Dirichlet Space Couplets Encode Higher Dimensions by Clifford Motor Operations
Article
Coincidence Points of a Sequence of Multivalued Mappings in Metric Space with a Graph

Muhammad Nouman Aslam Khan 1,2,†, Akbar Azam 2,† and Nayyar Mehmood 3,*‡

1 School of Chemical and Materials Engineering, National University of Sciences and Technology, H-12, Islamabad 44000, Pakistan; mnouman@scme.nust.edu.pk
2 Department of Mathematics, COMSATS Institute of Information Technology, Chak Shahzad, Islamabad 44000, Pakistan; akbarazam@yahoo.com
3 Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad 44000, Pakistan
* Correspondence: nayyar.mehmood@iiu.edu.pk; Tel.: +92-51-9019-948
† These authors contributed equally to this work.

Academic Editor: Lokenath Debnath
Received: 24 March 2017; Accepted: 22 May 2017; Published: 26 May 2017

Abstract: In this article the coincidence points of a self map and a sequence of multivalued maps are found in the settings of complete metric space endowed with a graph. A novel result of Asrifa and Vetrivel is generalized and as an application we obtain an existence theorem for a special type of fractional integral equation. Moreover, we establish a result on the convergence of successive approximation of a system of Bernstein operators on a Banach space.

Keywords: graphic contraction; coincidence points; sequence of multivalued maps; Bernstein operators

1. Introduction and Preliminaries

For the metric space \((X,d)\), using the notions of Nadler [1] and Hu [2], denote \(CB(X), C(X)\) and \(2^X\) by the collection of nonempty closed and bounded, compact and all nonempty subsets of \(X\) respectively. Consider \(A, B \in CB(X)\) the distance between sets \(A\) and \(B\) is defined by

\[
d(A,B) = \inf_{x \in A, y \in B} d(x,y),
\]

which does not allow to enjoy the properties of metric on \(CB(X)\) therefore a well known idea of Hausdorff–Pompeiu distance \(H\) on \(CB(X)\) induced by \(d\) is used to define a metric on \(CB(X)\) as follows:

\[
H(A,B) = \inf\{\epsilon > 0 : A \subseteq N(\epsilon, B), B \subseteq N(\epsilon, A)\},
\]

where:

\[
N(\epsilon, A) = \{x \in X : d(x, a) < \epsilon, \text{ for some } a \in A\}.
\]

In 1969, Nadler [1] proved fixed point results for multivalued mappings in complete metric spaces, using the Hausdorff distance \(H\), which was the generalization of Banach contraction principle in the settings of set-valued mappings. Covitz and Nadler [3] extended the idea of multivalued mappings in the generalized metric spaces. Reich [4] in 1972 published a fixed point result for the multivalued maps on the compact subsets of a complete metric space and posed the question, “can \(C(X)\) be replaced by \(CB(X)\)?”. In 1989, Mizoguchi and Takahashi answered this question in Theorem 5 of [5] and they also provide some Caristi type theorems for multivalued operators. Whereas Hu [2] in 1980 extended the multivalued fixed point results from complete metric space to complete \(\epsilon\)-chainable metric space. Azam and Arshad [6] have extended the Theorem 6 of [1] by finding the fixed points of a sequence of locally contractive multivalued maps in \(\epsilon\)-chainable metric space. Further Feng and Liu [7] used...
the concept of lower semi-continuity and a generalized contractive condition to extend the result of Nadler [1] and Caristi type theorems as defined in [5]. For more references the readers are referred to the work of Cirić [8], Klim and Wardowski [9,10], Nicolae [11].

Jachymski [12] in 2007 unified and extended the work of Nieto [13] and Ran and Reuring [14] by defining a new class of contractions (G-contraction) on metric space \((X,d)\) endowed with a graph. The connectivity of the graph brings more attractions regarding a necessary and sufficient condition for any G-contractive operator to be a Picard operator.

In the present article, fascinated by [6] the existence of coincidence points of a sequence of multivalued maps with a self map are taken into account with a generalized form of G-contraction. This provides a new way to generalize many existing results in the literature (see [1,6] and the references therein).

Let us recall some definitions from graph theory with the perspective of using them in our ideas and results. For a metric space \((X,d)\) let \(\triangle\) be the diagonal of the Cartesian product \(X \times X\). Consider a directed graph \(G\) such that \(X = V(G)\), where \(V(G)\) is the set of vertices of \(G\). The set \(E(G)\) of edges of \(G\) contains all the loops. If \(G\) has no parallel edge then we can identify \(G\) with the pair \((V(G), E(G))\).

Further, the graph \(G\) can be dealt with as a weighted graph if each edge is assigned by the distance between its edges.

Consider a directed graph \(G\), then \(G^{-1}\) denote the graph obtained from \(G\) by reversing the direction of edges and if we ignore the direction of edges in graph \(G\) we get an undirected graph \(\tilde{G}\). The pair \((V', E')\) is said to be a subgraph of \(G\) if \(V' \subseteq V(G)\) and \(E' \subseteq E(G)\) and for any edge \((a,b) \in E'\) for all \(a, b \in V'\).

Recall some fundamental definitions regarding the connectivity of graphs, which can be found in [15].

**Definition 1.** A path in \(G\) from the vertex \(p\) to \(q\) of length \(K\), is a sequence \(\{p_i\}\) of \(K + 1\) vertices such that \(p_0 = p, \ldots, p_K = q\) and \((p_{j-1}, p_j) \in E(G)\) for \(j = 1, 2, \ldots, K\).

**Definition 2.** A graph \(G\) is called connected if there is a path between any two vertices. Graph \(G\) is weakly connected if \(\tilde{G}\) is connected.

**Definition 3.** For \(a, b\) and \(c\) in \(V(G)\), \([a]_G\) denote the equivalence class of the relation \(\sim\) defined on \(V(G)\) by the rule:

\[ b \sim c \text{ if there is a path in } G \text{ from } b \text{ to } c. \]

For \(v \in V(G)\) and \(K \in \mathbb{N} \cup \{0\}\) by \([v]_G^K\) we denote the set

\[ [v]_G^K := \{ u \in V(G) : \text{there is a path of length } K \text{ from } v \text{ to } u \}. \]

Following is the definition of G-contraction by Jachymski [12].

**Definition 4.** [12] Let \((X,d)\) be a metric space endowed with a graph \(G\). We say that a mapping \(T : X \to X\) is a G-contraction if \(T\) preserves edges of \(G\) i.e.,

\[ \forall (x,y) \in E(G) \Rightarrow (Tx, Ty) \in E(G), \]

and there exists some \(\alpha \in [0, 1)\) such that:

\[ \forall (x,y) \in E(G) \Rightarrow d(Tx, Ty) \leq \alpha d(x,y). \]

Mizoguchi and Takahashi [5] had defined a MT-function as follows:
Definition 5. [16] A function \( \varphi: [0, +\infty) \to [0, 1) \) is said to be a MT-function if it satisfies Mizoguchi and Takahashi’s condition (i.e., \( \limsup_{r \to t^+} \varphi(r) < 1 \) for all \( t \in [0, +\infty) \)). Clearly, if \( \varphi: [0, +\infty) \to [0, 1) \) is a nondecreasing function or a nonincreasing function, then it is a MT-function.

Now we state some results from the basic theory of multivalued mappings.

Lemma 1. [17] Let \((X, d)\) be a metric space and \(A, B \in CB(X)\), with \(H(A, B) < \epsilon\), then for each \(a \in A\), there exists an element \(b \in B\) such that:
\[
d(a, b) < \epsilon.
\]

Lemma 2. [18] Let \((X, d)\) be a metric space and \(A, B \in CB(X)\), then for each \(a \in A\):
\[
d(a, B) \leq H(A, B).
\]

Lemma 3. [19] Let \(\{A_n\}\) be a sequence in \(CB(X)\) and there exists \(A \in CB(X)\) such that \(\lim_{n \to \infty} H(A_n, A) = 0\). If \(x_n \in A_n\) \((n = 1, 2, 3, ...)\) and there exists \(x \in X\) such that \(\lim_{n \to \infty} d(x_n, x) = 0\) then \(x \in A\).

2. Main Results

Definition 6. [20] A multivalued mapping \(F: X \to CB(X)\) is said to be Mizoguchi-Takahashi \(G\)-contraction if for all \(x, y \in X, x \neq y\) with \((x, y) \in E(G)\):

(i) \(H(F(x), F(y)) \leq \varphi(d(x, y))d(x, y)\);  
(ii) If \(u \in F(x)\) and \(v \in F(y)\) are such that \(d(u, v) \leq d(x, y)\), then \((u, v) \in E(G)\).

Motivated by the Definition 2.1 of [20], in a more general settings, we define the sequence of multivalued \(G\)-contraction as follows:

Definition 7. Let \(f : X \to X\) be a edge preserving surjection. A sequence of multivalued mappings \(\{T_q\}_{q=1}^\infty\) from \(X\) into \(CB(X)\) is said to be sequence of multivalued \(G\)-contraction if \((fu, fv) \in E(G)\), implies:
\[
H(T_q(u), T_r(v)) \leq \mu d(fu, fv)d(fu, fv), \text{ for all } q, r \in \mathbb{N}.
\]

For \(x \in T_q(u)\) and \(y \in T_r(v)\) satisfying \(d(fx, fy) \leq d(fu, fv)\) implies \((fx, fy) \in E(G)\), where \(\mu: [0, \infty) \to [0, 1)\) is a MT-function.

The next theorem provides the way to find the coincidence of a self map and a sequence of multivalued maps.

Theorem 1. Let \((X, d)\) a complete metric space, \(\{T_q\}_{q=1}^\infty\) a sequence of multivalued \(G\)-contraction from \(X\) into \(CB(X)\) and \(f : X \to X\) a surjection. If there exist \(m \in \mathbb{N}\) and \(v_0 \in X\), such that:

(i) \(T_1(v_0) \cap [fv_0^m]_G^m \neq \emptyset\);
(ii) For any sequence \(\{v_n\}\) in \(X\), if \(v_n \to v\) and \(v_n \in T_n(v_{n-1}) \cap [v_{n-1}]_G^m\) for all \(n \in \mathbb{N}\), then there exists a subsequence \(\{v_{n_k}\}\) such that \(v_{n_k}, v) \in E(G)\) for all \(k \in \mathbb{N}\).

Then \(f\) and sequence of mappings \(\{T_q\}_{q=1}^\infty\) have a coincidence point, i.e., there exists \(v^* \in X\) such that \(fv^* \in \bigcap_{q \in \mathbb{N}} T_q(v^*)\).

Proof. Choose any \(v_1 \in X\) such that \(fv_1 \in T_1(v_0) \cap [fv_0^m]_G^m\) then there exists a path from \(fv_0\) to \(fv_1\), i.e.,
\[
fv_0 = fu_0^{(1)}, \ldots, fu_{m}^{(1)} = fv_1 \in T_1(v_0), \text{ and } (fu_i^{(1)}, fu_{i+1}^{(1)}) \in E(G) \text{ for all } i = 0, 1, 2, ..., m - 1. \]
Without any loss of generality, assume that \( f u_k^{(1)} \neq f u_j^{(1)} \) for each \( k, j \in \{ 0, 1, 2, \ldots, m \} \) with \( k \neq j \).

Since \((f u_0^{(1)}, f u_1^{(1)}) \in E(G)\), so:
\[
H(T_1(u_0^{(1)}), T_2(u_1^{(1)})) \leq \mu(d(f u_0^{(1)}, f u_1^{(1)}))d(f u_0^{(1)}, f u_1^{(1)})
< \sqrt{\mu(d(f u_0^{(1)}, f u_1^{(1)}))d(f u_0^{(1)}, f u_1^{(1)})}
< d(f u_0^{(1)}, f u_1^{(1)})
\]

Rename \( f v_1 \) as \( f u_0^{(2)} \). As \( f u_0^{(2)} \in T_1(u_0^{(1)}) \), and using Lemma 1 one can find some \( f u_1^{(2)} \in T_2(u_1^{(1)}) \) such that:
\[
d(f u_0^{(2)}, f u_1^{(2)}) < d(f u_0^{(1)}, f u_1^{(1)}).
\]

Since \((f u_1^{(1)}, f u_2^{(1)}) \in E(G)\), so:
\[
H(T_2(u_1^{(1)}), T_2(u_2^{(1)})) \leq \mu(d(f u_1^{(1)}, f u_2^{(1)}))d(f u_1^{(1)}, f u_2^{(1)})
< d(f u_1^{(1)}, f u_2^{(1)}).
\]

Similarly since \( f u_1^{(2)} \in T_2(u_1^{(1)}) \), again using Lemma 1 one can find some \( f u_2^{(2)} \in T_2(u_2^{(1)}) \) such that:
\[
d(f u_1^{(2)}, f u_2^{(2)}) < d(f u_1^{(1)}, f u_2^{(1)}).
\]

Thus we obtain \( \{ f u_0^{(2)}, f u_1^{(2)}, f u_2^{(2)}, \ldots, f u_m^{(2)} \} \) of \( m + 1 \) vertices of \( X \) such that \( f u_0^{(2)} \in T_1(u_0^{(1)}) \) and \( f u_s^{(2)} \in T_2(u_s^{(1)}) \) for \( s = 1, 2, \ldots, m \), with:
\[
d(f u_s^{(2)}, f u_{s+1}^{(2)}) < d(f u_s^{(1)}, f u_{s+1}^{(1)}),
\]
for \( s = 0, 1, 2, \ldots, m - 1 \). As \((f u_s^{(1)}, f u_{s+1}^{(1)}) \in E(G)\) for all \( s = 0, 1, 2, \ldots, m - 1 \), thus \((f u_s^{(2)}, f u_{s+1}^{(2)}) \in E(G)\) for all \( s = 0, 1, 2, \ldots, m - 1 \).

Let \( f u_1^{(2)} = f v_1 \). Thus the set of points \( f v_1 = f u_0^{(2)}, f u_1^{(2)}, f u_2^{(2)}, \ldots, f u_m^{(2)} = f v_2 \in T_2(v_1) \) is a path from \( f v_1 \) to \( f v_2 \). Rename \( f v_2 \) as \( f u_0^{(3)} \). Then by the same procedure we obtain a path:
\[
f v_2 = f u_0^{(3)}, f u_1^{(3)}, f u_2^{(3)}, \ldots, f u_m^{(3)} = f v_3 \in T_3(v_2)
\]
from \( f v_2 \) to \( f v_3 \). Inductively, obtained:
\[
f v_h = f u_0^{(h+1)}, f u_1^{(h+1)}, f u_2^{(h+1)}, \ldots, f u_m^{(h+1)} = f v_{h+1} \in T_{h+1}(v_h)
\]
with:
\[
d(f u_0^{(h+1)}, f u_1^{(h+1)}) < d(f u_1^{(h)}, f u_1^{(h+1)}),
\]
\[
(2)
\]
hence \((f u_t^{(h)}, f u_t^{(h+1)}) \in E(G)\) for \( t = 0, 1, 2, \ldots, m - 1 \).

Consequently, construct a sequence \( \{ f v_h \}_{h=1}^{m+1} \) of points of \( X \) with:
\[
f v_1 = f u_0^{(1)} = f u_0^{(2)} \in T_1(v_0),
f v_2 = f u_0^{(3)} = f u_0^{(2)} \in T_2(v_1),
f v_3 = f u_0^{(4)} = f u_0^{(3)} \in T_3(v_2),
\]
\[
\vdots
\]
\[
f v_{h+1} = f u_0^{(h+1)} = f u_0^{(h+2)} \in T_{h+1}(v_h),
\]
for all \( h \in \mathbb{N} \).

For each \( t \in \{0, 1, 2, \ldots, m - 1\} \), and from (2), clearly \( \{d(fu_i^{(h)}, fu_{i+1}^{(h)})\}_{h=1}^n \) is a decreasing sequence of non-negative real numbers and so there exists \( a_t \geq 0 \) such that:

\[
\lim_{h \to \infty} d(fu_i^{(h)}, fu_{i+1}^{(h)}) = a_t.
\]

By assumption, \( \limsup_{t \to -\infty} \mu(t) < 1 \), so there exists \( k_t \in \mathbb{N} \) such that \( \mu(d(fu_i^{(h)}, fu_{i+1}^{(h)})) < \omega(a_t) \) for all \( h \geq k_t \) where \( \limsup_{t \to -\infty} \mu(t) < \omega(a_t) < 1 \).

Now put:

\[
\Theta_t = \max \left\{ \max_{r=1}^{t} \sqrt{\mu(d(fu_{t}^{(r)}, fu_{t+1}^{(r)})), \sqrt{\omega(a_t)} \right\}.
\]

Then, for every \( h > k_t \), consider:

\[
d(fu_i^{(h+1)}, fu_{i+1}^{(h+1)}) < \sqrt{\mu(d(fu_i^{(h)}, fu_{i+1}^{(h)}))d(fu_i^{(h)}, fu_{i+1}^{(h)})}
\]

\[
< \sqrt{\omega(a_t)d(fu_i^{(h)}, fu_{i+1}^{(h)})}
\]

\[
\leq \Theta_t d(fu_i^{(h)}, fu_{i+1}^{(h)})
\]

\[
\leq (\Theta_t)^2d(fu_i^{(h-1)}, fu_{i+1}^{(h-1)})
\]

\[
\leq \ldots
\]

\[
\leq (\Theta_t)^h d(fu_i^{(1)}, fu_{i+1}^{(1)}).
\]

Putting \( q = \max\{k_t : t = 0, 1, 2, \ldots, m - 1\} \), gives:

\[
d(fv_h, fv_{h+1}) = d(fu_0^{(h+1)}, fu_m^{(h+1)})
\]

\[
\leq \sum_{t=0}^{m-1} d(fu_t^{(h+1)}, fu_{t+1}^{(h+1)})
\]

\[
< \sum_{t=0}^{m-1} (\Theta_t)^h d(fu_t^{(1)}, fu_{t+1}^{(1)}) \quad \text{for all} \ h > q.
\]

Now for \( p > h > q \), consider:

\[
d(fv_h, fv_p) \leq d(fv_h, fv_{h+1}) + d(fv_{h+1}, fv_{h+2}) + \cdots + d(fv_{p-1}, fv_p)
\]

\[
< \sum_{t=0}^{m-1} (\Theta_t)^h d(fu_t^{(1)}, fu_{t+1}^{(1)}) + \cdots + \sum_{t=0}^{m-1} (\Theta_t)^p d(fu_t^{(1)}, fu_{t+1}^{(1)}).
\]

Since \( \Theta_t < 1 \) for all \( t \in \{0, 1, 2, \ldots, m - 1\} \), it follows that \( \{fv_h = fu_m^{(h)}\} \) is a Cauchy sequence. Using completeness of \( X \), find \( v^* \in X \) such that \( fv_h \to v^* \). Now using the fact that \( fv_h \in T(v_{n-1}) \cap [fv_{n-1}] \mathbb{G} \) for all \( n \in \mathbb{N} \), find a subsequence \( \{fv_{n_k}\} \) of \( \{fv_h\} \) such that \( (fv_{n_k}, f\bar{v}^*) \in E(G) \) for all \( k \in \mathbb{N} \).

Now for any \( q \in \mathbb{N} : \)

\[
d(fv^*, T_q(v^*)) \leq d(fv^*, fv_{h+1}) + d(fv_{h+1}, T_q(v^*))
\]

\[
\leq d(fv^*, fv_{h+1}) + H(T_{h+1}(v_h), T_q(v^*))
\]

\[
\leq d(fv^*, fv_{h+1}) + \mu(d(fv_h, f\bar{v}^*)) d(fv_h, f\bar{v}^*).
\]

Letting \( h \to \infty \) in the above inequality, gives \( d(fv^*, T_q(v^*)) \to 0 \), which implies \( f\bar{v}^* \in T_q(v^*) \) for all \( q \in \mathbb{N} \). Hence, \( f\bar{v}^* \in \bigcap_{q \in \mathbb{N}} T_q(v^*) \) as required.
Example 1. Let $X = \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \cup \{0\} \right\}$ for $q \in \mathbb{N}$. Consider the graph $G$ such that $V(G) = X$ and for all $x$ and $y$ in $X$:

$$E(G) = \{(x,y) : x \neq y\}.$$  

For $q \in \mathbb{N}$, let $T_q : X \to CB(X)$ be defined by:

$$T_q(x) = \begin{cases} \{0, \frac{1}{q} + 1, 1\} & \text{if } x = 0, \\ \left\{ \frac{1}{q^m} + 1, 1 \right\} & \text{if } x = \frac{1}{q^m}, n \in \mathbb{N}, \\ \left\{ \frac{1}{q^m} \right\} & \text{if } x = 1. \end{cases}$$

If we assume $f : X \to X$ as an identity map then sequence of multivalued mappings $(T_q)^\infty_{q=1}$ from $X$ into $CB(X)$ is a sequence of multivalued $G_f$-contraction.

It satisfies the conditions of Theorem 1 and $1 \in X$ is the fixed point of sequence of multivalued maps $T_q$ for $q \in \mathbb{N}$.

The next theorem provides a way to find the coincidence point of a hybrid pair.

Theorem 2. Let $(X,d)$ be a complete metric space, $T : X \to CB(X)$ and $f : X \to X$ a surjection. If $u, v \in X$ (with $u \neq v$) such that $(fu, fv) \in E(G)$, implies:

$$H(T(u), T(v)) \leq \mu(d(fu, fv))d(fu, fv),$$

(4)

where $\mu : [0, \infty) \to [0, 1)$ is a MT-function, if there exist $m \in \mathbb{N}$ and $v_0 \in X$, such that:

(i) $T(v_0) \cap [fv_0]^m_G \neq \emptyset$;

(ii) For any sequence $\{v_n\}$ in $X$, if $v_n \to v$ and $v_n = T(v_{n-1}) \cap [v_{n-1}]^m_G$ for all $n \in \mathbb{N}$ and $j = 1, 2, \ldots$, then there exists a subsequence $\{v_{n_k}\}$ such that $(v_{n_k}, v) \in E(G)$ for all $k \in \mathbb{N}$.

Then $f$ and $T$ have a coincidence point, i.e., there exists $v^* \in X$ such that $fv^* \in T(v^*)$.

Proof. Take $T_q := T$ for all $q \in \mathbb{N}$ in Theorem 1 and proof is following the same procedure. □

Corollary 1. Let $(X,d)$ be a complete metric space, $(T_q)_{q=1}^\infty$ a sequence of the self mappings on $X$ and $f : X \to X$ a surjection. If $u, v \in X$ (with $u \neq v$) such that $(fu, fv) \in E(G)$, implies:

$$d(T_q(u), T_q(v)) \leq \mu(d(fu, fv))d(fu, fv),$$

(5)

for all $q, r \in \mathbb{N}$, where $\mu : [0, \infty) \to [0, 1)$ is a MT function, if there exist $m \in \mathbb{N}$ and $v_0 \in X$, such that:

(i) $T_1(v_0) \cap [fv_0]^m_G \neq \emptyset$;

(ii) For any sequence $\{v_n\}$ in $X$, if $v_n \to v$ and $v_n = T(v_{n-1}) \cap [v_{n-1}]^m_G$ for all $n \in \mathbb{N}$,

then there exists a subsequence $\{v_{n_k}\}$ such that $(v_{n_k}, v) \in E(G)$ for all $k \in \mathbb{N}$.

Then $f$ and sequence of mappings $(T_q)_{q=1}^\infty$ have a coincidence point, i.e., there exists $v^* \in X$ such that $fv^* = \bigcap_{q \in \mathbb{N}} T_q(v^*)$.

Corollary 2. Let $(X,d)$ be a complete metric space, $T : X \to CB(X)$ and if $u, v \in X$ (with $u \neq v$) such that $(u,v) \in E(G)$, implies:

$$H(T(u), T(v)) \leq \mu(d(u,v))d(u,v),$$

(6)

where $\mu : [0, \infty) \to [0, 1)$ is a MT-function, if there exist $m \in \mathbb{N}$ and $v_0 \in X$, such that:

(i) $T(v_0) \cap [v_0]^m_G \neq \emptyset$. 

(ii) For any sequence \( \{ v_n \} \) in \( X \), if \( v_n \to v \) and \( v_n \in T(v_{n-1}) \cap [v_{n-1}]^m_G \) for all \( n \in \mathbb{N} \) and \( j = 1, 2, \ldots \), then there exists a subsequence \( \{ v_{n_k} \} \) such that \( (v_{n_k}, v) \in E(G) \) for all \( k \in \mathbb{N} \).

Then \( T \) has a fixed point, i.e., \( v^* = T(v^*) \).

The following are the consequence of the Theorem 1 and Theorem 2 for the case of self mappings.

**Corollary 3.** Let \( (X, d) \) be a complete metric space, \( T : X \to X \) and \( f : X \to X \) a surjection. If \( u, v \in X \) (with \( u \neq v \)) such that \( (f u, f v) \in E(G) \), implies:

\[
d(T(u), T(v)) \leq \mu(d(f u, f v))d(u, v),
\]

where \( \mu : [0, \infty) \to [0, 1) \) is a MT function, if there exist \( m \in \mathbb{N} \) and \( v_0 \in X \), such that:

(i) \( T(v_0) \cap [f v_0]^m_G \neq \emptyset \);

(ii) For any sequence \( \{ v_n \} \) in \( X \), if \( v_n \to v \) and \( v_n = T(v_{n-1}) \cap [v_{n-1}]^m_G \) for all \( n \in \mathbb{N} \) and \( j = 1, 2, \ldots \), then there exists a subsequence \( \{ v_{n_k} \} \) such that \( (v_{n_k}, v) \in E(G) \) for all \( k \in \mathbb{N} \).

Then \( f \) and \( T \) have a coincidence point, i.e., there exists \( v^* \in X \) such that \( f v^* = T(v^*) \).

**Corollary 4.** Let \( (X, d) \) be a complete metric space, \( T : X \to X \) and if \( u, v \in X \) (with \( u \neq v \)) such that \( (u, v) \in E(G) \), implies:

\[
d(T(u), T(v)) \leq \mu(d(u, v))d(u, v),
\]

where \( \mu : [0, \infty) \to [0, 1) \) is a MT function, if there exist \( m \in \mathbb{N} \) and \( v_0 \in X \), such that:

(i) \( T(v_0) \cap [v_0]^m_G \neq \emptyset \);

(ii) For any sequence \( \{ v_n \} \) in \( X \), if \( v_n \to v \) and \( v_n = T(v_{n-1}) \cap [v_{n-1}]^m_G \) for all \( n \in \mathbb{N} \) and \( j = 1, 2, \ldots \), then there exists a subsequence \( \{ v_{n_k} \} \) such that \( (v_{n_k}, v) \in E(G) \) for all \( k \in \mathbb{N} \).

Then \( T \) has a fixed point, i.e., \( v^* = T(v^*) \).

The next remark highlights the applications of all the above results in settings of complete metric spaces, complete metric spaces endowed with partial order and \( \epsilon \)-chainable complete metric spaces.

**Remark 1.** Consider the following cases:

R1. Let \( (X, d) \) be a complete metric space, consider the graph \( G_0 \) with:

\[
E(G_0) = X \times X.
\]

R2. Let \( (X, d) \) be a complete metric space with partial order \( \preceq \) on \( X \), consider the graphs \( G_1 \) and \( G_2 \) with:

\[
E(G_1) = \{(x, y) \in X \times X : x \preceq y \},
\]

and:

\[
E(G_2) = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x \}.
\]

R3. Let \( \epsilon > 0 \) and \( (X, d) \) be a complete \( \epsilon \)-chainable metric space, consider the graph:

\[
G_3 := \{(x, y) \in X \times X : 0 < d(x, y) < \epsilon, \text{ for } \epsilon > 0 \}.
\]

We remark that all above results are valid under the above construction of remarks (R1), (R2) and (R3).

Further, in an application of Theorem 1 we generalize the Theorem 6 of [20]. It establishes the convergence of successive approximations of operators on a Banach space, which consequently yields
the Kelisky-Rivlin theorem on iterates of Bernstein operators on the space $C(I)$, where $I$ is the closed unit interval.

**Theorem 3.** Let $X$ be a Banach space and $X_0$ be a closed subspace of $X$. Let $T, f : X → X$ be maps such that $f$ is surjection and:

$$\|Tx - Ty\| \leq \varphi(\|fx - fy\|) \|fx - fy\| \text{ whenever } fx - fy \in X_0, \; x \neq y.$$  \hspace{1cm} (9)

If $(I - f) (X) \subseteq X_0$ and $(f - T) (X) \subseteq X_0$, then for all $x \in X$, $\{T^n x\}$ converges to $\text{Coin} \{T, f\}$, where $\text{Coin} \{T, f\} = \{x \in X : Tx = fx\}$.

**Proof.** Consider the graph $G = (V(G), E(G))$ where $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : x - y \in X_0\}$. Clearly, $\Delta \subseteq E(G)$, $G = G$ and $G$ has no parallel edges. Consider $(x, y) \in E(G)$, then $fx - fy = (y - fy) - (x - fx) + (x - y) \in X_0$, since $(I - f) (X) \subseteq X_0$. Hence and by given contractive condition (9), we see that $\forall (x, y) \in E(G)$ with $x \neq y$, (6) holds. Also $Tx - Ty = (fy - Ty) - (fx - Tx) + (fx - fy) \in X_0$, since $(f - T) (X) \subseteq X_0$.

The use of $(f - T) (X) \subseteq X_0$ implies that $(fx, Tx) \in E(G)$ for $x \in X$. Therefore condition (i) of Corollary 4 holds with $x = x_0 = x_0$ and $N = 1$. Thus we are able to generate a sequence such that $Tx_{n-1} = fx_n$ for all $n \in \mathbb{N}$. Assume that $Tx_n \to v^* \in X$ but since $f$ is surjection so there exists some $v \in X$ such that $v^* = fv$. Here also $Tx_n \in [Tx_{n-1}]_G$ for all $n \in \mathbb{N}$, which implies that $(Tx_n, Tx_{n-1}) \in E(G)$ for all $n \in \mathbb{N}$. Now using the outline of the proof of Theorem 4.1 of [12], $(Tx_n, fv) \in E(G)$ for all $n \in \mathbb{N}$. Now assume:

$$\|fv - Tv\| \leq \|fv - fx_{n+1}\| + \|fx_{n+1} - Tv\| = \|fv - fx_{n+1}\| + \|Tx_n - Tv\|.$$  \hspace{1cm} (10)

Since $(Tx_n, fv) \in E(G)$ for all $n \in \mathbb{N}$, thus from (9) and (10) we have:

$$\|fv - Tv\| \leq \|fv - fx_{n+1}\| + \varphi(\|fx_n - fv\|) \|fx_n - fv\|.$$  \hspace{1cm} (11)

As $n \to \infty$, we get $fv = Tv$. Thus $v$ is the coincidence point of $f$ and $T$, by using Corollary 4. For the uniqueness of the coincidence point we let two coincidence points $u, v$ of $f$ and $T$, then:

$$\|Tu - Tv\| \leq \varphi(\|fu - fv\|) \|fu - fv\| \quad (1 - \varphi(\|fu - fv\|)) \|Tu - Tv\| \leq 0.$$  \hspace{1cm} (12)

This implies that $Tu = Tv$. \hspace{1cm} $\square$

In the next result, we discussed the generalization of fractional differential equation described in [21]. For the closed interval $I = [0, 1]$, assume function $g \in C(I, \mathbb{R})$ and $f : I \times \mathbb{R} → \mathbb{R}$ is a continuous function. The fractional differential equation is given as follows:

$$D^ax(t) + f(t, g(x(t))) = 0 \quad (0 \leq t \leq 1, \; \alpha > 1)$$  \hspace{1cm} (13)

with boundary conditions $x(0) = x(1) = 0$. It is to be noted that associated Green’s function with the problem (13) is:

$$G(t, s) = \begin{cases} \frac{(t(1-s))^{a-1} - (t-s)^{a-1}}{\Gamma(a)} & 0 \leq s \leq t \leq 1, \\ \frac{(t(1-s))^{a-1}}{\Gamma(a)} & 0 \leq t \leq s \leq 1. \end{cases}$$

where $\Gamma(.)$ represents the Gamma function.

**Theorem 4.** Consider the surjective function $g \in C(I, \mathbb{R})$ and $f : I \times \mathbb{R} → \mathbb{R}$ satisfies:

(i) $|f(s, g(x(s))) - f(s, g(y(s)))| \leq |g(x(s)) - g(y(s))|$ for all $s \in I$;
(ii) $\sup_{t \in I} \int_0^1 G(t, s) \, ds \leq k < 1.$

Then, problem (11) has a unique solution.

**Proof.** Assume space $X = C(I, \mathbb{R})$, and we have $d(x, y) = \max_{t \in [0,1]} |x(t) - y(t)|$ for $x$ and $y$ in $X$. It is well known that $x \in X$ is a solution of (11) if and only if it is a solution of the integral equation:

$$x(t) = \int_0^1 G(t, s) f(s, (gx)(s)) \, ds \text{ for all } t \in I.$$

Define the operator $F : X \to X$ by:

$$Fx(t) = \int_0^1 G(t, s) f(s, (gx)(s)) \, ds \text{ for all } t \in I,$$

and $S : X \to X$ by:

$$Sx = gx, \text{ with } (Sx)(t) = (gx)(t) \text{ for } t \in I.$$

Thus, for finding a solution of (11), it is sufficient to show that $F$ has a coincidence point with $g$. Now let $x, y \in C(I)$ for all $s \in I$. Here we have:

$$|Fx(t) - Fy(t)| = \left| \int_0^1 G(t, s) \left(f(s, (gx)(s)) - f(s, (gy)(s))\right) \, ds \right|$$

$$\leq \int_0^1 G(t, s) \left| \left(f(s, (gx)(s)) - f(s, (gy)(s))\right) \right| \, ds$$

$$\leq \int_0^1 G(t, s) \left| (gx)(s) - (gy)(s) \right| \, ds$$

$$\leq \int_0^1 G(t, s) \left| (Sx)(s) - (Sy)(s) \right| \, ds$$

$$\leq \int_0^1 G(t, s) d(Sx, Sy) \, ds$$

$$\leq d(Sx, Sy) \sup_{t \in I} \int_0^1 G(t, s) \, ds$$

$$\leq kd(Sx, Sy).$$

This implies that for each $x, y \in X$, we have:

$$d(Fx, Fy) \leq kd(Sx, Sy).$$

Now the use of Corollary 3 with graph $G = G_0$, we have $x^* \in X$ such that $Fx^* = Sx^*$ with $(Sx^*)(t) = (gx^*)(t)$ for $t \in I$. Thus $x^*$ is the required coincidence point of $F$ and $g$. \qed

**Author Contributions:** All authors contributed equally to the writing of this paper. All authors read and approve the final manuscript.
Conflicts of Interest: The authors declare no conflict of interest.

References

© 2017 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).