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On Diff(M)-Pseudo-Differential Operators and the Geometry of Non Linear Grassmannians

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Abstract: We consider two principal bundles of embeddings with total space $Emb(M, N)$, with structure groups $Diff(M)$ and $Diff_+(M)$, where $Diff_+(M)$ is the groups of orientation preserving diffeomorphisms. The aim of this paper is to describe the structure group of the tangent bundle of the two base manifolds:

$$B(M, N) = Emb(M, N)/Diff(M) \text{ and } B_+(M, N) = Emb(M, N)/Diff_+(M)$$

from the various properties described, an adequate group seems to be a group of Fourier integral operators, which is carefully studied. It is the main goal of this paper to analyze this group, which is a central extension of a group of diffeomorphisms by a group of pseudo-differential operators which is slightly different from the one developped in the mathematical litterature e.g. by H. Omori and by T. Ratiu. We show that these groups are regular, and develop the necessary properties for applications to the geometry of $B(M, N)$. A case of particular interest is $M = S^1$, where connected components of $B_+(S^1, N)$ are deeply linked with homotopy classes of oriented knots. In this example, the structure group of the tangent space $TB_+(S^1, N)$ is a subgroup of some group GL_{res}, following the classical notations of (Pressley, A., 1988). These constructions suggest some approaches in the spirit of one of our previous works on Chern-Weil theory that could lead to knot invariants through a theory of Chern-Weil forms.

Keywords: Fourier-Integral operators; non-liner Grassmannian; Chern-Weil forms; infinite dimensional frame bundle

1. Introduction

Given M and N two Riemannian manifolds without boundary, with M compact, the space of smooth embeddings $Emb(M, N)$ is currently known as a principal bundle with structure group $Diff(M)$, where $Diff(M)$ naturally acts by composition of maps. The base:

$$B(M, N) = Emb(M, N)/Diff(M)$$

is known as a Fréchet manifold, and there exists some local trivializations of this bundle. We focus here on the base manifold, which seems to carry a richer structure than $Emb(M, N)$ itself.

This paper gives the detailed description of the structure group of the tangent bundle of connected components of $TB(M, N)$. This structure group can be slightly different when changing of connected component of $B(M, N)$. It is viewed as an extension of the group of automorphisms $Aut(E)$ of a vector bundle E by some group of pseudo-differential operators. We show that this group is a regular Lie group (in the sense that it carries an exponential map), and that it is also a group of

Fourier integral operators, which explains the notations FIO_{Diff} and FCl_{Diff} ("Cl" for "classical"). All these groups are constructed along a short exact sequence of the type:

$$0 \to PDO \to FIO \to Diff \to 0$$

where PDO is a group of pseudo-differential operators, FIO is a group of Fourier integral operators, and $Diff$ is a group of diffeomorphisms; this sequence plays a central role in the proofs. The theorems described are general enough to be applied to many groups of diffeomorphisms: volume preserving diffeomorphisms, symplectic diffeomorphisms, hamiltonian diffeomorphisms, and to groups of pseudo-differential operators: classical or non-classical, bounded or unbounded, compact and so on, but we concentrate our efforts on $Diff(M)$ and $Diff_+(M)$, the group of orientation preserving diffeomorphisms. The constructions are made for operators acting on smooth sections of trivial or non trivial bundles. For a non trivial bundle E, the group of automorphisms of the bundle plays a central role in the description, because easy arguments suggest that there is no adequate embedding of the group of diffeomorphisms of the base manifold into the group of automorphisms of the bundle. Specializing to $M = S^1$, given a (real) vector bundle E over S^1, the groups $FIO_{Diff}(S^1, E)$ and in particular $FCl^{0,*}_{Diff_+}(S^1, E)$ is of particular interest, where $FCl^{0,*}_{Diff_+}(S^1, E)$ is defined through the short exact sequence:

$$0 \to Cl^{0,*}(S^1, E) \to FCl^{0,*}_{Diff_+}(S^1, E) \to Diff_+(S^1) \to 0$$

where $Cl^{0,*}(S^1, E)$ is the group of bounded classical pseudo-differential operators and $Diff_+(S^1)$ is the group of orientation-preserving diffeomorphisms. We have to notice that the necessary Fourier analysis on these operators naturally takes place in the complexification $E_{\mathbb{C}}$ of the vector bundle E, and that $E_{\mathbb{C}}$ as a complex vector bundle is trivial, but the real vector bundle E can be non trivial. Given any Riemannian connection on the bundle E, if ϵ is the sign of this connection (and this is a bounded pseudo-differential operators acting on smooth sections of E), it appears that $[FCl^{0,*}_{Diff_+}(S^1, E), \epsilon]$ is a set of smoothing operators. Thus, it is a subgroup of the group

$$Gl_{res} = \left\{ u \in Gl(L^2(S^1, E)) | [\epsilon, u] \text{ is Hilbert-Schmidt } \right\}$$

Even if the inclusion is not a bounded inclusion, this result extends the results given in [1] on the group: $Diff_+(S^1)$ (which inclusion map into Gl_{res} is not bounded too) and in [2] for the group $Cl^{0,*}(S^1, E)$. We get a non-trivial cocycle on the Lie algebra of $FCl^{0,*}_{Diff_+}(S^1, E)$ by the Schwinger cocycle, extending results obtained in [2,3] for a trivial complex bundle.

Coming back to $Emb(M, N)$, one could suggest that $Aut(E)$ is sufficient as a structure group, but we refer the reader to earlier works such as [4–6] to see how pseudo-differential operators can arise from Levi-Civita connections of Sobolev metrics when the adequate structure group for the L^2 metric is a group of multiplication operators. Moreover, especially for $M = S^1$, taking the quotient:

$$B_+(S^1, N) = Emb(S^1, N)/Diff_+(S^1)$$

we show that there is a sign operator $\epsilon(D)$, which is a pseudo-differential operator of order 0, and coming intrinsically from the geometry of $Emb(S^1, N)$, such that the recognized structure group of $TB_+(S^1, N)$ is $FCl^{0,*}_{Diff_+}(S^1, E) \subset Gl_{res}$. We finish with the starting point of this work, which was a suggestion of Claude Roger, saying that any well-defined Chern-Weil form of a connected component of $TB(S^1, N)$ can be understood as an invariant of a knot, whose homotopy class is exactly a connected component of $B(S^1, N)$. If one considers oriented knots, we get connected components of $B_+(S^1, N)$. The work begun has not been completely successful yet, but it is a pleasure to suggest some Chern-Weil froms that may lead to knot invariants, extending the approach of [6].

2. Preliminaries on Algebras and Groups of Operators

Now we fix M the source manifold, which is assumed to be Riemannian, compact, connected and without boundary, and the target manifold which is only assumed Riemannian. We note by $Vect(M)$ the space of vector fields on TM. Recall that the Lie algebra of the group of diffeomorphisms is $Vect(M)$, which is a Lie-subalgebra of the (Lie-)algebra of differential operators, which is itself a subalgebra of the algebra of classical pseudo-differential operators.

2.1. Differential and Pseudodifferential Operators on a Manifold M

Definition 1. Let $DO(M)$ be the graded algebra of operators, acting on $C^\infty(M, \mathbb{R})$, generated by:

- the multiplication operators: for $f \in C^\infty(M, \mathbb{R})$, we define the multiplication operator:

$$M_f : g \in C^\infty(M, \mathbb{R}) \mapsto f.g \text{ (by pointwise multiplication)}$$

- the vector fields on M: for a vector field $X \in Vect(M)$, we define the differentiation operator:

$$D_X : g \in C^\infty(M, \mathbb{R}) \mapsto D_X g \text{ (by differentiation, pointwise)}$$

Multiplication operators are operators of order 0, vector fields are operators of order 1. For $k \geq 0$, we note by $DO^k(M)$ the differential operators of order $\leq k$.

Differential operators are **local**, which means that:

$$\forall A \in DO(M), \forall f \in C_c^\infty(M, \mathbb{R}), supp(A(f)) \subset supp(f)$$

The inclusion $Vect(M) \subset DO(M)$ is an inclusion of Lie algebras. The algebra $DO(M)$, graded by the order, is a subalgebra of the algebra of classical pseudo-differential operators $Cl(M)$, which is an algebra that contains the square root of the Laplacian, and its inverse. This algebra contains trace-class operators on $L^2(M, \mathbb{R})$. Basic facts on pseudo-differential operators defined on a vector bundle $E \to M$ can be found in [7]. We assume known the definition of the algebra of pseudo-differential operators $PDO(M, E)$, classical pseudo-differential operators $Cl(M, E)$. When the vector bundle E is assumed trivial, *i.e.*, $E = M \times V$ or $E = M \times \mathbb{K}^p$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we use the notation $Cl(M, V)$ or $Cl(M, \mathbb{K}^p)$ instead of $Cl(M, E)$. These operators are **pseudolocal**, which means that:

$$\forall A \in PDO(M, E), \forall f \in L^2(M, E), \text{ if } f \text{ is smooth on } K, \text{ then } A(f) \text{ is smooth on } K$$

Definition 2. A pseudo-differential operator A is **log-polyhomogeneous** if and only if its formal symbol reads (locally) as:

$$\sigma(A)(x, \xi) \sim_{|\xi| \to +\infty} \sum_{j=0}^{o} \sum_{k=-\infty}^{o'} \sigma_{j,k}(x, \xi)(log(|\xi|))^j$$

where $\sigma_{j,k}$ is a positively $k-$homogeneous symbol.

The set of log-polyhomogenous pseudo-differential operators is an algebra.

A global symbolic calculus has been defined independently by two authors in [8,9], where we can see how the geometry of the base manifold M furnishes an obstruction to generalize local formulas of composition and inversion of symbols. We do not recall these formulas here because they are not involved in our computations. More interesting for this article is to precise when the local formulas of composition of formal symbols extend globally on the base manifold.

We assume that M is equipped with charts such that the changes of coordinates are translations and that the vector bundle $E \to M$ is trivial. This is in particular true when $M = S^1 = \frac{\mathbb{R}}{2\pi\mathbb{Z}}$, or when $M = T^n = \prod_{i=1}^n S^1$. In the case of S^1, we use the smooth atlas $\mathcal{ATL}$ of S^1 defined as follows:

$$\begin{array}{rcl} \mathcal{ATL} & = & \{\varphi_0, \varphi_1\} \\ \varphi_n & : & x \in]0; 2\pi[\mapsto e^{i(x+n\pi)} \subset S^1 \text{ for } n \in \{0; 1\} \end{array}$$

Associated to this atlas, we fix a smooth partition of the unit $\{s_0; s_1\}$. An operator $A : C^\infty(S^1, \mathbb{C}) \to C^\infty(S^1, \mathbb{C})$ can be described in terms of 4 operators:

$$A_{m,n} : f \mapsto s_m \circ A \circ s_n \text{ for } (m,n) \in \{0,1\}$$

Such a formula is a straightforward application of a localization formula in the case of an atlas $\{\varphi_i\}_{i\in I}$ of a manifold M with associated family of partitions of the unit $\{s_i\}_{i\in I}$, see e.g., [7] for details.

Notations. We note by $PDO(M,E)$ (resp. $PDO^o(M,E)$, resp. $Cl(M,E)$) the space of pseudo-differential operators (resp. pseudo-differential operators of order o, resp. classical pseudo-differential operators) acting on smooth sections of E, and by $Cl^o(M,E) = PDO^o(S^1,E) \cap Cl(S^1,E)$ the space of classical pseudo-differential operators of order o.

If we set:

$$PDO^{-\infty}(M,E) = \bigcap_{o\in\mathbb{Z}} PDO^o(M,E)$$

we notice that it is a two-sided ideal of $PDO(M,E)$, and we define the quotient algebra:

$$\mathcal{F}PDO(M,E) = PDO(M,E)/PDO^{-\infty}(M,E)$$

$$\mathcal{F}Cl(M,E) = Cl(M,E)/PDO^{-\infty}(M,E)$$

$$\mathcal{F}Cl^o(M,E) = Cl^o(M,E)/PDO^{-\infty}(M,E)$$

called the algebras of formal pseudo-differential operators. $\mathcal{F}PDO(M,E)$ is isomorphic to the set of formal symbols [8], and the identification is a morphism of $\mathbb{C}$-algebras, for the multiplication on formal symbols defined before (see e.g., [7]). At the level of kernels of operators, a smoothing operator has a kernel $K_\infty \in C^\infty(M \times M, \mathbb{C})$, where as the kernel of a pseudo-differential operator is in general smooth only on the off-diagonal region $(M \times M) - \Delta(M)$, where $\Delta(M)$ denotes here, very exceptionnally in this paper, the diagonal set (and not a Laplacian operator). We finish by mentioning that the last property is equivalent to pseudo-locality.

2.2. Fourier Integral Operators

With the notations that we have set before, a scalar Fourier-integral operator of order o is an operator:

$$A : C^\infty(M,\mathbb{C}) \to C^\infty(M,\mathbb{C})$$

such that, $\forall (i,j) \in I^2$,

$$A_{k,j}(f) = \int_{supp(s_j)} e^{i\phi(x,\xi)} \sigma_{k,j}(x,\xi) (s_j.\hat{f})(\xi) d\xi \quad (1)$$

where $\sigma_{k,j} \in C^\infty(supp(s_j) \times \mathbb{R}, \mathbb{C})$ satisfies:

$$\forall (\alpha,\beta) \in \mathbb{N}^2, \quad |D_x^\alpha D_\xi^\beta \sigma_{k,j}(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{o-\beta}$$

and where, on any domain U of a chart on M,

$$\phi(x,\xi) : T^*U - U \approx U \times \mathbb{R}^{dimM} - \{0\} \to \mathbb{R}$$

is a smooth map, positively homogeneous of degree 1 fiberwise and such that:

$$\det\left(\frac{\partial^2\phi}{\partial x \partial_\xi}\right) \neq 0$$

Such a map is called **phase** function. (In these formulas, the maps are read on local charts but we preferred to only mention this aspect and not to give heavier formulas and notations.) An operator A is pseudo-differential operator if the operators $A_{k,l}$ in Formula (1) can be written as Fourier integral operators with $\phi(x,\xi) = x.\xi$. Notice that, in order to define an operator A, the choice of ϕ and $\sigma_{k,l}$ is not a priori unique for general Fourier integral operators. Let $E = S^1 \times \mathbb{C}^k$ be a trivial smooth vector bundle over S^1. An operator acting on $C^\infty(M,\mathbb{C}^n)$ is Fourier integral operator (resp. a pseudo-differential operator) if it can be viewed as a $(n \times n)$-matrix of Fourier integral operators with same phase function (resp. scalar pseudo-differential operators).

We define also the algebra of formal operators, which is the quotient space:

$$\mathcal{F}FIO = FIO/PDO^{-\infty}$$

which is possible because $PDO^{-\infty}$ is a closed two-sided ideal. When we consider classical Fourier integral operators, noted FCl, that is operators with classical symbols, we add to this topology the topology on formal symbols [10,11] which is an ILH topology (see e.g., [12] for state of the art). We want to quote that if the symbols $\sigma_{m,n}$ are symbols of order 0, then we get Fourier integral operators that are L^2-bounded. We note this set FIO^0. This set is a subset of FIO, and we have:

$$Cl^0 \subset PDO^0 \subset FIO^0 \subset FIO$$

The techniques used for pseudo-differential operators are also used on Fourier integral operators, especially Kernel analysis. Let us consider a local coordinate operator $A_{m,n}$ then, using the notation of of the Formula (1), the operator $A_{m,n}$ is described by a kernel:

$$K_{m,n}(x,y) = \int_\xi e^{i(\phi(x,\xi))-y.\xi)}\sigma_{m,n}(x,\xi)d\xi$$

From this approach one derives the composition and inversion formulas that will not be used in this paper, see e.g., [13], but in the sequel we shall use the slightly restricted class of operators studied in [14–21] and also in [10,11] for formal operators.

2.3. Topological Structures and Regular Lie Groups of Operators

The topological structures can be derived both from symbols and from kernels, as we have quoted before, but principally because there is the exact sequence described below with slice. At the level of units of these sets, *i.e.*, of groups of invertible operators, the existence of the slice is also crucial. In the papers [10,11,14–22], the group of invertible Fourier integral operators receives first a structure of topological group, with in addition a differentiable structure, e.g., a Frölicher structure, which recognized as a structure of generalized Lie group, see e.g., [12].

We have to say that, with the actual state of knowledge, using [23], we can give a manifold structure (in the convenient setting described by Kriegl and Michor or in the category of Frölicher spaces following [24]) to the corresponding Lie groups. Let us recall the statement.

Theorem 3. *[23] Let G, H, K be convenient Lie groups or Frölicher Lie groups such that there is a short exact sequence of Lie groups:*

$$0 \to H \to G \to K \to 0$$

such that there is a local slice $K \to G$. Then:

$$G \text{ regular } \Leftrightarrow H \text{ and } K \text{ regular}$$

Remark 1. In [10,11,14–22], the group K considered is the group of 1-positively homogeneous symplectomorphisms $Diff_\omega(T^*M - M)$ where ω is the canonical symplectic form on the cotangent bundle. The local section considered enables to build up the phase function of a Fourier integral operator from such a symplectic diffeomorphism inside a neighborhood of Id_M. There is *a priori* no reason to restrict the constructions to classical pseudo-differential operators of order 0, and have groups of Fourier integral operators with symbols in wider classes. This remark appears important to us because the authors cited before restricted themselves to classical symbols.

2.4. $PDO(M, E)$, $Aut(E)$ and $Diff(M)$

We now get another group.

Theorem 4. *Let H be a regular Lie group of pseudo-differential operators acting on smooth sections of a trivial bundle $E \sim V \times M \to M$. The group $Diff(M)$ acts smoothly on $C^\infty(M, V)$, and is assumed to act smoothly on H by adjoint action. If H is stable under the $Diff(M)-$adjoint action, then there exists a corresponding regular Lie group G of Fourier integral operators through the exact sequence:*

$$0 \to H \to G \to Diff(M) \to 0$$

If H is a Frölicher Lie group, then G is a Frölicher Lie group. If H is a Fréchet Lie group, then G is a Fréchet Lie group.

Remark 2. The pseudo-differential operators can be classical, log-polyhomogeneous, or anything else. Applying the formulas of "changes of coordinates" (which can be understood as adjoint actions of diffeomorphisms) of e.g., [7], one easily gets the result.

Proof of Theorem 4. Let us first notice that the action:

$$(f, g) \in C^\infty(M, V) \times Diff(M) \mapsto f \circ g \in C^\infty(M, V)$$

can be read as, first a linear operator T_g with kernel:

$$K(x, y) = \delta(g(x), y) \quad \text{(Dirac } \delta - \text{function)}$$

or equivalently, on an adequate system of trivializations [7],

$$T_g(f)(x) = \int e^{ig(x).\xi} \hat{f}(\xi) d\xi$$

This operator is not a pseudo-differential operator because it is not pseudolocal (unless $g = Id_M$), but since:

$$det\left(\partial_x \partial_\xi (g(x).\xi)\right) = det(D_x g)$$

we get that T_g is a Fourier-integral operator. Notice that another way to see it is the expression of its kernel.

Now, given $(A, g) \in H \times Diff(M)$, we define:

$$A_g = T_g \circ A$$

We get here a set G of operators which is set-theorically isomorphic to $H \times Diff(M)$. Since H is invariant under the adjoint action of the group $Diff(M)$, G is a group, and from the beginning of this proof, we get that G is a group, and that there is the short exact sequence announced:

$$0 \to H \to G \to Diff(M) \to 0$$

with a global slice:

$$g \in Diff(M) \mapsto T_g \in G$$

Since the adjoint action of Diff(M) is assumed smooth on H, we can endow G with the product Frölicher structure to get a regular Frölicher Lie group. Since $Diff(M)$ is a Fréchet Lie group, if H is a Fréchet Lie group, then G is a Fréchet Lie group. □

Remark 3. Some restricted classed of such operators are already considered in the literature under the name of $G-$pseudo-differential operators, see e.g., [25], but the groups considered are discrete (amenable) groups of diffeomorphisms. This gives a class of FIOs with linear phase, see [13].

Definition 5. Let M be a compact manifold and E be a (finite rank) trivial vector bundle over M. We define:

$$FIO_{Diff}(M, E) = \{A \in FIO(M, E) | \phi_A(x, \xi) = g(x).\xi; g \in Diff(M)\}$$

The set of invertible operators $FIO^*_{Diff}(M, E)$ is obviously a group, that decomposes as:

$$0 \to PDO^*(M, E) \to FIO^*_{Diff}(M, E) \to Diff(M) \to 0$$

with global smooth section:

$$g \in Diff(M) \mapsto (f \in C^\infty(S^1, E) \mapsto f \circ g)$$

Hence, Theorem 4 applies trivially to the following context:

Proposition 6. *Let $FCl^{0,*}_{Diff}(M, E)$ be the set of operators $A \in FIO^*_{Diff}(M, E)$ such that A has a 0-order classical symbol. Then we get the exact sequence:*

$$0 \to Cl^{0,*}(M, E) \to FCl^{0;*}_{Diff}(M, E) \to Diff(M) \to 0$$

and $FCl^{0,*}_{Diff}(M, E)$ is a regular Frölicher Lie group, with Lie algebra isomorphic, as a vector space, to $Cl^0(M, E) \oplus Vect(M)$.

Notice that the triviality of the vector bundle E is here essential to make a $Diff(M)-$action on smooth section of $C^\infty(M, E)$. Let us assume now that E is not trivial. At the infinitesimal level, trying to extend straightway, one gets a first condition for the extension.

Lemma 7. *(see e.g., [26]) Let us fix a $0-$curvature connection ∇ on M. Then $X \in Vect(M) \mapsto \nabla_X \in DO^1(M, E)$ is a one-to-one Lie algebra morphism.*

We remark that the analogy with the setting of trivial bundles E stops here since the group $Diff(M)$ cannot be recovered in this group of operators. For example, when $M = S^1$, if E is non trivial, the (infinitesimally) flat connection ensures that the holonomy group $\mathcal{H}$ is discrete, but it cannot be trivial since the vector bundle E is not. On a non trivial bundle E, let us consider the group of bundle automorphism $Aut(E)$. The gauge group $DO^0(M, E)$ is naturally embedded in $Aut(E)$ and the bundle projection:

$$E \to M$$

induces a group projection

$$\pi : Aut(E) \to Diff(M)$$

Therefore ,we get a short exact sequence:

$$0 \to DO^{0,*}(M, E) \to Aut(E) \to Diff(M) \to 0$$

Following [27] there exists a local slice $U \subset Diff(M) \to Aut(E)$, where U is a C^0-open neighborhood on Id_M, which shows that $Aut(E)$ is a regular Fréchet Lie group. Therefore, the smallest group spanned by $PDO^*(M, E)$ and $Aut(E)$ is such that:

- the projection $E \to M$ induces a map $Aut(E) \to Diff(M)$ with kernel $DO^0(M, E) = Aut(E) \cap PDO(M, E)$
- $Ad_{Aut(E)}(PDO(M, E)) = PDO(M, E)$

therefore we can consider the space of operators on $C^\infty(M, E)$

$$FIO^*_{Diff}(M, E) = Aut(E) \circ PDO^*(M, E)$$

Lemma 8. *The map*

$$(B, A) \in Aut(E) \times PDO^*(M, E) \mapsto \pi(B) \in Diff(M)$$

induces a "phase map"

$$\tilde{\pi} : FIO^*_{Diff}(M, E) \to Diff(M)$$

Proof. Let $((B, A), (B', A')) \in Aut(E) \times PDO^*(M, E)$

$$\begin{aligned} B \circ A = B' \circ A' \quad &\Leftrightarrow \quad Id_E \circ A = B^{-1} \circ B' \circ A' \\ &\Leftrightarrow \quad B^{-1} \circ B' = A \circ A'^{-1} \in PDO^*(M, E) \\ &\Rightarrow \quad B^{-1} \circ B' \in DO^{0,*}(M, E) \\ &\Leftrightarrow \quad \pi(B^{-1} \circ B') = Id_M \\ &\Leftrightarrow \quad \pi(B) = \pi(B') \end{aligned}$$

□

The next lemma is obvious:

Lemma 9. $FIO^*_{Diff}(M, E)$ *is a group*

Lemma 10. $Ker(\tilde{\pi}) = PDO^*(M, E)$

Proof. Let $B \circ A \in FIO^*(M, E)$ such that:

$$\tilde{\pi}(B \circ A) = \pi(B) = Id_M$$

Then $B \in DO^{0,*}(M, E)$ and $B \circ A \in PDO^*(M, E)$ □

These results show the following theorem:

Theorem 11. *There is a short exact sequence of groups :*

$$0 \to PDO^*(M,E) \to FIO^*_{Diff}(M,E) \to Diff(M) \to 0$$

and, if $H \subset PDO^*(M,E)$ is a regular Fréchet or Frölicher Lie group of operators that contains the gauge group of E, if K is a regular Fréchet or Frölicher Lie subgroup of $Diff(M)$ such that there exists a local section $K \to Aut(E)$, the subgroup $G = K \circ H$ of $FIO^*_{Diff}(M,E)$ is a regular Fréchet Lie group from the short exact sequence:

$$0 \to H \to G \to K \to 0$$

2.5. Diffeomorphisms and kernel operators

Let $g \in Diff(M)$. Then a straightforward computation on local coordinates shows that the kernel of T_g is:

$$K_g = \delta(g(x), y)$$

where δ is the Dirac $\delta-$function. These operators also read locally as:

$$T_g(f) = \int_M e^{ig(x).\xi} \hat{f}(\xi) d\xi$$

on the same system of local trivializations used in [7], p.30-40.

2.6. Renormalized Traces on Diff(M)-Pseudodifferential Operators

Basics on renormalized traces are given in the appendix. Let us now investigate their extensions to the class of FIOs considered. Let us first explore the action of $Diff(M)$ and of $Aut(E)$ on $tr^Q(A)$.For this, we get:

Lemma 12. *Let $a \in \mathbb{Z}$. Let $A \in Cl^a(M,E)$ and let Q be a weight on E. Let B be an operator on $C^\infty(M,E)$ such that*

1. $Ad_B(Cl^a(M,E)) \subset Cl^a(M,E)$
2. *$Ad_B Q$ is a weight of the same order as Q*

Then

- $res(Ad_B A) = res(A)$
- $tr^{Ad_B Q}(Ad_B A) = tr^Q(A)$

The properties 1,2 are true in particular for operators $B \in Aut(E)$.

Proof.
Let Q be a weight on $C^\infty(M,E)$ and let $A \in Cl(M,E)$. Let $B \in Aut(E)$. Let $s \in \mathbb{R}^*_+$ then Ae^{-sQ} is trace class. By [7], we know that $Ad_B A$ (resp. $Ad_B Q$) is a classical pseudo-differential operator of the same order (resp. a weight of the same order). Then, since $e^{-\frac{s}{2}Q}$ is smoothing, $Ad_B(Ae^{-sQ})$, $BAe^{-\frac{s}{2}Q}$ and $e^{-\frac{s}{2}Q}B^{-1}$ are smoothing, and the following computations are fully justified:

$$\begin{aligned}
tr\left(Ad_B(Ae^{-sQ})\right) &= tr\left(\left(BAe^{-\frac{s}{2}Q}\right)\left(e^{-\frac{s}{2}Q}B^{-1}\right)\right) \\
&= tr\left(\left(e^{-\frac{s}{2}Q}B^{-1}\right)\left(BAe^{-\frac{s}{2}Q}\right)\right) \\
&= tr\left(e^{-\frac{s}{2}Q}Ae^{-\frac{s}{2}Q}\right) \\
&= tr\left(Ae^{-sQ}\right)
\end{aligned}$$

So that, we get the announced property. □

3. Splittings on the Set of S^1-Fourier Integral Operators

3.1. The Group $O(2)$ and the Diffeomorphism Group $Diff(S^1)$

Let us consider the $SO(2) = U(1)$-action on $S^1 = \mathbb{R}/\mathbb{Z}$ given by $(e^{2i\pi\theta}, x) \mapsto x + \theta$. This group acts on C^∞ by $(e^{2i\pi\theta}, f) \mapsto f(x + \theta)$ and we have:

$$\begin{aligned} f(x+\theta) &= \int e^{-i(x+\theta).\xi} \hat{f}(\xi) d\xi \\ &= \int e^{-i(x.\xi+\theta.\xi} \hat{f}(\xi) d\xi \end{aligned}$$

The term $e^{-i\theta.\xi}$ is oscillating in ξ and does not satisfies the estimates on the derivatives of symbols. So that, this operator is not a pseudo-differential operator but has obviously the form of a Fourier integral operator. The same is for the reflection $x \mapsto 1 - x$ which corresponds to the conjugate transformation $z \mapsto \bar{z}$ when representing S^1 as the set of complex numbers z such that $|z| = 1$. This is a spacial case of the properties already stated for a general manifold M given $g \in Diff(S^1)$, g acts on C^∞ by right composition of the inverse, namely, for $f \in C^\infty$,

$$\begin{aligned} g.f(x) &= f \circ g(x) \\ &= \int e^{-ig(x).\xi} \hat{f}(\xi) d\xi \end{aligned}$$

which is also obviously a Fourier-integral operator, and the kernel of this operator is:

$$K_g(x,y) = \delta\left(y, g(x)\right)$$

where δ is the Dirac δ-function. This is the construction already used in the proof of Theorem 4.

3.2. $\epsilon(D)$, Its Formal Symbol and the Splitting of $\mathcal{F}PDO$

The operator $D = -iD_x$ splits $C^\infty(S^1, \mathbb{C}^k)$ into three spaces:

- its kernel E_0, made of constant maps
- the vector space spanned by eigenvectors related to positive eigenvalues
- the vector space spanned by eigenvectors related to negative eigenvalues.

The following elementary result will be useful for the sequel, see [28] for the proof, and e.g., [3,6]:

Lemma 13.

(i) $\sigma(D) = \xi$

(ii) $\sigma(|D|) = |\xi|$ *where* $|D| = \left(\int_\Gamma \lambda^{1/2}(\Delta - \lambda Id)^{-1} d\lambda\right)$, *with* $\Delta = -D_x^2$.

(iii) $\sigma(D|D|^{-1}) = \frac{\xi}{|\xi|}$, *where* $D|D|^{-1} = |D|^{-1}D$ *is the sign of* D, *since* $|D|_{|E_0} = Id_{E_0}$

(iv) Let p_{E_+} *(resp.* p_{E_-}*) be the projection on* E_+ *(resp.* E_-*), then* $\sigma(p_{E_+}) = \frac{1}{2}(Id + \frac{\xi}{|\xi|})$ *and* $\sigma(p_{E_-}) = \frac{1}{2}(Id - \frac{\xi}{|\xi|})$

Let us now define two ideals of the algebra $\mathcal{F}PDO$, that we call $\mathcal{F}PDO_+$ and $\mathcal{F}PDO_-$, such that $\mathcal{F}PDO = \mathcal{F}PDO_+ \oplus \mathcal{F}PDO_-$. This decomposition is implicit in [29], section 4.4., p. 216, for classical pseudo-differential operators and we furnish the explicit description given in [28], extended to the whole algebra of (maybe non formal, non classical) pseudo-differential symbols here.

Definition 14. Let σ be a symbol (maybe non formal). Then, we define, for $\xi \in T^*S^1 - S^1$,

$$\sigma_+(\xi) = \begin{cases} \sigma(\xi) & \text{if } \xi > 0 \\ 0 & \text{if } \xi < 0 \end{cases} \text{ and } \sigma_-(\xi) = \begin{cases} 0 & \text{if } \xi > 0 \\ \sigma(\xi) & \text{if } \xi < 0 \end{cases}$$

At the level of formal symbols, we also define the projections: $p_+(\sigma) = \sigma_+$ and $p_-(\sigma) = \sigma_-$.

The maps $p_+ : \mathcal{F}PDO(S^1, \mathbb{C}^k) \to \mathcal{F}PDO(S^1, \mathbb{C}^k)$ and $p_- : \mathcal{F}PDO(S^1, \mathbb{C}^k) \to \mathcal{F}PDO(S^1, \mathbb{C}^k)$ are clearly algebra morphisms that leave the order invariant and are also projections (since multiplication on formal symbols is expressed in terms of pointwise multiplication of tensors).

Definition 15. We define $\mathcal{F}PDO_+(S^1, \mathbb{C}^k) = Im(p_+) = Ker(p_-)$ and $\mathcal{F}PDO_-(S^1, \mathbb{C}^k) = Im(p_-) = Ker(p_+)$.

Since p_+ is a projection, we have the splitting:

$$\mathcal{F}PDO(S^1, \mathbb{C}^k) = \mathcal{F}PDO_+(S^1, \mathbb{C}^k) \oplus \mathcal{F}PDO_-(S^1, \mathbb{C}^k)$$

Let us give another characterization of p_+ and p_-. Looking more precisely at the formal symbols of p_{E_+} and p_{E_-} computed in Lemma 13, we observe that:

$$\sigma(p_{E_+}) = \begin{cases} 1 & \text{if } \xi > 0 \\ 0 & \text{if } \xi < 0 \end{cases} \text{ and } \sigma(p_{E_-}) = \begin{cases} 0 & \text{if } \xi > 0 \\ 1 & \text{if } \xi < 0 \end{cases}$$

In particular, we have that $D_x^\alpha \sigma(p_{E_+})$, $D_\xi^\alpha \sigma(p_{E_+})$, $D_x^\alpha \sigma(p_{E_-})$, $D_\xi^\alpha \sigma(p_{E_-})$ vanish for $\alpha > 0$. From this, we have the following result:

Proposition 16. *Let $a \in \mathcal{F}PDO(S^1, \mathbb{C}^k)$. $p_+(a) = \sigma(p_{E_+}) \circ a = a \circ \sigma(p_{E_+})$ and $p_-(a) = \sigma(p_{E_-}) \circ a = a \circ \sigma(p_{E_-})$[28].*

3.3. The Case of Non Trivial (Real) Vector Bundle Over S^1

Let $\pi : E \to S^1$ be a non trivial real vector bundle over S^1 of rank k. Its bundle of frames is a $Gl(\mathbb{R}^k)-$ principal bundle, which means the following (see e.g., [30]):

Lemma 17. *Let $\varphi_1 :]a; b[\times \mathbb{R}^k \to E$ and $\varphi_2 :]a'; b'[\times \mathbb{R}^k \to E$ be two local trivializations of E. Let $\mathcal{D} = \pi(\varphi_1(]a; b[\times \mathbb{R}^k) \cap \varphi_2(]a'; b'[\times \mathbb{R}^k))$, let $\mathcal{D}_1 = \varphi_1^{-1}(\mathcal{D})$, and let $\mathcal{D}_2 = \varphi_2^{-1}(\mathcal{D})$. Then,*

$$\varphi_2^{-1} \circ \varphi_1 : \mathcal{D}_1 \times \mathbb{R}^k \to \mathcal{D}_2 \times \mathbb{R}^k$$

reads as:

$$\varphi_2^{-1} \circ \varphi_1 = \gamma \times M$$

where γ is a smooth diffeomorphism from $\mathcal{D}_1$ to $\mathcal{D}_2$, and where $M \in C^\infty(\mathcal{D}_1, Gl(\mathbb{R}^k))$.

Let us now turn to symbols of pseudo-differential operators acting on smooth sections of E. We first assume that we work with a system of local trivializations such that the diffeomorphisms γ are translations, and let us now look at the transformations of the symbols read on local trivializations. Under these assumptions, and with the notations of the previous lemma, a formal symbol σ_1 read on D_1 reads on D_2 as:

$$\sigma_2(\gamma(x), \xi) = M(x)\sigma_1(x, \xi)M(x)^{-1}$$

Proposition 18. *Let ∇ be a Riemannian covariant derivative on the bundle $E \to S^1$ and let $\frac{\nabla}{dt}$ be the associated first order differential operator, given by the covariant derivative evaluated at the unit vector field over S^1. We modify the operator $\frac{\nabla}{dt}$ into an injective operator $D = \frac{\nabla}{dt} + p_{ker\frac{\nabla}{dt}}$, where $p_{ker\frac{\nabla}{dt}}$ is the L^2 orthogonal projection on $ker\frac{\nabla}{dt} \subset C^\infty(S^1, E) \subset L^2(S^1, E)$, and we set:*

$$\epsilon(\nabla) = D \circ |D|^{-1}.$$

Then the formal symbol of $\epsilon(\nabla)$ is $\frac{i\xi}{|\xi|}$

Proof. Let us use the holonomy trivialization over an interval I. In this trivialization,

$$\frac{\nabla}{dt} = \frac{d}{dt}$$

and hence the formal symbol of $\frac{\nabla}{dt}$ reads as $i\xi$. Calculating exclusively on the algebra of formal operators on which composition and inversion governed by local formulas, we get $\sigma(|D|) = |\xi|$ and, by the same arguments as those of [28], we get the result. □

Proposition 19. *For each $A \in PDO(S^1, E)$, $[A, \epsilon(\nabla)] \in PDO^{-\infty}(S^1; E)$.*

Proof. We remark that, for any multiindex α such that $|\alpha| > 0$, $D_x^\alpha \sigma(\epsilon(\nabla)) = 0$ and $D_\xi^\alpha \sigma(\epsilon(\nabla)) = 0$. Hence, in $\mathcal{F}PDO(S^1, E)$,

$$\sigma([A, \epsilon(\nabla)]) = [\sigma(A), \sigma(\epsilon(\nabla))] = 0$$

so that $[A, \epsilon(\nabla)] \in PDO^{-\infty}(S^1, E)$. □

3.4. The Splitting Read on the Phase Function

The fiber bundle $T^*S^1 - S^1$ has two connected components and the phase function is positively homogeneous, so that we can make the same procedure as in the case of the symbols. However, we remark that we can split:

$$\phi = \phi_+ + \phi_-$$

where $\phi_+ = 0$ if $\xi < 0$ and $\phi_- = 0$ if $\xi > 0$. Unfortunately, ϕ_+ and ϕ_- are **not** phase functions of Fourier integral operators because there are some points where $\frac{\partial^2 \phi_+}{\partial_x \partial_\xi} = 0$ or $\frac{\partial^2 \phi_-}{\partial_x \partial_\xi} = 0$. However, we can have the following identities:

$$\begin{aligned}
\int_{\mathbb{R}} e^{i\phi(x,\xi)} \sigma(x,\xi) \hat{f}(\xi) d\xi &= \int_{\xi>0} e^{i\phi(x,\xi)} \sigma(x,\xi) \hat{f}(\xi) d\xi + \int_{\xi<0} e^{i\phi(x,\xi)} \sigma(x,\xi) \hat{f}(\xi) d\xi \\
&= \int_{\xi>0} e^{i\phi_+(x,\xi)} \sigma(x,\xi) \hat{f}(\xi) d\xi + \int_{\xi<0} e^{i\phi_-(x,\xi)} \sigma(x,\xi) \hat{f}(\xi) d\xi \\
&= \int_{\mathbb{R}} e^{i\phi_+(x,\xi)} \sigma_+(x,\xi) \hat{f}(\xi) d\xi + \int_{\mathbb{R}} e^{i\phi_-(x,\xi)} \sigma_-(x,\xi) \hat{f}(\xi) d\xi \\
&= \int_{\mathbb{R}} e^{i\phi(x,\xi)} \sigma_+(x,\xi) \hat{f}(\xi) d\xi + \int_{\mathbb{R}} e^{i\phi(x,\xi)} \sigma_-(x,\xi) \hat{f}(\xi) d\xi.
\end{aligned}$$

In the last line, we get the phase of a FIO.

3.5. The Schwinger Cocycle on $PDO(S^1, E)$ When E Is a Real Vector Bundle

The Scjwinger cocycle [31] is well-known in the theory of central extensions of algebras of pseudo-differential operators [3,6,32–34] are now analyzed from the viewpoint of operators acting on smooth sections of real vector bundles. Here, $\epsilon(\nabla)$ is not a sign operator, but an operator such that $\epsilon(\nabla)^2 = -Id$ up to a smoothing operator.

Theorem 20. *For any $A \in PDO(S^1, E)$, $[A, \epsilon(\nabla)] \in PDO^{-\infty}(S^1, E)$. Consequently,*

$$c_s^\nabla : A, B \in PDO(S^1, E) \mapsto \frac{1}{2}\mathrm{tr}\,(\epsilon(\nabla)[\epsilon(\nabla), A][\epsilon(\nabla), B])$$

is a well-defined $\mathbb{R}$-valued 2-cocycle on $PDO(S^1, E)$. Moreover, c_s^∇ is non trivial on any Lie algebra $\mathcal{A}$ such that $C^\infty(S^1, \mathbb{R}) \subset \mathcal{A} \subset PDO(S^1, E)$.

Notice that $C^\infty(S^1, \mathbb{R})$ is understood as an algebra acting on $C^\infty(S^1, E)$ by scalar multiplication fiberwise. The proof follows the same arguments as in [3].

Proof. First, c_s^∇ is the trace of operators acting on a real Hilbert space. so that, it is real valued. Since [1], see e.g., [6], if c_s^∇ was trivial on Hoschild cohomology, there would have a 1-form $\nu : \mathcal{A} \to \mathbb{R}$ such that:

$$c_s^D = \nu([.,.])$$

and hence it would be true on $C^\infty(S^1, \mathbb{R})$ which is a commutative algebra. Hence, since $c_s^\nabla \neq 0$ on $C^\infty(S^1, \mathbb{R})$, it is non trivial on A. □

4. Sets of Fourier Integral Operators

4.1. The set $FIO(S^1, E)$

Here, for the definitions, $\varepsilon = \epsilon(D)$ or $\varepsilon = \varepsilon(\nabla)$, depending on the fact that E is a complex or a real vector bundle. Let us now define:

$$FIO_{res}(S^1; E) = \{A \in FIO(S^1, E) \text{ such that } [A; \epsilon] \in PDO^{-\infty}(S^1, E)\}$$

Proposition 21. *$FIO_{res}(S^1, E)$ is a set, stable under composition, with unit element.*

Proof. $FIO(S^1, E)$ is stable under composition [13]. Since $Cl^0(S^1, E)$ is contained in $FIO_{res}(S^1, E)$ by Theorem 20 so that $FIO_{res}(S^1, E)$ contains the identity map.

Let $A, B \in FIO_{res}(S^1, E)$,

$$[AB, \epsilon] \quad = \quad A[B; \epsilon] + [A; \epsilon]B$$

Since $[A, \epsilon]$ and $[B; \epsilon]$ are smoothing, we get that $[AB, \epsilon]$ is smoothing. □

We use the natural notations,

$$FIO^0_{res} = FIO^0 \cap FIO_{res}$$

We shall note by $FIO^*_{res}(S^1, E)$ the group of units of this set, and by $FIO^{0,*}_{res}(S^1, E)$ the group of units of the set $FIO^0_{res}(S^1, E)$.

Proposition 22. *$FIO^*_{res}(S^1, E) = FIO^*(S^1, E) \cap FIO_{res}(S^1, E)$ and $FIO^{0,*}_{res}(S^1, E) = FIO^{0,*}(S^1, E) \cap FIO_{res}(S^1, E)$.*

Proof. We already have trivially $FIO^*_{res}(S^1,E) \subset (S^1,E) \cap FIO_{res}(S^1,E)$. Let $A \in FIO^*(S^1,E) \cap FIO_{res}(S^1,E)$. We have to check that $A^{-1} \in FIO_{res}(S^1,E)$.

$$\begin{aligned} A[A^{-1},\epsilon] &= [AA^{-1},\epsilon] - [A,\epsilon]A^{-1} \\ &= [Id,\epsilon] - [A,\epsilon]A^{-1} \\ &= -[A,\epsilon]A^{-1} \\ &\in PDO^{-\infty}(S^1,E). \end{aligned}$$

So that,

$$\begin{aligned} [A^{-1},\epsilon] &= A^{-1}A[A^{-1},\epsilon] \\ &\in PDO^{-\infty}(S^1,E) \end{aligned}$$

The proof is the same for $0-$order operators. □

By the way, since $FIO^{0,*}(S^1,E)$ is a "'generalized Lie group"' in the sense of Omori, it is a Frölicher Lie group. By the trace property of Frölicher spaces, using the last proposition, $FIO^{0,*}_{res}(S^1,E)$ is a Frölicher Lie group [24]. Now, since we have that:

$$FIO^{0,*}_{res} \subset GL_{res},$$

the determinant bundle defined over GL_{res} can be pulled-back on $FIO^{0,*}_{res}$. The same way, it is shown in [2,3] that the Schwinger cocycle extends to the Lie algebra $PDO^0(S^1,E) + PDO^1(S^1,\mathbb{C}) \otimes Id_E$.

4.2. Yet Some Subgroups of $FIO^*_{res}(S^1,E)$

Let us first gather and reformulate many known results:

Lemma 23. $Diff^+(S^1) \times C^\infty(S^1,\mathbb{C}^*) \subset FIO^{0,*}_{res}(S^1,\mathbb{C})$.

Proof. First, we have that:

$$C^\infty(S^1,\mathbb{C}^*) \subset Cl^{0,*}(S^1,\mathbb{C})$$

so that,

$$C^\infty(S^1,\mathbb{C}^*) \subset FIO^{0,*}(S^1,\mathbb{C})$$

Let $g \in Diff^+(S^1)$. Following [1], the map $f \mapsto |g'|^{1/2}.(f \circ g)$ describes an operator in $U_{res} \subset GL_{res}$. Since the map $f \mapsto |g'|^{1/2}.f$ is a multiplication operator in $C^\infty(S^1,\mathbb{C}^*)$, we get that:

$$f \mapsto f \circ g = \int e^{ig(.).\xi}\hat{f}(\xi)d\xi \in GL_{res} \cap FIO^{0,*}(S^1,\mathbb{C})$$

□

Theorem 24. *Assume that E be a trivial vector bundle over S^1. Let $\tilde{\pi}$ be the projection $FIO^*_{Diff}(S^1,E) \to Diff(S^1)$. Then,*

$$\pi^{-1}(Diff_+(S^1)) \subset FIO_{res}(S^1,E)$$

This is a simple consequence of the previous results.

Theorem 25. *Assume that E is non trivial and let ϵ defined as before. Let $\tilde{\pi}$ be the projection $FIO^*_{Diff}(S^1,E) \to Diff(S^1)$. Then,*

$$FIO^*_{Diff_+}(S^1,E) = \pi^{-1}(Diff_+(S^1)) \subset FIO_{res}(S^1,E)$$

and there is a global smooth section (in the sense of Frölicher spaces, not necessarily in the sense of groups)

$$Diff_+(S^1) \to FIO_{res}(S^1, E)$$

of the short exact sequence:

$$0 \to PDO^*(S^1, E) \to FIO^*_{Diff}(S^1, E) \cap FIO_{res}(S^1, E) \to Diff_+(S^1) \to 0$$

Proof. Let $g \in Diff_+(S^1)$. We fix on E a connection ∇ and we set $n = rank(E)$. Since $Diff_+(S^1)$is the connected component of Id_{S^1} in $Diff(S^1)$, given η the unit vector field defined by orientation on S^1, we can choose a path

$$\gamma \in C^\infty([0,1], Diff_+(S^1)) \subset C^\infty([0,1] \times S^1, S^1)$$

such that:

$$\gamma(0) = Id_{S^1}, \gamma(1) = g$$

and

$$\forall x \in S^1, \forall t \in [0;1], (\frac{d\gamma}{dt}(t)(x), \eta(x))_{T_xS^1} > 0$$

This path is unique up to parametrization since we impose also the condition of minimal length. Let

$$H_x = Hol(\gamma(.)(x)) \in Gl(E_x, E_{g(x)})$$

be the induced parallel transport map. We get, for each $g \in Diff_+(S^1)$, a map H_g which is smooth by the properties of parallel transport, linear on the fibers, invertible, and which projects on S^1 to g. Thus, $H_g \in Aut(E)$, and it easy to see that it is a bijection on the collection of smooth trivializations of E. Now, turning to the map

$$g \mapsto H_g$$

is appears as a smooth map $Diff(S^1) \to Aut(E)$, but it cannot ba a group morphism when E is non trivial. We have, moreover, that

$$\forall g \in Diff^+(S^1), [\nabla, H_g] = 0$$

since $dim(S^1) = 1$ and H_g is a parallel transport map. So that, since ϵ is derived from $\frac{\nabla}{dt} = \nabla_\eta$, we get that: $H_g \in GL_{res}$. Now, an operator in $FIO^*_{Diff_+}(S^1, E)$ reads as:

$$H_g \circ A,$$

where $A \in PDO^*(S^1, E) \subset GL_{res}$. Then $H_g \circ A \in FIO_{res}$ □

Theorem 26. *The group*

$$FCl^{0,*}_{Diff_+}(S^1, E) = FIO^*_{Diff_+}(S^1, E) \cap FCL^0(S^1, E)$$

is a regular Frölicher Lie group.

Proof. We get the obvious exact sequence of Lie groups:

$$0 \to Cl^{*,0}(S^1, E) \to FCl^{*,0}_{Diff}(S^1, E) \to Diff_+(S^1) \to 0$$

Both $Cl^{*,0}(S^1, E)$ and $Diff_+(S^1)$ are regular, and $Aut(E) \subset Cl^{*,0}(S^1, E)$, so that the smooth section $Diff_+(S^1) \to Aut(E)$ described in the proof of the previous theorem gives the result by Theorem 11. □

Let us now describe a subgroup of $FIO^*_{Diff_+}(S^1, E)$.

Definition 27. Let $FIO^*_{b,Diff_+}(S^1, E)$ be the space of operators $A \in FIO^*_{Diff_+}(S^1, E)$ such that

1. $\pi(A)$ is a diffeomorphism of $S^1 = \mathbb{R}/\mathbb{Z}$ such that $\pi(A)(0) = 0$;
2. if u is a smooth section of E such that $u(0) = 0$, then $(Au)(0) = 0$.

These operators are called based operators, and the set of sections u of E such that $u(0) = 0$ is the space of based sections, noted $C^\infty_b(S^1; E)$. We note by $Diff_{b,+}(S^1)$ the infinite dimensional Lie group of diffeomorphisms g such that $g(0) = 0$.

We recall that $Diff_{b,+}(S^1)$ is a regular Lie subgroup of $Diff_+(S^1)$. Its Lie algebra is noted $\mathfrak{diff}_b(S^1)$. Given $c(t)$ a smooth curve in $FIO^*_{b,Diff_+}(S^1, E)$, starting at Id_E, $\frac{dc(t)}{dt}|_{t=0} = U + X$, where $X \in \mathfrak{diff}_b(S^1)$ and $U \in PDO(S^1, E)$ which stabilize $C^\infty_b(S^1; E)$.

Theorem 28.

- *Let $G \subset PDO^*(S^1, E)$ be a regular Lie group of based operators, that contains the space of based invertible multiplication operators, with Lie algebra $\mathfrak{g}$.*
- *Let $D \subset Diff_{b,+}(S^1)$ be a regular Lie subgroup of based diffeomorphisms, with regular Lie algebra $\mathfrak{d}$.*

There is a regular Lie group $FG_D \subset FIO^*_{b,Diff_+}(S^1, E)$ for which the following sequence is exact:

$$0 \to G \to FG_D \to D \to O$$

Proof. We consider first the regular Lie group of automorphisms $\pi^{-1}(D) \subset Aut(E)$. Then, with the same arguments, G and generate a group that we note FG_D, and adapting the computations of Lemma 8, we obtain the above exact sequence. Finally, by Theorem 3, FG_D is a regular Lie group. □

5. Manifolds of Embeddings

Notation: Let $E \to M$ be a smooth vector bundle over M with typical fiber x. For $k \in \mathbb{N}^*$, we denote by

- $E^{\times k}$ the product bundle, of basis M, with typical fiber $F^{\times k}$;
- $\Omega^k(E)$ the space of $k - forms$ on M with values in E, that is, the set of smooth maps $(TM)^{\times k} \to E$ that are fiberwise k-linear and skew-symmetric $(T_xM)^{\times k} \to E_x$ for any $x \in M$. If $E = M \times F$, we note $\Omega^k(M, F)$ the space of k-forms instead of $\Omega^k(E)$.

Let M be a compact manifold without boundary; let N be a Riemannian manifold, equipped with the metric $(.,.)$. Let $Emb(M, N)$ be the manifold of smooth embeddings $M \to N$. References for principal bundles of embeddings are [35,36].

5.1. *$Emb(M, N)$ as a Principal Bundle*

The group of diffeomorphisms of M, $Diff(M)$, acts smoothly and on the right on $Emb(M, N)$, by composition. Moreover,

$$B(M, N) = Emb(M, N)/Diff(M)$$

is a smooth manifold [23], and $\pi : Emb(M, N) \to B(M, N)$ is a principal bundle with structure group $Diff(M)$ (see [23]). Then, $g \in Emb(M, N)$ is in the $Diff(M)-$orbit of f if and only if $g(M) = f(M)$. Let us now precise the vertical tangent space and a normal vector space of the orbits of $Diff(M)$ on $Emb(M, N)$. $T_f Emb(M, N)$, the tangent space at f, is identified with the space of smooth sections of f^*TN, which is the pull-back of TN by f. VT_fP, the vertical tangent space at f is the space of smooth sections of $Tf(M)$. Let $\mathcal{N}_f$ be the normal space to $f(M)$ with respect to the metric $(.,.)$ on N.

For any $x \in M$, $T_{f(x)}N = T_{f(x)}f(M) \oplus \mathcal{N}f(M)$. Hence, denoting $f * \mathcal{N}_f$ the pull back of $\mathcal{N}_f$ by f, we have that:

$$C^\infty(f^*TN) = C^\infty(TM) \oplus f^*\mathcal{N}_f$$

Moreover, for any volume form dx on M, if:

$$< .,. >: X, Y \in C^\infty(f^*TN) \mapsto < X, Y >= \int_M (X(x), Y(x))dx$$

is a L^2-inner product on $C^\infty(f^*TN)$, this splitting is orthogonal for $< .,. >$. We get here a fundamental difference between the inclusion $Emb(M,N) \subset C^\infty(M,N)$, where the model space of the type $C^\infty(f^*TN)$, and $Emb(M,N)$ as a $Diff(M)-$ principal bundle: sections of the vertical tangent vector bundle read as order 1 differential operators, where as the operators acting on the normal vector bundle reads as $0-$order differential operators, just like the structure group of $TC^\infty(M,N)$. To be more precise, let $X \in C^\infty(f^*TN)$ and let $p : f^*TN \to Tf(M)$ be the orthogonal projection. The vector field $p(X) \in C^\infty(Tf(M))$ is seen as a differential operator acting on smooth functions $f(M) \sim M \to \mathbb{R}$, and the normal component $(Id - p)(X)$ is a smooth section on $\mathcal{N}_f$. In the sequel we shall note:

$$\mathcal{N} = \coprod_{f \in Emb(M,N)} \mathcal{N}_f$$

We turn now to local trivializations. Let $f \in C^\infty_b(M,N)$. We define the map $Exp_f : C^\infty_0(M, f^*TN) \to C^\infty_b(M,N)$ defined by $Exp_f(v) = exp_{f(.)}v(.)$ where exp is the exponential map on N. Then Exp_f is a smooth local diffeomorphism. Restricting Exp_f to a C^∞ - neighborhood $\tilde{U}_f$ of the 0-section of f^*TN, we define a diffeomorphism, setting:

$$(Exp_f)_{|\tilde{U}_f} : \tilde{U}_f \to V_f = Exp_f(\tilde{U}_f) \subset C^\infty_b(M,N)$$

Then, setting $U_f = I_f^{-1}\tilde{U}_f$, we can define a chart Ξ^f on V_f by:

$$\Xi^f(g) = (I_f^{-1} \circ (Exp_f)^{-1}_{|\tilde{U}_f})(g) \in U_f \subset C^\infty_b(M,E)$$

Given f, g in $C^\infty_b(M,N)$ such that $V_{f,g} = V_f \cap V_g \neq \emptyset$, we compute the changes of charts $\Xi^{f,g}$ from $U^f_{f,g} = \Xi^f V_{f,g}$ to $U^g_{f,g} = \Xi^g V_{f,g}$. Let $u \in U^f_{f,g}$, $v = (\Xi^f)^{-1}(u) \in V_{f,g}$.

$$\Xi^{f,g}(u) = \Xi^g \circ (\Xi^f)^{-1}(u) = (I_g^{-1} \circ (Exp_g)^{-1} \circ Exp_f \circ I_f)(u)$$

Since, $\forall x \in M$, the transition maps:

$$\Xi^{f,g}(u)(x) = (I_g^{-1} \circ (exp_{g(x)})^{-1} \circ exp_{f(x)} \circ I_f)(u(x))$$

are smooth, $(V_f, \Xi^f, U_f)_{f \in C^\infty_b(M,N)}$ is a smooth atlas on $C^\infty_b(M,N)$. Moreover, let $w \in C^\infty_0(M,E)$, setting $v = (\Xi^f)^{-1}(u)$, the evaluation of the differential at $x \in M$ reads :

$$D_u\Xi^{f,g}(w)(x) = (I_g^{-1} \circ D_{v(x)}(exp_{g(x)})^{-1} \circ D_{u(x)}(exp_{f(x)} \circ I_f))(w(x))$$

Hence, for $u \in C^\infty$, $D_u\Xi^{f,g}$ is a multiplication operator acting on smooth sections of E for any isomorphism I_f and I_g we can choose. Since I_f and I_g are fixed, the family $u \mapsto D_u\Xi^{f,g}$ is a smooth family of 0- order differential operators; this construction is described carefully in [37]. Now, let $f \in Emb(M,N)$ and let us consider the map:

$$\Phi^{U,f} : (f, v, X) \in TU \sim (1-p)TU \oplus pTU \mapsto \Xi^f(v).exp_{Diff(M)}(X) \in Emb(M,N)$$

This map gives a local (fiberwise) trivialization of the principal bundles $Emb(M,N) \rightarrow B(M,N)$ following [23,38,39], and we see that the changes of local trivializations have $Aut(\mathcal{N})$ as a structure group.

If M is oriented, we note by $Diff_+(M)$ the group of orientation preserving diffeomorphisms and we have the following trivial lemma:

Lemma 29.
$$\frac{Diff(M)}{Diff_+(M)} = \mathbb{Z}_2$$

Then, defining
$$B_+(M,N) = \frac{Emb(M,N)}{Diff^+(M)}$$
we get:

Proposition 30. *$B_+(M,N)$ is a 2-cover of $B(M,N)$*

Now, taking basepoints $x_0 \in M$ and $y_0 \in N$, we define the principal bundle of based embeddings

Proposition 31.
Let
$$Emb_b(M,N) = \{f \in Emb(M,N) | f(x_0) = y_0\}$$
Let
$$Diff_b(M) = \{g \in Diff(M) | g(x_0) = x_0\}$$
Let
$$Diff_{b,+}(M) = Diff_b(M) \cap Diff_+(M)$$
Let
$$B_b(M,N) = Emb_b(M,N)/Diff_b(M,N)$$
and
$$B_{b,+}(M,N) = Emb_b(M,N)/Diff_{b,+}(M,N)$$

Then $Emb_b(M,N)$ is a principal bundle with base $B_b(M,N)$ (resp. $B_{b,+}(M,N)$) and with structure group $Diff_b(M)$ (resp. $Diff_{b,+}(M)$)

Proof. It follows from the fact that $Emb_b(M,N) = ev_{x_0}^{-1}(y_0)$ in $Emb(M,N)$, and $Diff_b(M) = ev_{x_0}^{-1}(x_0)$ in $Diff(M)$. □

5.2. Almost Complex Structure on Based Oriented Knots

Here, we consider $B_{b,+}(S^1,N)$, which can be understood as a space of unparametrized oriented knots. Let $f \in Emb_b(S^1,N)$. Following the decompositions of the previous section, the tangent space
$$T_f Emb(S^1,N) = \{X \in f^*TN | X(0) = 0\}$$
decomposed into the sum:
$$\mathcal{N}_{b,f} \oplus Df(C_b^\infty(TS^1))$$

Let us consider the operator:
$$J = i\epsilon(D)$$
where $D = -i\frac{\nabla}{dt}$. We get that $J^2 = -Id$, so that J is an almost complex structure of $TB_+(S^1,N)$.

6. Chern-Weil Forms on Principal Bundle of Embeddings and Homotopy Invariants

6.1. Chern Forms in Infinite Dimensional Setting

Let P be a principal bundle, of basis M and with structure group G. Let $\mathfrak{g}$ be the Lie algebra of G. Recall that G acts on P, and also on $P \times \mathfrak{g}$ by the action $((p,v),g) \in (P \times \mathfrak{g}) \times G \mapsto (p.g, Ad_{g^{-1}}(v)) \in (P \times \mathfrak{g})$. Let $AdP = P \times_{Adg} = (P \times \mathfrak{g})/G$ be the adjoint bundle of P, of basis M and of typical fiber $\mathfrak{g}$, and let $Ad^k P = (AdP)^{\times k}$ be the product bundle, of basis M and of typical fiber $\mathfrak{g}^{\times k}$.

Definition 32. Let k in $\mathbb{N}^*$. We define $\mathfrak{Pol}^k(P)$, the set of smooth maps $Ad^k P \to \mathbb{C}$ that are k-linear and symmetric on each fiber, equivalently as the set of smooth maps $P \times \mathfrak{g}^k \to \mathbb{C}$ that are k-linear symmetric in the second variable and G-invariants with respect to the natural coadjoint action of G on $\mathfrak{g}^k$.

Let $\mathfrak{Pol}(P) = \bigoplus_{k \in \mathbb{N}*} \mathfrak{Pol}(P)$.

Let $\mathcal{C}(P)$ be the set of connections on P. For any $\theta \in \mathcal{C}(P)$, we denote by $F(\theta)$ its curvature and ∇^θ (or ∇ when it carries no ambiguity) its covariant derivation. Given an algebra A, In this section, we study the maps, for $k \in \mathbb{N}^*$,

$$\begin{aligned} Ch \quad : \quad \mathcal{C}(P) \times \mathfrak{Pol}^k(P) &\to \Omega^{2k}(M,\mathbb{C}) && (2) \\ (\theta, f) &\mapsto Alt(f(F(\theta), ..., F(\theta))) && (3) \end{aligned}$$

where Alt denotes the skew-symmetric part of the form. Notice that, in the case of the finite dimensional matrix groups Gl_n with Lie algebra $\mathfrak{gl}_n$, the set $\mathfrak{Pol}(P)$ is generated by the polynomials $A \in \mathfrak{gl}_n \mapsto \mathrm{tr}(A^k)$, for $k \in 0, ..., n$. This leads to classical definition of Chern forms. However, in the case of infinite dimensional structure groups, most situations are still unknown and we do not know how to define a set of generators for $\mathfrak{Pol}(P)$.

Lemma 33. *Let $f \in \mathfrak{Pol}^k(P)$. Then*

$$\begin{aligned} f([a_1,v],a_2,...,a_k) + f(a_1,[a_2,v],...,a_k) + & \\ ... & \\ +f(a_1,a_2,...,[a_k,v]) &= 0 \end{aligned}$$

Proof. Let us notice first that f is symmetric. Let $v \in \mathfrak{g}$, and c_t a path in G such that $\{\frac{d}{dt}c_t\}_{t=0} = v$. Let $a_1, ..., a_k \in \mathfrak{g}^k$.

$$\begin{aligned} \{\frac{d}{dt}\{f(ad_{c_t^{-1}}a_1, ..., ad_{c_t^{-1}}a_k)\}_{t=0} &= f([a_1,v],a_2,...,a_k) + f(a_1,[a_2,v],...,a_k) + \\ & ... \\ & +f(a_1,a_2,...,[a_k,v]) \end{aligned}$$

Since f in G-invariant, we get:

$$\begin{aligned} f([a_1,v],a_2,...,a_k) + f(a_1,[a_2,v],...,a_k) + & \\ ... & \\ +f(a_1,a_2,...,[a_k,v]) &= 0 \end{aligned}$$

Lemma 34. *Let $f \in \mathfrak{Pol}^k(P)$ such that f, as a smooth map $P \times \mathfrak{g}^k \to \mathbb{C}$, satifies $d^M f = 0$ on a system of local trivializations of P. Then, the map*

$$Ch^f : \theta \in \mathcal{C}(P) \mapsto Ch^f(\theta) = Ch(\theta, f) \in \Omega^*(P, \mathbb{C})$$

takes values into closed forms on P. Moreover,

(i) it is vanishing on vertical vectors and defines a closed form on M.
(ii) the cohomology class of this form does not depend on the choice of the chosen connexion θ on P.

Proof. The proof runs as in the finite dimensional case, see e.g., [30]. First, it is vanishing on vertical vectors and G-invariant because the curvature of a connexion vanishes on vertical forms and is G-covariant for the coadjoint action. Let us now fix $f \in \mathfrak{Pol}^k(P)$. We compute $df(F(\theta), ..., F(\theta))$. We notice first that it vanishes on vertical vectors trivially. Let us fix $Y_1^h, ..., Y_{2k}^h, X^h$ $2k+1$ horizontal vectors on P at $p \in P$. On a local trivialization of P around p, these vectors read as:

$$\begin{array}{rcl} Y_1^h & = & Y_1 - \tilde{\theta}(Y_1) \\ & (...) & \\ Y_{2k}^h & = & Y_{2k} - \tilde{\theta}(Y_{2k}) \\ X^h & = & X - \tilde{\theta}(X) \end{array}$$

where $\tilde{\theta}$ stands here for the expression of θ in the local trivilization, and $Y_1, ..., Y_{2k}, X$ $2k+1$ tangent vectors on M at $\pi(p) \in M$. We extend these vector fields on a neighborhood of p

- by the action of G in the vertical directions
- setting the vectors fields constant on $U \times p$, where U is a local chart on M around $\pi(p)$.

Then, we have:

$$f(F(\theta), ..., F(\theta))(Y_1^h, ..., Y_{2k}^h) = f(F(\theta), ..., F(\theta))(Y_1, ..., Y_{2k})$$

since $F(\theta)$ is vanishing on vertical vectors.

Then, on a local trivialization with the notations defined before (the sign Alt is omitted for easier reading), and writing d^M for the differential of forms on any open subset of M,

$$\begin{array}{rcl} d^M f(F(\tilde{\theta}), ..., F(\tilde{\theta})) & = & \sum_{i=1}^{k} f(d^M F(\tilde{\theta}), F(\tilde{\theta}), ..., F(\tilde{\theta})) + f(F(\tilde{\theta}), d^M F(\tilde{\theta}), ..., F(\tilde{\theta})) + \\ & & ... + f(F(\tilde{\theta}), F(\tilde{\theta}), ..., d^M F(\tilde{\theta})) \end{array}$$

and then, using Lemma 33,

$$\begin{array}{rcl} \nabla^\theta f(F(\tilde{\theta}), ..., F(\tilde{\theta})) & = & \sum_{i=1}^{k} f(\nabla^\theta F(\tilde{\theta}), F(\tilde{\theta}), ..., F(\tilde{\theta})) + f(F(\tilde{\theta}), \nabla^\theta F(\tilde{\theta}), ..., F(\tilde{\theta})) + \\ & & ... + f(F(\tilde{\theta}), F(\tilde{\theta}), ..., \nabla^\theta F(\tilde{\theta})) \end{array}$$

Then, by Bianchi identity, we get that:

$$\begin{array}{rcl} d^M Ch(f, \theta) & = & \nabla^\theta Ch(f, \theta) \\ & = & 0 \end{array}$$

This proves (i) Then, following e.g., [30], if θ and θ' are connections, fix $\mu = \theta' - \theta$ and $\theta_t = \theta + t\nu$ for $t \in [0;1]$. We have:

$$\frac{dF(\theta_t)}{dt} = \nabla^{\theta^t}\mu$$

Moreover, μ is G-invariant and vanishes on vertical vectors. Thus,

$$\begin{aligned}\frac{dCh(f,\theta_t)}{dt} &= kf(F(\theta_t),...,F(\theta_t),\nabla^{\theta_t}\mu) \\ &= kd^M(f(F(\theta_t),...,F(\theta_t),\mu))\end{aligned}$$

Integrating in the t-variable, we get:

$$Ch(f,\theta_0) - Ch(f,\theta_1) = -kd^M\int_0^1 f(F(\theta_t),...,F(\theta_t),\mu)dt$$

Even if these computations are local, the two sides are global objects and do not depend on the chosen trivialization, which ends the proof. □

Important Remark. The condition $d^M f = 0$ is a **local** condition, checked in an (adequate) system of trivializations of the principal bundle, because it has to be checked on the vector bundle $Ad(P)^{\times k}$. This is in particular the case when we can find a 0-curvature connection θ on P such that:

$$[\nabla^\theta, f] = 0$$

In that case, since the structure group G is regular, we can find a system of local trivializations of P defined by θ and such that, on any local trivialization, $\nabla^\theta = d^M$ (see e.g., [23,40] for the technical tools that are necessary for this).

This technical remark can appear rather unsatisfactory first because it restricts the ability of application of the previous lemma, secondly because we need have a local (and rather unelegant) condition. This is why we give the following theorem, from Lemma 34.

Theorem 35. *Let $f \in \mathfrak{Pol}(P)$ for which there exists $\theta \in \mathcal{C}(P)$ such that $[\nabla^\theta, f] = 0$. We shall note this set of polynomials by $\mathfrak{Pol}_{reg}(P)$. Then, the map:*

$$Ch^f : \theta \in \mathcal{C}(P) \mapsto Ch^f(\theta) = Ch(\theta, f) \in \Omega^*(P,\mathbb{C})$$

takes values into closed forms on P. Moreover,

(i) it is vanishing on vertical vectors and defines a closed form on M.
(ii) The cohomology class of this form does not depend on the choice of the chosen connexion θ on P.

Moreover, $\forall(\theta, f) \in \mathcal{C}(P) \times \mathfrak{Pol}_{reg}(P), [\nabla^\theta, f] = 0$.

Proof. Let $f \in \mathfrak{Pol}_{reg}(P)$ and let $\theta \in \mathcal{C}(P)$ such that $[\nabla^\theta, f] = 0$. Let $\theta' \in \rfloor(P)$ and let $\nu = \theta' - \theta \in \Omega^1(M,\mathfrak{g})$. Let $(\alpha_1,...,\alpha_k) \in (\Omega^2(M,\mathfrak{g}))^k$.

$$\begin{aligned}[\nabla^{\theta'}, f](\alpha_1,...\alpha_k) &= [\nabla^\theta, f](\alpha_1,...\alpha_k) + f([\alpha_1,\nu],...,\alpha_n) + \\ &\quad ... + f(\alpha_1,...,[\alpha_n,\nu]) \\ &= f([\alpha_1,\nu],...,\alpha_n) + ... + f(\alpha_1,...,[\alpha_n,\nu]) \\ &= 0\end{aligned}$$

Then, $\forall(\theta, f) \in \mathcal{C}(P) \times \mathfrak{Pol}_{reg}(P)$, $[\nabla^\theta, f] = 0$. By the way, $\forall \theta' \in \mathcal{C}(P)$,

$$d^M f(\alpha_1, ..., \alpha_k) = f(\nabla^{\theta'}\alpha_1, ..., \alpha_k) + ... + f(\alpha_1, ..., \nabla^{\theta'}\alpha_k)$$

Applying this to $\alpha_1 = ... = \alpha_k = F(\theta')$, we get:

$$dCh(f, \theta') = f(\nabla^{\theta'}F(\theta'), ..., F(\theta')) + ... + f(F(\theta'), ..., \nabla^{\theta'}F(\theta')) = 0$$

by Bianchi identity. Thus $Ch(f, \theta')$ is closed. Then, mimicking the end of the proof of Lemma 34, we get that the difference $Ch(f, \theta) - Ch(f, \theta')$ is an exact form, which ends the proof.

Proposition 36. *Let $\phi : \mathfrak{g}^k \to \mathbb{C}$ be a $k-$linear, symmetric, $Ad-$invariant form. Let $f : P \times \mathfrak{g}^k \to \mathbb{C}$ be the map induced by ϕ by the formula: $f(x, g) = \phi(g)$. Then $f \in \mathfrak{Pol}_{reg}$.*

Proof. Obsiously, $f \in \mathfrak{Pol}$. Let $\varphi : U \times G \to P$ and $\varphi' : U \times G \to P$ be a local trivialisations of P, where U is an open subset of M. Then there exists a smooth map $g : U \to G$ such that $\varphi'(x, e_G) = \varphi(x, e_G).g(x)$. Then we remark that $\varphi^* f = \varphi'^* f$ is a constant map on horizontal slices since ϕ is Ad-invariant. Moreover, since $\varphi^* f$ in a constant (polynomial-valued) map on $\varphi(x, e_G)$ we get that $[\nabla^\theta, f] = 0$ for the (flat) connection θ such that $T\varphi(x, e_G)$ spans the horizontal bundle over U. □

6.2. Application to $Emb(M, N)$

Mimicking the approach of [6], the cohomology classes of Chern-Weil forms should give rise to homotopy invariants. Applying Theorem 35, we get:

Theorem 37. *The Chern-Weil forms Ch^f is a $H^*(B(M, N))-$valued invariant of the homotopy class of an embedding, $\forall k \in \mathbb{N}^*$.*

When $M = S^1$, $Emb(S^1, N)$ is the space of (parametrized) smooth knots on N, and $B(S^1, N)$ is the space of non parametrized knots. Its connected components are the homotopy classes of the knots, through classical results of differential topology, see e.g., [41]. We now apply the material of the previous section to manifolds of embeddings. For this, we can define invariant polynomials of the type of those obtained in [6] (for mapping spaces) by a field of linear functionnal λ with "good properties" that ensures that:

$$A \mapsto \lambda(A^k) \in \mathfrak{Pol}^k_{reg}$$

This approach is a straightforward generalization of the description of Chern-Weil forms on finite dimensional principal bundles where polynomials are generated by functionnals of the type $A \mapsto \mathrm{tr}(A^k)$ (tr is the classical trace) but as we guess that we can consider other classes of polynomials for spaces of embeddings. In this paper, let us describe how to replace the classical trace of matrices tr by a renormalized trace tr^Q. In the most general case, it is not so easy to define a family of weights $f \in Emb(M, N) \mapsto Q_f$ which satisfy the good properties. Indeed, we have two examples of constructions which match the necessary assumptions for $\mathfrak{Pol}_{reg}$ when $M = S^1$, and the first one is derived from the following example:

Knot Invariant Through Kontsevich and Vishik Trace

The Kontsevich and Vishik trace is a renormalized trace for which $tr^Q([A, B]) = 0$ for each differential operator A, B and does not depend on the weight chosen in the odd class. For example, one can choose $Q = Id + \nabla^*\nabla$, where ∇ is a connection induced on $\mathcal{N}_f$ by the Riemannian metric, as described in [6]. It is an order 2 injective elliptic differential operator (in the odd class), and the coadjoint action of $Aut(\mathcal{N}_f)$ will give rise to another order 2 injective elliptic differential operator [7]. When $Q = Id + \nabla^*\nabla$, this only changes ∇ into another connection on E. Thus, setting:

$$\phi(A, ..., A) = tr^Q(A^k)$$

we have:

$$f \in \mathfrak{Pol}_{reg}$$

Let us now consider a connected component of $B(M, N)$, *i.e.*, a homotopy class of an embedding among the space of embeddings. We apply now the construction to $M = S^1$. The polynomial:

$$\phi : A \mapsto tr^Q(A^k)$$

is $Diff(S^1)-$invariant, and gives rise to an invariant of non oriented knots, *i.e.*, a Chern form on the base manifold:

$$B(S^1, N) = Emb(S^1, N)/Diff(S^1)$$

by theorem 37. This approach can be extended to invariant of embeddings, replacing S^1 by another odd-dimensional manifold.

7. Conclusions

We have given here some groundbreaking properties for a theory of differential invariants on non-linear grassmannians. A work in progress intends to describe such Chern-Weil, or Chern-Simons, or Cheeger-Simons invariant which could lead to non trivial knot invariants.

Conflicts of Interest: The author declares no conflict of interest.

Appendix

Renormalized Traces of PDOs

E is equipped this an Hermitian products $< .,. >$, which induces the following L^2-inner product on sections of E:

$$\forall u, v \in C^\infty(S^1, E), \quad (u, v)_{L^2} = \int_{S^1} < u(x), v(x) > dx$$

where dx is the Riemannian volume. The main references are [42,43], see e.g., [6].

Definition A1. Q is a **weight** of order $s > 0$ on E if and only if Q is a classical, elliptic, admissible pseudo-differential operator acting on smooth sections of E, with an admissible spectrum.

Recall that, under these assumptions, the weight Q has a real discrete spectrum, and that all its eigenspaces are finite dimensional. For such a weight Q of order q, one can define the complex powers of Q [44], see e.g., [4] for a fast overview of technicalities. The powers Q^{-s} of the weight Q are defined for $Re(s) > 0$ using with a contour integral,

$$Q^{-s} = \int_\Gamma \lambda^s (Q - \lambda Id)^{-1} d\lambda$$

where Γ is an "angular" contour around the spectrum of Q. Let A be a log-polyhomogeneous pseudo-differential operator. The map $\zeta(A, Q, s) = s \in \mathbb{C} \mapsto \mathrm{tr}\left(AQ^{-s}\right) \in \mathbb{C}$, defined for $Re(s)$ large, extends on $\mathbb{C}$ to a meromorphic function with a pole of order $q + 1$ at 0 ([45]). When A is classical, $\zeta(A, Q, .)$ has a simple pole at 0 with residue $\frac{1}{q}\mathrm{res}A$, where res is the Wodzicki residue ([46], see also [29]). Notice that the Wodzicki residue extends the Adler trace [47] on formal symbols. Following [45], we define the renormalized trace, see e.g., [4,48] for the renormalized trace of classical operators.

Definition A2. $tr^Q A = lim_{z\to 0}(\mathrm{tr}(AQ^{-z}) - \frac{1}{qz} resA)$.

On the other hand, the operator e^{-tQ} is a smoothing operator for each $t > 0$, which shows that $trAe^{-tQ}$ is well-defined and finite for $t > 0$. From the function $t \mapsto trAe^{-tQ}$, we recover the function $z \mapsto \mathrm{tr}(AQ^{-z})$ by the Mellin transform (see e.g., [42], pp. 115–116), which shows the following lemma:

Lemma A1. *Let A, A' be classical pseudo-differential operators, let Q, Q' be weights.*

$$\forall t > 0, trAe^{-tQ} = trA'e^{-tQ'} \Rightarrow \left\{ \begin{array}{lcl} tr^Q(A) & = & tr^{Q'}(A') \\ res(A) & = & res(A') \end{array} \right.$$

If A is trace class, $\mathrm{tr}^Q(A) = \mathrm{tr}(A)$. The functional tr^Q is of course not a trace on $Cl(M,E)$. Notice also that, if A and Q are pseudo-differential operators acting on sections on a real vector bundle E, they also act on $E \otimes \mathbb{C}$. The Wodzicki residue res and the renormalized traces tr^Q have to be understood as functionals defined on pseudo-differential operators acting on $E \otimes \mathbb{C}$. In order to compute $\mathrm{tr}^Q[A,B]$ and to differentiate $\mathrm{tr}^Q A$, in the topology of classical pseudo-differential operators, we need the following ([4], see also [49] for the first point):

Proposition A1.

(i) *Given two (classical) pseudo-differential operators A and B, given a weight Q,*

$$\mathrm{tr}^Q[A,B] = -\frac{1}{q}\mathrm{res}(A[B, \log Q]) \tag{A1}$$

(ii) *Given a differentiable family A_t of pseudo-differential operators, given a differentiable family Q_t of weights of constant order q,*

$$\frac{d}{dt}\left(tr^{Q_t}A_t\right) = tr^{Q_t}\left(\frac{d}{dt}A_t\right) - \frac{1}{q}\mathrm{res}\left(A_t(\frac{d}{dt}\log Q_t)\right) \tag{A2}$$

The following "covariance" property of tr^Q ([4,48]) will be useful to define renormalized traces on bundles of operators,

Proposition A2. *Under the previous notations, if C is a classical elliptic injective operator of order 0, $tr^{C^{-1}QC}\left(C^{-1}AC\right)$ is well-defined and equals $\mathrm{tr}^Q A$.*

We moreover have specific properties for weighted traces of a more restricted class of pseudo-differential operators (see [4,50,51]), called odd class pseudo-differential operators following [50,51] :

Definition A3. A classical pseudo-differential operator A is called odd class if and only if:

$$\forall n \in \mathbb{Z}, \forall (x,\xi) \in T^*M, \sigma_n(A)(x,-\xi) = (-1)^n \sigma_n(A)(x,\xi).$$

We note this class Cl_{odd}.

Such a definition is consistent for pseudo-differential operators on smooth sections of vector bundles, and applying the local formula for Wodzicki residue, one can prove [4]:

Proposition A3. *If M is an odd dimensional manifold, A and Q lie in the odd class, then $f(s) = tr(AQ^{-s})$ has no pole at $s = 0$. Moreover, if A and B are odd class pseudo-differential operators,* $\mathrm{tr}^Q([A, B]) = 0$ *and* $\mathrm{tr}^Q A$ *does not depend on Q.*

This trace was first defined in the papers [50,51] by Kontesevich and Vishik. We remark that it is in particular a trace on $DO(M, E)$ when M is odd-dimensional.

Let us now describe a class of operators which is, in some sense, complementary to odd class:

Definition A4. A classical pseudo-differential operator A is called even class if and only if:

$$\forall n \in \mathbb{Z}, \forall (x, \xi) \in T^*M, \sigma_n(A)(x, -\xi) = (-1)^{n+1}\sigma_n(A)(x, \xi)$$

We note this class Cl_{even}.

Very easy properties are the following:

Proposition A4.
$Cl_{even} \circ Cl_{odd} = Cl_{odd} \circ Cl_{even} = Cl_{even}$ and
$Cl_{even} \circ Cl_{even} = Cl_{odd} \circ Cl_{odd} = Cl_{odd}$.

Now, following [6], we explore properties of tr^Q on Lie brackets.

Definition A5. Let E be a vector bundle over M, Q a weight and $a \in \mathbb{Z}$. We define :

$$\mathcal{A}_a^Q = \{B \in Cl(M, E); [B, \log Q] \in Cl^a(M, E)\}$$

Theorem A1. [6]

(i) *$\mathcal{A}_a^Q \cap Cl^0(M, E)$ is a subalgebra of $Cl(M, E)$ with unit.*
(ii) *Let $B \in Ell^*(M, E)$, $B^{-1}\mathcal{A}_a^Q B = A_a^{B^{-1}QB}$, where B^{-1} is the parametrix.*
(iii) *Let $A \in Cl^b(M, E)$, and $B \in \mathcal{A}^Q_{-dimM-b-1}$, then* $\mathrm{tr}^Q[A, B] = 0$.
(iv) *For $a < -\frac{dimM}{2}$, $\mathcal{A}_a^Q \cap Cl^{\frac{-dimM}{2}}(M, E)$ is an algebra on which the renormalized trace is a trace* (i.e., *vanishes on the brackets).*

We now produce non trivial examples of operators that are in $\mathcal{A}_a^Q$ when Q is scalar, and secondly we give a formula for some non vanishing renormalized traces of a bracket.

Lemma A2. *Let Q be a weight on $C_0^\infty(M, V)$ and let B be a classical pseudo-differential operator of order b. If B or Q is scalar, then $[B, \log Q]$ is a classical pseudo-differential operator of order $b - 1$.*

Proposition A5. *Let Q be a scalar weight on $C_0^\infty(M, V)$. Then*

$$Cl^{a+1}(M, V) \subset \mathcal{A}_a^Q$$

Consequently,

(i) *if $ord(A) + ord(B) = -dimM$,* $\mathrm{tr}^Q[A, B] = 0$.
(ii) *when $M = S^1$, if A and B are classical pseudo-differential operators, if A is compact and B is of order 0,* $\mathrm{tr}^Q[A, B] = 0$.

Lemma A3. *Let Q be a scalar weight on $C_0^\infty(M,V)$, and A, B two pseudo-differential operators of orders a and b on $C_0^\infty(M,V)$, such that $a+b=-m+1$ (m = dim M). Then*

$$\mathrm{tr}^Q[A,B] = -\frac{1}{q}\mathrm{res}\left(A[B,\log Q]\right) = -\frac{1}{q(2\pi)^n}\int_M\int_{|\xi|=1} tr(\sigma_a(A)\sigma_{b-1}([B,\log Q]))$$

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