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Viability for Semilinear Differential Equations with Infinite Delay

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Abstract: Let X be a Banach space, $A : D(A) \subset X \to X$ the generator of a compact C_0-semigroup $S(t) : X \to X, t \geq 0$, $D(\cdot) : (a,b) \to 2^X$ a tube in X, and $f : (a,b) \times \mathcal{B} \to X$ a function of Carathéodory type. The main result of this paper is that a necessary and sufficient condition in order that $D(\cdot)$ be viable of the semilinear differential equation with infinite delay $u'(t) = Au(t) + f(t,u_t), t \in [t_0, t_0+T], u_{t_0} = \phi \in \mathcal{B}$ is the tangency condition $\liminf_{h\downarrow 0} h^{-1} d(S(h)v(0) + hf(t,v); D(t+h)) = 0$ for almost every $t \in (a,b)$ and every $v \in \mathcal{B}$ with $v(0) \in D(t)$.

Keywords: viable domain; differential equation; infinite delay; tangency condition

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1. Introduction

The aim of this paper is to prove a necessary and sufficient condition in order that a given tube of a Banach space X be viable for a semilinear differential equation with infinite delay. Namely, let X be a real Banach space, $A : D(A) \subset X \to X$ the infinitesimal generator of a C_0-semigroup $S(t) : X \to X, t \geq 0$, $D(\cdot) : (a,b) \to 2^X$ be a tube in X with closed values, $-\infty \leq a < b \leq +\infty$. We consider the semilinear differential equation with infinite delay:

$$u^{'}(t) = Au(t) + f(t,u_t), t \in [t_0, t_0+T] \tag{1}$$

with the initial condition

$$u_{t_0} = \phi \in \mathcal{B} \tag{2}$$

where $\mathcal{B}$ is the phase space defined axiomatically, $u_t : (-\infty, 0] \to X$ defined by $u_t(\theta) = u(t+\theta)$ for all $\theta \in (-\infty, 0]$. We are interested to find necessary and sufficient conditions in order that $D(\cdot)$ be viable for (1), i.e., for each $t_0 \in (a,b)$ and $\phi \in \mathcal{B}$ with $\phi(0) \in D(t_0)$, there exist a $T > 0$ and at least a mild solution to (1) satisfying the initial condition $u_{t_0} = \phi$ and $u(t) \in D(t)$ for $t \in [t_0, t_0+T]$.

We recall that the function $u : (-\infty, t_0+T] \to X$ is a mild solution to (1) and (2) if $u_{t_0} = \phi$, u is continuous on $[t_0, t_0+T]$ and satisfying

$$u(t) = S(t-t_0)\phi(0) + \int_{t_0}^{t} S(t-s) f(s,u_s) ds \tag{3}$$

for $t \in [t_0, t_0+T]$.

The viability problem for the differential equation

$$u^{'}(t) = Au(t) + F(t,u(t)), t \in [t_0, t_0+T] \tag{4}$$

$$u(t_0) = x_0 \tag{5}$$

has been studied by many authors by using various frameworks and techniques. In this respect, we note the pioneering work of Nagumo [1] who considered the finite dimensional case, $A = 0$ and F is continuous. In this context, he showed that a necessary and sufficient condition in order that $D(t) \equiv D$ be a viable domain for (4) is the following tangency condition:

$$\liminf_{h \downarrow 0} h^{-1} d(x + hF(t, x); D) = 0$$

for each $(t, x) \in (a, b) \times D$. It is interesting to note that Nagumo's result (or some variant of it) has been rediscovered several times, among others by Brezis [2], Crandall [3], Hartman [4], and Martin [5]. For the development in this area, we refer the readers to Ursescu [6], Pavel [7] and [8], Pavel and Motreanu [9], Cârjă and Marques [10], Cârjă and Vrabie [11]. Viability for fractional differential equations was also discussed in [12,13]. Brief reviews of the main contributions in this area can be found in [10,11]. We emphasize Pavel's main contribution who was the first who formulated the corresponding tangency condition applying to the semilinear case. More precisely, Pavel [7] showed that, whenever A generates a compact C_0-semigroup and F is continuous on $(a, b) \times D$, where D is locally closed in X, a sufficient condition for viability is

$$\lim_{h \downarrow 0} h^{-1} d(S(h)x + hF(t, x); D) = 0$$

for each $(t, x) \in (a, b) \times D$.

As for the functional differential equations, the development was initialed about existence and stability by Travis and Webb [14,15] and Webb [16,17]. Since such equations are often more realistic to describe natural phenomena than those without delay, they have been investigated in variant aspects by many authors (see, e.g., [18–20] and references therein). Pavel and Iacob [21] discussed the viability problem for semilinear differential equations with finite delay (the case $\mathcal{B} = C([-q, 0]; X)$. They proved that, whenever A generates a compact C_0-semigroup and f is continuous from $(a, b) \times C([-q, 0]; X)$ into X, a necessary and sufficient condition for viability for (1) is

$$\lim_{h \downarrow 0} h^{-1} d(S(h)v(0) + hf(t, v); D) = 0 \tag{6}$$

for each $t \in (a, b)$, each $v \in C([-q, 0]; X)$ with $v(0) \in D$, where $D(t) \equiv D$ is a locally closed subset in X. Dong and Li [22] proved the same result when f is of Carathéodory type. Necula et al. studied the viability for delay evolution equations with nonlocal initial conditions in [23].

The purpose of this paper is to discuss the viable problem of the semilinear differential equation with infinite delay (1). In the study of equations with finite delay, the state space is the space of all continuous functions on $[-q, 0]$, $q > 0$, endowed with the uniform norm topology. When the delay is unbounded, the selection of the state space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato [24]. For a detailed discussion on the topic, we refer to the book by Hino et al. [25]. We prove that a necessary and sufficient condition in order that $D(t)$ be viable for (1) is the tangency condition. We only suppose that f is of Carathéodory type. The difficulty is that the semi-norm on $\mathcal{B}$ is defined axiomatically, and the convergence of a sequence in $\mathcal{B}$ cannot be obtained directly. Our result extends and improves that of Pavel and Iacob [21] who considered the case in which f is continuous, Dong and Li [22] for finite delay and $D(t) \equiv D$, and also extends the well-known existence result of Hale [26] who considered the case in which X is finite dimensional and $A = 0$. Moreover, using a standard argument based on Zorn's Lemma, we get the existence of noncontinuable (saturated) mild solutions.

2. Preliminaries

Throughout this paper X will be a real Banach space, $A : D(A) \subset X \to X$ the generator of a C_0-semigroup $S(t) : X \to X, t \geq 0$. Then $\{S(t); t \geq 0\}$ is exponentially bounded, i.e., there are constants $C \geq 1$ and $\omega \geq 0$ such that

$$\|S(t)\| \leq Ce^{\omega t}, \quad \forall t \geq 0$$

Moreover, if $S(t), t \geq 0$ is a compact semigroup (i.e., $S(t)$ maps bounded subsets into relatively compact subsets for $t > 0$), then $S(t)$ is continuous in the uniformly operator topology for $t > 0$ (see Pazy [27]) and X is separable (see [10]). For more details of semigroups of linear operators, we refer the readers to Pazy [27].

In this paper we will employ an axiomatic definition of the space $\mathcal{B}$ introduced by Hale and Kato [24]. To establish the axioms of space $\mathcal{B}$, we follow the terminology used in [25]. Thus, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$. We will assume that $\mathcal{B}$ satisfies the following axioms:

(A) If $x : (-\infty, \sigma + a) \to X, a > 0$, is continuous on $[\sigma, \sigma + a]$ and $x_\sigma \in \mathcal{B}$ then for every $t \in [\sigma, \sigma + a)$, the following conditions hold:

(i) x_t is in $\mathcal{B}$;
(ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$;
(iii) $\|x_t\|_{\mathcal{B}} \leq K(t-\sigma) sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$,

where $H \geq 0$ is a constant; $K, M : [0, +\infty) \to [0, +\infty)$, K is continuous and M is locally bounded and H, K and M are independent on $x(\cdot)$.

(A1) For the function $x(\cdot)$ in (A), x_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma + a)$.

(B) The space $\mathcal{B}$ is complete.

We first list the conditions here, for the convenience of reference.

(C1) $A : D(A) \subset X \to X$ is the infinitesimal generator of the C_0 semigroup $S(t)$. For $t > 0$, $S(t)$ is compact.

(C2) $D(\cdot) : (a, b) \to 2^X$ is closed valued and for each $t_0 \in (a, b)$ and $x \in D(t_0)$, there exist $r > 0$ and $\bar{T} \in (t_0, b)$ such that $B_X(x, r) \cap D(t)$is nonempty for all $t \in [t_0, \bar{T}]$, and the mapping $t \mapsto B_X(x, r) \cap D(t)$ is closed on $[t_0, \bar{T}]$. Here $B_X(x, r) = \{y \in X : \|y - x\| \leq r\}$ is the closed ball centered x with radius r.

(T) (Tangency condition)

$$\liminf_{h \downarrow 0} h^{-1} d(S(h)v(0) + hf(t, v); D(t + h)) = 0 \tag{7}$$

for a.e. $t \in (a, b)$ and all $v \in \mathcal{B}$ with $v(0) \in D(t)$, where $d(x, B)$ denotes the distance from $x \in X$ to the subset $B \subset X$.

Since the distance is non-expansive, i.e.,

$$|d(x, B) - d(y, B)| \leq \|x - y\|, \forall x, y \in X,$$

by standard arguments (see [8,21]), Condition (T) is equivalent to

$$\liminf_{h \downarrow 0} h^{-1} d(S(h)v(0) + h \int_t^{t+h} S(t + h - s) f(t, v) ds; D(t + h)) = 0 \tag{8}$$

for a.e. $t \in (a, b)$ and all $v \in \mathcal{B}$ with $v(0) \in D(t)$.

Remark 1. *If $D(t) \equiv D$ then the conditions (C2) means that D is locally closed, and the tangency condition (T) is reduced to (6).*

We say that a function $f:(a,b)\times\mathcal{B}\to X$ is of Caratheodary type if f satisfies

(1) for each $v\in\mathcal{B}$, the function $f(\cdot,v):(a,b)\to X$ is measurable on (a,b);
(2) for almost every (a.e.) $t\in(a,b)$, $f(t,\cdot):\mathcal{B}\to X$ is continuous on $\mathcal{B}$;
(3) for every $r>0$, there is a function $m_r\in L(a,b;X)$ such that

$$\|f(t,v)\|\leq m_r(t) \tag{9}$$

for a.e. $t\in(a,b)$ and every $v\in\mathcal{B}$ with $\|v\|_{\mathcal{B}}\leq r$.

A Carathéodory type function has the following Scorza Dragoni property which is nothing but the special case of [28,29]. We denote by λ the Lebesgue measure on $\mathbf{R}$ and by $\mathbf{L}$, the collection of all Lebesgue measurable sets in $\mathbb{R}$.

Theorem 1. *Let X,Y be separable metric spaces and $I=(a,b)$ or $I\in\mathbf{L}((a,b))$. Let $f:I\times X\to Y$ be a function such that $f(\cdot,x)$ is measurable for every $x\in X$ and $f(t,\cdot)$ is continuous for almost every $t\in I$. Then, for each $\varepsilon>0$, there exists a compact subset $K\subset I$ such that $\lambda(I\setminus K)<\varepsilon$ and the restriction of f to $K\times X$ is continuous.*

Suppose that $u:(-\infty,b)\to X$, $u_a\in\mathcal{B}$ and u is continuous on (a,b). Then the mapping $t\mapsto u_t$, from (a,b) into $\mathcal{B}$ is also continuous. The following result is a kind of variance of Lebesgue derivative type, which is useful in the sequel. We omit the proof since it is similar to that of [10], Theorem 2.

Theorem 2. *Assume that X is a separable Banach space, $D(\cdot):t\mapsto 2^X$ is closed valued and satisfying (C2), $S(t)$ is a C_0-semigroup on X and $f:(a,b)\times\mathcal{B}\to X$ is a function of Carathéodory type. Then there exists a negligible subset Z of (a,b) such that, for every $t\in(a,b)\setminus Z$, one has*

$$\lim_{h\downarrow 0}h^{-1}\int_t^{t+h}S(t+h-s)f(s,u_s)ds=f(t,u_t) \tag{10}$$

for all functions $u:(-\infty,b)\to X$ with $u_a\in\mathcal{B}$, $u(t)\in D(t)$ and u is continuous on (a,b).

3. Main Result

We are now ready to state our main result of this paper, the necessary and sufficient condition of the viability for semilinear differential equations with infinite delay.

Theorem 3. *Suppose that the conditions (C1) and (C2) hold, and f is of Carathéodory type. Then a necessary and sufficient condition in order that $D(\cdot)$ be viable for Equation (1) is the tangency condition (T).*

Proof of necessity. Let Z be given by Theorem 2, let $t_0\in(a,b)\setminus Z$. Let $v\in\mathcal{B}$ such that $v(0)\in D(t_0)$. By hypothesis, there exists $T=T(t_0,v)>0$ with $t_0+T<b$ and a function u continuous on $[t_0,t_0+T]$, satisfying (3) with $\phi=v$. Since $u(t_0+h)\in D(t_0+h)$ for all $h\in[0,T]$, we have

$$\begin{aligned}
&h^{-1}d(S(h)v(0)+hf(t_0,v);D(t_0+h))\\
\leq\;& h^{-1}\|S(h)v(0)+hf(t_0,v)-u(t_0+h)\|\\
\leq\;& \|f(t_0,v)-h^{-1}\textstyle\int_{t_0}^{t_0+h}S(t_0+h-s)f(s,u_s)ds\|
\end{aligned} \tag{11}$$

Letting $h\downarrow 0$, one obtains the condition (T). □

To prove the sufficiency, we need the following lemma.

Lemma 1. *Suppose that the hypotheses of Theorem 3 hold. Given $t_0\in(a,b),\phi\in\mathcal{B}$ with $\phi(0)\in D(t_0)$, there exists a $T>0$ with $t_0+T<b$, such that for every positive integer n, there exist an open subset $L_n\subset(a,b)$ with $\lambda(L_n)<\frac{1}{n}$, an increasing sequence $\{t_i^n\}_{i=1}^{\infty}\subset[t_0,t_0+T]$, $\bar{t}\in[t_0,t_0+T]\setminus Z$ and an approximate solution u^n on $[t_0,t_0+T]$ in the following sense:*

(i) $t_0^n = t_0, t_{i+1}^n - t_i^n = d_i^n \le \frac{1}{n}, \lim_{i\to\infty} t_i^n = t_0 + T$;

(ii) $u_{t_0}^n = \phi, u^n(t_i^n) = x_i^n \in D(t_i^n) \cap B_X(\phi(0), r)$;

(iii) $h_n(s) = f(t_i^n, u_{t_i^n}^n)$ *in case* $t_i^n \notin L_n$ *while* $h_n(s) = f(\bar{t}, u_{t_i^n}^n)$ *in case* $t_i^n \in L_n$ *for* $s \in [t_i^n, t_{i+1}^n)$;

(iv) $u^n(t) = S(t - t_i^n)x_i^n + \int_{t_i^n}^t S(t-s)h_n(s)ds + (t - t_i^n)p_i^n$
for $t \in [t_i^n, t_{i+1}^n]$, *where* $\|p_i^n\| \le \frac{1}{n}$. *Moreover,* $u_{t_i^n}^n \in B_{\mathcal{B}}(\phi, r) = \{v \in \mathcal{B} : \|v - \phi\|_{\mathcal{B}} \le r\}$.

Proof. Let $\phi \in \mathcal{B}$ with $\phi(0) \in D(t_0)$. Due to (C2), there exist $r > 0$ and $T > 0$ such that $B_X(\phi(0), r) \cap D(t) \neq \emptyset$ for $t \in [t_0, t_0 + T]$. Define $\bar{\phi} : (-\infty, t_0 + T]$ by $\bar{\phi}(\theta) = \phi(\theta - t_0)$ for $\theta \le t_0$ and $\bar{\phi}(\theta) = \phi(0)$ for $\theta \in [t_0, t_0 + T]$. Then $\bar{\phi}_{t_0} = \phi$ and $t \mapsto \bar{\phi}_t$ is continuous on $[t_0, t_0 + T]$ by the axiom (A1). Set $K = \sup\{K(t) : 0 \le t \le T\}$, $R = r + \phi(0)$ and $M = \int_{t_0}^{t_0+T} m_R(s)ds$, where K and m_R are the functions appeared in the axiom (A) and (9) respectively. We may assume that $K \ge 1$. Further, on the basis of the definition of $\mathcal{B}$ and the continuity of the semigroup $S(t)$, we may choose $T > 0$ small enough such that $t_0 + T < b$ and

$$\|\bar{\phi}_t - \bar{\phi}_{t_0}\|_{\mathcal{B}} = \|\bar{\phi}_t - \phi\|_{\mathcal{B}} \le \frac{1}{2}r, \quad t \in [t_0, t_0 + T], \tag{12}$$

$$K[\max_{0\le t\le T} \|S(t)\phi(0) - \phi(0)\| + N(M + T)] \le \frac{1}{2}r, \tag{13}$$

where $N = Ce^{\omega T}$.

In view of Theorems 1 and 2, we may choose an open set $L_n \subset (a, b)$, with $Z \subset L_n$ and $\lambda(L_n) \le \frac{1}{n}$, such that f is continuous on $((a, b) \setminus L_n) \times \mathcal{B}$, where Z is the set obtained in Theorem 2. We can also assume that for each $t \in (a, b) \setminus Z$, (8) and (9) hold. Fix $\bar{t} \in (a, b) \setminus L_n$. We shall construct u^n and t_i^n by induction. Set $t_0^n = t_0, u^n(t_0^n) = \phi(0) = x_0^n, u_{t_0^n}^n = \phi$. To simplify notation, we drop n as a superscript for t_i, x_i, u, p_i etc. Suppose that u is constructed on $(-\infty, t_i]$. Then we define t_{i+1} in the following manner. If $t_i = t_0 + T$, set $t_{i+1} = t_0 + T$, and if $t_i < t_0 + T$, then we define t_{i+1} as the following two cases.

Case 1 : $t_i \in L_n$. Set

$$\begin{aligned} \delta_i = \quad & \sup\{h \in (0, \tfrac{1}{n}] : t_i + h \le t_0 + T, [t_i, t_i + h) \in L_n, \\ & d(S(h)x_i + \textstyle\int_{t_i}^{t_i+h} S(t_i + h - s)f(\bar{t}, u_{t_i})ds; D(t_i + h)) \le \tfrac{h}{2n}\}. \end{aligned} \tag{14}$$

In view of (7) and the fact that

$$\lim_{h\downarrow 0} h^{-1} \int_{t_i}^{t_i+h} S(t_i + h - s)f(\bar{t}, u_{t_i})ds = f(\bar{t}, u_{t_i}), \tag{15}$$

one can easily see that $\delta_i > 0$. Choose a number $d_i \in (\frac{1}{2}\delta_i, \delta_i]$, such that

$$d(S(d_i)x_i + \int_{t_i}^{t_i+d_i} S(t_i + d_i - s)f(\bar{t}, u_{t_i})ds; D(t_i + d_i)) \le \frac{d_i}{2n}. \tag{16}$$

Define $t_{i+1} = t_i + d_i$. By (16), there is $x_{i+1} \in D(t_{i+1})$ such that

$$\|S(d_i)x_i + \int_{t_i}^{t_{i+1}} S(t_{i+1} - s)f(\bar{t}, u_{t_i})ds - x_{i+1}\| \le \frac{d_i}{n}.$$

Consequently, x_{i+1} can be written as

$$x_{i+1} = S(t_{i+1} - t_i)x_i + \int_{t_i}^{t_{i+1}} S(t_{i+1} - s)f(\bar{t}, u_{t_i})ds + (t_{i+1} - t_i)p_i \tag{17}$$

with $\|p_i\| \leq \frac{1}{n}$. In this case we define u on $[t_i, t_{i+1}]$ as

$$u(t) = S(t-t_i)x_i + \int_{t_i}^{t} S(t-s)f(\bar{t}, u_{t_i})ds + (t-t_i)p_i. \tag{18}$$

Case 2 : $t_i \notin L_n$. In this case we set

$$\begin{aligned} \delta_i = \quad & \sup\{h \in (0, \tfrac{1}{n}] : t_i + h \leq t_0 + T, \\ & d(S(h)x_i + \textstyle\int_{t_i}^{t_i+h} S(t_i + h - s)f(t_i, u_{t_i})ds; D(t_i + h)) \leq \tfrac{h}{2n}\}. \end{aligned} \tag{19}$$

By (8) we see that $\delta_i > 0$. Choose $d_i \in (\frac{1}{2}\delta_i, \delta_i]$, such that

$$d(S(d_i)x_i + \int_{t_i}^{t_i+d_i} S(t_i + d_i - s)f(t_i, u_{t_i})ds; D(t_i + d_i)) \leq \frac{d_i}{2n}. \tag{20}$$

Define $t_{i+1} = t_i + d_i$. By (20), there is $x_{i+1} \in D(t_{i+1})$ such that

$$\|S(d_i)x_i + \int_{t_i}^{t_{i+1}} S(t_{i+1} - s)f(t_i, u_{t_i})ds - x_{i+1}\| \leq \frac{d_i}{n}.$$

Consequently, x_{i+1} can be written as

$$x_{i+1} = S(t_{i+1} - t_i)x_i + \int_{t_i}^{t_{i+1}} S(t_{i+1} - s)f(t_i, u_{t_i})ds + (t_{i+1} - t_i)p_i \tag{21}$$

with $\|p_i\| \leq \frac{1}{n}$. In this case we define u on $[t_i, t_{i+1}]$ as

$$u(t) = S(t-t_i)x_i + \int_{t_i}^{t} S(t-s)f(t_i, u_{t_i})ds + (t-t_i)p_i. \tag{22}$$

Setting $h(s) = f(\bar{t}, u_{t_i})$ in case $t_i \in L_n$ and $h(s) = f(t_i, u_{t_i})$ in case $t_i \notin L_n$ for $s \in [t_i, t_{i+1}]$. Let us define the step functions α_n and β_n as $\alpha_n(s) = t_i$ in case $t_i \notin L_n$, $\alpha_n(s) = \bar{t}$ in case $t_i \in L_n$ and $\beta_n(s) = t_i$ for $s \in [t_i, t_{i+1})$. Then h_n can be written as $h(s) = f(\alpha(s), u_{\beta(s)})$. By the induction hypotheses, u can be written in the form

$$\begin{aligned} u(t) = \quad & S(t-t_0)\phi(0) + \sum_{m=0}^{i-1} \textstyle\int_{t_m}^{t_{m+1}} S(t-s)h(s)ds \\ & + \textstyle\int_{t_i}^{t} S(t-s)h(s)ds + \sum_{m=0}^{i-1} (t_{m+1} - t_m)S(t - t_{m+1})p_m \\ & + (t-t_i)p_i. \end{aligned} \tag{23}$$

Let us check that $u_{t_{i+1}} \in B(\phi, r)$. To do this, we first note that each $s \in [t_0, t_{i+1}]$, there is an integer k such that $t_k < s \leq t_{k+1}$. Due to (23), we have

$$\begin{aligned} \|u(s) - \phi(0)\| \quad & \leq \|S(s-t_0)\phi(0) - \phi(0)\| \\ & + N \sum_{m=0}^{k} \textstyle\int_{t_m}^{t_{m+1}} \|h(s)\|ds + \sum_{m=0}^{k} (t_{m+1} - t_m)N\|p_m\| \\ \leq \quad & \|S(s-t_0)\phi(0) - \phi(0)\| + N(M+T) \end{aligned}$$

On the basis of the definition of $\mathcal{B}$, (12), (13) and the above inequality, we have

$$\begin{aligned}\|u_{t_{i+1}} - \phi\|_{\mathcal{B}} &\leq \|u_{t_{i+1}} - \bar{\phi}_{t_{i+1}}\|_{\mathcal{B}} + \|\bar{\phi}_{t_{i+1}} - \phi\|_{\mathcal{B}} \\ &\leq K(t_{i+1} - t_0)\sup\{\|u(s) - \bar{\phi}(s)\| : t_0 \leq s \leq t_{i+1}\} + \|\bar{\phi}_{t_{i+1}} - \phi\|_{\mathcal{B}} \\ &\leq K\sup\{\|u(s) - \phi(0)\| : t_0 \leq s \leq t_{i+1}\} + \|\bar{\phi}_{t_{i+1}} - \phi\|_{\mathcal{B}} \\ &\leq K[[\max_{0\leq t\leq T} \|S(t)\phi(0) - \phi(0)\| + N(M+T)] + \tfrac{1}{2}r \\ &\leq \tfrac{1}{2}r + \tfrac{1}{2}r = r,\end{aligned}$$

hence $u_{t_{i+1}} \in B_{\mathcal{B}}(\phi, r)$. Using (23) again , we derive

$$\|u(t) - \phi(0)\| \leq \|S(t-t_0)\phi(0) - \phi(0)\| + N(M+T) \leq \frac{1}{2}r < r$$

for all $t \in [t_0, t_{i+1}]$, i.e., $u(t) \in B_X(\phi(0), r)$ for $t \in [t_0, t_{i+1}]$. Thus, properties (ii), (iii) and (iv) are verified.

To prove property (i), we first note that $\lim_{i\to\infty} t_i$ exists, since $\{t_i\}_{i=1}^{\infty}$ is increasing and $t_i \leq t_0 + T$ for all $i = 1, 2, \cdots$. Suppose that $\lim_{i\to\infty} t_i = t^*$, then $t^* \leq t_0 + T$. We have to prove $t^* = t_0 + T$. To do this, we first show that $\lim_{i\to\infty} x_i$ also exists. In fact, let $j \geq i$. Using (23) for $t = t_i$ and $t = t_j$, we derive

$$\begin{aligned}\|x_j - x_i\| \leq\ & \|S(t_i - t_0)(S(t_j - t_i)\phi(0) - \phi(0)\| \\ &+ \sum_{m=0}^{i-1} \int_{t_m}^{t_{m+1}} \|S(t_i - s)(S(t_j - t_i)h(s) - h(s))\| ds \\ &+ \sum_{m=0}^{i-1} (t_{m+1} - t_m)\|S(t_i - t_{m+1})(S(t_j - t_i)p_m - p_m)\| \\ &+ \sum_{m=i}^{j-1} \| \int_{t_m}^{t_{m+1}} S(t_j - s)h(s)ds\| \\ &+ \sum_{m=i}^{j-1} (t_{m+1} - t_m)\|S(t_j - t_{m+1})p_m\| \\ \leq\ & N\|S(t_j - t_i)\phi(0) - \phi(0)\| \\ &+ N\sum_{m=0}^{i-1} \int_{t_m}^{t_{m+1}} \|S(t_j - t_i)h(s) - h(s)\| ds \\ &+ N\sum_{m=0}^{i-1} (t_{m+1} - t_m)\|S(t_j - t_i)p_m - p_m\|. \\ &+ N\int_{t_i}^{t_j} m_R(s)ds + N(t_j - t_i)\frac{1}{n}.\end{aligned} \tag{24}$$

Now given $\varepsilon > 0$. Since $m_R \in L(a, b; X)$, there is $\eta > 0$ such that $\int_{t'}^{t''} m_R(s)ds \leq \varepsilon/(5N)$ for $t', t'' \in (a, b)$ with $|t'' - t'| < \eta$. By the existence of $\lim_{i\to\infty} t_i = t^*$, there is a positive integer k_0 such that

$$t_j - t_i < \min\{\frac{\varepsilon}{10N(N+1)M}, \frac{\varepsilon}{10(N+1)}, \eta\} \tag{25}$$

for all $j > i \geq k_0$. Choose $k_1 > k_0$ with the properties: for $j > i \geq k_1$,

- $\|S(t_j - t_i)\phi(0) - \phi(0)\| \leq \varepsilon/(5N)$;
- $\|S(t_j - t_i)p_m - p_m\| \leq \varepsilon/(10NT), \quad 1 \leq m \leq k_0 - 1$;
- $\|S(t_j - t_i)f(t_m, u_{t_m}) - f(t_m, u_{t_m})\| \leq \varepsilon/(10NT), \quad 1 \leq m \leq k_0 - 1$ with $t_m \notin L_n$;
- $\|S(t_j - t_i)f(\bar{t}, u_{t_m}) - f(\bar{t}, u_{t_m})\| \leq \varepsilon/(10NT), \quad 1 \leq m \leq k_0 - 1$ with $t_m \in L_n$.

Then we have

$$N\|S(t_j - t_i)\phi(0) - \phi(0)\| \leq N\frac{\varepsilon}{5N} = \frac{\varepsilon}{5}; \tag{26}$$

$$\begin{aligned}
& N\sum_{m=0}^{i-1}\int_{t_m}^{t_{m+1}}\|S(t_j-t_i)h(s)-h(s)\|ds \\
\leq\; & N(\sum_{m=0}^{k_0-1}\int_{t_m}^{t_{m+1}}\|S(t_j-t_i)h(s)-h(s)\|ds \\
& +\sum_{m=k_0}^{i-1}\int_{t_m}^{t_{m+1}}\|S(t_j-t_i)h(s)-h(s)\|ds) \\
\leq\; & N(t_{k_0}-t_0)\tfrac{\varepsilon}{10NT}+N(t_i-t_{k_0})(N+1)M \\
\leq\; & \tfrac{\varepsilon}{5};
\end{aligned} \tag{27}$$

$$N\int_{t_i}^{t_j} m_R(s)ds \leq N\frac{\varepsilon}{5N}=\frac{\varepsilon}{5}; \tag{28}$$

$$\begin{aligned}
& N\sum_{m=0}^{i-1}(t_{m+1}-t_m)\|S(t_j-t_i)p_m-p_m\| \\
\leq\; & N\sum_{m=0}^{k_0-1}(t_{m+1}-t_m)\|S(t_j-t_i)p_m-p_m\| \\
& +N\sum_{m=k_0}^{i-1}(t_{m+1}-t_m)\|S(t_j-t_i)p_m-p_m\| \\
\leq\; & N(t_{k_0}-t_0)\tfrac{\varepsilon}{10NT}+(t_i-t_{k_0})N(N+1) \\
\leq\; & \tfrac{\varepsilon}{5};
\end{aligned} \tag{29}$$

$$N(t_j-t_i)<\frac{\varepsilon}{5}. \tag{30}$$

From (24) to (30), we obtain that

$$\|x_j-x_i\|\leq\varepsilon \tag{31}$$

for all $j>i\geq k_1$, i.e., $\{x_i\}$ is a Cauchy sequence. Therefore $\lim_{i\to\infty}x_i=x^*$ exists, and $x^*\in B(\phi(0),r)\cap D(t^*)$ since $B(\phi(0),r)\cap D(t)\neq\emptyset$ is closed for all $t\in[t_0,t_0+T]$. We define $u(t^*)=x^*$. By (iv) we have

$$\|u(t)-x_i\|\leq\|S(t-t_i)x_i-x_i\|+(t_i-t)(M+1)$$

and therefore $\lim_{t\uparrow t^*}u(t)=x^*=u(t^*)$. Accordingly, u is continuous on $[t_0,t^*]$, and hence u_t is continuous on $[t_0,t^*]$. Therefore, $\lim_{i\to\infty}u_{t_i}=u_{t^*}\in B_{\mathcal{B}}(\phi,r)$.

We assert that $t^*\notin L_n$ for sufficiently large n. Indeed, if $t^*\in L_n$, then there are only finite many $t_i\notin L_n$ since $[t_0,t^*]\setminus L_n$ is closed. Therefore there is a positive integer i_0 such that $t_i\in L_n$ for all $i\geq i_0$. But then $[t_{i_0},t^*]\subset L_n$ by (15), which contradicts the fact that $\lambda(L_n)<\frac{1}{n}$ for sufficiently large n.

We now assume by contradiction that $t^*<t_0+T$. We choose $h^*\in(0,\frac{1}{n}]$ such that

$$d(S(h^*)x^*+\int_{t^*}^{t^*+h^*}S(t^*+h^*-s)f(t^*,u_{t^*})ds;D(t^*+h^*)\leq\frac{h^*}{4n}. \tag{32}$$

Since $\frac{1}{2}\delta_i<d_i$ and $d_i=t_{i+1}-t_i\to 0$ as $i\to\infty$, there is a positive integer i_0 such that $\delta_i<h^*$ for all $i>i_0$. On the basis of (19), we have

$$d(S(h^*)x^*+\int_{t_i}^{t_i+h^*}S(t_i+h^*-s)f(t^*,u_{t^*})ds;D(t_i+h^*))>\frac{h^*}{2n} \tag{33}$$

for $i>i_0$ and $t_i\notin L_n$. Letting $i\to\infty$ in (33), one obtains an inequality which contradicts (32). Hence $t^*=t_0+T$, which concludes the proof. □

Proof of sufficiency. Let $\{L_n\}$ be a sequence of open subsets of $\mathbb{R}$ such that $Z \subset L_n$ and $\lambda(L_n) < \frac{1}{n}$ for all $n \in \mathbb{N}$. Take $L = \cap_{n \geq 1} L_n$ and a sequence of n-approximate solutions $\{u^n\}$ and $\{t_i^n\}$ obtained in Lemma 1. Let us define

$$g_n(t) = \sum_{m=0}^{i-1} (t_{m+1}^n - t_m^n) S(t - t_{m+1}^n) p_m^n + (t - t_i^n) p_i^n$$

for $t \in [t_i, t_{i+1}]$. Then $\|g_n(t)\| \leq \frac{NT}{n}$ for all $t \in [t_0, t_0 + T]$ and u^n can be written in the form

$$u^n(t) = S(t - t_0)\phi(0) + \int_{t_0}^{t} S(t-s) h_n(s) ds + g_n(t) \tag{34}$$

for all $t \in [t_0, t_0 + T]$, $u_{t_0}^n = \phi$. Set

$$y^n(t) = \int_{t_0}^{t} S(t-s) h_n(s) ds, \quad t \in [t_0, t_0 + T].$$

Since the semigroup $S(t) : X \to X, t \leq 0$, is compact and $\{h_n\}$ is uniformly integrable on $[t_0, t_0 + T]$, by a standard argument involving a compactness result, it follows that there is a $y \in C([t_0, t_0 + T]; X)$ such that at least on a subsequence we have

$$\lim_{n \to \infty} y^n(t) = y(t)$$

uniformly in $t \in [t_0, t_0 + T]$. Since $\|g_n(t)\| \leq \frac{NT}{n}$ for all $t \in [t_0, t_0 + T]$, it follows that

$$\lim_{n \to \infty} u^n(t) = S(t - t_0)\phi(0) + y(t) \equiv u(t) \tag{35}$$

uniformly in $t \in [t_0, t_0 + T]$. Let us observe that if $s \notin L$, then $s \notin L_n$ for sufficiently large n, and then we have $\alpha_n(s) \to s$ as $n \to \infty$. Also we have $\beta_n(s) \to s$ as $n \to \infty$ for all $s \in [t_0, t_0 + T]$. Therefore $h_n(s) \to f(s, u_s)$ as $n \to \infty$ for a.e. $s \in [t_0, t_0 + T]$. Moreover, $u^n(\alpha_n(s)) \in D(\alpha_n(s)) \cap B(\phi(0), r)$ implies $u(s) \in D(s) \cap B(\phi(0), r)$ due to (C2). Finally, passing to limit in (34), one obtains (3), which completes the proof. □

Remark 2. *In [10], the function f is defined on $[a, b) \times D$, and not on the whole $[a, b) \times X$, which is more general. Here, if we define $\mathcal{K}_1 = \{v \in \mathcal{B} : v(0) \in \bigcup_{t \in (a,b)} D(t)\}$, and let $\mathcal{K} = \{v \in \mathcal{B} : d(v, \mathcal{K}_1) < r\}$, where $d(v, \mathcal{K}_1)$ denotes the distance between $v \in \mathcal{B}$ and $\mathcal{K}_1 \subset \mathcal{B}$ and $r > 0$ is the number appeared in the proof of Lemma 1. From the proof of Lemma 1 we can see that, if f is defined on $(a, b) \times \mathcal{K}$, then the result of Theorem 3 still holds.*

Concerning the continuation of the solution to (1) satisfying (2). Recall that a solution $v : [t_0, t_0 + T_1] \to X$ of (1), with $T_1 \geq T$ is said to be a continuation to the right of the solution $u : [t_0, t_0 + T] \to X$ to (1), if $v(t) = u(t)$ for all $t \in [t_0, t_0 + T]$. A solution u is said to be noncontinuable if it has no proper continuation. Using a standard argument based on Zorn's Lemma, one can easily verify that, if the hypotheses of Theorem 3 hold, and $u : [t_0, b_0) \to X$ is a noncontinuable mild solution to (1) satisfying (2), then either $b_0 = b$ or $\lim_{t \uparrow b_0} \|u(t)\| = +\infty$. Moreover, the tangency condition (T) is also necessary. Precisely, we have

Theorem 4. *Under the hypotheses of Theorem 3, a necessary and sufficient condition in order that for each $t_0 \in (a, b)$, and each $\phi \in \mathcal{B}$ with $\phi(0) \in D(t_0)$, there is a noncontinuable mild solution $u(t) \in D(t)$ to (1) satisfying (2) is the tangency condition (T).*

Remark 3. *Consider (1) with finite delay (i.e., the case $\mathcal{B} = C([-q, 0]; X)$). If $D(t) \equiv D$, then the condition (C2) reduce to "D is locally closed". We can obtain the following result [22].*

Theorem 5. *Let $D \subset X$ be a locally closed subset in a general Banach space, $f : (a,b) \times C([-q,0];X) \to X$ a function of Carathéodory type, and let $A : D(A) \to X$ be the infinitesimal generator of a compact C_0-semigroup $S(t) : X \to X, t \geq 0$. Then a necessary and sufficient condition in order that D be a viable domain of (1) is the tangency condition (T).*

Remark 4. *If D is open, then the tangency condition (T) is automatically satisfied. In this case, by Theorem 3, one obtains the locally existence result of problem (1) and (2), which extends the well-known result of J. K. Hale [26], who considered the case in which X is finite dimensional (i.e., $X = \mathbb{R}^n$) and $A = 0$.*

Theorem 6. *Let X be a real Banach space X, $f : (a,b) \times C([-q,0];X) \to X$ a function of Carathéodory type, and let A be the infinitesimal generator of a compact C_0-semigroup $S(t) : t \geq 0$. Then for each $t_0 \in (a,b)$, and each $\phi \in C([-q,0];X)$ with $\phi(0) \in D$, the problem (1) and (2) has a locally mild solution, for some $T = T(t_0,\phi) > 0$, with $T < b - t_0$.*

4. Conclusions

We have extended the main result of [22] on the viability of semilinear functional differential equations from the case of with finite delay to infinite delay, and from a single set D (viable domain) to a tube $D(\cdot)$. Our approach is still via constructing a sequence of approximate solutions. Such sequence of approximate solutions convergence uniformly to a viable solution due to the compactness of relevant semigroup $\{S(t) : t \geq 0\}$. So a further question is whether the result holds if the semigroup is not compact. This will be studied in the future.

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