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On Autonomy Imposition in Zero Interval Limit Perturbation Expansion for the Spectral Entities of Hilbert–Schmidt Integral Operators [†]

Süha Tuna * and Metin Demiralp

Informatics Institute, İstanbul Technical University, Ayazağa Campus, İstanbul 34469, Turkey; metin.demiralp@gmail.com

* Correspondence: suhatuna@itu.edu.tr; Tel.: +90-212-285-7194

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Abstract: In this work, we deal with the autonomy issue in the perturbation expansion for the eigenfunctions of a compact Hilbert–Schmidt integral operator. Here, the autonomy points to the perturbation expansion coefficients of the relevant eigenfunction not depending on the perturbation parameter explicitly, but the dependence on this parameter arises from the coordinate change at the zero interval limit. Moreover, the related half interval length is utilized as the perturbation parameter in the perturbative analyses. Thus, the zero interval limit perturbation for solving the eigenproblem under consideration is developed. The aim of this work is to show that the autonomy imposition brings an important restriction on the kernel of the corresponding integral operator, and the constructed perturbation series is not capable of expressing the exact solution approximately unless a specific type of kernel is considered. The general structure for the encountered constraints is revealed, and the specific class of kernels is identified to this end. Error analysis of the developed method is given. These analyses are also supported by certain illustrative implementations involving the kernels, which are and are not in the specific class addressed above. Thus, the efficiency of the developed method is shown, and the relevant analyses are confirmed.

Keywords: Hilbert–Schmidt integral operators; autonomy; eigenvalues; eigenfunctions; perturbation expansions; zero interval limit

MSC: 45P05; 47A55; 47A75

1. Introduction

Integral equations and integral operators are considered in one of the fundamental study areas in science and engineering. They can be encountered in chemical and physical applications [1–3], as well as in multivariate function approximation problems, such as Enhanced Multivariate Products Representation (EMPR) [4,5]. In EMPR, univariate elements, called support functions [4], are utilized in order to realize efficient approximations to an analytic multivariate function (the bivariate case is considered in [4]). Optimization of these support functions gains great importance to this end. Through the optimization process of these support functions, two spectral problems of two distinct integral operators both involving symmetric, thus self-adjoint (Hermitian) bivariate kernels are encountered. If these spectral problems are solved, the eigenfunctions accompanying the corresponding most dominant eigenvalues, possessed by both integral operators, can be assessed as the optimized univariate support

functions [4]. Thus, an efficient EMPR approximation to the bivariate function under consideration is obtained with the utilization of these support functions.

In order to solve the above-mentioned spectral problems, each of which belongs to an integral operator of Hilbert–Schmidt type, a numerical method based on the perturbation expansion involving the autonomy on perturbation coefficients of the relevant eigenfunctions is taken into consideration. This autonomy imposition brings quite a high level of simplicity in determining the perturbation entities even for the most dominant eigenvalue and for its accompanying eigenfunction of the Hilbert–Schmidt integral operator under consideration. On the other hand, this enforcement leads us to some restrictive equations, which involve some constraints amongst the perturbation terms of the kernel to be dealt with. In this paper, we focus on these issues and find a class of kernels for which the proposed method is suitable.

The content of the present paper is as follows. First, brief information about integral operators is given in the second section. Then, the details of the autonomy imposed zero interval limit perturbation expansion for the eigenpairs of the Hilbert–Schmidt integral operators are explained in the third section. The above-mentioned constraints encountered in almost every perturbation step are formulated in the fourth section. In addition to this, the fourth section contains the proof for the failure of the autonomy imposition on the eigenfunction perturbation coefficients, while the error analysis of the relevant approximation for the eigenfunction under consideration is given in the fifth one. The efficiency of the developed method for approximating the most dominant eigenvalue and its accompanying eigenfunction is confirmed via numerical implementations in the sixth section even for the kernels that do not satisfy the above-mentioned constraints and for the special type of kernels satisfying the restrictive equations revealed in the fourth section. The paper is finalized with the concluding remarks and discussions.

2. Integral Operators and Their Spectral Properties

Consider the following operator:

$$\mathcal{I}g(x) \equiv \int_a^b d\xi K(x, \xi) g(\xi) \quad (1)$$

where $K(x, \xi)$ is an analytic and bounded bivariate function whose domain is $[a, b]^2$ and will be called from now on “the kernel of the operator in Equation (1)”. On the other hand, g stands as a univariate and analytic function defined on the interval $[a, b]$. It is obvious that the operator \mathcal{I} in Equation (1) is a linear mathematical object and, in particular, is called “an integral operator” whose kernel is $K(x, \xi)$ [6–8]. Since $K(x, \xi)$ is continuous, hence bounded on $[a, b]^2$, the integral operator in Equation (1) is called the Hilbert–Schmidt integral operator. Hilbert–Schmidt integral operators are compact operators because of their boundedness property [6–11].

If an integral operator whose kernel is $K(x, \xi)$ satisfies the equation:

$$\int_a^b d\xi K(x, \xi) \psi(\xi) = \lambda \psi(x); \quad x, \xi \in [a, b] \quad (2)$$

where ψ is an unknown univariate function, while scalar λ is again an unknown. Then, the problem in Equation (2) is called “the spectral or the eigenvalue problem of the corresponding integral operator” [6–8]. Thus, the scalar λ is named “the eigenvalue” and ψ is called “the relevant eigenfunction” of the integral operator under consideration. If the kernel of the integral operator is symmetric, that is, $K(x, \xi) = K(\xi, x)$ for all x 's and ξ 's residing in the domain of the integral operator, then the corresponding integral operator is called a self-adjoint operator [6]. As an analogy to the classical linear algebra, all eigenvalues of a self-adjoint integral operator are real, which imply that the spectrum of the relevant operator is located on the real axis [6]. Moreover, in many methods based on the spectral properties of the linear operators, such as spectral decomposition [12] and principal

component analysis [13], the most important component of the spectrum is the greatest eigenvalue in absolute value sense and therefore its accompanying eigenfunction (eigenvector in matrix case). Thus, obtaining these entities even using an analytical or a numerical method accurately becomes crucial to this end.

3. Perturbation Scheme at the Zero Interval Limit

Consider the following equation:

$$\frac{1}{b-a} \int_a^b d\xi K(x, \xi) \psi(\xi) = \lambda \psi(x); \quad x, \xi \in [a, b] \tag{3}$$

where the operator in the focus is a Hilbert–Schmidt integral operator. By the brief knowledge given in the previous section, it is possible to say that Equation (3) defines a spectral problem of an integral operator whose kernel is $K(x, \xi)$. Then, λ and $\psi(x)$ can be named “the eigenvalue and its corresponding eigenfunction” of the relevant integral operator, respectively. In order to develop a general numerical solution method for the eigenproblem in Equation (3), a universal interval should be considered instead of the existing one for the problem in Equation (3). For this purpose, the following affine transformations can be put to use:

$$x \equiv \frac{a+b}{2} + \frac{b-a}{2} y, \quad \xi \equiv \frac{a+b}{2} + \frac{b-a}{2} \eta \tag{4}$$

It is obvious that the transformations in Equation (4) take any closed and bounded $[a, b]$ interval to $[-1, 1]$. By introducing the following shorthand notations:

$$x_{mp} \equiv \frac{a+b}{2}, \quad \varepsilon \equiv \frac{b-a}{2} \tag{5}$$

and applying the transformations in Equation (4) to Equation (3):

$$\frac{1}{2} \int_{-1}^1 d\eta K(x_{mp} + \varepsilon y, x_{mp} + \varepsilon \eta) \psi(x_{mp} + \varepsilon \eta) = \lambda(\varepsilon) \psi(x_{mp} + \varepsilon y) \tag{6}$$

is obtained. It is clear that the interval of the integral operator is transformed into a universalized bounded one, that is $[-1, 1]$. On the other hand, the a and b dependencies of the corresponding integral operator are transferred to its kernel accordingly. Thus, the integral operator and the related spectral problem in Equation (6) can be named the universalized Hilbert–Schmidt integral operator and universalized spectral problem, respectively.

In Equation (6), if ε is considered as a small positive entity whose value is close to zero, which also means that the original interval $[a, b]$ is small enough, then it becomes convenient to consider it as a perturbation parameter [9–11]. Beyond that, x_{mp} stands as the midpoint of the interval to be worked on and as an important issue that will be discussed a bit later. Thus, by assuming that the unknowns λ and ψ are analytic in the vicinity of x_{mp} , a perturbation equation can be achieved with the help of relevant infinite series, including the non-negative powers of ε . Therefore, since the bivariate kernel in Equation (6) is analytic in the vicinity of (x_{mp}, x_{mp}) , it can be expanded to a series in terms of the non-negative powers of the perturbation parameter ε , as well, as follows:

$$K(x_{mp} + \varepsilon y, x_{mp} + \varepsilon \eta) = \sum_{j=0}^{\infty} \sum_{k=0}^j K_{k,j-k} y^k \eta^{j-k} \varepsilon^j \tag{7}$$

where the corresponding expansion coefficients can be calculated as:

$$K_{k,j-k} \equiv \frac{1}{k!(j-k)!} \frac{\partial^j K}{\partial x^k \partial \xi^{j-k}}(x_{mp}, x_{mp}) \tag{8}$$

In other words, if the eigenvalue λ and the relevant eigenfunction ψ are assumed to be analytic in the vicinity of (x_{mp}, x_{mp}) , they can be represented as infinite series in terms of the non-negative powers of the perturbation parameter ϵ as follows:

$$\lambda(\epsilon) = \sum_{j=0}^{\infty} \lambda_j \epsilon^j \tag{9}$$

$$\psi(x_{mp} + \epsilon\eta) = \sum_{j=0}^{\infty} \psi_j \eta^j \epsilon^j, \quad \psi(x_{mp} + \epsilon y) = \sum_{j=0}^{\infty} \psi_j y^j \epsilon^j \tag{10}$$

If the infinite series in Equations (7), (9), (10) are embedded in the universalized spectral problem in Equation (6), then we have:

$$\frac{1}{2} \int_{-1}^1 d\eta \left(\sum_{j=0}^{\infty} \sum_{k=0}^j K_{k,j-k} y^k \eta^{j-k} \epsilon^j \right) \left(\sum_{j=0}^{\infty} \psi_j \eta^j \epsilon^j \right) = \left(\sum_{j=0}^{\infty} \lambda_j \epsilon^j \right) \left(\sum_{j=0}^{\infty} \psi_j y^j \epsilon^j \right) \tag{11}$$

from which the following equality can be obtained by rearranging the terms after the utilization of the Cauchy product:

$$\frac{1}{2} \int_{-1}^1 d\eta \sum_{m=0}^{\infty} \sum_{n=0}^m \sum_{k=0}^n K_{k,n-k} y^k \eta^{n-k} \psi_{m-n} \eta^{m-n} \epsilon^m = \sum_{m=0}^{\infty} \sum_{n=0}^m \lambda_n \psi_{m-n} y^{m-n} \epsilon^m \tag{12}$$

The above formula can be considered as the perturbation equation of the problem in Equation (3). Before starting to determine the unknown perturbation coefficients at the zero interval limit by considering ϵ as the perturbation parameter in Equation (12), the definite integral occurred on the left-hand side along the universalized interval with respect to η should be evaluated first. By doing so, the following equation can be obtained:

$$\frac{1}{2} \sum_{m=0}^{\infty} \sum_{n=0}^m \sum_{k=0}^n K_{k,n-k} \psi_{m-n} I_{m-k} y^k \epsilon^m = \sum_{m=0}^{\infty} \sum_{n=0}^m \lambda_n \psi_{m-n} y^{m-n} \epsilon^m \tag{13}$$

where:

$$I_n \equiv \frac{1 + (-1)^n}{n + 1}, \quad n = 0, 1, \dots \tag{14}$$

Thus, the equation in Equation (13) can be identified as the final form of the relevant perturbation recursion amongst the eigenpair components for the eigenvalue problem in Equation (3). By utilizing this equation, unknown λ_j and ψ_j ($j = 0, 1, \dots$) coefficients are expected to be determined uniquely in order to obtain unique approximations to the most dominant eigenvalue λ and its accompanying eigenfunction $\psi(x)$, respectively. To this end, if ϵ is taken to the zero limit in both sides of Equation (13) then:

$$\lambda_0 = K_{0,0} \tag{15}$$

is obtained. This result denotes that the zeroth coefficient of the perturbation expansion for the eigenvalue λ is equal to the value of the relevant kernel at the point (x_{mp}, x_{mp}) .

It is important to consider that the equation in Equation (13) holds for any ϵ . Thus, it can be assessable as an identity with respect to ϵ , instead of an equation, in its convergence domain. Due to this reason, any number of consecutive differentiations of this equation with respect to ϵ does not annihilate its validity. Hence, the r -times (r is a nonnegative integer) consecutive differentiation with respect to ϵ and then division by $r!$ for both sides of the equation in Equation (13) yields the following formula:

$$\frac{1}{2} \sum_{m=0}^{\infty} \sum_{n=0}^m \sum_{k=0}^n K_{k,n-k} \psi_{m-n} I_{m-k} y^k (\epsilon^m)^{(r)} = \sum_{m=0}^{\infty} \sum_{n=0}^m \lambda_n \psi_{m-n} y^{m-n} (\epsilon^m)^{(r)} \tag{16}$$

On the other hand, the following equalities can be easily written to facilitate the analyses:

$$\left\{ (\varepsilon^m)^{(r)} \right\}_{\varepsilon \rightarrow 0} = \delta_{m,r}, \quad r = 0, 1, \dots \tag{17}$$

Now, if ε is taken to the zero limit in Equations (16) and (17) is utilized to evaluate the limit, then the following result is achieved:

$$\frac{1}{2} \sum_{n=0}^r \sum_{k=0}^n K_{k,n-k} \psi_{r-n} I_{r-k} y^k = \sum_{n=0}^r \lambda_n \psi_{r-n} y^{r-n} \tag{18}$$

Utilizing the summation index transform over the triangle $0 \leq k \leq n \leq r$, that is:

$$\sum_{n=0}^r \sum_{k=0}^n c_{n,k} = \sum_{k=0}^r \sum_{n=k}^r c_{n,k}$$

and making the necessary arrangements over the indices:

$$\lambda_k \psi_{r-k} = \frac{1}{2} \sum_{n=r-k}^r K_{r-k,n-r+k} I_k \psi_{r-n}, \quad r = 0, 1, \dots, \quad k = 0, 1, \dots, r \tag{19}$$

is yielded. Thus, an explicit recursion relation amongst the relevant perturbation coefficients is obtained.

It is obvious that taking $r = 0$ in the newly-constructed explicit recursion in Equation (19) verifies the result in Equation (15) by assuming that $\psi_0 \neq 0$. If the current process is resumed by taking $r = 1$, then two values for the index k are encountered. These values are zero and one, respectively. Taking $k = 0$ yields:

$$K_{1,0} \psi_0 = \lambda_0 \psi_1 \tag{20}$$

which can be rewritten using the result in Equation (15) as:

$$\psi_1 = \frac{K_{1,0}}{K_{0,0}} \psi_0 \tag{21}$$

However, it is important to remark that ψ_0 has an arbitrary non-zero value. Instead of taking $k = 1$ at this stage, we will deal with a more general structure, which is valid for each perturbation step for the odd sub-indexed λ_j coefficients. Due to this aim, by having a glance at the explicit recursion in Equation (19) and considering the definitions in Equation (14), one can easily observe that the perturbation coefficients of the eigenvalue λ whose sub-indexes are odd can be obtained as:

$$\lambda_{2p+1} = 0, \quad p = 0, 1, \dots \tag{22}$$

Besides, it is known from the implicit recursion in Equation (18) that a general perturbation coefficient for the eigenvalue whose subindex is even, say $2p$ ($p = 0, 1, 2, \dots$), is encountered first in the $2p$ -th perturbation step. Thus, if r and k are taken as $2p$ in Equation (19), then:

$$\lambda_{2p} = \frac{1}{(2p+1)K_{0,0}} \sum_{n=0}^{2p} K_{0,n} K_{2p-n,0}; \quad p = 0, 1, \dots \tag{23}$$

is obtained, which stands as an explicit representation for odd sub-indexed λ_j coefficients. Thus, the explicit formulations to calculate all λ_r ($r = 0, 1, 2, \dots$) perturbation coefficients are obtained. After that stage, one group of coefficients remains to be determined, which are $\psi_{r,s}$ ($r = 0, 1, 2, \dots$). To be able to evaluate those coefficients, a general perturbation step should be considered. If we deal with the r -th

perturbation step, which means k is taken as zero in the explicit recursion given through Equation (19), the following is yielded:

$$\psi_r = \frac{K_{r,0}}{K_{0,0}} \psi_0, \quad r = 1, 2, \dots \tag{24}$$

Thus, all unknown perturbation coefficients can be determined uniquely with one arbitrariness coming from the value of ψ_0 . Although this arbitrariness can be thought of as an inconsistency, this undesired situation may be removed by applying a certain normalization procedure to the eigenfunction constructed using the coefficients in Equation (24).

4. Constraints on the Kernel and the Construction of the Perturbation Series

Although all of the unknown perturbation coefficients except ψ_0 are determined explicitly in the previous section, one can observe that some additional equations that need to be satisfied amongst the perturbation coefficients occur at almost each perturbation step. That means denumerable infinitely many inconsistencies amongst the perturbation coefficients, which should be taken into account as the constraints, are encountered. These constraints have arisen since the autonomy on the perturbation coefficients of the eigenfunction under consideration have been imposed.

With a careful examination, it is easy to observe that no constraint is encountered during the determination of the perturbation coefficients at the zeroth, first and second steps, respectively. This expected situation occurs since λ_0 is determined as $K_{0,0}$ at the zeroth perturbation step, and the first order perturbation terms do not involve any even-indexed λ_j coefficients. Besides, it is clear from Equations (14) and (19) that no constraint is encountered during the determination of an odd sub-indexed λ_j coefficient. If we keep on evaluating the higher order perturbation terms in order to construct the most dominant eigenvalue and its accompanying eigenfunction, only a single constraint is encountered while trying to evaluate both the third and the fourth order perturbation coefficients. At the fifth and sixth perturbation steps, this time, two more constraints occur in addition to these two single constraints mentioned above. Thus, four additional constraints come into the scene at this stage, and it can be easily observed that the total number of constraints grows as the perturbation order increases. If the perturbation order is denoted as r , $(n - 1)^2$ number of constraints are encountered where $r = 2n - 1$ while $(n - 1)n$ number of constraints are confronted where $r = 2n$, which n stands as a positive integer greater than one.

If r stands, again, as the general perturbation order and considering $k = 2m$ where $2m < r$ in Equation (19), the m -th constraint at the r -th perturbation step is obtained as follows:

$$\sum_{n=0}^{2m} K_{r-2m,n} \psi_{2m-n} = (2m + 1) \lambda_{2m} \psi_{r-2m}, \quad r = 0, 1, \dots, \quad m = 2, 3, \dots, \quad 2 < 2m < r \tag{25}$$

By utilizing the Equations (23) and (24) in the above equation:

$$\sum_{n=0}^{2m} K_{r-2m,n} K_{2m-n,0} = \frac{K_{r-2m,0}}{K_{0,0}} \sum_{n=0}^{2m} K_{0,n} K_{2m-n,0}, \quad m = 0, 1, \dots \tag{26}$$

are yielded and can be reorganized as follows:

$$\sum_{n=0}^{2m} K_{2m-n,0} \left(K_{r-2m,n} - \frac{K_{r-2m,0} K_{0,n}}{K_{0,0}} \right) = 0, \quad m = 0, 1, \dots \tag{27}$$

Equation (27) are satisfied for two cases at first glance. The first one of these circumstances is the following:

$$K_{j,0} = 0, \quad j = 0, 1, \dots \tag{28}$$

which leads us to the obvious zero solution for the problem in Equation (3), while the other option implies the following relations:

$$\frac{K_{p,q}}{K_{p,0}} = \frac{K_{0,q}}{K_{0,0}}, \quad p, q = 0, 1, \dots \tag{29}$$

which can also be represented using the following determinants:

$$\begin{vmatrix} K_{0,0} & K_{p,0} \\ K_{0,q} & K_{p,q} \end{vmatrix} = 0, \quad p, q = 0, 1, \dots \tag{30}$$

It is not quite a rational choice to proceed with the perturbation coefficients satisfying Equation (28) for the relevant eigenfunction since the zero solution is acquired in that sense. Thus, considering the relations amongst the kernel coefficients given by Equation (29) is more logical. If the Taylor coefficients of the bivariate kernel under consideration in Equation (8) are placed accordingly, it is possible to construct the following matrix:

$$\mathbf{K} = \begin{bmatrix} \boxed{K_{0,0}} & K_{1,0} & K_{2,0} & \cdots & \boxed{K_{p,0}} & \cdots \\ K_{0,1} & K_{1,1} & K_{2,1} & \cdots & K_{p,1} & \cdots \\ K_{0,2} & K_{1,2} & K_{2,2} & \cdots & K_{p,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ \boxed{K_{0,q}} & K_{1,q} & K_{2,q} & \cdots & \boxed{K_{p,q}} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots \end{bmatrix} \tag{31}$$

In Equation (31), it is easy to recognize that the boxed elements stand for the coefficients depicted in Equation (30). Since we assume that the value of $K_{0,0}$ is not zero, it becomes possible to say that the determinant of the first leading principal submatrix of \mathbf{K} is not zero. On the other hand, if the p and q values are taken as one in Equation (30), one can easily verify that the determinant of the second leading principal submatrix vanishes, which means the rows of the second leading principal submatrix are linearly dependent. On the other hand, if the q value is fixed at one and the p -value is changed from two to infinity in Equation (30), the first and the second row of the infinite dimensional matrix \mathbf{K} are revealed as being linearly dependent. By increasing q value this time, it becomes easy to conclude that all rows of the matrix \mathbf{K} are proportional to the first row. This important result tells us that \mathbf{K} is a one-rank matrix. To this end, the bivariate kernel that satisfies the constraints in Equation (29) must be a one-rank kernel, which means that this kernel must be of a purely multiplicative structure, as indicated below:

$$K(x, \xi) = A(x)A(\xi), \quad x, \xi \in [a, b] \tag{32}$$

where the function A is assumed to be bounded and analytic on the closed interval $[a, b]$. On the other hand, the structure in Equation (32) makes sense if we could deal with infinitely many perturbation terms. Since this situation is not possible in numerical applications, utilizing a kernel that is not purely multiplicative, but does approach this case asymptotically can be a significant option to this end.

If λ_j coefficients whose explicit definitions given through Equations (22) and (23) are embedded into the corresponding perturbation series in Equation (9) and assuming that the relevant kernel coefficients satisfy the relations in Equation (29):

$$\lambda(\epsilon) = \frac{1}{K_{0,0}} \sum_{j=0}^{\infty} \left(\frac{1}{2j+1} \sum_{n=0}^{2j} K_{0,n} K_{2j-n,0} \right) \epsilon^{2j} \tag{33}$$

is obtained. Thus, the perturbation series for the most dominant eigenvalue of the corresponding Hilbert–Schmidt integral operator whose kernel coefficients satisfy the relevant constraints is evaluated.

Similarly, by embedding the general structure of ψ_r given in Equation (24) into the perturbation expansion of the eigenfunction in Equation (10) and applying the inverse of the first affine transformation in Equation (4):

$$\psi(x) = \psi_0 \sum_{r=0}^{\infty} \frac{K_{r,0}}{K_{0,0}} (x - x_{mp})^r \quad (34)$$

is achieved as the power series representation of the corresponding eigenfunction. In other words, by using the developed perturbation method and assuming the autonomy on the eigenfunction perturbation coefficients, the corresponding univariate eigenfunction $\psi(x)$ is expressed as an infinite power series in ϵ whose general term involves the Taylor coefficients of the kernel satisfying the constraint equations in Equation (29). On the other hand, the zeroth coefficient, that is ψ_0 , is another important determining issue for the structure in Equation (34) and can be perceived as an arbitrary constant at this moment. As has been stated previously, no assumption has been made about the normalization of the corresponding eigenfunction. Thus, there is only one common flexibility ψ_0 to be determined in that sense. This freedom brings a common arbitrariness on the eigenfunction coefficients. This undesired status in fact is originated from the autonomy imposition on the eigenfunction and can be globally removed only by normalizing the corresponding eigenfunction. Thus, ψ_0 is computed as the normalization factor after truncating the series Equation (34) at a certain level. Hereby, the truncated and normalized series can be assessed as an approximation for the relevant eigenfunction $\psi(x)$.

Before closing the section, we need to emphasize the observed limitations and constraints mentioned above. The series in Equations (33) and (34) are capable of representing the most dominant eigenvalue and its accompanying eigenfunction, respectively, as long as the kernel coefficients satisfy all of the constraints given in Equation (29). This means, the relevant coefficients should satisfy denumerable infinitely many equations, which does not make sense practically. Thus, an asymptotically multiplicative kernel could be considered, which enables the initial perturbation coefficients to satisfy the corresponding initial restrictive equations instead of the purely multiplicative kernel depicted in Equation (32).

5. Error Analysis

Error estimation is one of the important issues in approximation problems. Here, in this paper, the eigenfunction under consideration is attempted to be approximated via a perturbation expansion in this work. This section is devoted to the concern on the error bound determination of the relevant perturbation series. Concordantly, one can easily verify that the series that will be utilized for approximation in Equation (34) is a power series with the coefficients involving the derivatives of the relevant kernel, which is of a purely multiplicative nature and whose Taylor coefficients satisfy the constraints in Equation (29). Although the kernel under consideration is bivariate and has differentiation possibilities with respect to both of its independent variables, the related coefficients in Equation (24) include only the derivatives with respect to its first variable, x (but of course, this is valid under the constraints mentioned above). This feature reveals the similarity between the perturbation series in Equation (34) and the well-known Taylor series. To this end, it is reasonable to make an error estimation for the developed perturbation series for the relevant eigenfunction by utilizing the error bound theorem for the Taylor series.

As is obvious from the univariate Taylor theorem, assuming that $g(x)$ has the derivatives up to $(n + 1)$ -th order, which are continuous on a closed interval and the last of them, that is, the $(n + 1)$ -th one, is bounded from above, such as:

$$\left| g^{(n+1)} \right| \leq G, \quad n = 0, 1, \dots \quad (35)$$

where G is an n -independent positive constant, then the remainder function $R(x)$ corresponding to the $(n + 1)$ -th degree Taylor polynomial for the function $g(x)$ can be bounded from above [14] as:

$$|R(x)| \leq \frac{G(x-a)^{n+1}}{(n+1)!}, \quad n = 0, 1, \dots \tag{36}$$

In light of the brief information given above, we can assume that the upper bound for the $(n + 1)$ -th order derivative with respect to x of $K(x, \xi)$ can be written as follows:

$$\left| \frac{\partial^{n+1}K}{\partial x^{n+1}} \right| \leq M, \quad n = 0, 1, 2, \dots \tag{37}$$

On the other hand, it is useful to remark that the zeroth coefficient of the expansion for the kernel in (7) denoted as $K_{0,0}$ should not vanish by assumption. Considering the series in Equation (34) and rewriting the kernel coefficients explicitly in accordance with the definition in Equation (8) gives:

$$\psi(x) = \sum_{r=0}^{\infty} \frac{\psi_0}{K_{0,0}} \frac{1}{r!} \frac{\partial^r K}{\partial x^r}(x_{mp}, x_{mp}) (x - x_{mp})^r \tag{38}$$

where $x_{mp} = (a + b)/2$. If the remainder function, occurring by retaining first n expansion terms in order to make an approximation to the relevant eigenfunction $\psi(x)$, is symbolized by $R(x)$, it is possible to write down the following:

$$|R(x)| \leq \frac{\psi_0 M}{|K_{0,0}| (n+1)!} |x - x_{mp}|^{n+1} \tag{39}$$

By recalling that $x - x_{mp} = \varepsilon y$ and $|y| \leq 1$ from Equation (4):

$$|R(x)| \leq \frac{\psi_0 M}{|K_{0,0}| (n+1)!} \varepsilon^{n+1} \tag{40}$$

can be obtained as the remainder bound for the approximation of the the relevant eigenfunction $\psi(x)$ where ψ_0 is a positive constant that normalizes the corresponding eigenfunction and, hence, is bounded. It is also obvious from Equation (40) that the norm of the remainder term depends on the perturbation parameter, ε , and the value of the kernel at the midpoint of the geometry under consideration, that is $K_{0,0}$.

6. Numerical Implementations

In this section, six numerical implementations are given in order to show the efficiency of the method described in the third section. To this end, the exact solution of the eigenvalue problem in Equation (3) and its perturbation-based numerical solutions at different truncation levels for various kernels will be compared through the plots. The first four of these kernels are taken as the well-known Pincherle–Goursat-type kernels, which are not of a purely multiplicative nature and, hence, do not need to satisfy the conditions presented in Equation (29). On the other hand, the last two implementations are considered to verify that the present method works efficiently for the kernels having a purely multiplicative nature. Thus, the integral operators given in the fifth and the sixth implementation, respectively, involve the Hilbert–Schmidt kernels satisfying the constraints in Equation (29).

The first four kernels that will be utilized in this section, all being the Pincherle–Goursat type, are as follows:

$$\begin{aligned} K_1(x, \xi) &= \exp(2x) + \exp(2\xi), \\ K_2(x, \xi) &= \exp(x + \xi) + \exp(-x - \xi), \\ K_3(x, \xi) &= \cos(\pi(x + \xi)), \\ K_4(x, \xi) &= \tan x \sin 3\xi + \sin 3x \tan \xi \end{aligned}$$

The reason for selecting this type of kernel is the simple solvability of their spectral problems. Hence, the spectral problems of the integral operators having Pincherle–Goursat-type kernels can be reduced into a matrix algebraic eigenvalue problem quite easily.

In each figure, the curves that are constructed using asterisks imply the exact solution of the spectral problem for the integral operator under consideration while the dashed curves dictate the truncated approximations to the corresponding exact eigenfunction. The truncation orders start from $n = 2$ and go to $n = 10$ with an increment of two, and can go further up to $n = 14$ when it is necessary. The n value standing in the legends of all figures implies the number of perturbation terms taken into account to construct the approximated eigenfunction. For example, $n = 2$ means that just two perturbation coefficients are evaluated, and the relevant approximation is just a straight line.

In Figures 1 and 2, one can easily verify that the increment in the n value brings better approximation quality for the eigenfunction under consideration. This fact can be observed since the difference between the dashed curves and asterisk curve in each figure starts to get smaller, even though the restriction equations in Equation (29) are not satisfied. However, the chosen function needs to satisfy the constraint equations to get an exact match. On the other hand, in Figures 3 and 4, the approximation curves diverge from the corresponding exact eigenfunction curves for each; hence, the constraints in Equation (29) are not satisfied.

On the other hand, each graphic is plotted over the interval $[0, 1]$ since the corresponding spectral problems are designed as their integration domains become $[0, 1]$. In addition to this, it becomes useful to state that all of the eigenfunctions plotted are the ones whose accompanying eigenvalues are the greatest in the absolute value sense. Furthermore, they are normalized over the interval $[0, 1]$.

After investigating the first four implementations given above, the last two will be revived as has already been mentioned above in this section. To this end, the eigenvalue problems involving Hilbert–Schmidt integral operators whose kernels satisfy the constraints in Equation (29) will be addressed. Those specific type of kernels that will be utilized in order to show the efficiency of the method developed in the present paper are as follows:

$$\begin{aligned} K_5(x, \zeta) &= \cos 5x \cos 5\zeta, \\ K_6(x, \zeta) &= \sin(\pi \exp(x)) \sin(\pi \exp(\zeta)) \end{aligned}$$

where $x, \zeta \in [0, 1]$.

In Figures 5 and 6, it is obvious that the truncated perturbation series converge to the corresponding exact eigenfunction for the kernels $K_5(x, \zeta)$ and $K_6(x, \zeta)$, respectively. These results occur since both kernels under consideration satisfy the restrictive equations in Equation (29) and the bounded property achieved in Equation (40).

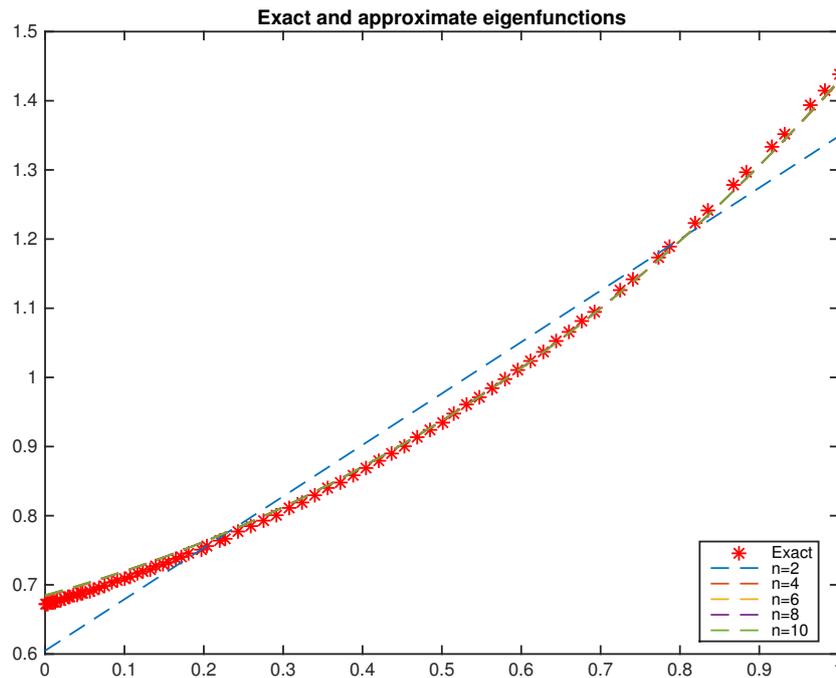


Figure 1. Exact and approximate eigenfunctions for the kernel $K(x, \xi) = \exp(2x) + \exp(2\xi)$ over $[0, 1]^2$ at various truncation orders.

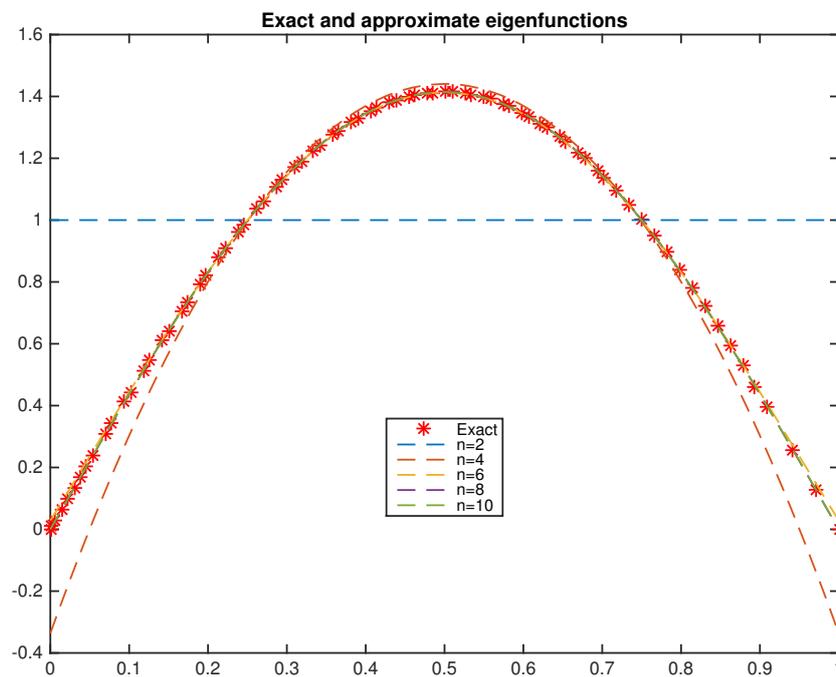


Figure 2. Exact and approximate eigenfunctions for the kernel $K(x, \xi) = \cos(\pi(x + \xi))$ over $[0, 1]^2$ at various truncation orders.

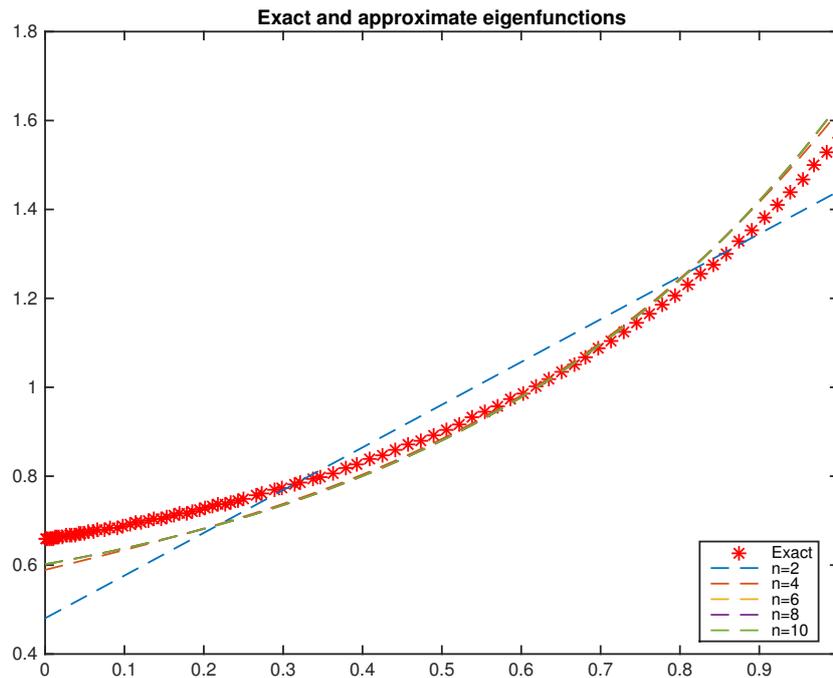


Figure 3. Exact and approximate eigenfunctions for the kernel $K(x, \xi) = \exp(x + \xi) + \exp(-x - \xi)$ over $[0, 1]^2$ at various truncation orders.

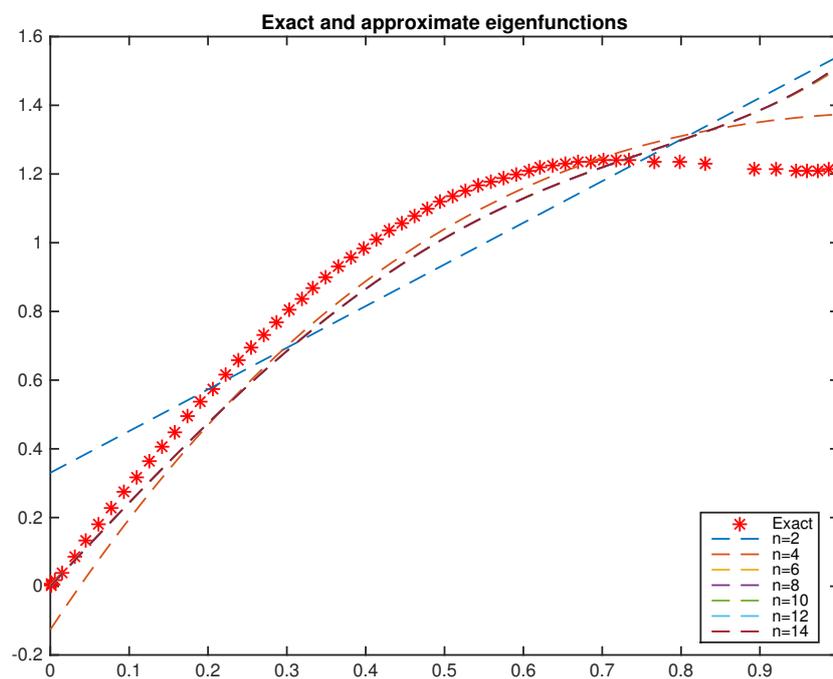


Figure 4. Exact and approximate eigenfunctions for the kernel $K(x, \xi) = \tan x \sin 3\xi + \sin 3x \tan \xi$ over $[0, 1]^2$ at various truncation orders.

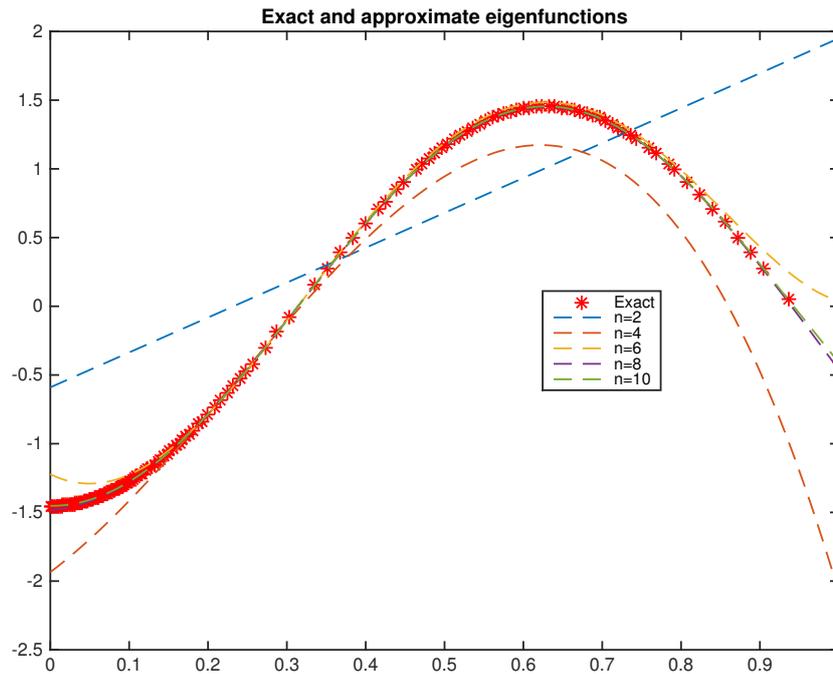


Figure 5. Exact and approximate eigenfunctions for the kernel $K(x, \xi) = \cos 5x \cos 5\xi$ over $[0, 1]^2$ at various truncation orders.

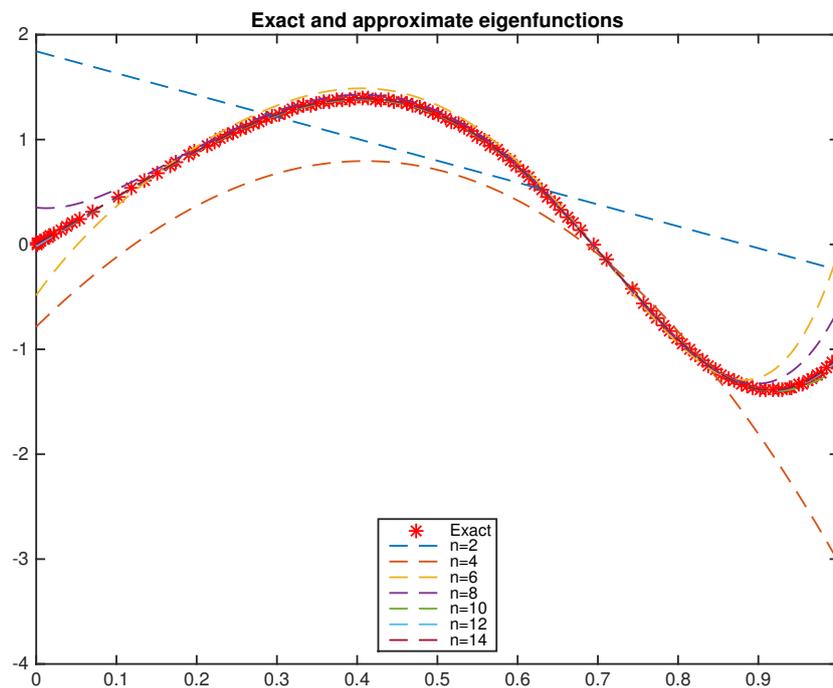


Figure 6. Exact and approximate eigenfunctions for the kernel $K(x, \xi) = \sin(\pi \exp(x)) \sin(\pi \exp(\xi))$ over $[0, 1]^2$ at various truncation orders.

7. Conclusions

In this work, a perturbation expansion-based approximation method is proposed for approximating the most dominant eigenvalue and its accompanying eigenfunction of a Hilbert–Schmidt integral operator having symmetric kernel. Throughout the development of this method, the half interval length of the domain of the integral operator is considered as the perturbation parameter, and all investigations are realized by assuming that this parameter is small enough, which means it approaches zero. Thus, this process can be interpreted as a perturbation method at the zero interval limit.

Since we have imposed autonomy on the perturbation coefficients of the eigenfunction under consideration, denumerable infinitely many constraints have arisen. The general structure for the mentioned constraints are obtained, and it is guaranteed that the developed method is valid for the compact operators involving purely multiplicative kernels.

With the help of numerical implementations, the efficiency of the present method is confirmed for the kernels that have a purely multiplicative nature. However, it is observed that the developed method may work efficiently for the kernels that are not of a purely multiplicative type. In addition to these, the effects of the integration interval, ψ_0 coefficient and truncation order to the approximation quality are investigated by error analysis.

As a final remark, if the restrictions amongst the kernel coefficients depicted in the fourth section are not satisfied, the generated perturbation series in Equation (34) may not converge to the exact solution for the relevant eigenfunction. Thus, it becomes wiser to work with kernels satisfying these restrictions or to deal with intervals that enable kernel coefficients to satisfy the restrictive relations. Even if the restriction equations are not exactly satisfied, their closeness to satisfiability may result in effective approximations. On the other hand, since we could only deal with a finite number of perturbation terms in practical cases, satisfying denumerable infinitely many constraints is not necessary to this end. Thus, working with the kernels whose initial Taylor coefficients satisfying initial constraints may empower the efficiency of the developed method even for the non-purely multiplicative kernels.

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