ell_2)} = \emptyset$. Let $s \neq t$, and consider the set $I_s(\ell_2) \setminus \overline{I_s(\ell_2)}$. If $I_s(\ell_2) \setminus \overline{I_s(\ell_2)} \neq \emptyset$, then it will come into play in the construction of the ℓ_2-degree generators of $\widetilde{M} \setminus \{J_1 e_1\}$, as we will see in the sequel. Otherwise, $I_s(\ell_2) \setminus \overline{I_s(\ell_2)}$ will not give any contribution for the computations of such generators. Let $I_s(\ell_2) \setminus \overline{I_s(\ell_2)} \neq \emptyset$, i.e., $\max I_s(\ell_2) > u_{1,\ell_2}$. The set $(I_s(\ell_2) \setminus \overline{I_s(\ell_2)}) \cup I_t(\ell_2)$ is a segment of monomials of degree ℓ_2. Indeed, one can observe that, if $I_t(\ell_2) \neq [\,]$, then $I_t(\ell_2) \subset \overline{I_s(\ell_2)}$ and $\max(\overline{I_s(\ell_2)}) = \max(I_t(\ell_2)) = u_{1,\ell_2}$.

In order to construct the ℓ_3-degree generators of J_1, let us consider the set:

$$S_{1,\ell_3} = \left\{ I_i(\ell_3) \setminus \left(\cup_{i=1}^2 \mathrm{Shad}^{\ell_3-\ell_i}(J_1(\ell_i)) \right), i = 1, \ldots, m \right\}.$$

If $S_{1,\ell_3} = \emptyset$, let $J_1(\ell_3) = [\,]$. Otherwise, if $S_{1,\ell_3} \neq \emptyset$, let $u_{1,\ell_3} \in \mathrm{Mon}(S)$ be the greatest monomial of degree ℓ_3 with $\mathrm{m}(u_{1,\ell_3}) \leq k_3 + 1$ not belonging to $\cup_{i=1}^2 \mathrm{Shad}^{\ell_3-\ell_i}(J_1(\ell_i))$. Setting $\overline{\overline{I_i(\ell_3)}} = I_i(\ell_3) \setminus \left(\cup_{i=1}^2 \mathrm{Shad}^{\ell_3-\ell_i}(J_1(\ell_i)) \right)$ ($i \in [m]$), we consider the elements $\overline{\overline{I_i(\ell_3)}} \in S_{1,\ell_3}$ with the greatest cardinality and such that $\max(\overline{\overline{I_i(\ell_3)}}) = u_{1,\ell_3}$. If $\overline{\overline{I_q(\ell_3)}} \in S_{1,\ell_3}$ is such an element, for some $q \in [m]$, we set $J_1(\ell_3) = \overline{\overline{I_q(\ell_3)}}$.

If the ℓ_3-degree sub-block $I_q(\ell_3) \setminus \overline{\overline{I_q(\ell_3)}}$ of the strongly stable ideal $I_q \in \mathcal{D}(M)$ is not empty, then it will come into play in the construction of $G(\widetilde{M} \setminus \{J_1 e_1\})_{\ell_3}$. More specifically, if $J_1(\ell_2) = \overline{I_s(\ell_2)}$, for $s \in [m]$, then the segment $(I_q(\ell_3) \setminus \overline{\overline{I_q(\ell_3)}}) \cup I_s(\ell_3)$, $I_s(\ell_3) \subset \overline{\overline{I_q(\ell_3)}}$ ($s \neq q$) will be considered in the construction of $G(\widetilde{M} \setminus \{J_1 e_1\})_{\ell_3}$, if it is not empty.

Proceeding in this way, we obtain a strongly stable ideal J_1 of S which is generated in at most $\ell_1 < \ell_2 < \cdots < \ell_r$ degrees and such that each ℓ_j-degree block $J_1(\ell_j)$ ($j \in \{4, \ldots, r\}$) is determined by the set:

$$S_{1,\ell_j} = \left\{ I_i(\ell_j) \setminus \left(\cup_{p=1}^{j-1} \mathrm{Shad}^{\ell_j-\ell_i}(J_1(\ell_p)) \right), i = 1, \ldots, m \right\}.$$

It is relevant to point out that in some degree ℓ_j, a certain ℓ_j-degree sub-block of $I_i(\ell_j)$ ($i \in [m]$; $j \in \{4, \ldots, r\}$) can arise, as in the $\{\ell_2, \ell_3\}$-degree cases. Such segments will be involved in the computation of $G(\widetilde{M} \setminus \{J_1 e_1\})_{\ell_j}$, as we will explain in a while.

Step 2. Construction of J_2.

In order to construct J_2, we manage the blocks and the sub-blocks not involved in the construction of J_1.

First, we examine all the blocks $I_i(\ell_1)$, with $i \neq t$, where t is the integer defined in Step 1. Among all these sets, we consider the ones that are maximal ℓ_1-degree blocks in M. If $I_a(\ell_1)$ is such a set, for some $a \in [m] \setminus \{t\}$, we choose $J_2(\ell_1) = I_a(\ell_1)$. If $I_i(\ell_1) = [\,]$ for all $i \in [m] \setminus \{t\}$, then we set $J_2(\ell_1) = 0$.

Let $J_2(\ell_1) = I_a(\ell_1)$. Consider the sets:

$$S_{2,\ell_2} = \{I_i(\ell_2) \setminus \text{Shad}^{\ell_2-\ell_1}(I_a(\ell_1)), i \in [m] \setminus \{s\}\},$$

$$\widetilde{S}_{2,\ell_2} = S_{2,\ell_2} \cup \left\{ \left((I_s(\ell_2) \setminus \overline{I_s(\ell_2)}) \cup I_t(\ell_2) \right) \setminus \text{Shad}^{\ell_2-\ell_1}(I_a(\ell_1)) \right\}, \tag{3}$$

where $I_s(\ell_2), \overline{I_s(\ell_2)}$ and $I_t(\ell_2)$ are the sets defined in Step 1. If the set defined in (3) is empty, let $J_2(\ell_2) = []$. Otherwise, if u_{2,ℓ_2} is the greatest monomial of degree ℓ_2 with $\text{m}(u_{2,\ell_2}) \leq k_2 + 1$ not belonging to $\text{Shad}^{\ell_2-\ell_1}(I_a(\ell_2))$, we test all $X \in \widetilde{S}_{2,\ell_2}$ with the greatest cardinality, and such that $\max(X) = u_{2,\ell_2}$. If $\widetilde{X}$ is such an element, let $J_2(\ell_2) = \widetilde{X}$. Reasoning as in Step 1, if $\widetilde{X} \in S_{2,\ell_2}$, i.e., $\widetilde{X} = I_b(\ell_2) \setminus \text{Shad}^{\ell_2-\ell_1}(I_a(\ell_1))$, for some $b \in [m] \setminus \{s\}$, and, $I_b(\ell_2) \setminus \widetilde{X} \neq \emptyset$ (it has to be $b \neq a$), then the segment $(I_b(\ell_2) \setminus \widetilde{X}) \cup I_a(\ell_2)$ comes out. Such a set will be considered in the construction of the ℓ_2-degree generators of $\widetilde{M} \setminus \{J_1e_1, J_2e_2\}$. Similarly, if $\widetilde{X} = ((I_s(\ell_2) \setminus \overline{I_s(\ell_2)}) \cup I_t(\ell_2)) \setminus \text{Shad}^{\ell_2-\ell_1}(I_a(\ell_1))$ and $((I_s(\ell_2) \setminus \overline{I_s(\ell_2)}) \cup I_t(\ell_2)) \setminus \widetilde{X} \neq \emptyset$, then the set $((I_s(\ell_2) \setminus \overline{I_s(\ell_2)}) \cup I_t(\ell_2) \setminus \widetilde{X}) \cup I_a(\ell_2)$ will come into play in the construction of $G(\widetilde{M} \setminus \{J_1e_1, J_2e_2\})_{\ell_2}$, if it is a non-empty set.

Finally, if $I_i(\ell_1) = []$, for all $i \neq t$, then $\alpha(J_2) \geq \ell_2$, and we can construct $J_2(\ell_2)$ using the above arguments on $J_2(\ell_1) \neq []$.

In order to get $J_2(\ell_3)$, setting $S_{2,\ell_3} = \{I_i(\ell_3) \setminus (\cup_{j=1}^2 \text{Shad}^{\ell_3-\ell_j}(J_2(\ell_j))), i \in [m] \setminus \{q\}\}$, we consider the set:

$$\widetilde{S}_{2,\ell_3} = S_{2,\ell_3} \cup \left\{ \left((I_q(\ell_3) \setminus \overline{\overline{I_q(\ell_3)}}) \cup I_s(\ell_3) \right) \setminus \left(\cup_{j=1}^2 \text{Shad}^{\ell_3-\ell_j}(J_2(\ell_j)) \right) \right\}, \tag{4}$$

where $I_q(\ell_3)$, $\overline{\overline{I_q(\ell_3)}}$ and $I_s(\ell_3)$ are the sets defined in the ℓ_3-degree case of Step 1.
If the set in (4) is empty, let $J_2(\ell_3) = []$. Otherwise, if u_{2,ℓ_3} is the greatest monomial of degree ℓ_3 with $\text{m}(u_{2,\ell_3}) \leq k_3 + 1$ not belonging to $\cup_{j=1}^2 \text{Shad}^{\ell_3-\ell_j}(J_2(\ell_j))$, we test all $Y \in \widetilde{S}_{2,\ell_3}$ with the greatest cardinality and such that $\max(Y) = u_{2,\ell_3}$. If $\widetilde{Y} \in \widetilde{S}_{2,\ell_3}$ is such an element, we set $J_2(\ell_3) = \widetilde{Y}$. Let $\widetilde{Y} \in S_{2,\ell_3}$, i.e., $\widetilde{Y} = I_c(\ell_3) \setminus \cup_{j=1}^2 \text{Shad}^{\ell_3-\ell_j}(J_2(\ell_j))$, for some $c \in [m] \setminus \{q\}$. If $I_c(\ell_3) \setminus \widetilde{Y} \neq \emptyset$, then such a set will contribute to the construction of the ℓ_3-degree generators of $\widetilde{M} \setminus \{J_1e_1, J_2e_2\}$ (see Step 1, construction of $G(J_1)_{\ell_3}$). Otherwise, it will not give any contribution for such generators. A similar reasoning, follows as in the previous ℓ_2-degree case, if $\widetilde{Y} = ((I_q(\ell_3) \setminus \overline{\overline{I_q(\ell_3)}}) \cup (I_s(\ell_3)) \setminus (\cup_{j=1}^2 \text{Shad}^{\ell_3-\ell_j}(J_2(\ell_j)))$.

Going on this way, we obtain a strongly stable ideal J_2 of S, which is generated in at most $\ell_1 < \ell_2 < \cdots < \ell_r$ degrees and such that each block $J_2(\ell_j)$ ($j \in \{4 \ldots, r\}$) is determined either by the set:

$$S_{2,\ell_j} = \left\{ I_i(\ell_j) \setminus \left[\cup_{p=1}^{j-1} \text{Shad}^{\ell_j-\ell_i}(J_2(\ell_p)) \right], i \in [m] \right\},$$

where the ℓ_j-degree blocks $I_i(\ell_j)$ have not been involved in the construction of J_1, or by a certain ℓ_j-degree sub-block arising in the construction of $G(J_1)_{\ell_j}$. Moreover, the nonempty sub-blocks of $I_i(\ell_j)$ ($j \in \{4, \ldots, r\}$) that will arise during the creation of J_2 will be involved in the calculation of $G(\widetilde{M} \setminus \{J_1e_1, J_2e_2\})_{\ell_j}$.

Now, let us examine the special segments that can appear during the construction of $\widetilde{M}$. Let us consider the ℓ_1-degree case described in Step 1. The set $(I_s(\ell_2) \setminus \overline{I_s(\ell_2)}) \cup I_t(\ell_2)$, with $s \neq t$, gives a contribution to the construction of the ℓ_2-degree generators of the ideal $J_v \in \mathcal{D}(\widetilde{M})$ ($v \in \{2, \ldots, m\}$) for which $J_v(\ell_1) = I_s(\ell_1)$. In other words, we can construct a strongly stable ideal $J_v \in \mathcal{D}(\widetilde{M})$ such that $\alpha(J_v) \in \{\ell_1, \ell_2\}$, with $J_v(\ell_1) = I_s(\ell_1)$ and $J_v(\ell_2) = (I_s(\ell_2) \setminus \overline{I_s(\ell_2)}) \cup I_t(\ell_2)$. Note that $I_s(\ell_1) = []$ means that $\max((I_s(\ell_2) \setminus \overline{I_s(\ell_2)}) \cup I_t(\ell_2)) = x_1^{\ell_2}$.

Assume $J_v(\ell_2) = (I_s(\ell_2) \setminus \overline{I_s(\ell_2)}) \cup I_t(\ell_2)$, $I_t(\ell_2) \neq []$. In such a case, $I_t(\ell_3) \neq []$ may give a contribution to the ℓ_3-degree generators of J_v (i.e., $J_v(\ell_3) = I_t(\ell_3)$). Note that such a case is achieved if $I_t(\ell_3)$ has the greatest cardinality among all the blocks, the sub-blocks and the segments Z of M that are not yet involved in the construction of the ℓ_3-degree generators of $\mathcal{D}(\widetilde{M})$, and such that $\max(Z)$ is equal to the greatest monomial $z \in \mathrm{Mon}_{\ell_3}(S)$ with $\mathrm{m}(z) \le k_3+1$ not belonging to $\cup_{j=1}^{2}\mathrm{Shad}^{\ell_3-\ell_j}(J_v(\ell_j))$. If $I_t(\ell_3) = [\,]$, or $I_t(\ell_3) \neq [\,]$ does not satisfy the conditions above, then we look for a block, a sub-block or a segment of M not yet involved in the construction of the ideals $J_1, \dots, J_{v-1} \in \mathcal{D}(\widetilde{M})$ and satisfying the conditions above. If it does not exist, we set $J_v(\ell_3) = [\,]$; and so on; similarly if $J_v(\ell_2) = I_s(\ell_2) \setminus \overline{I_s(\ell_2)}$. Furthermore, the same reasoning can be iterated for the segments arising in degrees ℓ_j, $j \ge 3$.

Finally, proceeding in the same way as in Steps 1 and 2, due to the structure of M, all the monomial generators of M are swapped in a suitable way so that the monomial submodule $\widetilde{M} = \oplus_{i=1}^{m} J_i e_i$ is an SBF submodule such that $\mathrm{Corn}(\widetilde{M}) = \mathrm{Corn}(M)$ and $b(\widetilde{M}) = b(M)$.

Now, we consider the second case. Let $\mathrm{Corn}(\mathcal{D}(M)) = \{I_{j_1}, \dots, I_{j_t}\} \subset \mathcal{D}(M)$.

We construct an SBF submodule $\widetilde{M} = M_1 \oplus M_2$, such that $M_1 = \oplus_{i=1}^{m-t} J_i e_i$ with $J_i = \langle x_{k_1+1}^{\ell_1-1} \rangle$, for all $i = 1, \dots, m-t$, and $M_2 = \oplus_{i=1}^{t} J_{m-t+i} e_{m-t+i}$, with $\mathcal{D}(M_2) = \mathrm{Corn}(\mathcal{D}(\widetilde{M}))$. The monomial submodule M_2 will be obtained by using the criterion described in Steps 1 and 2. Note that M_1 does not give any contribution to the computation of the extremal Betti numbers of $\widetilde{M}$, and $J_1 = \cdots = J_{m-t} \supsetneq J_{m-t+1}$. $\square$

We close this section by considering some examples where the algorithm in Theorem 1 is used. First, we consider a complicated example suitably chosen in order to show that all the cases considered in Theorem 1 can really occur in a single concrete situation.

For a pair (k, d) of positive integers with $d \ge 1$ and $1 \le k \le n-1$, let us define:

$$A_<(k,d) = \{u \in \mathrm{Mon}_d(S) : \mathrm{m}(u) \le k+1\}.$$

Example 6. *Let* $S = K[x_1, \dots, x_6]$. *Set* $k_1 = 5$, $k_2 = 4$, $k_3 = 3$, $\ell_1 = 2$, $\ell_2 = 3$, $\ell_3 = 5$, *and* $\mathcal{C} = \{(k_1,\ell_1), (k_2,\ell_2), (k_3,\ell_3)\}$. *Moreover, let* $a_1 = 4$, $a_2 = 8$ *and* $a_3 = 39$. *Consider the monomial submodule* $M = \oplus_{i=1}^{4} I_i e_i$ *of* S^4 *generated in degrees* $2,3,5$, *where the ideals* $I_i \in \mathcal{D}(M)$ $(i = 1,2,3,4)$ *are described in Table 5:*

Table 5. The non-SBF submodule M.

I_1	$I_1(2) = \{x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6\}$ $I_1(3) = \{x_2^3, x_2^2x_3, x_2^2x_4, x_2^2x_5, x_2x_3^2, x_2x_3x_4, x_2x_3x_5, x_2x_4^2, x_2x_4x_5, x_2x_5^2, x_3^3, x_3^2x_4, x_3^2x_5\}$ $I_1(5) = \{x_3x_4^4\}$
I_2	$I_2(2) = \{x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_2^2, x_2x_3, x_2x_4, x_2x_5, x_2x_6\}$ $I_2(3) = [\,]$ $I_2(5) = \{x_3^5, x_3^4x_4, x_3^3x_4^2\}$
I_3	$I_3(2) = \{x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6\}$ $I_3(3) = \{x_2^3, x_2^2x_3, x_2^2x_4, x_2^2x_5, x_2x_3^2, x_2x_3x_4, x_2x_3x_5, x_2x_4^2, x_2x_4x_5\}$ $I_3(5) = \{x_3^5, x_3^4x_4\}$
I_4	$I_4(2) = [\,]$ $I_4(3) = [\,]$ $I_4(5) = A_<(3,5)$

M is a not SBF submodule with $\mathrm{Corn}(M) = \mathcal{C}$, $b(M) = (a_1, a_2, a_3) = (4, 8, 39)$ *and the corner matrix is the following one:*

$$C_M = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 5 & 0 & 3 & 0 \\ 1 & 2 & 1 & 35 \end{pmatrix}.$$

From Table 5, we can observe that the required inclusions $I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4$ *do not hold.*

By the method described in Theorem 1, we will construct an SBF submodule $\widetilde{M} = \oplus_{i=1}^{4} J_i e_i$ *of* S^4 *generated in degrees* $2,3,5$, *which preserves the extremal Betti numbers of M, both values and positions, i.e.,* $\mathrm{Corn}(\widetilde{M}) = \mathrm{Corn}(M)$ *and* $b(\widetilde{M}) = b(M)$.

Construction of J_1. Let us consider the two-degree blocks of M with the greatest cardinality. From Table 5, there exists $I_2(2)$ such that $|I_2(2)| > |I_i(2)|$, for $i \in \{1,3,4\}$. Hence, let

$$J_1(2) = I_2(2).$$

Now, let us consider the set of monomials $S_{1,3} = \{I_i(3) \setminus \mathrm{Shad}(I_2(2)), i = 1,2,3,4\}$. Denote by $u_{1,3}$ the greatest monomial of $\mathrm{Mon}(S)$ of degree three with $\mathrm{m}(u_{1,3}) \le k_2 + 1 = 5$ not belonging to $\mathrm{Shad}(I_2(2))$. It is $u_{1,3} = x_3^3$. By direct computation, we can see that the set $I_1(3) \setminus \mathrm{Shad}(I_2(2)) = \{x_3^3, x_3^2x_4, x_3^2x_5\} \in S_{1,3}$ has the greatest cardinality and $\max(I_1(3) \setminus \mathrm{Shad}(I_2(2))) = u_{1,3}$. Let:

$$J_1(3) = I_1(3) \setminus \mathrm{Shad}(I_2(2)) := \overline{I_1(3)}.$$

Note that $I_1(3) \setminus \overline{I_1(3)} \neq \emptyset$. Indeed,

$$I_1(3) \setminus \overline{I_1(3)} = \{x_2^3, x_2^2x_3, x_2^2x_4, x_2^2x_5, x_2x_3^2, x_2x_3x_4, x_2x_3x_5, x_2x_4^2, x_2x_4x_5, x_2x_5^2\}. \tag{5}$$

This segment will be used for the construction of $G(J_i)_3$, $2 \le i \le 4$.

In order to construct $J_1(5)$, let us consider the set:

$$S_{1,5} = \left\{ I_i(5) \setminus \left(\mathrm{Shad}^3(J_1(2)) \cup \mathrm{Shad}^2(J_1(3)) \right), i = 1,2,3,4 \right\}.$$

Let $u_{1,5} \in \mathrm{Mon}(S)$ be the greatest monomial of degree five with $\mathrm{m}(u_{1,5}) \le k_3 + 1 = 4$ not belonging to $\mathrm{Shad}^3(J_1(2)) \cup \mathrm{Shad}^2(J_1(3))$. It is $u_{1,5} = x_3x_4^4$.

The set $I_4(5) \setminus \left(\mathrm{Shad}^3(J_1(2)) \cup \mathrm{Shad}^2(J_1(3))\right) = \{x_3x_4^4, x_4^5\}$ has the greatest cardinality among all the sets in $S_{1,5}$ and $\max(I_4(5) \setminus (\mathrm{Shad}^3(J_1(2)) \cup \mathrm{Shad}^2(J_1(3)))) = u_{1,5}$. Let:

$$J_1(5) = I_4(5) \setminus \left(\mathrm{Shad}^3(J_1(2)) \cup \mathrm{Shad}^2(J_1(3)) \right) := \overline{\overline{I_4(5)}}.$$

Note that $I_4(5) \setminus \overline{\overline{I_4(5)}} \neq \emptyset$. In fact, $I_4(5) \setminus \overline{\overline{I_4(5)}} = \{u \in A_<(3,5) : u \ge_{\mathrm{lex}} x_3^2x_4^3\}$. Moreover,

$$(I_4(5) \setminus \overline{\overline{I_4(5)}}) \cup I_1(5) = \{u \in A_<(3,5) : u \ge_{\mathrm{lex}} x_3x_4^4\} \tag{6}$$

is a segment. Such a set will come into play in the characterization of the five-degree generators of the ideals J_i, for $i \in \{2,3,4\}$. Table 6 summarizes the finitely-generated Borel ideal J_1:

Table 6. The ideal J_1.

J_1	$J_1(2) = I_2(2)$
	$J_1(3) = \overline{I_1(3)}$
	$J_1(5) = \overline{\overline{I_4(5)}}$

Construction of J_2: Let us consider the non-zero two-degree blocks of M not involved in the construction of J_1, i.e., $I_1(2)$, $I_3(2)$. We have that $|I_1(2)| = |I_3(2)|$. Let:

$$J_2(2) = I_1(2).$$

In order to determine the three-degree generators (five-degree generators, respectively) of J_2, we will take into account the sets in (5) (in (6), respectively).

Setting $S_{2,3} = \{I_i(3) \setminus \mathrm{Shad}(I_1(2)), i = 2,3,4\}$, let us consider the set of monomials:

$$\widetilde{S}_{2,3} = S_{2,3} \cup \left\{ (I_1(3) \setminus \overline{I_1(3)}) \setminus \mathrm{Shad}(I_1(2)) \right\}.$$

Denote by $u_{2,3}$ the greatest monomial of degree three with $\mathrm{m}(u_{1,3}) \leq k_2 + 1 = 5$ *not belonging to* $\mathrm{Shad}(J_1(2))$. *It is* $u_{2,3} = x_2^3$.

It is clear that $(I_1(3) \setminus \overline{I_1(3)}) \setminus \mathrm{Shad}(J_1(2)) = I_1(3) \setminus \overline{I_1(3)} \in \widetilde{S}_{2,3}$ *has the greatest cardinality (see (5)) and* $\max(I_1(3) \setminus \overline{I_1(3)}) = u_{2,3}$. *Let:*

$$J_2(3) = I_1(3) \setminus \overline{I_1(3)} := \widetilde{X}.$$

In order to construct $J_2(5)$, we consider the following sets:

$$S_{2,5} = \left\{ I_i(5) \setminus \left(\mathrm{Shad}^3(J_2(2)) \cup \mathrm{Shad}^2(J_2(3)\right), i \in \{1,2,3\} \right\},$$

$$\widetilde{S}_{2,5} = S_{2,5} \cup \left\{ ((I_4(5) \setminus \overline{\overline{I_4(5)}}) \cup I_1(5)) \setminus \left(\mathrm{Shad}^3(J_2(2)) \cup \mathrm{Shad}^2(J_2(3))\right) \right\}.$$

Let $u_{2,5}$ be the greatest monomial of degree five with $\mathrm{m}(u_{1,5}) \leq k_3 + 1 = 4$ *not belonging to* $\mathrm{Shad}^3(J_2(2)) \cup \mathrm{Shad}^2(J_2(3))$. *It is* $u_{2,5} = x_3^5$. *One can quickly check that the set:*

$$((I_4(5) \setminus \overline{\overline{I_4(5)}}) \cup I_1(5)) \setminus \left(\mathrm{Shad}^3(J_2(2)) \cup \mathrm{Shad}^2(J_2(3))\right) = \{x_3^5, x_3^4x_4, x_3^3x_4^2, x_3^2x_4^3, x_3x_4^4\} := \widetilde{Y}$$

has the greatest cardinality among all the sets in $\widetilde{S}_{2,5}$ *and* $\max(\widetilde{Y}) = u_{2,5}$. *Let:*

$$J_2(5) = \widetilde{Y}.$$

Note that $((I_4(5) \setminus \overline{\overline{I_4(5)}}) \cup I_1(5)) \setminus \widetilde{Y} \neq \emptyset$. *Indeed,*

$$((I_4(5) \setminus \overline{\overline{I_4(5)}}) \cup I_1(5)) \setminus \widetilde{Y} = \{u \in A_<(3,5) : u \geq_{\mathrm{lex}} x_2x_4^4\}. \tag{7}$$

Table 7 represents the ideal J_2:

Table 7. The ideal J_2.

J_2	$J_2(2) = I_1(2)$
	$J_2(3) = \widetilde{X}$
	$J_2(5) = \widetilde{Y}$

Construction of J_3: Let us consider the non-zero two-degree blocks of M not involved in the construction of J_1 and J_2. Since $I_3(2)$ is the only non-zero two-degree block, let:

$$J_3(2) = I_3(2).$$

Moreover, since the only non-zero three-degree block of M is $I_3(3)$, and there is not a three-degree sub-block arising during the construction of J_2, we set:

$$J_3(3) = I_3(3).$$

In order to determine $J_3(5)$, we have to take into account the following sets:

$$S_{3,5} = \left\{ I_i(5) \setminus (\mathrm{Shad}^3(J_3(2)) \cup \mathrm{Shad}^2(J_3(3)), i \in \{2,3\} \right\},$$

$$\widetilde{S}_{3,5} = S_{3,5} \cup \left\{ ((I_4(5) \setminus \overline{\overline{I_4(5)}}) \cup I_1(5)) \setminus \widetilde{Y}) \setminus (\mathrm{Shad}^3(J_3(2)) \cup \mathrm{Shad}^2(J_3(3)) \right\}.$$

Note that $I_2(5)$ and $I_3(5)$ are the only five-degree blocks of M not yet involved in the construction of J_1 and J_2. Let $u_{3,5} \in \mathrm{Mon}(S)$ be the greatest monomial of degree five with $\mathrm{m}(u_{3,5}) \leq k_3 + 1 = 4$ not belonging to $\mathrm{Shad}^3(J_3(2)) \cup \mathrm{Shad}^2(J_3(3))$. It is $u_{3,5} = x_3^5$.

By direct computation, one has that the set $I_2(5) \setminus (\mathrm{Shad}^3(J_3(2)) \cup \mathrm{Shad}^2(J_3(3)) = I_2(5)$ satisfies the required properties, i.e., $\max(I_2(5) \setminus (\mathrm{Shad}^3(J_3(2)) \cup \mathrm{Shad}^2(J_3(3))) = u_{3,5}$, and moreover, it has the greatest cardinality among all the sets in $\widetilde{S}_{3,5}$. Hence, we set:

$$J_3(5) = I_2(5),$$

and J_3 is described in Table 8:

Table 8. The ideal J_3.

J_3	$J_3(2) = I_3(2)$
	$J_2(3) = I_3(3)$
	$J_3(5) = I_2(5)$

Construction of J_4: In order to determine J_4, we can manage only monomials of degree five and, more precisely, the block $I_3(5)$ and the set in (7). We notice that:

$$((I_4(5) \setminus \overline{\overline{I_4(5)}}) \cup I_1(5)) \setminus \widetilde{Y}) \cup I_3(5) = \{u \in A_<(3,5) : u \geq_{\mathrm{lex}} x_3^4 x_4\} \tag{8}$$

is a segment. More in detail, the ideal J_4 is shown in Table 9:

Table 9. The ideal J_4.

J_4	$J_4(2) = [\,]$
	$J_4(3) = [\,]$
	$J_4(5) = ((I_4(5) \setminus \overline{\overline{I_4(5)}}) \cup I_1(5)) \setminus \widetilde{Y}) \cup I_3(5)$

We have obtained a monomial submodule $\widetilde{M} = \oplus_{i=1}^4 J_i e_i$ of S^4 (Table 10), where the ideals $J_i \in \mathcal{D}(M)$ $(i = 1, \ldots, 4)$ are:

Table 10. The SBF submodule $\widetilde{M}$.

J_1	$J_1(2) = \{x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7, x_2^2, x_2x_3, x_2x_4, x_2x_5, x_2x_6\}$
	$J_1(3) = \{x_3^3, x_3^2x_4, x_3^2x_5\}$
	$J_1(5) = \{x_3x_4^4, x_4^5\}$
J_2	$J_2(2) = \{x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7\}$
	$J_2(3) = \{x_2^3, x_2^2x_3, x_2^2x_4, x_2^2x_5, x_2x_3^2, x_2x_3x_4, x_2x_3x_5, x_2x_4^2, x_2x_4x_5, x_2x_5^2\}$
	$J_2(5) = \{x_3^5, x_3^4x_4, x_3^3x_4^2, x_3^2x_4^3, x_3x_4^4\}$
J_3	$J_3(2) = \{x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6\}$
	$J_3(3) = \{x_2^3, x_2^2x_3, x_2^2x_4, x_2^2x_5, x_2x_3^2, x_2x_3x_4, x_2x_3x_5, x_2x_4^2, x_2x_4x_5\}$
	$J_3(5) = \{x_3^5, x_3^4x_4, x_3^3x_4^2\}$
J_4	$J_4(2) = [\,]$
	$J_4(3) = [\,]$
	$J_4(5) = \{u \in A_<(3,5) : u \geq {}_{\mathrm{lex}} x_3^4 x_4\}$

$\widetilde{M}$ *is an SBF submodule generated in degrees* $2,3,5$ *such that* $\mathrm{Corn}(\widetilde{M}) = \mathrm{Corn}(M)$ *and* $b(\widetilde{M}) = b(M)$. *Indeed, the corner matrix of* $\widetilde{M}$ *is:*

$$C_{\widetilde{M}} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 4 & 3 & 0 \\ 2 & 4 & 2 & 31 \end{pmatrix}$$

and $J_1 \supseteq J_2 \supseteq J_3 \supseteq J_4$.

Example 7. *Let* $S = K[x_1, \ldots, x_6]$. *Set* $k_1 = 5$, $k_2 = 3$, $k_3 = 2$, *and* $\ell_1 = 2$, $\ell_2 = 3$, $\ell_3 = 5$, *and* $\mathcal{C} = \{(k_1, \ell_1), (k_2, \ell_2), (k_3, \ell_3)\}$. *Consider the monomial submodule* $M = \oplus_{i=1}^{4} I_i e_i$ *of* S^4 *generated in degrees* $2,3,5$ *(Table 11), where:*

Table 11. The non-SBF submodule M.

I_1	$I_1(2) = \{x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_2^2, x_2x_3, x_2x_4, x_2x_5, x_2x_6\}$ $I_1(3) = [\,]$ $I_1(5) = \{x_3^5\}$
I_2	$I_2(2) = \{x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6\}$ $I_2(3) = \{x_2^3, x_2^2x_3, x_2^2x_4\}$ $I_2(5) = \{x_2x_3^4, x_3^5\}$
I_3	$I_3(2) = \{x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_2^2, x_2x_3, x_2x_4, x_2x_5, x_2x_6\}$ $I_3(3) = \{x_3^3, x_3^2x_4, x_3x_4^2\}$ $I_3(5) = [\,]$
I_4	$I_4(2) = [\,]$ $I_4(3) = \{x_1^3, x_1^2x_2, x_1^2x_3, x_1^2x_4, x_1x_2^2, x_1x_2x_3, x_1x_2x_4, x_1x_3^2, x_1x_3x_4,$ $x_1x_4^2, x_2^3, x_2^2x_3, x_2^2x_4, x_2x_3^2, x_2x_3x_4, x_2x_4^2\}$ $I_4(5) = \{x_3^5\}$

M *is a strongly stable submodule with* $\mathrm{Corn}(M) = \mathcal{C}$ *and corner values sequence given by* $b(M) = (5, 10, 4)$. *Indeed, its corner matrix is the following one:*

$$C_M = \begin{pmatrix} 2 & 1 & 2 & 0 \\ 0 & 1 & 2 & 7 \\ 1 & 2 & 0 & 1 \end{pmatrix}.$$

Applying the algorithm described in Theorem 1, we obtain an SBF submodule $\widetilde{M} = \oplus_{i=1}^{4} J_i e_i$ *of* S^4, *generated in degrees* $2,3,5$ *with* $\mathrm{Corn}(\widetilde{M}) = \mathrm{Corn}(M)$ *and* $b(\widetilde{M}) = b(M)$.

Construction of J_1: *Let us consider the two-degree blocks of* M *with the greatest cardinality. From Table 11, there exists* $I_3(2)$ *such that* $|I_3(2)| > |I_i(2)|$, *for* $i \in \{1, 2, 4\}$. *Hence, let:*

$$J_1(2) = I_3(2).$$

Now, let us consider the set of monomials $S_{1,3} = \{I_i(3) \setminus \mathrm{Shad}(J_1(2)), i = 1,2,3,4\}$. *One can observe that* $I_i(3) \setminus \mathrm{Shad}(J_1(2)) = \emptyset$, *for* $i = 2,4$; *whereas* $I_3(3) \setminus \mathrm{Shad}(J_1(2)) = I_3(3) = \{x_3^3, x_3^2x_4, x_3x_4^2\}$.

Denote by $u_{1,3}$ *the greatest monomial of* S *of degree three with* $\mathrm{m}(u_{1,3}) \leq k_2 + 1 = 4$ *not belonging to* $\mathrm{Shad}(I_3(2))$. *It is* $u_{1,3} = x_3^3 = \max(I_3(3) \setminus \mathrm{Shad}(J_1(2)))$. *Let:*

$$J_1(3) = I_3(3).$$

In order to construct $J_1(5)$, *let us consider the set:*

$$S_{1,5} = \left\{ I_i(5) \setminus \left(\mathrm{Shad}^3(J_1(2)) \cup \mathrm{Shad}^2(J_1(3) \right), i = 1,2,3,4 \right\}.$$

Since $S_{1,5} = \emptyset$, we set:

$$J_1(5) = [\,].$$

Table 12 summarizes the finitely generated Borel ideal J_1:

Table 12. The ideal J_1.

J_1	$J_1(2) = I_3(2)$
	$J_1(3) = \overline{I_3(3)}$
	$J_1(5) = I_3(5) = [\,]$

Note that $J_1 = I_3$.

Construction of J_2: Let us consider the non-zero two-degree blocks of M not involved in the construction of J_1, i.e., $I_i(2)$, for $i = 1,2,4$. Since $|I_1(2)| > |I_2(2)|$ and $I_4(2) = [\,]$, let:

$$J_2(2) = I_1(2).$$

In order to determine the three-degree generators of J_2, we have to take into account the set:

$$S_{2,3} = \{I_i(3) \setminus \mathrm{Shad}(J_2(2)), i = 1,2,4\}.$$

Since $S_{2,3} = \emptyset$, we set:

$$J_2(3) = [\,].$$

In order to construct $J_2(5)$, we consider the following set:

$$S_{2,5} = \left\{I_i(5) \setminus (\mathrm{Shad}^3(J_2(2)), i \in \{1,2,4\}\right\}.$$

It is $S_{2,5} \neq \emptyset$. Let $u_{2,5}$ be the greatest monomial of S of degree five with $\mathrm{m}(u_{1,5}) \leq k_3 + 1 = 3$ not belonging to $\mathrm{Shad}^3(J_2(2))$. It is $u_{2,5} = x_3^5 = \max(I_1(5) \setminus (\mathrm{Shad}^3(J_2(2)))$. Moreover, one can verify that:

$$I_1(5) \setminus (\mathrm{Shad}^3(J_2(2)) = I_1(5) = I_2(5) \setminus (\mathrm{Shad}^3(J_2(2)) = I_4(5) \setminus (\mathrm{Shad}^3(J_2(2)) = \{x_3^5\}.$$

Hence, let:

$$J_2(5) = I_1(5).$$

Finally, J_2 (Table 13) can be chosen equal to the ideal I_1:

Table 13. The ideal J_2.

J_2	$J_2(2) = I_1(2)$
	$J_2(3) = [\,]$
	$J_2(5) = I_1(5)$

Construction of J_3: Let us consider the non-zero two-degree blocks of M not involved in the construction of J_1 and J_2. Since $I_4(2) = [\,]$, we choose:

$$J_3(2) = I_2(2).$$

In order to compute $J_3(3)$, let us consider the set:

$$S_{2,3} = \{I_i(3) \setminus \mathrm{Shad}(J_3(2)), i = 2,4\}.$$

One has:

$$I_2(3) \setminus \mathrm{Shad}(J_3(2)) = \{x_2^3, x_2^2x_3, x_2^2x_4\},$$

$$I_4(3) \setminus \mathrm{Shad}(J_3(2)) = \{x_2^3, x_2^2x_3, x_2^2x_4, x_2x_3^2, x_2x_3x_4, x_2x_4^2\} := \overline{I_4(3)} \supset I_2(3) \setminus \mathrm{Shad}(J_3(2)).$$

Moreover, the greatest monomial of S *of degree three with* $\mathrm{m}(u_{3,3}) \leq k_2 + 1 = 4$ *not belonging to* $\mathrm{Shad}(J_3(2))$ *is* $u_{3,3} = x_2^3 = \max(I_4(3) \setminus \mathrm{Shad}(J_3(2)))$*. Let:*

$$J_3(3) = \overline{I_4(3)}.$$

Note that, setting:

$$\overline{\overline{I_4(3)}} = I_4(3) \setminus \overline{I_4(3)} = \{x_1^3, x_1^2x_2, x_1^2x_3, x_1^2x_4, x_1x_2^2, x_1x_2x_3, x_1x_2x_4, x_1x_3^2, x_1x_3x_4, x_1x_4^2\},$$

then $\overline{\overline{I_4(3)}} \cup I_2(3)$ *is a segment of degree three. As we will see, it comes into play in the characterization of the three-degree generators of the ideal* J_4*.*

In order to determine $J_3(5)$*, we have to analyze the following set:*

$$S_{3,5} = \left\{ I_i(5) \setminus \left(\mathrm{Shad}^3(J_3(2)) \cup \mathrm{Shad}^2(J_3(3) \right), i \in \{2,4\} \right\},$$

Note that $I_2(5)$ *and* $I_4(5)$ *are the only five-degree blocks of* M *not yet involved in the construction of* J_1 *and* J_2*.* $S_{3,5} \neq \emptyset$*. Indeed,* $I_i(5) \setminus \left(\mathrm{Shad}^3(J_3(2)) \cup \mathrm{Shad}^2(J_3(3) \right) = \{x_3^5\}$*, for* $i = 2, 4$*, and* x_3^5 *is the greatest monomial of* S *of degree five with* $\mathrm{m}(u_{3,5}) \leq k_3 + 1 = 3$ *not belonging to* $\mathrm{Shad}^3(J_3(2)) \cup \mathrm{Shad}^2(J_3(3)$*. Hence, we set:*

$$J_3(5) = I_4(5),$$

and J_3 *is shown in Table 14:*

Table 14. The ideal J_3.

J_3	$J_3(2) = I_2(2)$
	$J_2(3) = \overline{I_4(3)}$
	$J_3(5) = I_4(5)$

Construction of J_4*: We note that the only two-degree block not involved in the construction of* J_1*,* J_2 *and* J_3 *is* $I_4(2)$*, which is empty. Hence, we set:*

$$J_4(2) = [\,] = I_4(2).$$

Moreover, since the set $S_{4,3} = \emptyset$ *and* $\widetilde{S}_{4,3} = \{\overline{\overline{I_4(3)}} \cup I_2(3)\}$*, let:*

$$J_4(3) = \overline{\overline{I_4(3)}} \cup I_2(3),$$

and:

$$J_4(5) = I_2(5).$$

Indeed, $I_2(5) \setminus \mathrm{Shad}((\overline{\overline{I_4(3)}} \cup I_2(3))) = I_2(5)$*. More in detail, the ideal* J_4 *is described in Table 15:*

Table 15. The ideal J_4.

J_4	$J_4(2) = [\,]$
	$J_4(3) = \overline{\overline{I_4(3)}} \cup I_2(3)$
	$J_4(5) = I_2(5)$

We have obtained a monomial submodule $\widetilde{M} = \oplus_{i=1}^4 J_ie_i$ *of* S^4*, where the ideals* $J_i \in \mathcal{D}(M)$ *(*$i = 1, \ldots, 4$*) are described in Table 16:*

Table 16. The SBF submodule $\widetilde{M}$.

J_1	$J_1(2) = I_3(2)$ $J_1(3) = I_3(3)$ $J_1(5) = I_3(5) = [\,]$
J_2	$J_2(2) = I_1(2)$ $J_2(3) = I_1(3) = [\,]$ $J_2(5) = I_1(5)$
J_3	$J_3(2) = I_2(2)$ $J_3(3) = \{x_2^3, x_2^2x_3, x_2^2x_4, x_2x_3^2, x_2x_3x_4, x_2x_4^2\}$ $J_3(5) = I_4(5)$
J_4	$J_4(2) = I_4(2) = [\,]$ $J_4(3) = \{x_1^3, x_1^2x_2, x_1^2x_3, x_1^2x_4, x_1x_2^2, x_1x_2x_3, x_1x_2x_4, x_1x_3^2, x_1x_3x_4, x_1x_4^2, x_2^3, x_2^2x_3, x_2^2x_4\}$ $I_4(5) = I_2(5)$

Moreover, $\mathrm{Corn}(M) = \mathrm{Corn}(\widetilde{M})$, $b(M) = b(\widetilde{M})$, *and the corner matrix of* $\widetilde{M}$ *is the following:*

$$C_{\widetilde{M}} = \begin{pmatrix} 2 & 2 & 1 & 0 \\ 2 & 0 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

Example 8. Let $S = K[x_1, \ldots, x_6]$. Set $k_1 = 5$, $k_2 = 3$, $k_3 = 2$, and $\ell_1 = 3$, $\ell_2 = 4$, $\ell_3 = 5$, and $\mathcal{C} = \{(k_1, \ell_1), (k_2, \ell_2), (k_3, \ell_3)\}$. Consider the monomial submodule $M = \oplus_{i=1}^{5} I_i e_i$ of S^3 in Table 17:

Table 17. The non-SBF submodule M.

I_1	$I_1(3) = \{x_1^3, x_1^2x_2, x_1^2x_3\}$ $I_1(4) = [\,]$ $I_1(5) = [\,]$
I_2	$I_2(3) = \{x_1^3, x_1^2x_2, x_1^2x_3, x_1^2x_4, x_1^2x_5, x_1^2x_6, x_1x_2^2, x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_1x_2x_6\}$ $I_2(4) = \{x_1x_3^3, x_1x_3^2x_4, x_1x_3x_4^2\}$ $I_2(5) = [\,]$
I_3	$I_3(3) = [\,]$ $I_3(4) = \{x_1^4, x_1^3x_2\}$ $I_3(5) = [\,]$
I_4	$I_4(3) = \{x_1^3, x_1^2x_2, x_1^2x_3, x_1^2x_4, x_1^2x_5, x_1^2x_6\}$ $I_4(4) = [\,]$ $I_4(5) = \{x_1x_2^4, x_1x_2^3x_3, x_1x_2^2x_3^2, x_1x_2x_3^3, x_1x_3^4, x_2^5, x_2^4x_3\}$
I_5	$I_5(3) = 0$ $I_5(4) = \{x_1^4, x_1^3x_2, x_1^3x_3, x_1^3x_4, x_1^2x_2^2, x_1^2x_2x_3, x_1^2x_2x_4, x_1^2x_3^2, x_1^2x_3x_4, x_1^2x_4^2, x_1x_2^3, x_1x_2^2x_3,$ $x_1x_2^2x_4, x_1x_2x_3^2, x_1x_2x_3x_4, x_1x_2x_4^2, x_1x_3^3, x_1x_3^2x_4, x_1x_3x_4^2, x_1x_4^3\}$ $I_5(5) = \{x_2^5, x_2^4x_3, x_2^3x_3^2\}$

M is a strongly stable submodule with $\mathrm{Corn}(M) = \mathcal{C}$, $b(M) = (3, 12, 7)$ and:

$$C_M = \begin{pmatrix} 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 10 \\ 0 & 0 & 0 & 5 & 2 \end{pmatrix}.$$

Indeed, the ideals I_1 and I_3 do not give any contribution to the computation of the extremal Betti numbers of M, i.e., $\mathrm{Corn}(\mathcal{D}(M)) = \{I_2, I_4, I_5\} \subset \mathcal{D}(M)$.

Using Theorem 1, we construct a monomial module $\widetilde{M} = \oplus_{i=1}^{5} J_i e_i$, with $\mathrm{Corn}(\widetilde{M}) = \mathrm{Corn}(M)$ and $b(\widetilde{M}) = b(M)$.

First, let:

$$J_1 = J_2 = \langle x_6^2 \rangle.$$

Therefore, in order to construct J_3, J_4, J_5, we manage the blocks and sub-blocks of the ideals in $\mathrm{Corn}(\mathcal{D}(M)) = \{I_2, I_4, I_5\}$. More specifically, when we speak about the blocks (or sub-blocks) of M, we refer to the blocks (or sub-blocks) of the corner ideals I_2, I_4, I_5.

Construction of J_3: Let us consider the three-degree blocks of M with the greatest cardinality. From Table 17, there exists $I_2(3)$ such that $|I_2(3)| > |I_4(3)| > |I_5(3)|$. Hence, let:

$$J_3(3) = I_2(3).$$

Now, let us consider the set of monomials:

$$S_{3,4} = \{I_i(4) \setminus \mathrm{Shad}(J_3(3)), i = 2,4,5\}.$$

One can observe that $I_4(4) \setminus \mathrm{Shad}(J_3(3)) = \emptyset$; whereas:

$$I_2(4) \setminus \mathrm{Shad}(J_3(3)) = I_2(4) = \{x_1x_3^3, x_1x_3^2x_4, x_1x_3x_4^2\},$$

$$I_5(4) \setminus \mathrm{Shad}(J_3(3)) = \{x_1x_3^3, x_1x_3^2x_4, x_1x_3x_4^2, x_1x_4^3\} := \overline{I_5(4)}.$$

Denote by $u_{3,3}$ the greatest monomial of S of degree three with $\mathrm{m}(u_{3,3}) \leq k_2 + 1 = 4$ not belonging to $\mathrm{Shad}(J_3(3))$. It is $u_{3,3} = x_1x_3^3 = \max \overline{I_5(4)}$.

Let

$$J_3(4) = \overline{I_5(4)}.$$

Note that, setting:

$$\overline{\overline{I_5(4)}} := I_5(4) \setminus \overline{I_5(4)} = \{x_1^4, x_1^3x_2, x_1^3x_3, x_1^3x_4, x_1^2x_2^2, x_1^2x_2x_3, x_1^2x_2x_4, x_1^2x_3^2,$$

$$x_1^2x_3x_4, x_1^2x_4^2, x_1x_2^3, x_1x_2^2x_3, x_1x_2^2x_4, x_1x_2x_3^2, x_1x_2x_3x_4, x_1x_2x_4^2\}$$

the set $\overline{\overline{I_5(4)}} \cup I_2(4)$ is a segment of degree four. It will come into play in the characterization of the four-degree generators of the ideals J_4, J_5.

In order to construct $J_3(5)$, let us consider the set:

$$S_{3,5} = \left\{I_i(5) \setminus \left(\mathrm{Shad}^2(J_3(3)) \cup \mathrm{Shad}(J_3(4)\right), i = 2,4,5\right\}.$$

One has:

$$I_2(5) \setminus \left(\mathrm{Shad}^2(J_3(3)) \cup \mathrm{Shad}(J_3(4)\right) = \emptyset,$$

$$I_4(5) \setminus \left(\mathrm{Shad}^2(J_3(3)) \cup \mathrm{Shad}(J_3(4)\right) = \{x_2^5, x_2^4x_3\} := \overline{I_4(5)},$$

$$I_5(5) \setminus \left(\mathrm{Shad}^2(J_3(3)) \cup \mathrm{Shad}(J_3(4)\right) = \{x_2^5, x_2^4x_3, x_2^3x_3^2\} = I_5(5).$$

Hence, since the greatest monomial of S of degree five with $\mathrm{m}(u_{3,5}) \leq k_3 + 1 = 3$ not belonging to $\mathrm{Shad}^2(J_3(3)) \cup \mathrm{Shad}(J_3(4)$ is $u_{3,5} = x_2^5 = \max(I_5(5) \setminus (\mathrm{Shad}^2(J_3(3)) \cup \mathrm{Shad}(J_3(4))$, we set:

$$J_3(5) = I_5(5).$$

Observe that:

$$\overline{\overline{I_4(5)}} := I_4(5) \setminus \overline{I_4(5)} = \{x_1x_2^4, x_1x_2^3x_3, x_1x_2^2x_3^2, x_1x_2x_3^3, x_1x_3^4\} \tag{9}$$

will be used for the construction of the five-degree generators of J_4 and J_5. Table 18 represents the finitely generated Borel ideal J_3:

Table 18. The ideal J_3.

J_3	$J_3(3) = I_2(3)$
	$J_3(4) = \overline{I_5(4)}$
	$J_3(5) = I_5(5)$

Construction of J_4: Let us consider the non-zero three-degree blocks of M not involved in the construction of J_3, i.e., $I_i(3)$, for $i = 2,4$. Since $|I_4(3)| > |I_2(3)|$, let:

$$J_4(3) = I_4(3).$$

In order to determine the four-degree generators of J_4, we have to take into account the sets:

$$S_{4,4} = \{I_i(4) \setminus \mathrm{Shad}(J_4(3)), i = 2,4\},$$

$$\widetilde{S}_{4,4} = S_{4,4} \cup \{\overline{\overline{I_5(4)}} \cup I_2(4) \setminus \mathrm{Shad}(J_4(3))\}.$$

Let $u_{4,4}$ be the greatest monomial of S of degree four with $\mathrm{m}(u_{4,4}) \leq k_2 + 1 = 4$ not belonging to $\mathrm{Shad}(J_4(3))$. It is:

$$u_{4,4} = x_1x_2^3 = \max((\overline{\overline{I_5(4)}} \cup I_2(4)) \setminus \mathrm{Shad}(J_4(3))).$$

Hence, setting:

$$\widetilde{X} = (\overline{\overline{I_5(4)}} \cup I_2(4)) \setminus \mathrm{Shad}(J_4(3))\} = \{x_1x_2^3, x_1x_2^2x_3, x_1x_2^2x_4, x_1x_2x_3^2, x_1x_2x_3x_4, x_1x_2x_4^2\},$$

Let:

$$J_4(4) = \widetilde{X}.$$

Note that the set:

$$\widetilde{Y} = \left(\overline{\overline{I_5(4)}} \cup I_2(4)\right) \setminus \widetilde{X} = \{x_1^4, x_1^3x_2, x_1^3x_3, x_1^3x_4, x_1^2x_2^2, x_1^2x_2x_3, x_1^2x_2x_4, x_1^2x_3^2, x_1^2x_3x_4, x_1^2x_4^2\} \tag{10}$$

is a segment of degree four, which comes into play for determine the four-degree generators of J_5.

In order to construct $J_4(5)$, we consider the following set:

$$S_{4,5} = \left\{I_i(5) \setminus \left(\mathrm{Shad}^2(J_4(3)) \cup \mathrm{Shad}(J_4(4)\right), i \in \{2,4\}\right\}.$$

Since, $I_2(5) \setminus \left(\mathrm{Shad}^2(J_4(3)) \cup \mathrm{Shad}(J_4(4)\right) = \emptyset$,

$$I_4(5) \setminus \left(\mathrm{Shad}^2(J_4(3)) \cup \mathrm{Shad}(J_4(4)\right) = \{x_2^5, x_2^4x_3\} := \overline{I_4(5)},$$

and moreover, $\max(I_4(5) \setminus (\mathrm{Shad}^2(J_4(3)) \cup \mathrm{Shad}(J_4(4))) = x_2^5$ is the greatest monomial of S of degree five with $\mathrm{m}(u_{4,5}) \leq k_3 + 1 = 3$ not belonging to $\mathrm{Shad}^2(J_4(3)) \cup \mathrm{Shad}(J_4(4))$; let:

$$J_4(5) = \overline{I_4(5)}.$$

Finally, Table 19 represents J_2:

Table 19. The ideal J_2.

J_4	$J_4(3) = I_4(3)$ $J_4(4) = \widetilde{X}$ $J_4(5) = \overline{I_4(5)}$

Construction of J_5: In order to determine the three-degree generators (four-degree generators, five-degree generators, respectively) of J_5, we have to consider the non-zero three-degree blocks (four-degree blocks, five-degree blocks, respectively) of M not involved in the construction of J_3 and J_4, and moreover, in the case of the $\{4,5\}$-degree generators we should also consider the sub-blocks arising during the construction of J_3 (see (9), (10)).

Hence, J_5 is described in Table 20:

Table 20. The ideal J_5.

J_5	$J_5(3) = [\,]$ $J_5(4) = \widetilde{Y}$ $J_5(5) = \overline{\overline{I_4(5)}}$

We have obtained a monomial submodule $\widetilde{M} = \oplus_{i=1}^{5} J_i e_i$ of S^5 (Table 21) , where the ideals $J_i \in \mathcal{D}(M)$ $(i = 1, \ldots, 4)$ are:

Table 21. The SBF submodule $\widetilde{M}$.

J_1	$\langle x_6^2 \rangle$
J_2	$\langle x_6^2 \rangle$
J_3	$J_3(3) = I_2(3)$ $J_3(4) = \{x_1x_3^3, x_1x_3^2x_4, x_1x_3x_4^2, x_1x_4^3\}$ $J_3(5) = I_5(5)$
J_4	$J_4(3) = I_4(3)$ $J_4(4) = \{x_1x_2^3, x_1x_2^2x_3, x_1x_2^2x_3, x_1x_2^2x_4, x_1x_2x_3^2, x_1x_2x_3x_4, x_1x_2x_4^2\}$ $J_4(5) = \{x_2^5, x_2^4x_3\}$
J_5	$J_5(3) = I_3(3) = [\,]$ $J_5(4) = \{x_1^4, x_1^3x_2, x_1^3x_3, x_1^3x_4, x_1^2x_2^2, x_1^2x_2x_3, x_1^2x_2x_4, x_1^2x_3^2, x_1^2x_3x_4, x_1^2x_4^2\}$ $J_5(5) = \{x_1x_2^4, x_1x_2^3x_3, x_1x_2^2x_3^2, x_1x_2x_3^3, x_1x_3^4\}$

$\widetilde{M}$ is an SBF submodule of S^5 generated in degrees $2,3,4,5$ with $\mathrm{Corn}(M) = \mathrm{Corn}(\widetilde{M})$ and $b(M) = b(\widetilde{M})$. Indeed, the corner matrix of $\widetilde{M}$ is:

$$C_{\widetilde{M}} = \begin{pmatrix} 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 5 & 4 \\ 0 & 0 & 2 & 1 & 4 \end{pmatrix}.$$

5. Conclusions and Perspectives

In this paper, given a strongly stable submodule M of the finitely generated graded free S-module S^m, $m \geq 1$, we have constructed an SBF submodule of S^m preserving the extremal Betti numbers (values, as well as positions) of M. Due to Theorem 1 and taking into account what has been done in [1], Theorem 4.6, we are able to obtain a numerical characterization of all possible extremal Betti numbers of any SBF submodule of a finitely generated graded free S-module S^m.

Remarkably, the constructive nature of the main theorem proved in this paper (Theorem 1) may allow for the implementation of a symbolic package ([15,16]) doing almost automatically all the lengthy and tedious calculations involved. Work in this direction is in progress.

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