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Three Identities of the Catalan-Qi Numbers

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Abstract: In the paper, the authors find three new identities of the Catalan-Qi numbers and provide alternative proofs of two identities of the Catalan numbers. The three identities of the Catalan-Qi numbers generalize three identities of the Catalan numbers.

Keywords: identity; Catalan number; Catalan-Qi number; Catalan-Qi function; alternative proof; hypergeometric series; generalization

MSC: Primary 05A19; Secondary 11B75, 11B83, 33B15, 33C05, 33C20

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1. Introduction

It is stated in [1] that the Catalan numbers C_n for $n \ge 0$ form a sequence of natural numbers that occur in tree enumeration problems such as "In how many ways can a regular n-gon be divided into $n-2$ triangles if different orientations are counted separately?" (for other examples, see [2,3]) the solution of which is the Catalan number C_{n-2}. The Catalan numbers C_n can be generated by

$$\frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n \tag{1}$$

Three of explicit equations of C_n for $n \ge 0$ read that

$$C_n = \frac{(2n)!}{n!(n+1)!} = \frac{4^n \Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)} = {}_2F_1(1-n,-n;2;1)$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \,\mathrm{d}\, t, \quad \Re(z) > 0$$

is the classical Euler gamma function and

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{z^n}{n!}$$

is the generalized hypergeometric series defined for complex numbers $a_i \in \mathbb{C}$ and $b_i \in \mathbb{C} \setminus \{0,-1,-2,\dots\}$, for positive integers $p,q \in \mathbb{N}$, and in terms of the rising factorials $(x)_n$ defined by

$$(x)_n = \begin{cases} x(x+1)(x+2)\dots(x+n-1), & n \ge 1 \\ 1, & n = 0 \end{cases}$$

and

$$(-x)_n = (-1)^n (x-n+1)_n$$

A generalization of the Catalan numbers C_n was defined in [4–6] by

$$_pd_n = \frac{1}{n}\binom{pn}{n-1} = \frac{1}{(p-1)n+1}\binom{pn}{n}$$

for $n \ge 1$. The usual Catalan numbers $C_n = {}_2d_n$ are a special case with $p = 2$.

In combinatorial mathematics and statistics, the Fuss-Catalan numbers $A_n(p,r)$ are defined in [7,8] as numbers of the form

$$A_n(p,r) = \frac{r}{np+r}\binom{np+r}{n} = r\frac{\Gamma(np+r)}{\Gamma(n+1)\Gamma(n(p-1)+r+1)}$$

It is obvious that

$$A_n(2,1) = C_n, \quad n \ge 0 \quad \text{and} \quad A_{n-1}(p,p) = {}_pd_n, \quad n \ge 1$$

There have existed some literature such as [8–20] on the investigation of the Fuss-Catalan numbers $A_n(p,r)$.

In (Remark 1 [21]), an alternative and analytical generalization of the Catalan numbers C_n and the Catalan function C_x was introduced by

$$C(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^z \frac{\Gamma(z+a)}{\Gamma(z+b)}, \quad \Re(a), \Re(b) > 0, \quad \Re(z) \ge 0$$

In particular, we have

$$C(a,b;n) = \left(\frac{b}{a}\right)^n \frac{(a)_n}{(b)_n}$$

For the uniqueness and convenience of referring to the quantity $C(a,b;x)$, we call the quantity $C(a,b;x)$ the Catalan-Qi function and, when taking $x = n \ge 0$, call $C(a,b;n)$ the Catalan-Qi numbers. It is clear that

$$C\left(\frac{1}{2}, 2; n\right) = C_n, \quad n \ge 0$$

In (Theorem 1.1 [22]), among other things, it was deduced that

$$A_n(p,r) = r^n \frac{\prod_{k=1}^{p} C\left(\frac{k+r-1}{p}, 1; n\right)}{\prod_{k=1}^{p-1} C\left(\frac{k+r}{p-1}, 1; n\right)}$$

for integers $n \ge 0$, $p > 1$, and $r > 0$. In the recent papers [21–31], some properties, including the general expression and a generalization of an asymptotic expansion, the monotonicity, logarithmic convexity, (logarithmically) complete monotonicity, minimality, Schur-convexity, product and determinantal inequalities, exponential representations, integral representations, a generating function, and connections with the Bessel polynomials and the Bell polynomials of the second kind, of the Catalan numbers C_n, the Catalan function C_x, and the Catalan-Qi function $C(a,b;x)$ were established.

In 1928, J. Touchard ([32] p. 472) and ([33] p. 319) derived an identity

$$C_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} 2^{n-2k} C_k \tag{2}$$

where $\lfloor x \rfloor$ denotes the floor function the value of which is the largest integer less than or equal to x. For the proof of Equation (2) by virtue of the generating function (1), see ([33] pp. 319–320).

In 1987, when attending a summer program at Hope College, Holland, Michigan in USA, D. Jonah ([34] p. 214) and ([33] pp. 324–326) presented that

$$\binom{n+1}{m} = \sum_{k=0}^{m} \binom{n-2k}{m-k} C_k, \quad n \geq 2m, \quad n \in \mathbb{N} \tag{3}$$

In 1990, Hilton and Pedersen ([34] p. 214) and ([33] p. 327) generalized Identity (3) for an arbitrary real number n and any integer $m \geq 0$.

In 2009, J. Koshy ([33] p. 322) provided another recursive equation

$$C_n = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^{k-1} \binom{n-k+1}{k} C_{n-k} \tag{4}$$

We observe that Identity (4) can be rearranged as

$$\sum_{k=\lceil \frac{n-1}{2} \rceil}^{n} (-1)^k \binom{k+1}{n-k} C_k = 0$$

where $\lceil x \rceil$ stands for the ceiling function which gives the smallest integer not less than x.

The aims of this paper are to generalize Identities (2)–(4) for the Catalan numbers C_n to ones for the Catalan-Qi numbers $C(a, b; n)$.

Our main results can be summarized up as the following theorem.

Theorem 1. *For $a, b > 0$, $n \in \mathbb{N}$, and $n \geq 2m \geq 0$, the Catalan-Qi numbers $C(a, b; n)$ satisfy*

$$ {}_3F_2\left(a, \frac{1-n}{2}, -\frac{n}{2}; b, \frac{1}{2}; 1\right) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \left(\frac{a}{b}\right)^k C(a, b; k) \tag{5}$$

$$ {}_4F_3\left(1, a, -m, m-n; b, \frac{1-n}{2}, -\frac{n}{2}; \frac{b}{4a}\right) = \frac{1}{\binom{n}{m}} \sum_{k=0}^{m} \binom{n-2k}{m-k} C(a, b; k) \tag{6}$$

and

$$ {}_3F_2\left(1-b-n, -\frac{n+1}{2}, -\frac{n}{2}; -n-1, 1-a-n; \frac{4a}{b}\right) = \frac{1}{C(a, b; n)} \sum_{k=\lceil \frac{n-1}{2} \rceil}^{n} (-1)^{n-k} \binom{k+1}{n-k} C(a, b; k) \tag{7}$$

As by-products, alternative proofs for Identities (2) and (4) are also supplied in next section.

2. Proofs

We are now in a position to prove Theorem 1 and Identities (2) and (4).

Proof of Identity (5). By the definition (1), we have

$$ {}_3F_2\left(a, \frac{1-n}{2}, -\frac{n}{2}; b, \frac{1}{2}; 1\right) = \sum_{k=0}^{\infty} \frac{(a)_k \left(\frac{1-n}{2}\right)_k \left(-\frac{n}{2}\right)_k}{(b)_k \left(\frac{1}{2}\right)_k k!}$$

Using the relations

$$\left(\frac{1-n}{2}\right)_k = 0, \quad k > \left\lfloor \frac{n}{2} \right\rfloor, \quad n = 1, 3, 5, \ldots$$

and

$$\left(-\frac{n}{2}\right)_k = 0, \quad k > \left\lfloor \frac{n}{2} \right\rfloor, \quad n = 2,4,6,\ldots$$

we obtain

$${}_3F_2\left(a, \frac{1-n}{2}, -\frac{n}{2}; b, \frac{1}{2}; 1\right) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\left(\frac{1-n}{2}\right)_k \left(-\frac{n}{2}\right)_k}{\left(\frac{1}{2}\right)_k k!} \left(\frac{a}{b}\right)^k C(a,b;k)$$

Further using the relations

$$\left(\frac{z}{2}\right)_r \left(\frac{z+1}{2}\right)_r = 4^{-r}(z)_{2r}, \quad (-z)_r = (-1)^r r! \binom{z}{r}, \quad \text{and} \quad \left(\frac{1}{2}\right)_r = \frac{(2r)!}{r! 4^r}$$

we acquire

$$\frac{\left(\frac{1-n}{2}\right)_k \left(-\frac{n}{2}\right)_k}{\left(\frac{1}{2}\right)_k k!} = \binom{n}{2k}$$

The proof of Identity (5) is thus complete. □

Proof of Identity (6). By the definition (1), we have

$${}_4F_3\left(1, a, -m, m-n; b, \frac{1-n}{2}, -\frac{n}{2}; \frac{b}{4a}\right) = \sum_{k=0}^{m} \frac{(-m)_k (m-n)_k}{4^k \left(\frac{1-n}{2}\right)_k \left(-\frac{n}{2}\right)_k} C(a,b;k)$$

Since

$$4^k \left(\frac{1-n}{2}\right)_k \left(-\frac{n}{2}\right)_k = \frac{n!}{(n-2k)!}$$

and

$$(-m)_k (m-n)_k = \frac{m!(n-m)!}{(m-k)!(n-m-k)!}$$

it follows that

$$\frac{(-m)_k (m-n)_k}{4^k \left(\frac{1-n}{2}\right)_k \left(-\frac{n}{2}\right)_k} = \frac{\binom{n-2k}{m-k}}{\binom{n}{m}}$$

Hence, we can derive Identity (6). □

Proof of Identity (7). By the definition (1), we have

$$\begin{aligned} &{}_3F_2\left(1-b-n, -\frac{n+1}{2}, -\frac{n}{2}; -n-1, 1-a-n; \frac{4a}{b}\right) - 1 \\ &\qquad = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{(1-b-n)_k \left(-\frac{n+1}{2}\right)_k \left(-\frac{n}{2}\right)_k}{(-n-1)_k (1-a-n)_k k!} \left(\frac{4a}{b}\right)^k \end{aligned}$$

where

$$\left(-\frac{n}{2}\right)_k = 0, \quad k > \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor, \quad n = 2,4,6,\ldots$$

and

$$\left(-\frac{n+1}{2}\right)_k = 0, \quad k > \left\lfloor \frac{n+1}{2} \right\rfloor, \quad n = 1,3,5,\ldots$$

Using the relations

$$(-z)_r = (-1)^r (z-r+1)_r \quad \text{and} \quad (z)_{r+s} = (z)_r (z+r)_s$$

we have

$$(1-a-n)_k = (-1)^k \frac{(a)_n}{(a)_{n-k}}$$

As a result, it follows that

$$\begin{aligned} {}_3F_2\left(1-b-n,-\frac{n+1}{2},-\frac{n}{2};-n-1,1-a-n;\frac{4a}{b}\right) - 1 \\ = \frac{1}{C(a,b;n)} \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^k \binom{n-k+1}{k} C(a,b;n-k) \end{aligned}$$

which can be reformulated as Identity (7). The proof of Identity (7) is complete. □

Proof of Identity (2). Putting $a=\frac{1}{2}$ and $b=2$ in Equation (5) results in

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} 2^{-2k} C_k = {}_3F_2\left(\frac{1}{2},\frac{1-n}{2},-\frac{n}{2};2,\frac{1}{2};1\right) = {}_2F_1\left(\frac{1-n}{2},-\frac{n}{2};2;1\right)$$

Now applying Kummer's transformation equation

$${}_2F_1(\alpha,\beta;1+\alpha-\beta;z) = (1+z)^{-\alpha} {}_2F_1\left(\frac{\alpha}{2},\frac{\alpha+1}{2};1+\alpha-\beta;\frac{4z}{(z+1)^2}\right)$$

to $\alpha=-n$, $\beta=-n-1$, and $z=1$ leads to

$${}_2F_1\left(\frac{1-n}{2},-\frac{n}{2};2;1\right) = 2^{-n} {}_2F_1(-1-n,-n;2;1) = 2^{-n} C_{n+1}$$

The proof of Identity (2) is complete. □

Proof of Identity (4). Putting $a=\frac{1}{2}$ and $b=2$ in Equation (7) gives

$$C_n\left[1 - {}_3F_2\left(-1-n,-\frac{n+1}{2},-\frac{n}{2};-n-1,\frac{1}{2}-n;1\right)\right] = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^{k-1} \binom{n-k+1}{k} C_{n-k}$$

that is,

$${}_3F_2\left(-1-n,-\frac{n+1}{2},-\frac{n}{2};-n-1,\frac{1}{2}-n;1\right) = {}_2F_1\left(-\frac{n+1}{2},-\frac{n}{2};\frac{1}{2}-n;1\right)$$

Applying the summation equation

$${}_2F_1(\ell,h;c;1) = \frac{\Gamma(c)\Gamma(c-\ell-h)}{\Gamma(c-\ell)\Gamma(c-h)}, \quad \Re(c-\ell-h) > 0$$

to $c=\frac{1}{2}-n$, $\ell=-\frac{n+1}{2}$, and $h=-\frac{n}{2}$ yields

$${}_2F_1\left(-\frac{n+1}{2},-\frac{n}{2};\frac{1}{2}-n;1\right) = \frac{\Gamma(\frac{1}{2}-n)}{\Gamma(1-\frac{n}{2})\Gamma(\frac{1-n}{2})}$$

Further employing the duplication equation

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\, 2^{1-2z}\Gamma(2z)$$

at $z = \frac{1}{2} - n$ gives us

$$ {}_2F_1\left(-\frac{n+1}{2}, -\frac{n}{2}; \frac{1}{2} - n; 1\right) = \frac{\Gamma\left(\frac{1}{2} - n\right)}{2^n \sqrt{\pi}\,\Gamma(1-n)} = 0, \quad n \in \mathbb{N} $$

where $\frac{1}{\Gamma(m)}$ has zeros at $m = 0, -1, -2, \dots$. Identity (4) is thus proved. $\square$

Remark 1. From Equations (3) and (6), we can conclude

$$ {}_4F_3\left(1, \frac{1}{2}, -m, m-n; 2, \frac{1-n}{2}, -\frac{n}{2}; 1\right) = \frac{n+1}{n+1-m} $$

and

$$ {}_3F_2\left(-\frac{1}{2}, -m-1, m-n-1; -1-\frac{n}{2}, -\frac{n+1}{2}; 1\right) = \frac{n-2m}{n+m} $$

for $n \ge 2m$ and $n \in \mathbb{N}$.

Remark 2. Please note, we recommend a newly-published paper [35] which is closely related to the Catalan numbers C_n.

Remark 3. This paper is a slightly revised version of the preprint [36] and has been reviewed by the survey article [37].

3. Conclusions

Three new identities for the Catalan-Qi numbers are discovered and alternative proofs of two identities for the Catalan numbers are provided. The three identities for the Catalan-Qi numbers generalize three identities for the Catalan numbers.

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