

Article

Quantum Measurements, Stochastic Networks, the Uncertainty Principle, and the Not So Strange “Weak Values”

Dmitri Sokolovski ^{1,2}

¹ Departamento de Química-Física, Universidad del País Vasco, UPV/EHU, Leioa 48940, Bizkaia, Spain; dgsokol15@gmail.com; Tel.: +34-94-601-5974; Fax: +34-94-601-3500

² IKERBASQUE, Basque Foundation for Science, Maria Diaz Haroko Kalea, 3, Bilbao 48013, Bizkaia, Spain

Academic Editors: Paul Busch, Takayuki Miyadera and Teiko Heinosaari

Received: 6 May 2016; Accepted: 23 August 2016; Published: 15 September 2016

Abstract: Suppose we make a series of measurements on a chosen quantum system. The outcomes of the measurements form a sequence of random events, which occur in a particular order. The system, together with a meter or meters, can be seen as following the paths of a stochastic network connecting all possible outcomes. The paths are shaped from the virtual paths of the system, and the corresponding probabilities are determined by the measuring devices employed. If the measurements are highly accurate, the virtual paths become “real”, and the mean values of a quantity (a functional) are directly related to the frequencies with which the paths are traveled. If the measurements are highly inaccurate, the mean (weak) values are expressed in terms of the relative probabilities’ amplitudes. For pre- and post-selected systems they are bound to take arbitrary values, depending on the chosen transition. This is a direct consequence of the uncertainty principle, which forbids one from distinguishing between interfering alternatives, while leaving the interference between them intact.

Keywords: quantum probabilities; uncertainty principle; transition amplitudes; “weak values”

PACS: 03.65.Ta; 02.50.Cw; 03.67.-a

1. Introduction

A sequence of measurements made on a quantum system leads to a number of random outcomes, observed by an experimentalist. In this sense, the composite system + meter(s) may be seen as following a path across a stochastic network, connecting all possible measurement results. The same can be said about a purely classical stochastic system, and the quantum nature of the experiment reveals itself only in the manner in which the transition probabilities are calculated. While a classical theory allows for non-invasive observations, a quantum meter plays an active role in “shaping” the network. In the simplest case of von Neumann (vN) measurements [1], the transition probabilities, as well as the observed events themselves are determined by the virtual paths available to the studied system, the nature of the measured quantity and the accuracy of the meter(s).

Perhaps the most studied network is produced by an accurate vN measurement of a projector on a state $|\psi\rangle$ (preparation or pre-selection), followed by a not necessarily accurate vN measurement of an operator \hat{A} , followed by an accurate measurement of a projector on a final state $|\phi\rangle$ (post-selection) [2]. Post-selection may succeed or fail, and a sub-ensemble is selected by collecting the statistics only in the case it has been successful. If \hat{A} has N distinct eigenvalues, there are N virtual paths connecting the two states, for which quantum mechanics provides probability amplitudes, but not probabilities, if the system is considered in isolation. If the measurement of \hat{A} is accurate (ideal, strong), the meter destroys the interference between the paths, which can be now equipped with probabilities (typically,

squares of the moduli of the corresponding amplitudes) and become “real”. Each time the experiment is repeated, the pointer points at one of the eigenvalues of the operator \hat{A} , whose mean value is then calculated using the relative frequencies of these occurrences. The probability to end up in $|\phi\rangle$ is not what it would be, had the meter not perturbed the system’s evolution.

More controversial is the opposite limit, where the intermediate measurement is made inaccurate (weak), in order to avoid the said perturbation. The weakness of the measurement inevitably causes the meter’s readings to be spread, covering the whole real axis in the limit when the perturbation is sent to zero [2]. This gives an operational meaning to the uncertainty principle, which states that the value of \hat{A} in a superposition of its eigenstates must be indeterminate. Indeed, trying to measure it without destroying the superposition may yield any reading at all.

Much of the controversy resides, however, in the description of the expectation of a weak meter’s reading (weak value) as a “new concept” [2], capable of shedding new light on quantum phenomena. In the years that followed it gave rise a number of extraordinary notions. These include “negative kinetic energy” [3], “negative number of particles” [4], “having one particle in several places simultaneously” [5], “photons disembodied from its polarisation” [6], “electrons with disembodied charge and mass” [6] and “an atom with the internal energy disembodied from the mass” [6], all supported by the “evidence” of weak measurements.

Recently, we have shown [7] that the above statements amount to over-interpretations, easily dismissed once the weak values are identified with the transition amplitudes on the paths connecting the initial and final states, and are not seen as the values of the measured quantities. In this paper, we will continue to look at the problem of consecutive quantum measurements, focusing on the classical statistical ensembles (networks) produced whenever a meter, or meters, interact with the observed quantum system. We will pay special attention to the importance of the quantum uncertainty principle and emphasize the role that quantum interference plays in the loss of information about the system’s past. Throughout the paper, we will use the simplest case of a two-level system as an example and refer the reader to [8–15] for a more general analysis.

The rest of the paper is organized as follows. In Section 2, we compare two of the best known formulations of the uncertainty principle. In Section 3, we briefly describe the work of a von Neumann meter. In Section 4, we introduce a simple classical toy model, to be compared with its quantum counterparts later. Section 5 returns to the simplest quantum network produced by a vN measurement of a quantity, with pre- and post-selection. In Section 6, we use our approach to describe a measurement of the difference of two physical quantities, in such a way that the values themselves remain indeterminate. Section 7 sets the rules for combining virtual paths. Accurate (strong) and inaccurate (weak) limits of the measurements are analyzed in Sections 8 and 9, respectively. In Section 10, the uncertainty principle is used to explain the properties of the weak values. Common misconceptions related to the issue are discussed in Section 11. Section 12 contains our conclusions.

2. Quantum Uncertainty Principle(s)

There are several formulations of the uncertainty principle (UP), covering various aspects of the problem (for a recent review, see [16]). Perhaps best known is the one relating the respective standard deviations, σ_A and σ_B , of two variables, represented by non-commuting operators \hat{A} and \hat{B} , measured in the same state. The so-called Robertson’s uncertainty relation [17] reads:

$$\sigma_A \sigma_B \geq \frac{|\langle [\hat{A}, \hat{B}] \rangle|}{2}, \quad (1)$$

where $\sigma_X^2 \equiv \langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2$, $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$ is the commutator, and the angular brackets indicate averaging in the pure state $|\Psi\rangle$ that the system is supposed to be in, $\langle \hat{X} \rangle = \langle \Psi | \hat{X} | \Psi \rangle$. Equation (1) is often interpreted as an indication of the fact that a measurement of \hat{A} must disturb a measurement of \hat{B} , and vice versa.

A somewhat different approach to the UP can be found in [18], where one reads: “Any determination of the alternative taken by a process capable of following more than one alternative destroys the interference between alternatives”. This principle is complemented by the rule for assigning probabilities to alternative scenarios [19]: “When an event can occur in several alternative ways, the probability amplitude for the event is the sum of the probability amplitudes for each way considered separately. If an experiment is performed which is capable of determining when one or another alternative is actually taken, the probability of the event is the sum of the probabilities of each alternative”.

From the above statements, to which we will refer as “Feynman’s uncertainty principle”, one may conclude that interfering alternatives cannot be told apart and must, therefore, form a single indivisible pathway [20]. Another corollary to the principle is that interference must be destroyed by a physical agent, e.g., a meter coupled to the observed system. Thus, construction of probabilities by means of squaring the moduli of the corresponding amplitudes pre-supposes the existence of a suitable meter, as well as the fact that the meter has already been deployed [15].

It is the Feynman’s UP to which we will appeal in what follows. Despite the obvious difference between the formulations, Feynman’s principle is not all that different from the UP expressed by Equation (1). Consider, for example, a system with a zero Hamiltonian, $\hat{H} = 0$, prepared at $t = 0$ in a state $|\psi\rangle$. An accurate measurement of \hat{B} at $t = T$ yields an eigenvalue b_{i_B} with the probability $p(i_B|\psi) = |\langle i_B|\psi\rangle|^2$, where $|i_B\rangle$ is the corresponding eigenstate. Suppose we also want to learn something about the value of \hat{A} at $t = T/2$. The system must be, in some sense, in one of the eigenstates of \hat{A} . To see in what sense, precisely, we insert the unity $\sum_{i_A} |i_A\rangle\langle i_A| = 1$ into the amplitude $\langle i_B|\psi\rangle$. The result is $\sum_{i_A} \langle i_B|i_A\rangle\langle i_A|\psi\rangle$. There are now different ways (paths, pathways, routes) to reach the final state by “passing through one of the states $|i_A\rangle$ ”. Without a meter, we cannot determine which of them is actually taken. Thus, despite the abundance of virtual paths, there is only a single “real” one, “traveled” with the probability $p(i_B)$. If, on the other hand, we employ a meter, capable of telling us which of the eigenvalues a_{i_A} has occurred at $t = T/2$, the virtual paths will become real, with the i_A -th path traveled with the probability $p(i_B|i_A|\psi) = |\langle i_B|i_A\rangle\langle i_A|\psi\rangle|^2$. We also find that the measurement of \hat{A} has disturbed the result of measuring \hat{B} , since:

$$\sum_{i_A} p(i_B|i_A|\psi) \neq p(i_B|\psi). \quad (2)$$

Quantifying the disturbance would lead to estimates similar to (1). We will not follow this matter any further, and in the next section we discuss possible realizations of quantum measurements.

3. Von Neumann Measurements and Meters

In the following, we will often refer to von Neumann (vN) measurements [1], whose definition we will revisit in this section. A vN meter, designed to measure a variable \hat{A} of a system governed by a Hamiltonian \hat{H} , consists of a pointer (e.g., a massive particle on a line) whose position is ξ and which is coupled to the observed system via ($\hbar = 1$):

$$\hat{H}_{int} = -i\beta(t)\frac{\partial}{\partial\xi}\hat{A}. \quad (3)$$

If the value of \hat{A} is measured at some $t = t_0$, the function $\beta(t)$ is chosen to be $\beta(t) = \delta(t - t_0)$, where $\delta(z)$ is the Dirac delta. We will assume that \hat{A} has discrete eigenvalues a_i , and $|i\rangle$ are the corresponding eigenstates, $\hat{A} = \sum_i |i\rangle a_i \langle i|$. Prior to the interaction, at $t = 0$, the system is described by a state $|\psi\rangle = \sum c_i(0)|i\rangle$, and the pointer is in the state $|M\rangle$, such that:

$$G(\xi) \equiv \langle \xi|M\rangle \quad (4)$$

is a real function (e.g., a Gaussian), peaked around $\xi = 0$ and rapidly tending to zero as $\xi \rightarrow \pm\infty$. (The choice of $G(\xi)$ as a real function ensures that both the mean momentum and the mean position of the pointer are zero. In the classical limit, a vN meter does not perturb the motion of the monitored system, provided its momentum is set exactly to zero. Quantally, this condition can be met only approximately. The spread of the momenta around the zero mean accounts, therefore, for the inevitable perturbation the meter would incur. Making G broad in the coordinate space would minimise this perturbation, at the expense of the measurement's accuracy.) Immediately after the interaction has taken place, the pointer becomes entangled with the system, and their wave function is given by:

$$\langle \xi | \Psi(t_0 + 0) \rangle = \sum_i c_i(t_0) G(\xi - a_i) |i\rangle, \quad (5)$$

where $c_i(t_0) = \langle i | \exp[-i\hat{H}(t_0)] \psi \rangle$. We assume further that the final position of the pointer can be determined precisely, leaving out the fundamental question of how exactly this is done and whether a collapse of the wave function has occurred [1]. In one way or another, after the interaction has taken place, we may “look” at the pointer and find a meter's reading ξ . Perhaps the best known practical realization of a vN measurement is the Stern–Gerlach apparatus (see, for example, [2,21]). Other vN meters can be constructed by using the spin-orbit interaction [22–24] or Bose–Einstein condensates trapped in double-well potentials [25].

If nothing else is done to the system, the probability (density) to find a reading ξ_0 is ($tr[\dots]$ stands for the trace of the operator inside the brackets):

$$P(\xi_0) = tr[|\xi_0\rangle \langle \xi_0 | \Psi(t) \rangle \langle \Psi(t) |] = \sum_i |c_i(t_0)|^2 G^2(\xi_0 - a_i), \quad (6)$$

where $|\Psi(t)\rangle = \exp[-i\hat{H}(t - t_0)] |\Psi(t_0 + 0)\rangle$. Alternatively, at some $t = T > t_0$, we may try to find (post-select) the system in some state $|\phi\rangle$, and keep the meter's readings only if the post-selection is successful. With this additional condition, the probability to find a reading ξ_0 becomes:

$$P(\xi_0) = tr[|\xi_0\rangle \langle \xi_0 | \Psi(T) \rangle \langle \Psi(T) | \phi \rangle \langle \phi |] = \left| \sum_i \langle \phi | \exp[-i\hat{H}(T - t_0)] |i\rangle G(\xi_0 - a_i) \langle i | \exp(-i\hat{H}t_0) | \psi \rangle \right|^2, \quad (7)$$

Thus, apart from the preparation (pre-selection) of the system in a state $|\psi\rangle$, two other events have occurred: the pointer was found pointing at ξ_0 , and the system was later found in a final state $|\phi\rangle$. Quantum mechanics provides the probabilities of such occurrences by means of expressions similar to Equations (6) and (7). We can imagine, for example, performing more measurements between $t = 0$ and $t = T$, thus creating a network of scenarios, each of which would occur with certain frequency if the experiment is repeated many times. Before discussing such “quantum” networks, we briefly turn to their classical counterparts.

4. Classical Networks and Functionals

We start with a simple toy model, which, we hope, will be helpful for the discussion of the following sections. Suppose one has at his/her disposal a kit containing purely classical elements, little balls, tubes and connectors, which can be joined together. A connector x has two entrances, and $w_x(i|j)$ is the probability to enter via the inlet j and subsequently leave via the outlet i . If one of the outlets is blocked, the probability to exit via the remaining one is unity for both entrances. Connecting the elements as shown in Figure 1, one can “engineer” a network of paths, which a ball inserted at the top will travel at random, until ending up in one of the final states f_1 and f_2 . Ascribing variables (numbers) a_1, a_2, b_1, b_2 to each of the connectors, one can also associate numbers (functionals) with each path. For example, in Figure 1a, there are eight paths $\{i\}$, $i = 1, \dots, 8$. Four of them end in the final state f_1 ,

- {1}: $f_1 \leftarrow b_1 \leftarrow a_1 \leftarrow in$, traveled with the probability $p[1] = w_{b_1}(1|1)w_{a_1}(1|1)w_{in}(1|1)$,
- {2}: $f_1 \leftarrow b_1 \leftarrow a_2 \leftarrow in$, traveled with the probability $p[2] = w_{b_1}(1|2)w_{a_2}(2|1)w_{in}(2|1)$,
- {3}: $f_1 \leftarrow b_2 \leftarrow a_1 \leftarrow in$, traveled with the probability $p[3] = w_{b_2}(2|2)w_{a_1}(2|1)w_{in}(1|1)$,
- {4}: $f_1 \leftarrow b_2 \leftarrow a_2 \leftarrow in$, traveled with the probability $p[4] = w_{b_2}(2|1)w_{a_2}(1|1)w_{in}(2|1)$,

while four more paths, $\{j\}$, $j = 5, 6, 7, 8$, end in f_2 . It is always possible to consider sub-networks containing only few paths, regardless of what is happening at later times, or elsewhere in the diagram.

For example, we may limit our attention only to the cases where the ball goes to b_1 . As a result of this “post-selection”, there are only two paths: {1} through $b_1 \leftarrow a_1 \leftarrow in$ and {2} through $b_1 \leftarrow a_2 \leftarrow in$, traveled with probabilities $p[1] = w_{a_1}(1|1)w_{in}(1|1)$ and $p[2] = w_{a_2}(2|2)w_{in}(2|1)$, respectively. There is a single random variable, a functional $F_1[i]$ taking the values a_i , $i = 1, 2$, and whose mean value, evaluated over many runs by counting the times the ball passes through a_1 and a_2 , is: $\langle F_1 \rangle = \{p[1]F[1] + p[2]F[2]\} / \{p[1] + p[2]\}$.

Alternatively, we may focus only on those cases where the ball ends up in f_1 , and ask for the difference between the values of b_i and a_j , $b_i - a_j$, which is recorded for each run. Let $a_1 = b_1 = -1$ and $a_2 = b_2 = 1$. Now, the said difference is a functional $F_2[path]$ defined on the four paths introduced above and taking the values $F_2[1] = F_2[4] = 0$, $F_2[2] = -2$ and $F_2[3] = 2$. Since two of the values of F_2 are the same, we can reduce the number of paths by one, by combining {1} and {4} into a single pathway, $\{1+4\}$, traveled with a probability $p[1+4] = p[1] + p[4]$, and on which $F_2[1+4]$ is understood to be zero. The resulting equivalent system is shown in Figure 1b, and for $\langle F_2 \rangle$, evaluated by writing down the value of F_2 for each run of the ball, adding up the results and dividing by the number of runs, we find:

$$\langle F_2 \rangle = \frac{p[2]F[1] + p[3]F[2]}{p[1+4] + p[2] + p[3]}. \quad (8)$$

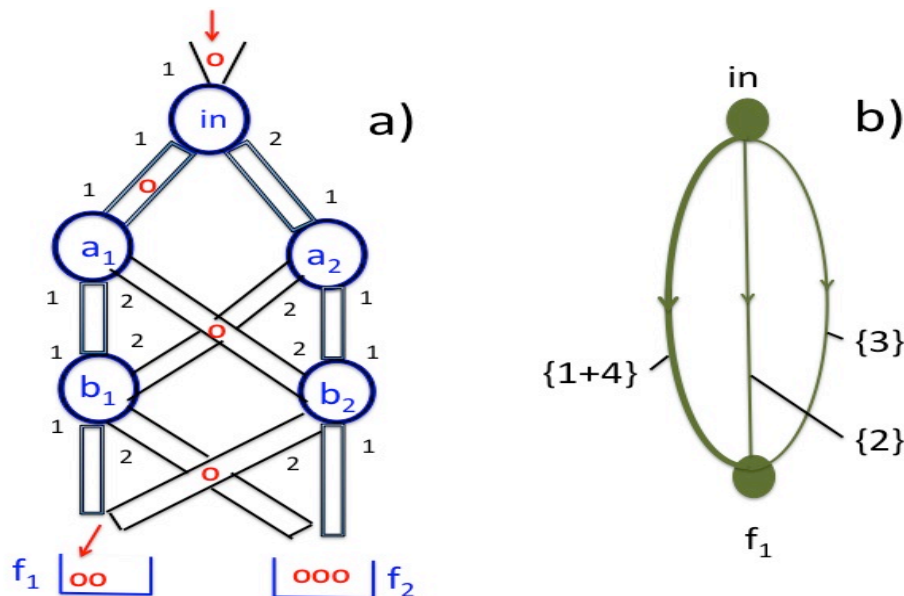


Figure 1. (a) Schematic diagram of the toy model of Section 4. A ball is introduced at the top and follows the path {3} before ending up in the receptacle f_1 . The numbers mark the inlets and outlets of a connector, with the closed ones not shown; (b) A reduced model for the system post-selected in f_2 , with only F_2 measured.

We have invoked the simple toy model in Figure 1 in order to emphasize its resemblances to and differences from the quantum case outlined at the end of the previous section. As in the classical case,

a set of vN measurements creates sequences of events occurring with certain probabilities, which can be seen as classical pathways “traveled” with certain frequencies, if the the experiment is repeated many times.

The quantum origin of the experiment becomes evident in the way the probabilities of the events are calculated, as well as in the nature of these events. A meter recombines and divides the system’s virtual paths, endowed with probability amplitudes only, thereby producing “real” ones, to which probabilities can be assigned. In particular, two interfering paths are combined into a single one in a much stronger sense, than in the classical case. For example, in Figure 1b, the paths $\{1\}$ and $\{4\}$ are combined into $\{1 + 4\}$, since we chose to be interested only in the difference b_i of a_i and not in their actual values. We will show that in the quantum case, these actual values may not even exist. Finally, in the classical case, it is understood that two different setups built from the elements of the same kit can be completely different and need not share any statistical properties. In the quantum case, this is less clear, and we will return to the problem of “contextuality” later.

5. A Simple Quantum Network: Measuring a Quantity with Post-Selection

As the first example, we consider a vN measurement of an operator $\hat{A} = \sum_{i=1}^2 |i\rangle a_i \langle i|$, for a two-level system, pre- and post-selected in the states $|\psi\rangle$ and $|\phi\rangle$, at $t = 0$ and $t = T$, respectively. For simplicity, we put the system’s Hamiltonian to zero, $\hat{H} = 0$, so that nothing happens between $t = 0$ and $t = t_0$, as well as between $t = t_0$ and $t = T$. Our purpose is to show that this creates a classical statistical ensemble not too dissimilar from the one built from the classical elements in the previous section. To see how this is done, we consider the two virtual paths connecting $|\psi\rangle$ and $|\phi\rangle$, $\{i\}$, $i = 1, 2$. These are

$$\begin{aligned} \{1\}: |\phi\rangle &\leftarrow |1\rangle \leftarrow |\psi\rangle, \text{ with an amplitude } A[1] = \langle\phi|1\rangle\langle 1|\psi\rangle, \text{ and} \\ \{2\}: |\phi\rangle &\leftarrow |2\rangle \leftarrow |\psi\rangle, \text{ with an amplitude } A[2] = \langle\phi|2\rangle\langle 2|\psi\rangle. \end{aligned}$$

We also have $\sum_{i=1}^2 A[i] = \langle\phi|\psi\rangle$.

The value of \hat{A} at $t = t_0 < T$ is a functional $F_1[path]$ defined on the two paths and, obviously, taking the values a_1 and a_2 :

$$F_1[1] = a_1, \quad \text{and} \quad F_1[2] = a_2. \quad (9)$$

Without a meter, we may define an amplitude distribution $\Phi(f)$ for the value f taken by F_1 ,

$$\Phi(f) = A[1]\delta(f - a_1) + A[2]\delta(f - a_2), \quad \int \Phi(f)df = \sum_{i=1}^2 A[i] = \langle\phi|\psi\rangle \quad (10)$$

but cannot say what is the probability for having, e.g., the value a_1 .

With a meter (3) coupled to the system, the final pointer’s state $|M'\rangle$ is easily found to be given by:

$$M'(\xi) \equiv \langle\xi|M'\rangle = \int G(\xi - f)\Phi(f)df = A[1]G(\xi - a_1) + A[2]G(\xi - a_2), \quad (11)$$

and the meter reads ξ_0 with a (unnormalized) probability:

$$P(\xi_0) = |M'(\xi_0)|^2 = |A[1]G(\xi_0 - a_1) + A[2]G(\xi_0 - a_2)|^2. \quad (12)$$

Equation (12) may be interpreted in the following way. By turning the meter on, we have created a continuum of real paths, all labeled by ξ . These connect the three observed events, *system in $|\phi\rangle \leftarrow$ pointer at $\xi_0 \leftarrow$ system in $|\psi\rangle$* , assuming the meter is read before the post-selection is performed.

The range of possible scenarios is determined by the width of the initial pointer's state $G(\xi)$ in Equation (4), which also determines the accuracy of the measurement. For example, if $G(\xi)$ is chosen to be:

$$G(\xi) = 1/\sqrt{\Delta\xi} \quad \text{for } |\xi| \leq \Delta\xi/2, \quad \text{and } 0 \quad \text{otherwise,} \quad (13)$$

the possible values of ξ would lie inside the interval $[a_1 - \Delta\xi/2, a_2 + \Delta\xi/2]$, assuming $a_2 > a_1$ (see Figure 2). The path labeled ξ is traveled with a probability (density) $P(\xi)$. Far from passively observing the motion of the system, a meter actively participates in shaping the observed statistical ensemble. Accordingly, the probabilities in Equation (12) contain both the amplitudes $A[i]$, which characterize the unobserved system, and the meter's own $G(\xi)$.

In the general case, we may learn little about the quantity of interest, \hat{A} . Instead of extracting its value, we can only conclude that if the meter reading is ξ_0 , the system's state after interaction with the meter was (the next Equation follows upon multiplication of the pure state $|\Psi(t+0)\rangle\langle\Psi(t+0)|$ in Equation (5) by a projector $|\xi_0\rangle\langle\xi_0|$, and tracing out the pointer's variable):

$$|\Psi(\xi_0)\rangle = G(\xi_0 - a_1)\langle 1|\psi\rangle|1\rangle + G(\xi_0 - a_2)\langle 2|\psi\rangle|2\rangle, \quad (14)$$

for which the precise value of \hat{A} remains indeterminate. It is, however, possible to say that the measurement “drives” the system through a state $|\Psi(\xi_0)\rangle$, with a probability $P(\xi_0)$. To see that this is, indeed, the case, we could make an accurate measurement of the projector onto $|\Psi(\xi_0)\rangle$ just after t_0 . Repeating the procedure many times will show that whenever the reading of the first meter lies in a narrow interval around ξ_0 , the additional projection will also succeed. Thus, while every attempt to measure \hat{A} leads to a particular stochastic network, not every measurement allows us to determine the value of \hat{A} . We will return to the question of the accuracy shortly after considering yet another example.

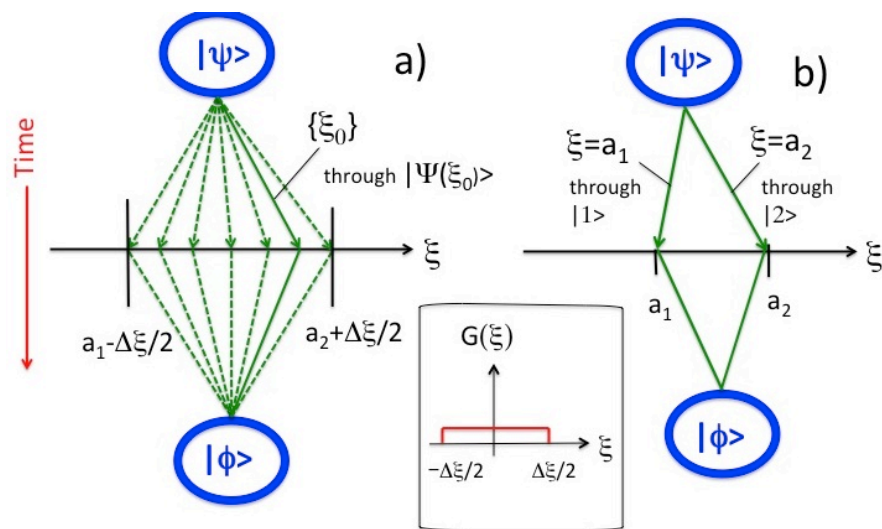


Figure 2. (a) Schematic diagram showing a continuum of real pathways produced in a measurement of an operator \hat{A} with eigenvalues $a_1 < a_2$. A pathway connects three observable events: preparation in $|\psi\rangle$ at $t = 0$, obtaining a meter reading ξ_0 at $t = T/2$ and finding the system in $|\phi\rangle$ at $t = T$. If a reading ξ_0 is obtained, the system “passes” through the state $|\Psi(\xi_0)\rangle$ in Equation (14). For $G(\xi)$ in Equation (13) (see the inset), the pointer readings are contained in the interval $[a_1 - \Delta\xi/2, a_2 + \Delta\xi/2]$; (b) The two real pathways surviving in the high accuracy limit, $\Delta\xi \rightarrow 0$.

6. A More Elaborate Network: Measuring the Difference of Two Quantities

Suppose that, as in the previous section, we have a two-level system pre- and post-selected in the states $|\psi\rangle$ and $|\phi\rangle$ at $t = 0$ and $t = T$, respectively. Now, we want to learn something about the difference of the quantities represented by operators $\hat{A} = \sum_{i_A=1}^2 |i_A\rangle a_{i_A} \langle i_A|$ and $\hat{B} = \sum_{i_B=1}^2 |i_B\rangle b_{i_B} \langle i_B|$, which may or may not commute, at $t = t_1$ and $t = t_2$, such that $0 < t_1 < t_2 < T$ (classically, we could consider, for example, the x - and y -components of the momentum and ask for $p_y(t_2) - p_x(t_1)$). Quantumly, there is more than one way to approach this task.

- (i) We could employ two vN meters, set to measure \hat{A} and \hat{B} separately, and evaluate the difference of their readings.
- (ii) We could also construct a Hermitian operator formally representing the difference of interest, $\hat{C} = \exp[-i\hat{H}(t_2 - t_1)]\hat{B}\exp[i\hat{H}(t_2 - t_1)] - \hat{A}$, and use a single vN meter to measure it at $t = t_1$. As discussed in [26], a classical analog of this procedure is a predictive measurement, which uses the fact that knowing the system's coordinates and momenta at $t = t_1$ allows one to restore the system's subsequent trajectory and, with it, all future values of all quantities.
- (iii) A procedure unique to quantum mechanics allows one to evaluate the difference of interest, leaving the actual values of \hat{A} and \hat{B} indeterminate.

In general, all three methods give different results [26], and here, we are only interested in the option (iii). With this in mind, we couple a vN pointer (position ξ), to \hat{A} and \hat{B} at $t = t_1$ and $t = t_2$, respectively, so that the interaction with the meter is given by:

$$H_{int} = -i \frac{\partial}{\partial \xi} [\hat{B} \delta(t - t_2) - \hat{A} \delta(t - t_1)], \quad (15)$$

and the pointer's initial state, $G(\xi)$, has, as before, a Gaussian shape, centered at $\xi = 0$. The measurement works as follows: after the first coupling, the pointer is shifted by one of the eigenvalues of \hat{A} and after the second one, by an eigenvalue of \hat{B} , albeit in the opposite sense. The resulting shift is, thus, the difference of the two values. This is not exactly the scheme used by von Neumann [1], and we will call this meter von Neumann-like [12].

Without a meter, there are four virtual paths, $\{i\}$, $i = 1, 2, 3, 4$, which connect $|\psi\rangle$ and $|\phi\rangle$ and pass through different intermediate states. For example, the path going through $|1_A\rangle$ and then through $|2_B\rangle$ has a probability amplitude $\langle \phi | \exp[-i\hat{H}(T - t_2)] | 2_B \rangle \langle 2_B | \exp[-i\hat{H}(t_2 - t_1)] | 1_A \rangle \langle 1_A | \exp(-i\hat{H}t_1) | \psi \rangle$. Putting, as before, $\hat{H} = 0$, we, therefore, have the paths

$$\begin{aligned} \{1\}: & |\phi\rangle \leftarrow |1_B\rangle \leftarrow |1_A\rangle \leftarrow |\psi\rangle, \text{ with an amplitude } A[1] = \langle \phi | 1_B \rangle \langle 1_B | 1_A \rangle \langle 1_A | \psi \rangle, \\ \{2\}: & |\phi\rangle \leftarrow |1_B\rangle \leftarrow |2_A\rangle \leftarrow |\psi\rangle, \text{ with an amplitude } A[2] = \langle \phi | 1_B \rangle \langle 1_B | 2_A \rangle \langle 2_A | \psi \rangle, \\ \{3\}: & |\phi\rangle \leftarrow |2_B\rangle \leftarrow |1_A\rangle \leftarrow |\psi\rangle, \text{ with an amplitude } A[3] = \langle \phi | 2_B \rangle \langle 2_B | 1_A \rangle \langle 1_A | \psi \rangle, \\ \{4\}: & |\phi\rangle \leftarrow |2_B\rangle \leftarrow |2_A\rangle \leftarrow |\psi\rangle, \text{ with an amplitude } A[4] = \langle \phi | 2_B \rangle \langle 2_B | 2_A \rangle \langle 2_A | \psi \rangle \end{aligned}$$

with the amplitudes satisfying a condition $\sum_{i=1}^4 A[i] = \langle \phi | \psi \rangle$. The four paths are easily identified in the diagram in Figure 3a.

The difference between \hat{A} and \hat{B} is a functional $F_2[path]$ defined on the four paths, where it may take four different values:

$$F_2[1] = b_1 - a_1, \quad F_2[2] = b_1 - a_2, \quad F_2[3] = b_2 - a_1, \quad \text{and} \quad F_2[4] = b_2 - a_2. \quad (16)$$

Let \hat{A} and \hat{B} represent the components of spin-1/2 (without a factor of 1/2) along two different axes, so that $a_1 = b_1 = -1$ and $a_2 = b_2 = 1$. Now, F_2 takes three different values, $-2, 0$ and 2 . With no meter employed, we cannot say what is the chance of having, e.g., $F = 2$, but, as before, may write down an amplitude distribution $\Phi(f)$ for the value f of the functional F_2 ,

$$\Phi(f) = \{A[1] + A[4]\} \delta(f) + A[2] \delta(f + 2) + A[3] \delta(f - 2), \quad (17)$$

normalized by the condition $\int \Phi(f)df = \sum_{i=1}^4 A[i] = \langle \phi | \psi \rangle$.

As in the previous section, this so far purely cosmetic re-arrangement of the virtual paths is useful for describing the effects of a meter, once it is coupled to the system as prescribed by Equation (15). After a successful post-selection, the pointer is in a pure state $|M'\rangle$, and it is a simple matter to check that:

$$M'(\xi) \equiv \langle \xi | M' \rangle = \int G(\xi - f) \Phi(f) df = \{A[1] + A[4]\}G(\xi) + A[2]G(\xi + 2) + A[3]G(\xi - 2). \quad (18)$$

We can now look at Equation (18) from the point of view adopted in the previous section. As in Section 4, by external manipulation (connecting the system to a meter), we have created a classical statistical ensemble, where the system can reach the final state via a continuum of real paths, labeled by the the variable ξ . Again, the meter plays an active role, and the (unnormalized) probability to travel the ξ_0 -th path is determined by the shape of G , which is the property of the pointer, as well as the amplitudes of the virtual paths, which represent the system “on its own”,

$$P(\xi_0) = |\{A[1] + A[4]\}G(\xi_0) + A[2]G(\xi_0 + 2) + A[3]G(\xi_0 - 2)|^2. \quad (19)$$

With this, the mean reading of the meter over many trials is readily seen to be given by:

$$\bar{\xi} = \frac{\int \xi P(\xi) d\xi}{\int P(\xi) d\xi}. \quad (20)$$

As in the previous example, the value of F_2 remains indeterminate, if there is more than one term in the r.h.s. of Equation (19). To establish a good correlation between the pointer readings ξ and the values of F_2 , we must, therefore, try to avoid the overlap between different G 's in (19). We will do so after discussing the general rules for combining virtual paths in the next section.

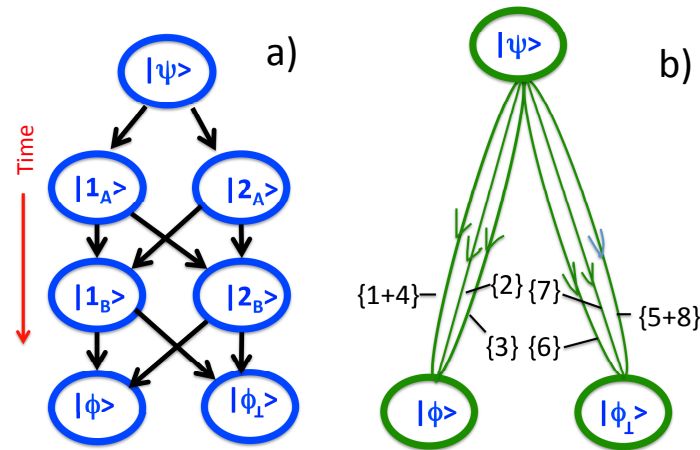


Figure 3. (a) Eight virtual paths connecting the initial state of a two-level system, $|\psi\rangle$, with its final state $|\phi\rangle$, and its orthogonal complement $|\phi_\perp\rangle$; (b) real paths produced in an accurate measurement of the functional F_2 in Equation (16), which takes the values $-2, 0$ and 2 .

7. Superposition Principle for Virtual Paths

Feynman’s uncertainty principle, cited in Section 2, implies that virtual paths can be combined and recombined into new sets of paths, with interference being responsible for the loss of information about certain aspects of quantum motion. Next, we briefly discuss the rules for creating such combinations.

We start with the well-known case of changing the basis functions in which a state of the system can be expanded at a given time. Let a two-state system be in the state $|\Psi\rangle = a_1|\varphi_1\rangle + a_2|\varphi_2\rangle$, where $\langle \varphi_m | \varphi_{m'} \rangle = \delta_{mm'}$, and a_m is the amplitude with which $|\varphi_m\rangle$ enters in $|\Psi\rangle$. Consider a

different basis, $|\phi_i\rangle$, $\langle\phi_i|\phi_{i'}\rangle = \delta_{ii'}$, which is related to the first basis by a unitary transformation, $|\phi_i\rangle = u_{i1}|\phi_1\rangle + u_{i2}|\phi_2\rangle$. Now, the amplitude b_i , with which $|\phi_i\rangle$ enters in $|\Psi\rangle$, is $b_i = \langle\phi_i|\Psi\rangle = \sum_j u_{ji}a_j$, $i = 1, 2$. These equations can be read as follows: if a new basis state is a weighted sum of the old ones, the corresponding amplitude is the weighted sum of the corresponding amplitudes.

The same rules can be applied to virtual paths, and we can describe the action of the meter in Section 5 in a similar manner. If the pointer reading is ξ_0 , the meter combines virtual paths $\{1\}$ and $\{2\}$ with amplitudes $A[1]$ and $A[2]$, into a single real path $G(\xi_0 - a_1)\{1\} + G(\xi_0 - a_2)\{2\}$ with an amplitude $A[G(\xi_0 - a_1)\{1\} + G(\xi_0 - a_2)\{2\}] = G(\xi_0 - a_1)A[1] + G(\xi_0 - a_2)A[2]$. The path is travelled with the probability $P(\xi_0) = |G(\xi_0 - a_1)A[1] + G(\xi_0 - a_2)A[2]|^2$.

The same can be said about the measurement in Section VI. There the meter reading ξ_0 combines four available virtual paths into a single real one, $\{\xi_0\}$

$$\{\xi_0\} = G(\xi_0)\{1\} + G(\xi_0 - 2)\{2\} + G(\xi_0 + 2)\{3\} + G(\xi_0)\{4\}, \quad (21)$$

with the amplitude

$$A[\xi_0] = \sum_{i=1}^4 G(\xi_0 - F_2[i])A[i] = G(\xi_0)\{A[1] + A[4]\} + G(\xi_0 - 2)A[2] + G(\xi_0 + 2)A[3], \quad (22)$$

and $|A(\xi_0)|^2$ gives the probability to obtain a meter reading ξ_0 if the measurement is repeated many times. In general, it is understood that if paths $\{1\}$ and $\{2\}$ with the amplitudes $A[1]$ and $A[2]$ are combined into a new path, $\alpha\{1\} + \beta\{2\}$, with complex valued weights α and β , then the amplitude for the new pathway is:

$$A[\alpha\{1\} + \beta\{2\}] = \alpha A[1] + \beta A[2]. \quad (23)$$

If a functional $F[i]$ takes different definite values on $\{1\}$ and $\{2\}$, $F[1] \neq F[2]$, its value on $\alpha\{1\} + \beta\{2\}$ is indeterminate (we will elaborate on this statement in Section 10).

That the width of $G(\xi)$, $\Delta\xi$, determines the resolution, or the accuracy, of a measurement is readily seen if $G(\xi)$ is chosen to have the form of a “rectangular window” (13). Inserting (13) in Equation (18) yields:

$$P(\xi_0) = \Delta\xi^{-1} \left| \int_{\xi_0 - \Delta\xi/2}^{\xi_0 + \Delta\xi/2} \Phi(f) df \right|^2, \quad (24)$$

and we can be sure that if a reading is ξ_0 , the value of the measured functional, f , is no smaller than $\xi_0 - \Delta\xi/2$ and no greater than $\xi_0 + \Delta\xi/2$. We are, however, “banned” from looking inside the interval $[\xi_0 - \Delta\xi/2, \xi_0 + \Delta\xi/2]$, where all information about f is lost to the interference, which the meter has failed to destroy. Next, we look at the question of accuracy in more detail.

8. Most Accurate Measurements, Real Pathways and Contextuality

Suppose we can change the width $\Delta\xi$ of the initial pointer state, e.g., by choosing:

$$\langle\xi|M\rangle \equiv G(\xi|\Delta\xi) = (\Delta\xi)^{-1/2} G(\xi/\Delta\xi), \quad (25)$$

where the factor $(\Delta\xi)^{-1/2}$ ensures that the state is properly normalized, $\int G(\xi|\Delta\xi)^2 d\xi = 1$. Since, by assumption, the pointer’s final position is evaluated accurately, it is the uncertainty of its initial setting that determines the precision of the measurement. For a highly accurate measurement, we should choose $\Delta\xi$ as small as possible, thus requiring:

$$\Delta\xi \rightarrow 0, \quad G(\xi|\Delta\xi)^2 \rightarrow \delta(\xi). \quad (26)$$

If so, in the example of Section 5, Equation (12) reduces to:

$$P(\xi) = |A[1]|^2 \delta(\xi - a_1) + |A[2]|^2 \delta(\xi - a_2) \equiv p[1] \delta(\xi - a_1) + p[2] \delta(\xi - a_2), \quad (27)$$

and of all real paths, only two survive. Equivalently, we can say that the action of the meter has destroyed all interference between the virtual paths $\{1\}$ and $\{2\}$, which can now be equipped with the probabilities $p[1]$ and $p[2]$, respectively. The paths are “real” in the sense that each time the experiment is repeated, the pointer will indicate which of the two paths has been traveled, with $p[1]/(p[1] + p[2])$ and $p[2]/(p[1] + p[2])$ giving the relative frequencies of each occurrence. Note that the appearance of $|A[i]|^2$, later interpreted as probabilities, is not postulated, but is a consequence of the change in the physical state of the system, caused by its interaction with the meter [15].

Recalling that whenever post-selection in the state $|\phi\rangle$ fails, the system ends up in its orthogonal complement, $|\phi_\perp\rangle$, $|\phi\rangle\langle\phi| + |\phi_\perp\rangle\langle\phi_\perp| = 1$, we can add two new paths, $\{3\}$ and $\{4\}$, connecting $|\psi\rangle$ with $|\phi_\perp\rangle$, and traveled with probabilities:

$$p[3] = |\langle\phi_\perp|1\rangle\langle 1|\psi\rangle|^2, \quad (28)$$

and:

$$p[4] = |\langle\phi_\perp|2\rangle\langle 2|\psi\rangle|^2, \quad (29)$$

respectively. We now have an exact equivalent of the (upper part of the) model in Figure 1. The only difference is that this time, it is built from quantum “elements”, rather than from pieces of a classical kit. The rest is a simple exercise in classical statistics. In both cases, we can calculate the averages of the functional F_1 . For example, conditioned by successful post-selection in $|\phi\rangle$, the mean value of F_1 is given by $\langle F_1 \rangle = (F_1[1]p[1] + F_1[2]p[2])/(p[1] + p[2])$. We can also choose the operator \hat{A} to be the projector on the state $|1\rangle$, so that its eigenvalues are one and zero, and we have $F_1[1] = 1$ and $F_1[2] = 0$. Now, the mean value of F_1 yields precisely the probability to travel the first path,

$$\langle F_1 \rangle = \frac{p[1]}{p[1] + p[2]}. \quad (30)$$

Classically, we would arrive at the same result by writing down one whenever the ball passes through a_1 , zero if it passes through a_2 , add up what is written and divide by the number of balls we used.

The case of measuring the difference of operators $\hat{B}(t_2)$ and $\hat{A}(t_1)$, discussed in Section 6, can be analyzed in a similar manner. There, an accurate meter creates three real paths connecting $|\psi\rangle$ and $|\phi\rangle$. The three pathways are

- $\{1 + 4\}$, traveled with a probability $p[1 + 4] = |A[1] + A[4]|^2$,
- $\{2\}$, traveled with a probability $p[2] = |A[2]|^2$, and
- $\{3\}$, traveled with a probability $p[3] = |A[3]|^2$.

Adding three more paths connecting $|\psi\rangle$ with $|\phi_\perp\rangle$, we obtain a classical stochastic system shown in Figure 3, similar to that built from the tubes and connectors in Figure 1. The mean value of the functional F_2 defined in Equation (16), conditioned by arriving in $|\phi\rangle$, is now twice the difference of the relative frequencies with which the paths $\{2\}$ and $\{3\}$ are traveled,

$$\langle F_2 \rangle = \frac{F_2[1]p[1 + 4] + F_2[2]p[2] + F_2[3]p[3]}{p[1 + 4] + p[2] + p[3]} = \frac{2(p[2] - p[3])}{p[1 + 4] + p[2] + p[3]}. \quad (31)$$

To conclude, we return to the remark made at the end of Section 4. Joining the connectors and tubes in different ways, one may construct completely different statistical networks, which have nothing in common, except their constituent parts. In a similar way, by applying different measurement

schemes and measuring different operators to different accuracies, one may engineer equally different stochastic ensembles, having nothing in common except their parent quantum system. Ignoring this fact may lead to various counterfactual paradoxes, some of which disappear once the above remark is taken into account (for more detail, see [7] and the references therein).

9. Most Inaccurate Measurements and the Meaning of “Indeterminate”

Next, we look at what happens in the opposite limit,

$$\Delta\xi \rightarrow \infty. \quad (32)$$

There are several reasons for doing this. Firstly, as in the case of the accurate limit considered in the previous section, the result is universal in the sense that it no longer depends on the particular choice of $G(\xi)$ and appears to represent some “intrinsic” property of the studied system. Secondly, the observation of the system’s history changes, in general, the probability to end up in the state $|\phi\rangle$, which, when an accurate meter is employed, is different from that without a meter,

$$\sum_i p[i] \neq |\langle\phi|\exp(-i\hat{H}T)|\psi\rangle|^2. \quad (33)$$

This is a well-known example of “one measurement perturbing the other”, and it is tempting to try to get rid of the perturbation and to see what will then occur.

The perturbation disappears if, and only if, all G ’s in Equation (18) have approximately the same value, $G(\xi) \approx G(\xi + 2) \approx G(\xi - 2)$, so that $P(\xi)$ in Equation (19) is $\approx |G(\xi)|^2 |\langle\phi|\exp(-i\hat{H}T)|\psi\rangle|^2$. This can only happen in the limit (32), when $G(\xi)$ is made very broad. One immediate consequence of taking the limit (32) is that the range of possible meter’s readings has widened to occupy the whole real axis,

$$-\infty \leq \xi \leq \infty. \quad (34)$$

This gives an operational meaning to the word “indeterminate” that we have often used above. If one wishes to keep interference between paths intact, he/she must choose the meter so inaccurate, that its reading will be arbitrary, leaving the researcher completely ignorant of the value of the functional in question.

It appears that not much can be learned from an individual inaccurate measurement, and we follow [2] to look instead at the average meter reading, $\bar{\xi}$, obtained as the number of trials N tends to infinity. Apparently, $\bar{\xi}$ must be expressed in terms of the path amplitudes $A[i]$ since, in the absence of probabilities, these are the only quantities quantum mechanics provides for the description of the transition. The authors of [2] have shown that in the case of Section 5, the correct expression is:

$$\bar{\xi} = \text{Re} \frac{\sum_i a_i \langle\phi|i\rangle \langle i|\psi\rangle}{\sum_i \langle\phi|i\rangle \langle i|\psi\rangle} = \text{Re} \frac{\sum_i F_1[i] A[i]}{\sum_i A[i]} = \text{Re}\{a_1\alpha[1] + a_2\alpha[2]\}, \quad (35)$$

where:

$$\alpha[i] = A[i] / \sum_i A[i] \quad (36)$$

is the relative amplitude for the path number i . In the special case of \hat{A} being the projector onto the state $|1\rangle$, $\hat{A} = |1\rangle\langle 1|$, we also have:

$$\bar{\xi} = \text{Re}\alpha[1]. \quad (37)$$

In [12], the result was extended to any real functional for which, in the absence of a meter, the amplitude for taking a value f is $\Phi(f)$,

$$\bar{\xi} = \text{Re} \frac{\int f \Phi(f) df}{\int \Phi(f) df}. \quad (38)$$

For example, the average reading of a very inaccurate meter measuring the difference of $\hat{B}(t_2)$ and $\hat{A}(t_1)$ in the example of Section 6 is given by:

$$\bar{\xi} = \text{Re} \frac{F[1]A[1+4] + F[2]A[2] + F[3]A[3]}{A[1+4] + A[2] + A[3]} = 2\text{Re}(\alpha[2] - \alpha[3]). \quad (39)$$

In general, an average reading of a very inaccurate meter is expressed in terms of the relative amplitudes for the virtual paths connecting the initial and final states. Equations (37) and (39) bear uncanny resemblance to Equations (30) and (31), in which the probabilities $p[i]$ have been replaced by the corresponding probability amplitudes $A[i]$, and the real part was chosen to make the result real valued. This may be taken as a hint that the quantity under the Re sign, the weak value of a functional F ,

$$\bar{F} \equiv \frac{\sum_i F[i] A[i]}{\sum_i A[i]} \quad (40)$$

has the universal importance of its accurate (“strong”) counterpart in Equation (31),

$$\langle F \rangle \equiv \frac{\sum_i F[i] p[i]}{\sum_i p[i]}, \quad (41)$$

in Equations (30) and (31). In the next section, we will show that this cannot be the case and for a good reason.

10. Weak Values and the Uncertainty Principle

As discussed in Section 8, an accurately measured mean value of the projector onto an intermediate state gives the relative frequency with which the path created by the meter is traveled and, so, serves to distinguish it from other real paths present. What would the result of such measurement mean in the most inaccurate limit, where the path in question exists only as a virtual one? The uncertainty principle states that two virtual paths cannot be distinguished by physical means inside the single real pathway into which they are combined.

There appears to be a logical rather than a mathematical difficulty. To address it, consider a purely classical conundrum. Suppose there are two drops of water on a glass surface, and we put a grain of sand in Drop Number 1. The drops move closer to each other, and coalesce. In which of the two drops is the sand grain now? If asked, one would certainly reject the question as meaningless, given the new circumstances. However, if an answer is required under pain of death, the individual might reply “inside the Water Drop Number 100, for all I care”. The answer to a meaningless question may be “anything at all”.

The above passage, illustrated in Figure 4, is not entirely out of place in the present discussion. With two real paths created by an accurate meter, in each trial, the pointer would point at zero or one, indicating the realization of one of the two real possibilities. If the meter is highly inaccurate and the interference between two virtual paths is not destroyed, the pointer is equally likely to point at any number. If asked “was the first or the second virtual path traveled in this transition?”, the meter, unable to politely decline to answer a meaningless question, points at, say, 100. Importantly, this behavior is a necessity, rather than a paradox. It is the Uncertainty Principle that makes the question meaningless, and any other kind of answer would lead to a conflict with the Principle.

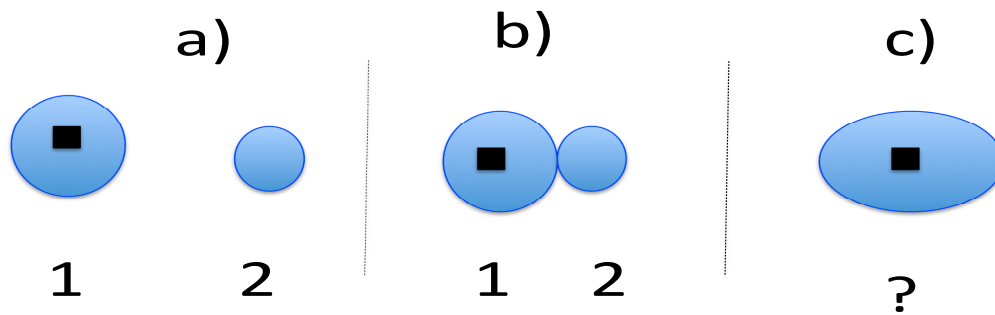


Figure 4. Two water drops on a smooth surface approach each other and coalesce. (a) The grain of sand is in the first drop; (b) the grain is still in the first drop; (c) in which of the two drops is the grain now?

Fully aware of the randomness of an individual inaccurate result, the authors of [2] focused on the meter's average reading, which, given the same initial and final states, is a uniquely defined quantity, which should come up the same in every long series of trials. This is given by the real part (or the imaginary part, if a slightly different meter is used [2,7]) of what looks like an average (31), but obtained with a complex valued distribution, constructed from the path amplitudes $A[i]$. We wish to examine its meaning, bearing in mind the restrictions imposed by the UP.

Firstly, it should be remembered that Equation (38) expresses the average $\bar{\xi}$ defined in a standard way by Equation (20), in terms of the system's amplitudes $\alpha[i]$. Secondly, as discussed in Section 8, the mean pointer reading in an accurate measurement of a projector yields the relative frequency with which the real path in question is traveled. Thus, with $N \gg 1$ trials performed, $\bar{\xi}N$ gives the number of trials in which the system has chosen the path. The result makes sense, since $p[i] \geq 0$ and $\bar{\xi}N \leq N$. The same cannot be said about the amplitudes $A[i]$. Choosing $A[1] = 1$ and $A[2] = -1.01$ and inserting them into Equation (40), we obtain $\bar{\xi} = -100$. An attempt to interpret $\bar{\xi}N$ as the number of cases where the first virtual path is traveled in a series of 10^6 trials gives a bizarre value of -10^8 , another arbitrary answer to a meaningless question.

We need not limit ourselves to measuring projectors. Every functional taking the values $F[i]$ can be written as a sum $F[path] = \sum_j F[j]F^j[path]$, where $F^j[path]$ takes the values $F^j[i] = \delta_{ij}$. In an accurate measurement, the mean value of $F^j[path]$ gives the relative frequency with which the j -th path is traveled. In an inaccurate measurement, its value will remain indeterminate and with it, also the value of $F[path]$.

Suppose a group of researchers measure the z -components of the spin of an electron, σ_z , at $t = T/2$, in a setup where the spin is pre- and post-selected at $t = 0$ and $t = T$. Moreover, all researchers have different initial and final states $|\psi\rangle$ and $|\phi\rangle$. If the measurement is accurate, all researchers will be obtaining the results -1 and $+1$, albeit in different proportions. They can all agree that the quantity they measure may take only these two values and use this knowledge to develop theories of spin-1/2 particles, if needed. If the measurement is weak, they can only agree that the spin they measure is able to take any real value.

Suppose further that the equipment is such that each researcher only has access to the average values and not to the distributions. If the measurements are accurate, all choices of $|\psi\rangle$ and $|\phi\rangle$ $\bar{\xi}$ would lie between -1 and $+1$. This gives the knowledge of a useful fact that σ_z represents a quantity whose value cannot be larger than one or less than minus one, although its discreet nature must be learned from some other source. If the measurement is weak, it can be shown [27] that there is always a choice of $|\psi\rangle$ and $|\phi\rangle$ which would give $\bar{\xi}$ any real value, positive, negative or zero. Thus, after trying all possible $|\psi\rangle$ and $|\phi\rangle$, the experimentators will remain in the state of maximal ignorance regarding the properties of the measured quantity.

The mathematical trick that allows the UP to conceal the information about individual virtual paths is based on replacing the path probabilities with path amplitudes when calculating a weak value. It is not so much the fact that $A[i]$ are in general complex valued, but rather the absence of restrictions on the signs of their real and imaginary parts, which can cause the weak value of a projector to be -100 [12]. The mean and variance of a non-negative distribution give the estimates for the center and the width of its region of support. The mean and variance of a distribution that can change sign do not have this property, allowing $\bar{\xi}$ to lie anywhere on the real axis or obtain a zero variance even when the distribution is not a Dirac delta [12].

11. Weak Values: Over-Interpretations and Misconceptions

The conclusions of the previous section can be summarized as follows [7]. A highly inaccurate “weak” measurement (WM) on a pre- and post-selected system is a perturbative scheme, in which an additional system (a vN meter) is weakly coupled to the observed one (e.g., a spin). The coupling is such that the mean value of one of the meter’s variables (pointer position ξ) coincides with the real (or imaginary, if necessary [2,7]) part of a weighted sum of the relative amplitudes on the virtual paths connecting the system’s initial and final states,

$$\bar{\xi} \sim \text{Re(Im)} \sum_i F[i] \alpha[i], \quad \alpha[i] \equiv A[i] / \sum_i A[i]. \quad (42)$$

In Equation (42), the paths $\{i\}$, $i = 1, 2, \dots$ and the weights $F[i]$ are determined by the nature of the measured quantities. If it is possible (it may not be) to measure directly a functional F^j , whose values are zero on all paths except the path number j , $F^j[i] = \delta_{ij}$, the mean reading of the weak pointer will yield precisely the real (imaginary) part of the amplitude α_j . In short, weak measurements measure probability amplitudes.

A certain amount of confusion, which led to a plethora of recent publications on this still fashionable subject (for a recent review, see [28] and references therein), has its roots in [2], where the weak values, defined in [28] as “...complex numbers that one can assign to the powers of a quantum observable operator \hat{A} using two states, an initial state $|i\rangle$ called the preparation or preselection, and a final state $|f\rangle$ called the postselection”, were first introduced. The authors of [2] considered a weak measurement of the z-component of a spin-1/2, similar to that described in Section 5, and concluded that “... the usual measurement procedure for preselected and postselected ensembles of quantum systems gives unusual results.” The title question of the [2], “How the result of measurement of a component of the spin of a spin-1/2 particle can turn out to be 100?”, has, however, a simple answer. This would happen if the meter measures the difference of relative amplitudes $\alpha[1] - \alpha[2]$, for an improbable transition, for which $|A[1] + A[2]|^2 = |\langle\phi|\psi\rangle|^2$ is small. Surprise at the “unusual results” is also misplaced: the authors of [2] might as well obtain any other value, large or small, negative, positive or zero. Indeed, the opposite would be surprising, as finding only the values between -1 and 1 would contradict the uncertainty principle, as discussed in the previous section.

With weak values described only vaguely as a “new concept” [2], some authors proposed to use them as a tool for resolving “counterfactual paradoxes”. One simple example is the so-called “three-box case” [5,20,29] (for a path analysis of the so-called Hardy’s paradox [4] see, for example, [30]). Here, the final state can be reached via three virtual pathways with amplitudes:

$$A[1] = C, \quad A[2] = -C, \quad \text{and} \quad A[3] = C, \quad (43)$$

where C is a complex number. An accurate measurement of a functional F^1 ,

$$F^1[1] = 1, \quad F^1[2] = F^1[3] = 0, \quad (44)$$

always finds the system taking the first path. On the other hand, an accurate measurement of a functional F^3 ,

$$F^3[3] = 1, \quad F^3[1] = F^3[2] = 0, \quad (45)$$

always finds the system taking the third path, and the authors of [5] alert the reader to "... the peculiarity of having one particle in several places simultaneously even in the stronger sense than in the double slit experiment...". Admitting that the two results were obtained in different physical circumstances, the authors of [4] rely on the WM "... to test... the assertions otherwise regarded as counterfactual". Their reasoning is as follows. Since WM perturb the system only slightly, it is possible to conduct several of them at the same time [4] (or, we add, even perform full tomography of the studied transition [7]). Simultaneous WM of F_1 and F_3 by two meters yield $\bar{\xi}_1 = 1$ and $\bar{\xi}_2 = 1$ and seemingly confirm the particle's presence in two places at the same time.

A closer look suggests a far simpler explanation of this "peculiarity". Different strong measurements "engineer" two completely different classical ensembles (cf. the passage at the end of Section 8). This becomes clear if one adds all remaining final states and the real pathways leading to them [20]. The fact that in two different setups the same final state is always reached via paths of a different kind is not surprising at all (we would not give it a second thought if the setups were built from the tubes and connectors of Section 4). Therefore, what is tested by the simultaneous WM of F_1 and F_3 ? We have seen that their results would be:

$$\bar{\xi}_1 = \text{Re}\alpha[1] = 1, \quad \text{and} \quad \bar{\xi}_2 = \text{Re}\alpha[3] = 1. \quad (46)$$

Since the relative amplitudes always add up to one, $\alpha[1] + \alpha[2] + \alpha[3] = 1$, we also learn that $\text{Re}\alpha[2] = -1$. Far from asserting that the particle is in "two places", the weak values give us the values of the real parts of the relative amplitudes. This information is not yet sufficient even for predicting the results of the two accurate measurements, since for this, we also need the imaginary parts of $\alpha[i]$. Suppose that by making a WM of a different kind, we established that $\text{Im}\alpha[i] = 0, i = 1, 2, 3$. Now, we know that all α 's are real, and $\alpha[1] = -\alpha[2] = \alpha[3]$ and the standard rules of quantum mechanics tell us what would happen in two mutually-exclusive situations, where the paths $\{2\}$ and $\{3\}$ are combined into a real pathway $\{2 + 3\}$ or $\{1\}$ and $\{2\}$ are combined into $\{1 + 2\}$, as happens in the two accurate measurements mentioned above. The last shred of mystery, if there were one, disappears when we recall that the perturbation theory typically gives information about both the moduli and the phases of amplitudes and the wave function, and the WM scheme is the rule rather than the exception [7].

Finally, in a much criticized [27,31–35] paper entitled "How the result of a single coin toss can turn out to be 100 heads" [36], the authors argue that "weak values are not inherently quantum but rather a purely statistical feature of pre- and postselection with disturbance". Without going into the details of the author's reasoning or into the arguments of their critics, one sees that the claim of [36] must be wrong. To obtain a mean value that lies outside the region of support of a distribution, the distribution must change sign [12]. Such distributions naturally appear in quantum mechanics, whose basic building blocks are complex valued probability amplitudes [19]. They may not, however, appear in a classical theory where all physically meaningful averages are taken with non-negative probabilities.

The parallel between the titles of [2] and [36], which followed it twenty six years on, are not coincidental. The fallacy of weak classical values in [36] stems naturally from the suggestion [2] that the combinations of the transitions amplitudes, which arise when a measurement is extremely inaccurate, amount to a "new concept" of a quantum variable, capable of shedding light on a new physical reality.

12. Conclusions

In summary, we have shown that the application of a sequence of quantum measurements creates a stochastic network. A network is classical in the sense that the probabilities for all observable events have, of course, all the usual properties of probabilities in a classical theory. A network is quantum in

its origin, and the action of a meter, or meters, can be seen as a way of “engineering” real scenarios, endowed with probabilities, from the virtual paths of the system, for which only probability amplitudes are available. A network is conveniently described in terms of functionals, which take numerical values on the virtual paths and whose values the measurements are set to determine. In the case of von Neumann and von Neumann-like meters considered here, the real pathways are determined by:

- (i) the set of virtual paths available for the system if no meter is present.
- (ii) the values of the measured functional. Two paths sharing the same value cannot be distinguished by the meter(s).
- (iii) the form and width of the initial state of the pointer, $G(\xi)$, which determine the accuracy of the measurement.

Highly accurate measurements turn all virtual paths, that can be distinguished, into the real ones, traveled with the probabilities proportional to the squared moduli of the corresponding probability amplitudes. If a functional whose value is 1 on one of the virtual paths and 0 on all others is measured accurately, its mean value yields the relative frequency with which the path is traveled.

If the same functional is measured in a highly inaccurate manner, its mean (weak) value coincides with the real (or imaginary) part of the relative amplitude, corresponding to the selected path. Since there are no a priori restrictions on the values of the relative amplitudes, it is always possible to find a transition in which the result of a weak measurement would be any desired number, large, small, positive or negative. This is a necessary consequence of the uncertainty principle, and the “unusual” weak values lying outside the spectrum of the operator \hat{A} in the example given in Section 5 are by no means unusual. Many of exotic interpretations of the weak measurements’ results stem from the failure to identify the weak values with the path amplitudes, or their linear combinations, and are easily dismissed once the problem is reformulated in the language of standard quantum mechanics.

Acknowledgments: Support of the Spanish ministry MINECO and the European Regional Development Fund FEDER through the Grant FIS2015-67161-P (MINECO/FEDER), as well as useful discussions with E. Akhmatkaya are gratefully acknowledged.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Von Neumann, J. *Mathematical Foundations of Quantum Mechanics*; Princeton University Press: Princeton, NJ, USA, 1955; p. 183.
2. Aharonov, Y.; Albert, D.Z.; Vaidman, L. How the Result of a Measurement of Component of the Spin of a Spin-1/2 Particle Can Turn out to be 100. *Phys. Rev. Lett.* **1988**, *60*, 1351–1354.
3. Aharonov, Y.; Popescu, S.; Rohrlich, D.; Vaidman, L. Measurements, Errors, and Negative Kinetic Energy. *Phys. Rev. A* **1993**, *48*, 4084–4090.
4. Aharonov, Y.; Botero, A.; Popescu, S.; Reznik, B.; Tollaksen, J. Revisiting Hardy’s paradox: Counterfactual statements, real measurements, entanglement and weak values. *Phys. Lett. A* **2002**, *301*, 130–138.
5. Aharonov, Y.; Vaidman, L. *Time in Quantum Mechanics*, 2nd ed.; Muga, G., Mayato, R.S., Egusquiza, I., Eds.; Springer: Berlin, Germany; Heidelberg, Germany, 2008; Volume 1, pp. 399–447.
6. Aharonov, Y.; Popescu, S.; Rohrlich, D.; Skrzypczyk, P. Quantum Cheshire cats. *New J. Phys.* **2013**, *15*, 113015–113023.
7. Sokolovski, D. Weak measurements measure probability amplitudes (and very little else). *Phys. Lett. A* **2016**, *380*, 1593–1599.
8. Sokolovski, D.; Mayato, R.S. Path summation and von Neumann-like quantum measurements. *Phys. Rev. A* **2005**, *71*, 042101.
9. Sokolovski, D.; Mayato, R.S. Simultaneous histories, path sums, and measurements for noncommuting variables. *Phys. Rev. A* **2006**, *73*, 052115.

10. Sokolovski, D.; Mayato, R.S. Erratum: Simultaneous histories, path sums, and measurements for noncommuting variables [Phys. Rev. A 73, 052115 (2006)]. *Phys. Rev. A* **2007**, *76*, 039903.
11. Sokolovski, D. Zeno effect and ergodicity in finite-time quantum measurements. *Phys. Rev. A* **2011**, *84*, 062117.
12. Sokolovski, D. Weak values, “negative probability,” and the uncertainty principle. *Phys. Rev. A* **2007**, *76*, 042125.
13. Sokolovski, D.; Akhmatskaya, E. Hartman effect and weak measurements that are not really weak. *Phys. Rev. A* **2011**, *84*, 022104.
14. Sokolovski, D.; Akhmatskaya, E. “Superluminal paradox” in wave packet propagation and its quantum mechanical resolution. *Ann. Phys.* **2013**, *339*, 307–327.
15. Sokolovski, D. Path integral approach to space-time probabilities: A theory without pitfalls but with strict rules. *Phys. Rev. D* **2013**, *84*, 076001.
16. Sen, D. The uncertainty relations in quantum mechanics. *Curr. Sci.* **2014**, *107*, 203–218.
17. Robertson, H.P. The Uncertainty Principle. *Phys. Rev. A* **1929**, *34*, 163–164.
18. Feynman, R.P.; Hibbs, A.R. *Quantum Mechanics and Path Integrals*; McGraw-Hill: New York, NY, USA, 1965; Chapter 7.
19. Feynman, R.P.; Leighton, R.; Sands, M. *The Feynman Lectures on Physics III*; Dover Publications, Inc.: New York, NY, USA, 1989; Chapter 1.
20. Sokolovski, D.; Gimenez, I.P.; Mayato, R.S. Path integrals, the ABL rule and the three-box paradox. *Phys. Lett. A* **2008**, *372*, 6578–6583.
21. Bohm, D. *Quantum Theory*; Addison-Wesley: Reading, MA, USA, 1965.
22. Sokolovski, D.; Sherman, E.T. Measurement of noncommuting spin components using spin-orbit interaction. *Phys. Rev. A* **2011**, *84*, 030101.
23. Sokolovski, D.; Sherman, E.T. Spin measurements and control of cold atoms using spin-orbit fields. *Phys. Rev. A* **2014**, *89*, 043614.
24. Sherman, E.T.; Sokolovski, D. Von Neumann spin measurements with Rashba fields. *New J. Phys.* **2014**, *16*, 015013.
25. Sokolovski, D. Qubit Residence Time Measurements with a Bose-Einstein Condensate. *Phys. Rev. Lett.* **2009**, *102*, 230405.
26. Sokolovski, D. Residence time of a two-level quantum system. *Proc. R. Soc. Lond. A* **2004**, *460*, 1505–1517.
27. Sokolovski, D. The meaning of “anomalous weak values” in quantum and classical theories. *Phys. Lett. A* **2015**, *379*, 1097–1101.
28. Dressel, J.; Malik, M.; Miatto, F.M.; Jordan, A.N.; Boyd, R.W. Colloquium: Understanding quantum weak values: Basics and applications. *Rev. Mod. Phys.* **2014**, *86*, 307–316.
29. Aharonov, Y.; Vaidman, L. Complete description of a quantum system at a given time. *J. Phys. A Math. Gen.* **1991**, *24*, 2315–2328.
30. Sokolovski, D.; Gimenez, I.P.; Mayato, R.S. Feynman-path analysis of Hardy’s paradox: Measurements and the uncertainty principle. *Phys. Lett. A* **2008**, *372*, 3784–3791.
31. Hoffmann, H.F.; Inuma, M.; Shikano, Y. Why the “Classical” Explanation of Weak Values by Ferrie and Combes Does Not Work: A Comment on Phys. Rev. Lett. 113, 120404 (2014). arXiv:1410.7126v2. arXiv.org e-Print archive. 2014. Available online: <https://arxiv.org/abs/1410.7126v2> (accessed on 21 November 2014).
32. Vaidman, L. Comment on “How the Result of a Single Coin Toss Can Turn Out to Be 100 Heads”. arXiv:1409.5386v1. arXiv.org e-Print archive. 2014. Available online: <http://arxiv.org/abs/1409.5386> (accessed on 18 September 2014).
33. Aharonov, Y.; Rohrlich, D. Comment on “How the Result of A Single Coin Toss Can Turn Out to Be 100 Heads”. arXiv:1410.0381v1. arXiv.org e-Print archive. 2014. Available online: <http://arxiv.org/abs/1410.0381> (accessed on 1 October 2014).
34. Dressel, J. Weak Values are Interference Phenomena. *Phys. Rev. A* **2015**, *91*, 032116.

35. Sokolovski, D. A comment on the paper “How can a Result of a Single Coin Toss Turn Out to be 100 Heads” by C. Ferrie and J. Combes. arXiv:1410.0570v1. arXiv.org e-Print archive. 2014. Available online: <http://arxiv.org/abs/1410.0570> (accessed on 2 October 2014).
36. Ferrie, C.; Combes, J. How can a Result of a Single Coin Toss Turn Out to be 100 Heads. *Phys. Rev. Lett.* **2014**, *113*, 120404.



© 2016 by the author; licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC-BY) license (<http://creativecommons.org/licenses/by/4.0/>).