

Article

Quantum Incompatibility in Collective Measurements

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Abstract: We study the compatibility (or joint measurability) of quantum observables in a setting where the experimenter has access to multiple copies of a given quantum system, rather than performing the experiments on each individual copy separately. We introduce the index of incompatibility as a quantifier of incompatibility in this multi-copy setting, as well as the notion of the compatibility stack representing various compatibility relations present in a given set of observables. We then prove a general structure theorem for multi-copy joint observables and use it to prove that all abstract compatibility stacks with three vertices have realizations in terms of quantum observables.

Keywords: quantum incompatibility; collective measurements; joint measurability hypergraphs; noisy spin observables

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1. Introduction

The laws of quantum physics dictate that there are certain tasks that are mutually exclusive, meaning that they cannot be performed simultaneously with a single device. This quantum incompatibility is usually encountered in the context of mutually-exclusive measurements: one cannot measure two orthogonal spin directions σ_x and σ_y with a single measurement setup. From the modern quantum information point of view, incompatibility has been identified as a genuine resource gained from switching from classical to quantum protocols. As such, it is at the heart of many typically quantum applications, such as secure quantum key distributions or the possibility to steer remote quantum systems. Due to its importance in these applications, as well as its status as a fundamental quantum feature, it is essential to gain a deeper understanding of quantum incompatibility.

Even though quantum theory gives predictions of outcomes in statistical experiments, the essence of incompatibility is manifested on the level of single experimental runs. More specifically, for compatible observables, it is possible to find a single measurement setup, such that on each experimental run, the reading of the measurement outcome allows one to assign the values of the outcomes for the compatible observables. The prototypical example of this is the *joint measurement* of a pair of compatible observables A and B: if Ω_A and Ω_B are the outcome sets for A and B, respectively, then a joint observable will have the outcome set $\Omega_A \times \Omega_B$. The measurement outcome in each experimental run is therefore a pair of numbers (a, b) , from which we assign the values a and b as the

outcomes of A and B. By repeating this procedure multiple times, the resulting distributions should then correspond to those obtained from the separate statistical experiments of A and B. This should highlight the distinction between the joint measurement of a pair of observables and any scenario where the outcome distributions are reconstructed from the full distribution of a third measurement.

In this paper, we take a step away from this usual framework and study joint measurements of observables in a setting where the experimenter has access to multiple copies of a given quantum system, rather than performing the experiments on each individual copy separately. At first sight, it may seem that the whole phenomenon of incompatibility is lost in such an approach: if two copies of the same system are available, then by measuring σ_x on one system and σ_y on the other, one has in a sense measured these incompatible observables jointly. However, things change drastically when one looks at more than two observables. In fact, by including also a third spin direction, σ_z , one gets a triple of incompatible observables, which cannot be jointly measured even with two copies of the same system.

This approach leads to a new way of treating and quantifying the incompatibility of larger sets of observables, by looking at the minimal number of system copies needed to be able to measure all of them with a single collective measurement. We will define this number to be the *index of incompatibility* of the set. On a more detailed level, we define the *compatibility stack* of a set of observables as a list of hypergraphs expressing various multi-copy compatibility relations between the observables. This definition naturally generalizes the joint measurability hypergraphs introduced in [1]. After these general treatments, we focus on the qubit case, where we prove the results for the multi-copy joint measurability of triples of noisy qubit observables. In particular, we demonstrate that all compatibility stacks of order three have a quantum realization.

The paper is organized as follows. In Section 2, we recall the definition of a quantum observable. Section 3 presents the notion of k -compatibility. The definition is then expanded in Section 4 to define the compatibility stack, a mathematical way of describing k -compatibility relations within a given set of observables. Section 5 goes deeper into the notion of k -compatibility and provides a necessary and sufficient condition for the k -compatibility of n observables. The general content of the previous parts is then exemplified in Section 6 in the case of three qubit observables. The conclusion and future outlooks are given in Section 7.

2. Quantum Observables

We start by recalling the definition of a quantum observable as a positive operator valued measure. In this paper, we will restrict our investigation to observables with a finite number of measurement outcomes. We refer to [2] for an exhaustive presentation of the properties of quantum observables.

The quantum mechanical description of a physical system is based on a complex Hilbert space \mathcal{H} , which we assume to be finite dimensional throughout the paper. We denote by $\mathcal{L}(\mathcal{H})$ the vector space of linear operators on \mathcal{H} , in which we let $\mathbb{1}$ be the identity operator.

Definition 1. Let Ω be a finite set. A map $A : \Omega \rightarrow \mathcal{L}(\mathcal{H})$ is an $\mathcal{L}(\mathcal{H})$ -valued observable on Ω if:

- (i) $A(x) \geq 0$ for all $x \in \Omega$;
- (ii) $\sum_{x \in \Omega} A(x) = \mathbb{1}$.

The states of the system are represented by positive trace one operators on \mathcal{H} , and for a state ρ , the number $\text{tr}[\rho A(x)]$ is the probability of obtaining an outcome x in a measurement of A .

As an example, consider the x -component of the spin of a spin-1/2 system. The corresponding observable is then the map $X : \{-1, +1\} \rightarrow \mathcal{L}(\mathbb{C}^2)$ defined on the two-outcome set $\Omega = \{+1, -1\}$ and having as its values the two orthogonal projections:

$$X(\pm 1) = \frac{1}{2}(\mathbb{1} \pm \sigma_x).$$

We can add white noise to this observable, and this results in a noisy spin observable $X_a: \{+1, -1\} \rightarrow \mathcal{L}(\mathbb{C}^2)$ defined as:

$$X_a(\pm 1) = \frac{1}{2}(\mathbb{1} \pm a\sigma_x),$$

where the parameter $1 - a$ is the noise intensity. The y - and z -components of the spin are of course treated in the same manner giving rise to the corresponding observables Y and Z and their noisy versions Y_b and Z_c .

It is occasionally convenient to view an observable A as a map on the power set 2^Ω rather than the set Ω . For any $X \subseteq \Omega$, we denote $A(X) = \sum_{x \in X} A(x)$, so that:

- (i') $A(X) \geq 0$ for all $X \subseteq \Omega$;
- (ii') $A(\Omega) = \mathbb{1}$;
- (iii') $A(X \cup Y) = A(X) + A(Y)$ for all $X, Y \subseteq \Omega$, such that $X \cap Y = \emptyset$.

The two definitions are clearly equivalent, and we will switch between them whenever it is convenient.

3. k -Compatibility of Observables

3.1. Definition

Let A_1, \dots, A_n be $\mathcal{L}(\mathcal{H})$ -valued observables with outcome sets $\Omega_1, \dots, \Omega_n$. The compatibility of these observables means that we can simultaneously implement their measurements, even if only one input state is available. Generalizing the usual formulation of joint measurements, we assume that we have access to k copies of the initial state. We can hence make a collective measurement on a state $\varrho^{\otimes k}$ (for any $A \in \mathcal{L}(\mathcal{H})$, we use the notation $A^{\otimes k} \in \mathcal{L}(\mathcal{H}^{\otimes k})$ for the k -fold tensor product $A^{\otimes k} = A \otimes \dots \otimes A$, and we set $\mathcal{H}^{\otimes 0} = \mathbb{C}$ and $A^{\otimes 0} = \mathbb{1}$). This measurement should give a measurement outcome for each observable A_1, \dots, A_n , so we are looking for an $\mathcal{L}(\mathcal{H}^{\otimes k})$ -valued observable G on the product set $\Omega_1 \times \dots \times \Omega_n$. In order for G to serve as a joint measurement, it is required that if we ignore other than the i -th component x_i of a measurement outcome (x_1, \dots, x_n) , the probability must agree with the probability of getting x_i in a measurement of A_i .

For this reason, we introduce the i -th marginal $G^{[i]}$ of G . For all $X \subseteq \Omega_i$, $G^{[i]}$ is the observable given by:

$$\begin{aligned} G^{[i]}(X) &= G(\Omega_1 \times \dots \times \Omega_{i-1} \times X \times \Omega_{i+1} \times \dots \times \Omega_n) \\ &= G(\pi_i^{-1}(X)), \end{aligned}$$

where $\pi_i: \Omega_1 \times \dots \times \Omega_n \rightarrow \Omega_i$ is the projection $\pi_i(x_1, \dots, x_n) = x_i$. This definition of a marginal can also be written in an equivalent form as:

$$G^{[i]}(x) = \sum_{\substack{x_1, \dots, x_{i-1}, \\ x_{i+1}, \dots, x_n}} G(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n)$$

required to hold for all $x \in \Omega_i$.

Definition 2. $\mathcal{L}(\mathcal{H})$ -valued observables A_1, \dots, A_n on the outcome sets $\Omega_1, \dots, \Omega_n$, respectively, are k -compatible if there exists an $\mathcal{L}(\mathcal{H}^{\otimes k})$ -valued observable G on the product set $\Omega_1 \times \dots \times \Omega_n$, such that:

$$\text{tr}[\varrho^{\otimes k} G^{[i]}(x)] = \text{tr}[\varrho A_i(x)] \tag{1}$$

for all $i = 1, \dots, n$, $x \in \Omega_i$ and all states ϱ . The observable G is called a k -copy joint observable of A_1, \dots, A_n .

If $k = 1$ in Definition 2, we have the usual definition of compatibility, also called joint measurability [3].

3.2. Basic Properties

Let us observe some basic properties of the k -compatibility relation. Firstly, for observables A_1, \dots, A_n , we can define:

$$G(x_1, \dots, x_n) = A_1(x_1) \otimes \dots \otimes A_n(x_n) \tag{2}$$

and this $\mathcal{L}(\mathcal{H}^{\otimes n})$ -valued observable clearly satisfies (1) for $k = n$. We thus conclude that:

- Any collection of n observables is n -compatible.

Secondly, if G is a k -copy joint observable of A_1, \dots, A_n , we get a k -copy joint observable of A_1, \dots, A_{n-1} by simply summing over the outcomes in Ω_n . More generally, we have that:

- Any subset of a k -compatible set of observables is k -compatible.

Finally, if G is a k -copy joint observable, we can trivially extend it to a higher dimensional Hilbert space by setting $G' = G \otimes \mathbb{1}$. Therefore, we conclude that:

- Any collection of k -compatible observables is k' -compatible for all $k' \geq k$.

The fourth simple, but important property of the k -compatibility relation is the following additivity.

Proposition 1. *Let \mathcal{A} be a finite collection of observables and $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ for two nonempty subsets \mathcal{A}_1 and \mathcal{A}_2 . If \mathcal{A}_1 is k_1 -compatible and \mathcal{A}_2 is k_2 -compatible, then \mathcal{A} is $(k_1 + k_2)$ -compatible.*

Proof. First, if $\mathcal{A}_i = \mathcal{A}$ for some i , then the claim is trivial. Hence, we assume that $\mathcal{A}_i \neq \mathcal{A}$ for all $i \in \{1, 2\}$. We denote $\mathcal{A}_3 = \mathcal{A}_2 \setminus \mathcal{A}_1$. As \mathcal{A}_3 is a subset of \mathcal{A}_2 , it is k_2 -compatible. The set \mathcal{A} is a disjoint union of \mathcal{A}_1 and \mathcal{A}_3 , and we can label the observables, so that $\mathcal{A}_1 = \{A_1, \dots, A_m\}$ and $\mathcal{A}_3 = \{A_{m+1}, \dots, A_n\}$. We denote by G_1 and G_3 the k_1 - and k_2 -copy joint observables of \mathcal{A}_1 and \mathcal{A}_3 , respectively, and then define:

$$G(x_1, \dots, x_n) = G_1(x_1, \dots, x_m) \otimes G_3(x_{m+1}, \dots, x_n).$$

This observable is a $(k_1 + k_2)$ -copy joint observable of \mathcal{A} . \square

3.3. Index of Incompatibility

For any set of n observables, the smallest integer $1 \leq k \leq n$, such that the collection is k -compatible, is well defined and can be used to determine the “strength” of incompatibility; the more copies of the system you need on the input to measure the given set of observables, the more incompatible they are. This leads to the following notion.

Definition 3. *The index of incompatibility is the minimal number of copies that is needed in order to make a given set of observables compatible. Hence, for a set of observables \mathcal{A} , the index of incompatibility $i(\mathcal{A})$ is:*

$$i(\mathcal{A}) = \min_k \{\mathcal{A} \text{ is } k\text{-compatible}\}.$$

The usual compatibility corresponds to 1-compatibility; hence, the index of incompatibility of a compatible set of observables is 1. The index of incompatibility can be taken as an integer valued quantification of the incompatibility of a given set. Our earlier observations and Proposition 1 imply the following.

- (i) $1 \leq i(\mathcal{A}) \leq \#\mathcal{A}$;
- (ii) if $\mathcal{A} \subseteq \mathcal{B}$, then $i(\mathcal{A}) \leq i(\mathcal{B})$;
- (iii) $i(\mathcal{A}_1 \cup \mathcal{A}_2) \leq i(\mathcal{A}_1) + i(\mathcal{A}_2)$;

(iv) $i(\mathcal{A}) = 1$ if and only if \mathcal{A} is compatible.

It is not clear from this definition whether for each integer $n \geq 2$ there exists a set \mathcal{A} of n observables, such that the index $i(\mathcal{A})$ has the maximal value n . In Section 6, we will show that there exists a triplet of observables whose index of incompatibility is 3.

4. Compatibility Stack

4.1. Definition

Although the index of incompatibility gives a simple quantification of the incompatibility of a set of observables, it does not take into account the finer compatibility structures present in the set. This calls for a more refined description of the various compatibility relations between the observables. In the usual single copy scenario, this can be conveniently done in terms of joint measurability hypergraphs [1].

In general, a hypergraph is a pair (V, E) consisting of a set V and a set E of non-empty subsets of V . The elements of V are called *vertices*, and the elements of E are *edges* (when subsets of more than two vertices are involved, these are actually hyperedges). Following [1], we say that a hypergraph (V, E) is a joint measurability hypergraph (or compatibility hypergraph) if all non-empty subsets of edges are also edges, i.e.,

$$\emptyset \neq \mathcal{A}' \subseteq \mathcal{A} \in E \Rightarrow \mathcal{A}' \in E.$$

Every set of observables gives rise to a joint measurability hypergraph where the vertices represent the observables, and the edges linking some particular vertices represent the compatibility of the corresponding observables. The above condition then states that the compatibility of some set of observables implies the compatibility of any subset of these observables. Furthermore, it was shown in [1] that every abstract joint measurability hypergraph where all of the singleton sets are edges has such a realization in terms of quantum observables.

The generalization of this approach to the case of k -compatibility is given by the following notion.

Definition 4. Let V be a finite set with n elements, and let E_k be a set of non-empty subsets of V for $k = 1, \dots, n$. We denote $H_k = (V, E_k)$. The list (H_1, \dots, H_n) of hypergraphs is a compatibility stack if:

- (S1) each $H_k = (V, E_k)$ is a joint measurability hypergraph,
- (S2) E_1 contains all singleton sets and $E_n = 2^V$ and
- (S3) if $\mathcal{A} \in E_k$ and $\mathcal{B} \in E_l$, then $\mathcal{A} \cup \mathcal{B} \in E_{k+l}$.

The motivation for the previous definition is that any finite set of observables gives rise to a compatibility stack. Namely, let $V = \{A_1, \dots, A_n\}$ be a finite set of observables. We take these observables as vertices, and a set of edges E_k is defined in a way that a subset $\mathcal{A} \subseteq V$ belongs to E_k if \mathcal{A} is k -compatible. Conditions (S1)–(S3) hold by our earlier discussion; these are necessary conditions for the k -compatibility relations of any set V of observables. First, every subset of a k -compatible set of observables is also k -compatible; hence, (V, E_k) is a joint measurability hypergraph. Second, each set made of one observable is 1-compatible; hence, E_1 contains all singleton sets. The condition $E_n = 2^V$ follows from the facts that any set of n observables is n -compatible and any subset of n -compatible observables is also n -compatible. Finally, Proposition 1 is reflected in Condition (S3).

Proposition 2. For a compatibility stack (H_1, \dots, H_n) , the following hold:

- (1) $E_1 \subseteq E_2 \subseteq \dots \subseteq E_n$;
- (2) For each $k = 1, \dots, n$, the set E_k contains all subsets of V of order k .

Proof. (1) Fix $k = 2, \dots, n$. Suppose $\mathcal{A} \in E_{k-1}$, and pick $A \in \mathcal{A}$. We have $\{A\} \in E_1$ by (S2). Then, $\mathcal{A} = \mathcal{A} \cup \{A\}$, and hence, $\mathcal{A} \in E_k$ by (S3). Therefore, $E_{k-1} \subseteq E_k$.

(2) This follows by induction. Indeed, by (S2), the claim is true for E_1 . If \mathcal{A} is an order $k + 1$ subset of V and $A \in \mathcal{A}$, then $\mathcal{A} \setminus \{A\} \in E_k$ by the inductive hypothesis; hence, $\mathcal{A} = \{A\} \cup (\mathcal{A} \setminus \{A\}) \in E_{k+1}$ by (S3). \square

Item (1) abstractly reflects the understanding that if a set of observables is k -compatible, it is $(k + 1)$ -compatible, as well. Item (2) is, on the other hand, saying that any collection of k observables is k -compatible.

For a compatibility stack (H_1, \dots, H_n) consisting of hypergraphs $H_k = (V, E_k)$, we say that the *index* of a non-empty subset $\mathcal{A} \subseteq V$ is the smallest integer j , such that $\mathcal{A} \in E_j$. If the compatibility stack represents the k -compatibility relations of a set of observables, then the index of \mathcal{A} is exactly the index of incompatibility as given by Definition 3. It is clear that the normalization, monotonicity and subadditivity Properties (i)–(iii) of the index of incompatibility are still retained in this abstract setting.

For $n \leq 4$, a compatibility stack can be visually represented as an n -dimensional object with different colors assigned to different hypergraphs H_k . This is exemplified in Figure 1 for $n = 4$ and the set of vertices (i.e., observables) $V = \{A, B, C, D\}$. A compact version (left) shows only indexes of given subsets, while on the right, all elements are shown. Different hypergraphs $H_k = (V, E_k)$ are represented by the same color; e.g., H_2 is determined by the set of all (hyper-)edges E_2 , which is composed of all blue elements from the figure. These elements visually show for the exemplified case that 2-compatible are not only all pairs of observables from V , but also all triples from V , except for $\{A, B, C\}$.

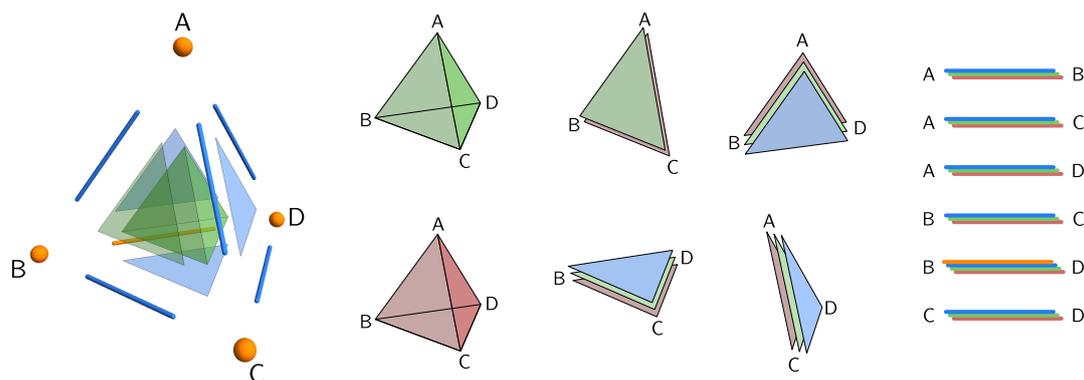


Figure 1. Visualization of an incompatibility stack for four observables A, B, C and D. On the left is a compact representation in terms of indexes, i.e., the smallest possible k -compatibility for a given subset of observables. On the right is the same stack in terms of hypergraphs H_k , where each hypergraph is represented by different color ($k = 1$ is orange; $k = 2$ is blue; $k = 3$ is green; and $k = 4$ is red).

The visualization also follows Item (1) of Proposition 2, which states that, if there is an element with some color (e.g., blue), then the same element must be present also with colors “below” it (i.e., green, red). For this reason, it is better to use just the index of a given subset of observables, as this allows the neater visualization on the left side of Figure 1. Item (2) of the proposition just means that the color corresponding to some k will mark all subsets with k elements. For example, in the figure, blue corresponds to $k = 2$, and thus, all pairs of observables will have a representation via a blue element.

4.2. Compatibility Stacks with Three Vertices

The simplest (non-trivial) example of a compatibility stack is the case of three vertices A, B, C. A graphical representation of a compatibility stack (H_1, H_2, H_3) is a triangle, where the edges and the area can be colored according to the corresponding index. The situation is depicted in Figure 2 for all possible compatibility stacks and in Figure 3 for some impossible cases.

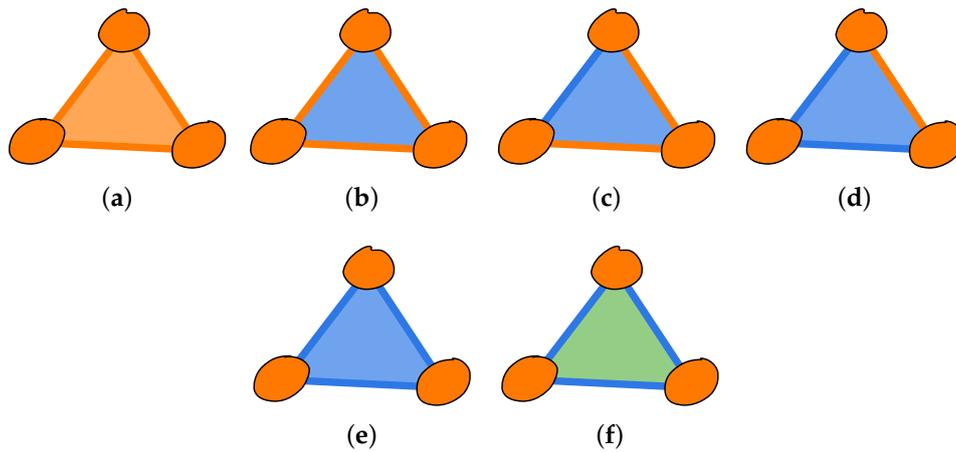


Figure 2. All possible compatibility stacks with three vertices. The orange color marks index 1; blue marks index 2; and green marks index 3. Whereas (a) depicts the most compatible case, when all of the measurements can be performed on a single copy of a state; (f) depicts the worst case where for each measurement, we need an extra copy of the state. The cases (b)–(e) represent all the intermediate possibilities.

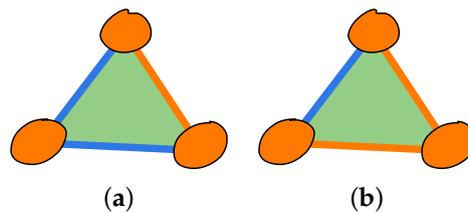


Figure 3. Figures (a) and (b) depict two examples of impossible k -compatibility relations for three observables. The orange color marks index 1; blue marks index 2; and green marks index 3.

For example, in the case (a) of Figure 2, the compatibility stack is given by $H_1^{(a)} = H_2^{(a)} = H_3^{(a)} = (V, 2^V)$ with $V = \{A, B, C\}$ being the set of vertices. For the cases (b)–(e), we still have $H_2^{(b)-(e)} = H_3^{(b)-(e)} = (V, 2^V)$, but for $H_1^{(b)-(e)}$, its set E_1 has fewer and fewer elements. In the case (e), $H_1^{(e)}$ becomes simply $H_1^{(e)} = (V, \{\{A\}, \{B\}, \{C\}\})$. Finally, in the case (f), the compatibility stack is given as:

$$\begin{aligned}
 H_1^{(f)} &= (V, \{\{A\}, \{B\}, \{C\}\}), \\
 H_2^{(f)} &= (V, \{\{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}\}), \\
 H_3^{(f)} &= (V, 2^V).
 \end{aligned}$$

The case (a) of Figure 3 is on first sight representable by a stack:

$$\begin{aligned}
 H_1^{(a)'} &= (V, \{\{A\}, \{B\}, \{C\}, \{A, B\}\}), \\
 H_2^{(a)'} &= (V, \{\{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}\}), \\
 H_3^{(a)'} &= (V, 2^V).
 \end{aligned}$$

However, its impossibility comes from the fact that E_1 contains both $\{C\}$ and $\{A, B\}$, which by (S3) would require E_2 to contain the set $\{A, B, C\}$, which is not the case. A similar discussion is valid also for the case (b).

The fact that some collections of hypergraphs are not compatibility stacks comes from the fact that Definition 4 puts limitations on the indexes of the hypergraph edges. In particular, Condition (S3) reduces the number of possible compatibility stacks more than Proposition 2 alone. We shall make this clear in the discussion of four vertices below.

4.3. Compatibility Stacks with Four Vertices

Having four vertices increases the number of possible compatibility stacks considerably. Let the four vertices be denoted as A, B, C and D. For any pair of these vertices, the remaining pair will be called *reciprocal*, e.g., {B, C} is reciprocal to {A, D}.

The four vertices can be illustrated as the vertices of a tetrahedron. In this representation, different types of graph (hyper-)edges correspond to different elements of the tetrahedron (vertices, edges, sides and bulk), with each of these possibly having a different index (in this section, edges of the graph will be always denoted as *graph edges*, while the physical edges of the tetrahedron will be just *edges*). We will abuse the language a bit by saying that particular elements of the tetrahedron are *k*-compatible, meaning that the corresponding graph edges have index $\leq k$.

The coarsest classification of compatibility stacks is by the index of the bulk. The case of index 1 bulk is possible only when also all sides and edges have index 1 (Figure 4a), since from Item (1) of Proposition 2, we have $2^V = E_1 \subseteq E_2 \subseteq \dots \subseteq E_n$, implying the equality of all of the sets E_k . On the other hand, the bulk can have index 4 only when all sides have index 3 and edges have index 2 (Figure 4b). Indeed, if on the contrary, some edge would have index 1, then, by 2-compatibility of its reciprocal edge, the bulk would be 3-compatible by (S3). A similar consideration holds for the side indexes.

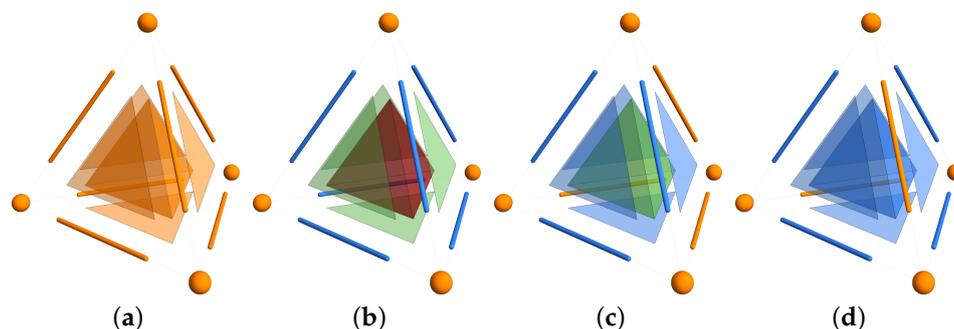


Figure 4. Compatibility stacks with four vertices can be represented by colored tetrahedrons. As before, orange color marks index 1; blue marks index 2; and green marks index 3. In addition, index 4 is marked by red color. The case (a) represents the most compatible case, where we need only a single copy of a state to measure all four observables; whereas the case (b) is in this respect the worst one, as we need a new copy of a state for each measurement. The cases (c) and (d) show some intermediate possibilities of compatibility stacks.

The cases in between (as, e.g., Figure 4c,d) are more populated, and their number is reduced by the compatibility stack condition (S3). Particularly useful is the following consequence of the subadditivity of the index.

- If two sets composed of reciprocal pairs have index 1, then the set of all four vertices has index ≤ 2 .

Visually on the tetrahedron, this means that if two opposing edges are compatible, then the bulk is 2-compatible (see, e.g., Figure 4d). This observation is also easily intuitively grasped, as, if we have two pairs of compatible observables, let us say {B, C} and {A, D}, then there exist corresponding joint observables F and G, respectively. These two observables are always 2-compatible, and hence, also the four observables A, B, C and D are 2-compatible.

Definition 4 of the compatibility stack and Proposition 2 lead to the possibilities enumerated in Table 1. The existence of a given compatibility stack is, however, just necessary for such a

combination of compatibility indexes. As we do not have a systematic way of finding realizations for the compatibility stacks yet, it is an open question whether all of them may be associated with sets of quantum observables.

Table 1. All possible compatibility stacks enumerated by their bulk index (number of copies of a system required to measure all four observables; colors in parentheses represent indexes from previous figures) and the number of edges with index 2 (how many pairs of observables are not 1-compatible). Altogether, 34 different stacks (up to trivial permutations) are possible.

# of Index 2 Edges ▶	0	1	2	3	4	5	6
Bulk Index ▼							
1 (orange)	1	–	–	–	–	–	–
2 (blue)	5	3	3	4	2	1	1
3 (green)	–	–	–	3	2	3	5
4 (red)	–	–	–	–	–	–	1

5. Structure of k -Copy Joint Observables

In this section, we show that, in order to find the index of incompatibility of any collection of observables $\mathcal{A} = \{A_1, \dots, A_n\}$, it is enough to characterize all symmetric k -copy joint observables of A_1, \dots, A_n . In particular, we prove that the k -compatibility of A_1, \dots, A_n is equivalent to the usual compatibility of their symmetrized versions $\tilde{A}_1, \dots, \tilde{A}_n$ in $\mathcal{L}(\mathcal{H}^{\otimes k})$. This reduces the k -compatibility problem to a standard compatibility problem on an enlarged quantum system.

5.1. Symmetric Product

The symmetric group S_k acts in a natural way on the tensor product $\mathcal{H}^{\otimes k}$ of k copies of \mathcal{H} : if $p \in S_k$ is any permutation, its action on a decomposable element $\psi_1 \otimes \dots \otimes \psi_k \in \mathcal{H}^{\otimes k}$ is defined as:

$$\sigma(p)(\psi_1 \otimes \dots \otimes \psi_k) = \psi_{p^{-1}(1)} \otimes \dots \otimes \psi_{p^{-1}(k)}.$$

The map $\sigma : S_k \rightarrow \mathcal{L}(\mathcal{H}^{\otimes k})$ is a unitary representation of S_k on $\mathcal{H}^{\otimes k}$.

Using this unitary representation, we then define the *symmetrizer channel* Σ_k on $\mathcal{L}(\mathcal{H}^{\otimes k})$ (in the Heisenberg picture) as:

$$\Sigma_k(A) = \frac{1}{k!} \sum_{p \in S_k} \sigma(p) A \sigma(p)^* \quad A \in \mathcal{L}(\mathcal{H}^{\otimes k}).$$

This map is completely positive and unital; hence, it is a quantum channel. On decomposable operators $A = A_1 \otimes \dots \otimes A_k$, we have:

$$\Sigma_k(A_1 \otimes \dots \otimes A_k) = \frac{1}{k!} \sum_{p \in S_k} A_{p(1)} \otimes \dots \otimes A_{p(k)},$$

hence Σ_k is an idempotent projection onto the linear subspace $\text{Sym}(k, \mathcal{L}(\mathcal{H}))$ of the k -symmetric tensor operators in $\mathcal{L}(\mathcal{H}^{\otimes k}) = \mathcal{L}(\mathcal{H})^{\otimes k}$. It becomes an orthogonal projection by endowing $\mathcal{L}(\mathcal{H}^{\otimes k})$ with the Hilbert–Schmidt inner product $\langle A | B \rangle_{HS} = \text{tr}[A^* B]$.

The *symmetric product* of two operators $A_1 \in \text{Sym}(k_1, \mathcal{L}(\mathcal{H}))$ and $A_2 \in \text{Sym}(k_2, \mathcal{L}(\mathcal{H}))$ is the operator $A_1 \odot A_2 \in \text{Sym}(k_1 + k_2, \mathcal{L}(\mathcal{H}))$ with:

$$A_1 \odot A_2 = \Sigma_{k_1+k_2}(A_1 \otimes A_2).$$

The symmetric product is associative and commutative.

We will constantly use the following, easily verifiable, formula: if $A_1, \dots, A_k, B_1, \dots, B_k \in \mathcal{L}(\mathcal{H})$, then:

$$\langle A_1 \otimes \dots \otimes A_k | B_1 \otimes \dots \otimes B_k \rangle_{HS} = \frac{1}{k!} \sum_{p \in \mathcal{S}_k} \text{tr} [A_1 B_{p(1)}] \dots \text{tr} [A_k B_{p(k)}].$$

5.2. Structure Theorem

It is immediate to verify that G is a k -copy joint observable of the n observables A_1, \dots, A_n if and only if its symmetric version $\tilde{G} = \Sigma_k \circ G$ is such. Hence, for a k -compatible set of observables, the set of k -copy joint observables always contains a symmetric element. The following theorem will tell us even more and show that the k -compatibility of A_1, \dots, A_n is equivalent to the usual compatibility of some $\mathcal{L}(\mathcal{H}^{\otimes k})$ -valued symmetric observables derived from A_1, \dots, A_n .

Theorem 1. *The $\mathcal{L}(\mathcal{H})$ -valued observables A_1, \dots, A_n on $\Omega_1, \dots, \Omega_n$ are k -compatible if and only if there exists a $\mathcal{L}(\mathcal{H}^{\otimes k})$ -valued observable \tilde{G} on $\Omega_1 \times \dots \times \Omega_n$, such that:*

$$\tilde{G}^{[i]}(x) = \mathbb{1}^{\otimes(k-1)} \otimes A_i(x) \quad \forall i = 1, \dots, n, x \in \Omega_i. \tag{3}$$

In this case, we can choose \tilde{G} , such that $\tilde{G}(x_1, \dots, x_n) \in \text{Sym}(k, \mathcal{L}(\mathcal{H}))$ for all x_1, \dots, x_n .

Proof. Sufficiency is easy, because any observable \tilde{G} as in (3) satisfies:

$$\begin{aligned} \text{tr} [\varrho^{\otimes k} \tilde{G}^{[i]}(x)] &= \langle \varrho^{\otimes k} | \Sigma_k(\mathbb{1}^{\otimes(k-1)} \otimes A_i(x)) \rangle_{HS} \\ &= \langle \Sigma_k(\varrho^{\otimes k}) | \mathbb{1}^{\otimes(k-1)} \otimes A_i(x) \rangle_{HS} \\ &= \langle \varrho^{\otimes k} | \mathbb{1}^{\otimes(k-1)} \otimes A_i(x) \rangle_{HS} \\ &= \text{tr} [\varrho A_i(x)], \end{aligned}$$

which is (1).

Conversely, suppose that (1) holds for G , and let $\tilde{G} = \Sigma_k \circ G$. Then, \tilde{G} is an observable on $\Omega_1 \times \dots \times \Omega_n$, which is such that $\tilde{G}(x_1, \dots, x_n) \in \text{Sym}(k, \mathcal{L}(\mathcal{H}))$ for all x_1, \dots, x_n . Denote $G = \tilde{G}^{[i]}(x)$. We have:

$$\begin{aligned} \text{tr} [\varrho^{\otimes k} G] &= \langle \varrho^{\otimes k} | \Sigma_k(G^{[i]}(x)) \rangle_{HS} = \langle \varrho^{\otimes k} | G^{[i]}(x) \rangle_{HS} \\ &= \text{tr} [\varrho A_i(x_i)] \end{aligned}$$

by (1). Choosing the state $\varrho = \mathbb{1}/d + t\Delta$, where $|t| \leq 1/(d \|\Delta\|)$ and $\Delta = \Delta^*$ with $\text{tr} [\Delta] = 0$, the last equation gives:

$$\sum_{j=0}^k \binom{k}{j} \text{tr} [((\mathbb{1}/d)^{\otimes(k-j)} \otimes \Delta^{\otimes j}) G] t^j = \text{tr} [(\mathbb{1}/d) A_i(x)] + \text{tr} [\Delta A_i(x)] t.$$

Comparing the coefficients of the same degree in t , we obtain the system of equations:

$$d^{-k} \text{tr} [G] = d^{-1} \text{tr} [A_i(x)] \tag{4}$$

$$d^{-(k-1)} \langle \mathbb{1}^{\otimes(k-1)} \otimes \Delta | G \rangle_{HS} = \langle \Delta | A_i(x) \rangle_{HS} \tag{5}$$

$$\binom{k}{j} d^{-(k-j)} \langle \mathbb{1}^{\otimes(k-j)} \otimes \Delta^{\otimes j} | G \rangle_{HS} = 0 \quad \forall j \in \{2, 3, \dots, k\} \tag{6}$$

which must hold for all $\Delta \in \mathcal{L}(\mathcal{H}^{\otimes k})$ with $\Delta^* = \Delta$ and $\text{tr}[\Delta] = 0$. Now, take a set of $D = d^2 - 1$ operators $T_1, \dots, T_D \in \mathcal{L}(\mathcal{H})$ such that $T_r^* = T_r$, $\text{tr}[T_r] = 0$ and $\text{tr}[T_r T_s] = \delta_{rs}$ for all $r, s = 1, \dots, D$, and write $\Delta = x_1 T_1 + \dots + x_D T_D$ for $x_1, \dots, x_D \in \mathbb{R}$. Then, (6) yields:

$$\sum_{j_1 + \dots + j_D = j} \binom{j}{j_1 \dots j_D} x_1^{j_1} \dots x_D^{j_D} \left\langle \mathbb{1}^{\otimes(k-j)} \otimes T_1^{\otimes j_1} \otimes \dots \otimes T_D^{\otimes j_D} \mid G \right\rangle_{HS} = 0$$

for all $j \in \{2, 3, \dots, k\}$. This equality holds for all $x_1, \dots, x_D \in \mathbb{R}$; hence, the coefficient of any monomial $x_1^{j_1} \dots x_D^{j_D}$ must vanish. Since the operators:

$$\left\{ \binom{j}{j_1 \dots j_D}^{\frac{1}{2}} d^{\frac{j-k}{2}} \mathbb{1}^{\otimes(k-j)} \otimes T_1^{\otimes j_1} \otimes \dots \otimes T_D^{\otimes j_D} \mid 0 \leq j \leq k, j_1 + \dots + j_D = j \right\}$$

constitute an orthonormal basis of $\text{Sym}(k, \mathcal{L}(\mathcal{H}))$, it follows that:

$$G = a \mathbb{1}^{\otimes k} + \sum_{r=1}^D b_r \mathbb{1}^{\otimes(k-1)} \otimes T_r = a \mathbb{1}^{\otimes k} + \mathbb{1}^{\otimes(k-1)} \otimes T,$$

where $a \in \mathbb{R}$ and $T \in \mathcal{L}(\mathcal{H})$ is a trace zero self-adjoint operator. By (4),

$$a = d^{-1} \text{tr}[A_i(x)],$$

and, by (5),

$$\text{tr}[\Delta T] = \text{tr}[\Delta A_i(x)].$$

The last equation holds for all trace zero self-adjoint operators Δ ; hence, $T = A_i(x) - d^{-1} \text{tr}[A_i(x)] \mathbb{1}$. In conclusion,

$$G = \mathbb{1}^{\otimes(k-1)} \otimes A_i(x),$$

which is (3). \square

Equation (3) should be compared with the usual compatibility, which requires that:

$$G^{[i]}(x) = A_i(x) \quad \forall i = 1, \dots, n, x \in \Omega_i. \tag{7}$$

There is one essential difference. While in the case of compatibility, every joint observable satisfies (7), in the case of k -compatibility, not every joint observable satisfies (3), but there is always at least one that does.

Corollary 1. *The $\mathcal{L}(\mathcal{H})$ -valued observables A_1, \dots, A_n are k -compatible if and only if the $\mathcal{L}(\mathcal{H}^{\otimes k})$ -valued observables $\tilde{A}_1, \dots, \tilde{A}_n$ are compatible, where:*

$$\tilde{A}_i(x) = \mathbb{1}^{\otimes(k-1)} \otimes A_i(x).$$

Example 1. *Let us consider two two-outcome observables A_1 and A_2 defined by positive operators A_1 and A_2 , respectively. That is, $\Omega_1 = \Omega_2 = \{+1, -1\}$, and $A_i(+1) = A_i$ for $i = 1, 2$. These are always 2-compatible, and a possible choice for their 2-copy joint observable is given by (2). By Theorem 1, one can also find a $\mathcal{L}(\mathcal{H}^{\otimes 2})$ -valued symmetric joint observable \tilde{G} on $\Omega_1 \times \Omega_2$. Indeed, if $A_i^c = \mathbb{1} - A_i$, \tilde{G} is defined by:*

$$\begin{aligned} \tilde{G}(+1, +1) &= \frac{1}{2} (A_1 \otimes A_2 + A_2 \otimes A_1) \\ \tilde{G}(-1, +1) &= \frac{1}{2} (A_1^c \otimes A_2 + A_2 \otimes A_1^c) \\ \tilde{G}(+1, -1) &= \frac{1}{2} (A_1 \otimes A_2^c + A_2^c \otimes A_1) \\ \tilde{G}(-1, -1) &= \frac{1}{2} (A_1^c \otimes A_2^c + A_2^c \otimes A_1^c) \end{aligned}$$

6. Three Qubit Observables

In this section, we concentrate on the case of three observables. Up to the permutation of observables, there are six different compatibility stacks, depicted in Figure 2. We will now show that all compatibility stacks in Figure 2 have a realization in terms of qubit observables.

6.1. 2-Copy Joint Observables from Mixing

Let X, Y and Z be the three sharp spin-1/2 observables on \mathbb{C}^2 , with outcome spaces $\{+1, -1\}$. We further denote $X_a(\pm 1) = \frac{1}{2}(\mathbb{1} \pm a\sigma_x)$ for $0 \leq a \leq 1$, and similarly, $Y_b(\pm 1) = \frac{1}{2}(\mathbb{1} \pm b\sigma_y)$ and $Z_c(\pm 1) = \frac{1}{2}(\mathbb{1} \pm c\sigma_z)$ for $0 \leq b, c \leq 1$. These are considered as noisy (unsharp) versions of the sharp observables X, Y and Z , with noise intensities $1 - a, 1 - b$, and $1 - c$, respectively.

Furthermore, let us define three observables:

$$\begin{aligned} G_\alpha^{1,2}(x, y) &= \frac{1}{4}(\mathbb{1} + x \sin \alpha \sigma_x + y \cos \alpha \sigma_y), \\ G_\beta^{2,3}(y, z) &= \frac{1}{4}(\mathbb{1} + y \sin \beta \sigma_y + z \cos \beta \sigma_z), \\ G_\gamma^{1,3}(x, z) &= \frac{1}{4}(\mathbb{1} + x \cos \gamma \sigma_x + z \sin \gamma \sigma_z). \end{aligned}$$

These observables, parametrized by the angles α, β and γ , will be useful in constructing joint observables later.

We recall the following results on joint measurability of noisy spin-1/2 observables [4].

Theorem 2. *The following facts hold.*

- (1) X_a and Y_b are compatible if and only if $a^2 + b^2 \leq 1$.
- (2) X_a, Y_b and Z_c are compatible if and only if $a^2 + b^2 + c^2 \leq 1$.

In particular, we see that the marginals of the observables $G_\alpha^{1,2}, G_\beta^{2,3}$ and $G_\gamma^{1,3}$ are noisy versions of the couples $(X, Y), (Y, Z)$ and (X, Z) , respectively, with noise intensities attaining the upper bound of Item (1) of Theorem 2.

From these results, we already find realizations of the cases (a)–(d) in Figure 2. For instance, with the following choices of the parameters a, b and c , we get suitable triples of observables:

- (a) $a = b = c = 1/\sqrt{3}$, with joint observable

$$G(x, y, z) = \frac{1}{8} \left[\mathbb{1} + \frac{1}{\sqrt{3}}(x\sigma_x + y\sigma_y + z\sigma_z) \right]$$

- for the triple X_a, Y_b and Z_c ;
- (b) $a = b = c = 1/\sqrt{2}$, with joint observables $G_{\pi/4}^{1,2}, G_{\pi/4}^{2,3}$ and $G_{\pi/4}^{1,3}$ for the corresponding pairs of observables;

- (c) $a = b = 4/5$ and $c = 3/5$, with joint observable $G_\gamma^{1,3}$ (having $\sin \gamma = 3/5$ and $\cos \gamma = 4/5$) for observables X_a and Z_c and joint observable $G_\beta^{2,3}$ (having $\sin \beta = 4/5$ and $\cos \beta = 3/5$) for observables Y_b and Z_c ;
- (d) $a = 4/5, b = 1$ and $c = 3/5$, with joint observable $G_\gamma^{1,3}$ (having $\sin \gamma = 3/5$ and $\cos \gamma = 4/5$) for observables X_a and Z_c .

The cases (e) and (f) are more involved, and the rest of the section is dedicated to them; let us see how far we can get by mixing joint observables of two observables. The method is as follows. We choose randomly either X, Y or Z , measure the chosen observable, say X , on the first system and then an optimal joint observable of the noisy versions of the remaining observables Y and Z on the second system. This gives the following sufficient condition for 2-compatibility.

Proposition 3. (Sufficient condition for 2-compatibility) X_a, Y_b and Z_c are 2-compatible if there are numbers $\lambda_1, \lambda_2, \lambda_3 \in [0, 1]$ and $\alpha, \beta, \gamma \in [0, \pi/2]$, such that:

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

and:

$$\begin{cases} a \leq \lambda_1 + \lambda_2 \cos \gamma + \lambda_3 \sin \alpha \\ b \leq \lambda_1 \sin \beta + \lambda_2 + \lambda_3 \cos \alpha \\ c \leq \lambda_1 \cos \beta + \lambda_2 \sin \gamma + \lambda_3 \end{cases} \quad (8)$$

Proof. We choose randomly either X, Y or Z , measure the chosen observable, say X , on the first system and then the optimal joint observables $G_\beta^{2,3}$ of the noisy versions of the remaining observables Y and Z on the second system. The total procedure leads to an observable:

$$G(x, y, z) = \lambda_1 X(x) \otimes G_\beta^{2,3}(y, z) + \lambda_2 Y(y) \otimes G_\gamma^{1,3}(x, z) + \lambda_3 Z(z) \otimes G_\alpha^{1,2}(x, y),$$

where $\lambda_1, \lambda_2, \lambda_3$ represent the probabilities for the choice of the measurement X, Y, Z , respectively. It is easy to check that G is a 2-copy joint observable of the three observables X_a, Y_b and Z_c , with a, b and c attaining the upper bounds from (8). \square

Using Proposition 3, we can cook up a realization of Figure 2e. The 2-compatibility of X_a, Y_b and Z_c can be achieved by setting $\alpha = \beta = \gamma = \pi/4$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1/3$. This choice provides a possible setting (in Figure 2):

(e) $a = b = c = (1 + \sqrt{2})/3$.

6.2. Optimal 2-Copy Joint Observable

To find a realization of the compatibility stack depicted in Figure 2f, we need to show that X_a, Y_b, Z_c are not 2-compatible for some values of noise intensities a, b, c . The fact that these kind of parameters exist follows from the next theorem.

Theorem 3. X_a, Y_a and Z_a are 2-compatible if and only if $0 \leq a \leq \sqrt{3}/2$.

By Theorem 1, the observables X_a, Y_a and Z_a are 2-compatible if and only if the observables \tilde{X}_a, \tilde{Y}_a and \tilde{Z}_a have a symmetric joint observable \tilde{G} , where:

$$\begin{aligned} \tilde{X}_a(\pm 1) &= \mathbb{1} \odot X_a(\pm 1) = \frac{1}{2}(\mathbb{1} \otimes X_a(\pm 1) + X_a(\pm 1) \otimes \mathbb{1}) \\ &= \frac{1}{4}(2 \mathbb{1} \otimes \mathbb{1} \pm a(\mathbb{1} \otimes \sigma_x + \sigma_x \otimes \mathbb{1})) \end{aligned} \quad (9)$$

and similarly for \tilde{Y}_a and \tilde{Z}_a . We will now show that \tilde{G} can be chosen to be covariant with respect to the transitive action of a suitable group on the joint outcome space $\Omega = \{+1, -1\}^3$ of the three observables \tilde{X}_a, \tilde{Y}_a and \tilde{Z}_a . Covariance will then drastically decrease the freedom in the choice of \tilde{G} , actually reducing it to only fixing two parameters. To exploit covariance, we start from the following simple fact.

Proposition 4. *Suppose G is a finite group, Ω is a G -space and U is a unitary representation of G in the Hilbert space \mathcal{K} . Let \mathcal{F} be a collection of subsets of Ω , such that:*

$$g.X = \{g.x \mid x \in X\} \in \mathcal{F} \quad \text{for all } g \in G \text{ and } X \in \mathcal{F}.$$

Then, for any observable $G: \Omega \rightarrow \mathcal{L}(\mathcal{K})$ satisfying the relation:

$$G(g.X) = U(g)G(X)U(g)^* \quad \forall X \in \mathcal{F}, g \in G,$$

the observable $G^\wedge: \Omega \rightarrow \mathcal{L}(\mathcal{K})$ given by:

$$G^\wedge(x) = \frac{1}{\#G} \sum_{g \in G} U(g)^* G(g.x) U(g) \quad \forall x \in \Omega \tag{10}$$

is such that:

- (i) $G^\wedge(g.x) = U(g)G^\wedge(x)U(g)^*$ for all $x \in \Omega$ and $g \in G$;
- (ii) $G^\wedge(X) = G(X)$ for all $X \in \mathcal{F}$.

Proof. Direct verification. \square

According to (10), we call the observable G^\wedge the *U-covariant version* of G .

The choice of the covariance group G and its action on the outcomes Ω for a joint observable of \tilde{X}_a, \tilde{Y}_a and \tilde{Z}_a is prescribed by the covariance properties of X, Y and Z . Namely, the set of effects $\{X(\pm 1), Y(\pm 1), Z(\pm 1)\}$ is invariant for the rotations in the octahedron subgroup $O \subset SO(3)$. Moreover, O acts transitively on this set. We therefore expect that the proper covariance group for our problem is $G = O$. We now explain this statement in more detail.

The *octahedron group* O is the order 24 group of the 90° rotations around the three coordinate axes $\vec{i}, \vec{j}, \vec{k}$, together with the 120° rotations around the axes $(\pm\vec{i} \pm \vec{j} \pm \vec{k})/\sqrt{3}$ and the 180° rotations around $(\pm\vec{i} \pm \vec{j})/\sqrt{2}, (\pm\vec{j} \pm \vec{k})/\sqrt{2}$ and $(\pm\vec{i} \pm \vec{k})/\sqrt{2}$. It preserves the set $\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \in \{+1, -1\}\}$ and acts transitively on it. Moreover, the stabilizer subgroup of any $\vec{u} \in \Omega$ is just the subgroup $O_{\vec{u}}$ of the three 120° rotations around $\vec{u}/\sqrt{3}$.

The octahedron group also acts on the spin-1/2 Hilbert space $\mathcal{H} = \mathbb{C}^2$ by restriction of the usual two-valued $SU(2)$ -representation of $SO(3)$. This gives an ordinary representation $U(g) = \tilde{g} \otimes \tilde{g}$ of O on the 2-copy Hilbert space $\mathcal{H}^{\otimes 2} = \mathbb{C}^2 \otimes \mathbb{C}^2$, where \tilde{g} is any of the two elements of $SU(2)$ corresponding to the rotation $g \in O$.

Finally, let $\pi_i: \Omega \rightarrow \{+1, -1\}$ be the projection onto the i -th component ($i = 1, 2, 3$) and define the collection of subsets:

$$\mathcal{F} = \{\pi_1^{-1}(x), \pi_2^{-1}(y), \pi_3^{-1}(z) \mid x, y, z \in \{+1, -1\}\}.$$

Clearly, the collection \mathcal{F} is O -invariant. Moreover, if $\tilde{G}: \Omega \rightarrow \mathcal{L}(\mathcal{H})$ is any symmetric joint observable of \tilde{X}_a, \tilde{Y}_a and \tilde{Z}_a , then:

$$\tilde{G}(\pi_1^{-1}(x)) = \tilde{X}_a(x) \quad \tilde{G}(\pi_2^{-1}(y)) = \tilde{Y}_a(y) \quad \tilde{G}(\pi_3^{-1}(z)) = \tilde{Z}_a(z).$$

The covariance properties of the observables X, Y and Z then imply that $\tilde{G}(g.X) = U(g)\tilde{G}(X)U(g)^*$ for all $X \in \mathcal{F}$ and $g \in O$. Hence, by Proposition 4, the U -covariant version $(\tilde{G})^\wedge$ of \tilde{G} defined in (10) yields the same margins \tilde{X}_a, \tilde{Y}_a and \tilde{Z}_a . Since the representations U of O and σ of S_2 commute, the joint observable $(\tilde{G})^\wedge$ is both U -covariant and symmetric.

In summary, in order to find the maximal value of a for which the observables X_a, Y_a and Z_a are 2-compatible, we are led to classify the family of symmetric U -covariant observables on Ω . This is done in the next proposition.

Proposition 5. *A map $G : \Omega \rightarrow \mathcal{L}(\mathcal{H}^{\otimes 2})$ is a symmetric and U -covariant observable if and only if there exist real numbers α and β with $\alpha \geq 0, \beta \geq 0$ and $\alpha + \beta \leq 3/8$, such that:*

$$G(\vec{u}) = \frac{4(\alpha + \beta) - 1}{16} [\vec{u} \cdot \vec{\sigma} \otimes \vec{u} \cdot \vec{\sigma} - (\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z)] + \frac{\alpha - \beta}{4\sqrt{3}} (\vec{u} \cdot \vec{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{u} \cdot \vec{\sigma}) + \frac{1}{8} \mathbb{1} \otimes \mathbb{1} \tag{11}$$

for all $\vec{u} \in \Omega$.

Proof. We will proceed in several steps.

(I) Since the action of O on Ω is transitive, a U -covariant observable G is completely determined by its value at $\vec{u}_0 = (+1, +1, +1)$ by the relation:

$$G(g.\vec{u}_0) = U(g)G(\vec{u}_0)U(g)^* \quad \forall g \in O. \tag{12}$$

This equation implies that $G(\vec{u}_0)$ must commute with the representation U restricted to the stabilizer $O_{\vec{u}_0}$ of \vec{u}_0 . This happens if and only if $G(\vec{u}_0)$ is in the commutant $(e^{i(\pi/3)\vec{n}\cdot\vec{\sigma}} \otimes e^{i(\pi/3)\vec{n}\cdot\vec{\sigma}})'$ of the operator $e^{i(\pi/3)\vec{n}\cdot\vec{\sigma}} \otimes e^{i(\pi/3)\vec{n}\cdot\vec{\sigma}}$, where $\vec{n} = \vec{u}_0/\sqrt{3} = (\vec{i} + \vec{j} + \vec{k})/\sqrt{3}$. The eigenvalues of $e^{i(\pi/3)\vec{n}\cdot\vec{\sigma}} \otimes e^{i(\pi/3)\vec{n}\cdot\vec{\sigma}}$ are $e^{i(2\pi/3)}$ and $e^{-i(2\pi/3)}$ with multiplicity one, and 1 with multiplicity two. Hence, $\dim(e^{i(\pi/3)\vec{n}\cdot\vec{\sigma}} \otimes e^{i(\pi/3)\vec{n}\cdot\vec{\sigma}})' = 6$. A linear basis of $(e^{i(\pi/3)\vec{n}\cdot\vec{\sigma}} \otimes e^{i(\pi/3)\vec{n}\cdot\vec{\sigma}})'$ is made up of the self-adjoint operators:

$$\begin{aligned} M_0 &= \mathbb{1} \otimes \mathbb{1} & M_1 &= \vec{n} \cdot \vec{\sigma} \otimes \vec{n} \cdot \vec{\sigma} \\ M_2 &= \frac{1}{3} (\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z) \\ M_3 &= \vec{n} \cdot \vec{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{n} \cdot \vec{\sigma} & M_4 &= \vec{n} \cdot \vec{\sigma} \otimes \mathbb{1} - \mathbb{1} \otimes \vec{n} \cdot \vec{\sigma} \\ M_5 &= \sigma_x \otimes \sigma_y - \sigma_y \otimes \sigma_x + \sigma_y \otimes \sigma_z - \sigma_z \otimes \sigma_y + \sigma_z \otimes \sigma_x - \sigma_x \otimes \sigma_z. \end{aligned}$$

Among them, M_0, M_1, M_2 and M_3 are symmetric, and M_4 and M_5 are antisymmetric. Thus, G is symmetric only if $G(\vec{u}_0)$ is a real linear combination of M_0, M_1, M_2, M_3 . This is also a sufficient condition for the symmetry of G by (12) and the symmetry of the $U(g)$'s.

(II) It is easy to check that the operators M_0, M_1, M_2 and M_3 all commute among themselves. Moreover,

$$\begin{aligned} M_0^2 &= M_1^2 = M_0, & M_2^2 &= \frac{1}{3} (M_0 - 2M_2), & M_3^2 &= 2(M_0 + M_1), \\ M_1M_2 &= \frac{1}{3} (M_0 + M_1 - 3M_2), & M_1M_3 &= M_3, & M_2M_3 &= \frac{1}{3} M_3, \\ & & M_0M_i &= M_i \quad \forall i. \end{aligned}$$

Thus, the four self-adjoint operators:

$$\begin{aligned}
 P_+ &= \frac{1}{4}(M_0 + M_1 + M_3) & P_- &= \frac{1}{4}(M_0 + M_1 - M_3) \\
 Q_+ &= \frac{1}{4}(M_0 - 3M_2) & Q_- &= \frac{1}{4}(M_0 - 2M_1 + 3M_2)
 \end{aligned}$$

are mutually commuting orthogonal projections summing up to the identity of $\mathcal{H}^{\otimes 2}$. It follows that P_+ , P_- , Q_+ , Q_- are rank-one mutually orthogonal projections. Since they span the same linear space as $\{M_0, M_1, M_2, M_3\}$, we can rewrite:

$$G(\vec{u}_0) = \alpha P_+ + \beta P_- + \gamma Q_+ + \delta Q_- \tag{13}$$

where:

$$\alpha \geq 0, \quad \beta \geq 0, \quad \gamma \geq 0, \quad \delta \geq 0 \tag{14}$$

by the positivity condition $G(\vec{u}_0) \geq 0$. This is also a sufficient condition for the positivity of G by (12).

(III) By taking the trace of the normalization condition:

$$\sum_{g \in O/O_{\vec{u}_0}} U(g)G(\vec{u}_0)U(g)^* = \mathbb{1} \otimes \mathbb{1} \tag{15}$$

and observing that $\#O/O_{\vec{u}_0} = \#\Omega = 8$, we obtain:

$$\text{tr}[G(\vec{u}_0)] = \frac{1}{2}. \tag{16}$$

Moreover, the operators M_0 and M_2 commute with the representation U , hence so does the rank-one projection Q_+ . Multiplying both the sides of (15) by Q_+ and taking again the trace, we then get:

$$\text{tr}[Q_+G(\vec{u}_0)] = \frac{1}{8}. \tag{17}$$

Inserting (13) into (16) and (17) yields the conditions:

$$\gamma = \frac{1}{8} \quad \delta = \frac{3}{8} - \alpha - \beta.$$

The positivity requirement (14) thus translates into:

$$\alpha \geq 0, \quad \beta \geq 0, \quad \alpha + \beta \leq \frac{3}{8},$$

and (13) is rewritten as:

$$\begin{aligned}
 G(\vec{u}_0) &= \alpha(P_+ - Q_-) + \beta(P_- - Q_-) + \frac{1}{8}Q_+ + \frac{3}{8}Q_- \\
 &= \frac{3}{4}\left(\alpha + \beta - \frac{1}{4}\right)(M_1 - M_2) + \frac{1}{4}(\alpha - \beta)M_3 + \frac{1}{8}M_0 \\
 &= \frac{4(\alpha + \beta) - 1}{16} [\vec{u}_0 \cdot \vec{\sigma} \otimes \vec{u}_0 \cdot \vec{\sigma} - (\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z)] \\
 &\quad + \frac{\alpha - \beta}{4\sqrt{3}} (\vec{u}_0 \cdot \vec{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{u}_0 \cdot \vec{\sigma}) + \frac{1}{8} \mathbb{1} \otimes \mathbb{1}.
 \end{aligned}$$

We have already seen that $3M_2 = \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z$ commutes with $U(g)$ for all $g \in O$. Therefore, the formula (11) follows from the previous equation by the relation (12). We still need to check that G given by (11) is normalized, and this easily follows from:

$$\sum_{\vec{u} \in \Omega} \vec{u} \cdot \vec{\sigma} \otimes \vec{u} \cdot \vec{\sigma} = 8(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z) \quad \sum_{\vec{u} \in \Omega} \vec{u} \cdot \vec{\sigma} \otimes \mathbb{1} = \sum_{\vec{u} \in \Omega} \mathbb{1} \otimes \vec{u} \cdot \vec{\sigma} = 0.$$

□

Remark 1. The choice of the covariance group $G = O$ and its natural action on the joint outcome space Ω is the minimal possible in order to construct a transitive action of G on Ω preserving the set of effects $\{\tilde{X}_a(\pm 1), \tilde{Y}_a(\pm 1), \tilde{Z}_a(\pm 1)\}$. Transitivity is needed in order to label all of the covariant joint observables by means of the single operator $G(\vec{u}_0)$ as in (12) and, thus, reduce the many free parameters of the problem to the only choice of such an operator.

Now, we need to take the three margins of the most general U -covariant observable found in Proposition 5 and compare it with the observables \tilde{X}_a, \tilde{Y}_a and \tilde{Z}_a . By the covariance property, it is sufficient to consider only the first margin $G^{[1]}$. We have:

$$\sum_{y,z \in \{+1,-1\}} (x\sigma_x + y\sigma_y + z\sigma_z) \otimes (x\sigma_x + y\sigma_y + z\sigma_z) = 4(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z)$$

$$\sum_{y,z \in \{+1,-1\}} [(x\sigma_x + y\sigma_y + z\sigma_z) \otimes \mathbb{1} + \mathbb{1} \otimes (x\sigma_x + y\sigma_y + z\sigma_z)] = 4x(\sigma_x \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_x)$$

and hence:

$$G^{[1]}(x) = \sum_{y,z \in \{+1,-1\}} G(x, y, z) = \frac{(\alpha - \beta)x}{\sqrt{3}} (\sigma_x \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_x) + \frac{1}{2} \mathbb{1} \otimes \mathbb{1}.$$

Comparing this formula with (9) yields:

$$\alpha - \beta = \frac{\sqrt{3}}{4} a.$$

By the positivity conditions $\alpha \geq 0, \beta \geq 0$ and $\alpha + \beta \leq 3/8$, we thus see that the maximal value of a is $a = \sqrt{3}/2$. This completes the proof of Theorem 3.

As a result, the case (f) of Figure 2 can now be achieved for example by setting:

(f) $a = b = c = 1,$

though any choice larger than $\sqrt{3}/2$ suffices.

By combining the result of Theorem 3 with that of Theorem 2 in the case $a = b = c$, we get a complete characterization of the index of incompatibility for three equally noisy orthogonal qubit observables X_a, Y_a and Z_a . In Figure 5, the index of incompatibility $i(\{X_a, Y_a, Z_a\})$ is plotted as a function of the noise parameter a .

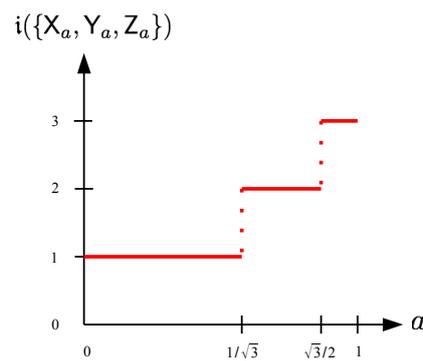


Figure 5. The index of incompatibility $i(\{X_a, Y_a, Z_a\})$ as a function of the noise parameter a for three noisy orthogonal qubit observables.

7. Conclusions

The incompatibility of quantum observables can be evaluated and measured in various ways. In this paper, we introduce a measure of incompatibility based on the number of system copies needed to be able to measure given observables simultaneously. We call this number the index of incompatibility. It quantifies the incompatibility of a set of observables as a whole, but leaves out the finer details regarding the various compatibility relations between the observables.

In [1], it was shown that every conceivable joint measurability combination of a set of observables is realizable. Such combinations are representable by joint measurability hypergraphs where the vertices are observables and edges mark the compatibility relations. By translating this approach to our multi-copy setting, we have analogously defined the notion of the compatibility stack that represents the potential multi-copy compatibility relations present in a set of observables. Namely, whereas in [1] the hypergraph was binary (the presence of graph edges represented compatibility between the observables of the corresponding subsets), here, we have such a graph for each possible compatibility index. We demonstrate that every compatibility stack with three vertices has a realization in terms of quantum observables. However, it remains an open question if all compatibility stacks have such a realization.

Supplementary Materials: The following are available online at www.mdpi.com/2227-7390/4/3/54/s1: *Mathematica* code together with .cdf files allowing for a visualization of the possible compatibility stacks with four vertices.

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