Mathematics **2015**, *3*, 398-411; doi:10.3390/math3020398

OPEN ACCESS

mathematics

ISSN 2227-7390

www.mdpi.com/journal/mathematics

Article

Implicit Fractional Differential Equations via the Liouville–Caputo Derivative

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Academic Editor: Hari M. Srivastava

Received: 16 March 2015 / Accepted: 13 May 2015 / Published: 25 May 2015

Abstract: We study an initial value problem for an implicit fractional differential equation with the Liouville–Caputo fractional derivative. By using fixed point theory and an approximation method, we obtain some existence and uniqueness results.

Keywords: fractional differential equations; fractional integral; fractional derivative; Liouville–Caputo derivative; implicit

1. Introduction

Differential equations of fractional order have recently been proven to be valuable tools in the modeling of many physical phenomena [1–3]. There has also been a significant theoretical development in fractional differential equations in recent years; see the monographs of Abbas *et al.* [4], Kilbas *et al.* [5], Miller and Ross [6], Podlubny [7] and Samko *et al.* [8].

The basic theory for initial value problems for fractional differential equations involving the Riemann–Liouville and Liouville–Caputo differential operator was discussed by Diethelm [9].

Recently, fractional functional differential equations, fractional differential inclusions and impulsive fractional differential equations with different conditions were studied, for example, by Aghajani *et al.* [10], Ahmad and Nieto [11], Benchohra *et al.* [12], Chalishajar and Karthikeyan [13,14], Henderson and Ouahab [15,16], Ouahab [17,18] and in the references therein.

In this paper, we study the existence of solutions of the following implicit fractional differential equation with initial condition:

$$\begin{cases} {}^cD^\alpha y(t) & = & f(t, y(t), {}^c D^\alpha y(t)), \quad t \in J \\ \\ y(0) & = & y_0 \end{cases} \tag{1}$$

where $J = [0, b],\ 0 < \alpha < 1$ and $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

This problem is motivated by the importance of implicit ordinary differential equations of the form:

$$f(t, x(t), x'(t), \ldots, x^{(n-1)}(t)) = 0. \tag{2}$$

under various initial and boundary conditions. Implicit equations have been considered by many authors [19–26]. Furthermore, our intention is to extend the results to implicit differential equations of fractional order.

Very recently, some existence results for an implicit fractional differential equation on compact intervals were investigated [27–29].

Our goal in this work is to give some existence and uniqueness results for implicit fractional differential equations.

2. Fractional Calculus

According to the Riemann–Liouville approach to fractional calculus, the notation of the fractional integral of order α ($\alpha > 0$) is a natural consequence of the well-known formula (usually attributed to Cauchy) that reduces the calculation of the $n-$fold primitive of a function f to a single integral of the convolution type. The Cauchy formula reads:

$$[I^n f](t) = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s)\, ds,\ t > 0 \quad , n \in \mathbb{N}.$$

Definition 1. *The fractional integral of order $\alpha > 0$ of a function $f \in L^1(a, b)$ is defined by:*

$$I^\alpha_{a^+} f(t) := \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds,$$

where Γ is the classical gamma function. When $a = 0$, we write $I^\alpha f(t) = f(t) * \phi_\alpha(t)$, where $\phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, $\phi_\alpha(t) = 0$ for $t \leq 0$ and $\phi_\alpha \to \delta$ as $\alpha \to 0^+$, where δ is the Dirac delta function. For consistency, $I^0 = Id$ (Identity operator), i.e., *$I^0 f(t) = f(t)$. Furthermore, by $I^\alpha f(0^+)$, we mean the limit (if it exists) of $I^\alpha f(t)$ for $t \to 0^+$; this limit may be infinite.*

After the notion of the fractional integral, that of the fractional derivative of order α ($\alpha > 0$) becomes a natural requirement, and one is attempted to substitute α with $-\alpha$ in the above formulas. However, this generalization needs some care in order to guarantee the convergence of the integral and to preserve the well-known properties of the ordinary derivative of integer order. Denoting by D^n, with $n \in \mathbb{N}$, the operator of the derivative of order n, we first note that:

$$D^n I^n = Id, \quad I^n D^n \neq Id, \quad n \in \mathbb{N},$$

i.e., D^n is the left inverse (and not the right inverse) to the corresponding integral operator I^n. We can easily prove that:

$$I^n D^n f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{t^k}{k!},\ t > 0.$$

As a consequence, we expect that D^α is defined as the left inverse to I^α. For the fractional derivative of order $\alpha > 0$ with integer n, such that $n-1 < \alpha \leq n$, we have:

Definition 2. *For a function f given on interval $[a,b]$, the Riemann–Liouville fractional-order derivative of order α of f is defined by:*

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{-\alpha+n-1} f(s)ds,$$

provided the right-hand side is defined.

Defining for consistency, $D^0 = I^0 = Id$, then we easily recognize that:

$$D^\alpha I^\alpha = Id, \quad \alpha \geq 0, \tag{3}$$

and

$$D^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha},\ \alpha > 0, \gamma > -1,\ t > 0. \tag{4}$$

Of course, Properties (3) and (4) are a natural generalization of those known when the order is a positive integer.

Note the remarkable fact that the fractional derivative $D^\alpha f$ is not zero for the constant function $f(t) = 1$, if $\alpha \notin \mathbb{N}$. In fact, Equation (4) with $\gamma = 0$ illustrates that:

$$D^\alpha 1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)},\ \alpha > 0,\ \ t > 0. \tag{5}$$

It is clear that $D^\alpha 1 = 0$, for $\alpha \in \mathbb{N}$, due to the poles of the gamma function at the points $0, -1, -2, \ldots$.

We now observe an alternative definition of fractional derivative, introduced by Caputo [30,31] in the late 1960s and adopted by Caputo and Mainardi [32] in the framework of the theory of linear viscoelasticity (see a review in [2]).

Definition 3. *Let $f \in AC^n([a,b])$. The Liouville–Caputo fractional-order derivative of f is defined by:*

$$({}^cD_a^\alpha f)(t) := \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds.$$

This definition is of course more restrictive than the Riemann–Liouville definition, in that it requires the absolute integrability of the derivative of order n. Whenever we use the operator ${}^cD^\alpha$, we (tacitly) assume that this condition is met. We easily recognize that in general:

$$D^\alpha f(t) := D^n I^{n-\alpha} f(t) \neq I^{n-\alpha} D^n f(t) := {}^cD^\alpha f(t), \tag{6}$$

unless the function $f(t)$, along with its first $n-1$ derivatives, vanishes at $t = 0^+$. In fact, assuming that the passage of the $n-$derivative under the integral is legitimate, we recognize that, for $n-1 < \alpha < n$ and $t > 0$,

$$D^\alpha f(t) ={}^c D^\alpha f(t) + \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0^+), \tag{7}$$

and therefore, recalling the fractional derivative of the power Function (4):

$$D^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{\Gamma(k+1)} f^{(k)}(0^+) \right) = {}^cD^\alpha f(t). \tag{8}$$

The alternative definition, that is Definition 3, for the fractional derivative thus incorporates the initial values of the function and of lower order. The subtraction of the Taylor polynomial of degree $n-1$ at $t = 0^+$ from $f(t)$ means a sort of regularization of the fractional derivative. In particular, according to this definition, the relevant property for which the fractional derivative of a constant is still zero is satisfied, *i.e.*,

$${}^cD^\alpha 1 = 0, \quad \alpha > 0. \tag{9}$$

At the end of this section, we present some properties of a special function. Denote $E_{\alpha,\beta}$ the generalized Mittag–Leffler special function defined by:

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}.$$

Also,

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_\Upsilon \frac{\lambda^{\alpha-\beta} e^\lambda}{\lambda^\alpha - z} d\lambda$$

where Υ is a contour, which starts and ends at $-\infty$ and encircles the disc $|\lambda| \leq |z|^{\frac{1}{\alpha}}$ counterclockwise.

Lemma 1. *[9] Let $\alpha > 0,\ n = \lceil \alpha \rceil$ and $\lambda \in \mathbb{R}$. The solution of the initial value problem:*

$$\begin{cases} {}^cD^\alpha y(t) &= \lambda y(t) + q(t) \\ \\ y^{(k)}(0) &= y_k,\ k = 0, 1, \ldots, n-1, \end{cases} \tag{10}$$

where $q \in C[0, b]$ is a given function, can be expressed in the form:

$$y(t) = \sum_{k=0}^{n-1} y_k u_k(t) + y_*(t)$$

with:

$$y_*(t) = \begin{cases} I_0^\alpha q(t) & if\ \lambda = 0 \\ \\ \frac{1}{\lambda}\int_0^t q(t-s)u_0'(s)ds & if\ \lambda \neq 0, \end{cases}$$

where $u_k(t) = I_0^k e_\alpha(t),\ k = 0, 1, \ldots, n-1$ *and* $e_\alpha(t) := E_\alpha(\lambda t^\alpha)$.

Remark 1. *In the case* $0 < \alpha < 1$*, we can rewrite the solution of Problem (10) in the form:*

$$y(t) = y(0)E_\alpha(\lambda t^\alpha) + \alpha \int_0^t q(t-s)s^{\alpha-1}E_\alpha'(\lambda s^\alpha)ds$$

Lemma 2. *Let* $v : [0, b] \to [0, \infty)$ *be a real function, and* w *is a nonnegative, locally-integrable function on* $[0, b]$*. Assume that there are constants* $a > 0$ *and* $0 < \beta < 1$*, such that:*

$$v(t) \leq w(t) + a \int_0^t \frac{v(s)}{(t-s)^\beta}ds,$$

then there exists a constant $K = K(\beta)$*, such that:*

$$v(t) \leq w(t) + Ka \int_0^t \frac{w(s)}{(t-s)^\beta}ds,$$

for every $t \in [0, b]$.

Proof. From Gronwall's lemma for singular kernels, whose proof can be found in Lemma 7.1.1 on page 188 of [33], we know that:

$$v(t) \leq w(t) + c \int_0^t E_{1-\beta}'(c(t-s))w(s)ds$$

where $c \in \mathbb{R}$ is a constant dependent on β. Now, $E_{1-\beta}'(s)$ is bounded for $s \in [0, b]$ and for $s \to t^-$:

$$E_{1-\beta}'(c(t-s)) \approx \frac{a}{c^{-\beta}(t-s)^\beta}$$

where $a > 0$ is a constant. This implies that:

$$v(t) \leq w(t) + Ka \int_0^t \frac{w(s)}{(t-s)^\beta}ds$$

with constant $K = K(\beta)$. $\square$

Lemma 3. *Let* (X, d) *be a non-empty complete metric space with a contraction mapping* $T : X \to X$*. Then,* T *admits a unique fixed-point* x^* *in* X*(i.e.,* $T(x^*) = x^*$*).*

Lemma 4. *If* K *is a convex subset of a topological vector space* V *and* T *is a continuous mapping of* K *into itself, so that* $T(K)$ *is contained in a compact subset of* K*, then* T *has a fixed point.*

Lemma 5. *Consider a sequence of real-valued continuous functions* $\{f_n\}_{n\in\mathbb{N}}$ *defined on a closed and bounded interval* $[0, b]$*. If this sequence is uniformly bounded and equicontinuous, then there exists a subsequence* (f_{n_k}) *that converges uniformly.*

For further readings and details on fractional calculus, we refer to the books and papers by Kilbas *et al.* [5], Podlubny [7] and Samko *et al.* [8].

3. Existence and Uniqueness

In this section, we prove some existence results and describe the structure of the solution set. We first note that if $y \in C[0, b]$ is an absolutely continuous function on $[0, b]$ satisfying Equation (1), then:

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), {}^cD^\alpha y(s))ds.$$

Theorem 1. *Assume that there exist $K_1, K_2 > 0$, such that:*

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| \leq K_1|x - \bar{x}| + K_2|y - \bar{y}|, \textit{ for each } x, \bar{x}, y, \bar{y} \in \mathbb{R}.$$

If $\frac{b^\alpha K_1}{\alpha} + K_2 < 1$, then there exists unique $y \in C(J, \mathbb{R})$, which satisfy Equation (1).

Proof. Consider $N : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ defined by:

$$N(z(t)) = f\left(t, y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s)ds, z(t)\right), \quad t \in J.$$

Let $z_1, z_2 \in C(J, \mathbb{R})$, then:

$$\begin{aligned} |N(z_1(t)) - N(z_2(t))| &\leq \frac{K_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z_1(s) - z_2(s)|ds + K_2|z_1(t) - z_2(t)| \\ &\leq \left(\frac{K_1 t^\alpha}{\Gamma(\alpha)}\right) ||z_1 - z_2||_\infty + K_2|z_1(t) - z_2(t)| \end{aligned}$$

Hence:

$$\|N(z_1) - N(z_2)\|_\infty \leq \left(\frac{K_1 b^\alpha}{\Gamma(\alpha)} + K_2\right) \|z_1 - z_2\|_\infty, \quad \text{for each } z_1, z_2 \in C(J, \mathbb{R}).$$

From the Banach fixed point theorem, Lemma 3, there exists a unique $z \in C(J, \mathbb{R})$, such that $z = N(z)$. Therefore:

$$z(t) = f\left(t, y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s)ds, z(t)\right), \quad t \in J.$$

Set

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s)ds.$$

This implies that ${}^cD^\alpha y(t) = z(t)$, and hence:

$${}^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t)), \quad \text{for every } t \in J.$$

□

The goal of the second result of this section is to apply the Schauder fixed point theorem, Lemma 4. For the study of this problem, we present some auxiliary lemmas.

Lemma 6. *Let $\epsilon, \epsilon' \in (0,1)$ and $\mathcal{L} : C^1(J,\mathbb{R}) \to C(J,\mathbb{R})$ be an operator defined by:*

$$\mathcal{L}(z) = \epsilon\, {}^cD^\alpha z + \epsilon' z.$$

Then, $\mathcal{L}$ is a linear, continuous and invertible operator.

Proof. It is clear that $\mathcal{L}$ is a linear operator. For every $z \in C^1(J,\mathbb{R})$, we have:

$$\|\mathcal{L}(z)\|_\infty \le \left(\frac{\epsilon b^{-\alpha+1}}{\Gamma(\alpha)} + \epsilon'\right)(\|z'\|_\infty + \|z\|_\infty) \quad \text{for every } z \in C^1(J,\mathbb{R}).$$

Hence, $\mathcal{L}$ is continuous. Now, we show that if $\mathcal{L}(z) = 0$, then $z = 0$, that is $\mathcal{L}$ is injective. Indeed, let $z \in C(J,\mathbb{R})$, such that $\mathcal{L}(z) = 0$, then:

$$\epsilon\, {}^cD^\alpha z(t) + \epsilon' z(t) = 0,\ t \in J \Rightarrow z(0) = 0.$$

From Lemma 1, we have: $z = 0$.

Let $h \in C(J,\mathbb{R})$; we consider the fractional Cauchy problem:

$${}^cD^\alpha z + \frac{\epsilon'}{\epsilon} z = \frac{h(t)}{\epsilon}, \quad z(0) = h(0).$$

Again from Lemma 1, we obtain that:

$$\epsilon z(t) = \epsilon h(0) E_\alpha\left(-\frac{\epsilon'}{\epsilon} t^\alpha\right) + \alpha \int_0^t (t-s)^{\alpha-1} E'_\alpha\left(-\frac{\epsilon'}{\epsilon}(t-s)^\alpha\right) h(s)ds.$$

This implies that $\mathcal{L}$ is bijective, and from the Banach isomorphism theorem, $\mathcal{L}^{-1}$ is a continuous operator. □

Lemma 7. *Let $F : C^1(J,\mathbb{R}) \to C(J,\mathbb{R})$ be an operator defined by:*

$$F(z) = \epsilon\, {}^cD^\alpha z + \epsilon'(y_0 + I^\alpha f(.,z,{}^cD^\alpha z)).$$

We note

$$[L(z)](t) := y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s)ds, \quad t \in J.$$

Assume:

(H_1) there exist $M_1, M_3 > 0, M_2 \in (0,1)$, such that:

$$|f(t,x,y)| \le M_1|x| + M_2|y| + M_3, \textit{ for each } x, y, \in \mathbb{R},\ t \in J.$$

Then, F is continuous and compact.

Proof. The proof will be given in three steps.

Step 1: F is continuous.

Let $(z_n)_{n\in\mathbb{N}} \subset C^1(J,\mathbb{R})$ be a sequence, such that $(z_n)_{n\in\mathbb{N}} \to z \in C^1(J,\mathbb{R})$.

Then:

$$\left|\int_0^t (t-s)^{\alpha-1}(z_n(s)-z(s))ds\right| \;\leq\; \frac{t^\alpha}{\alpha}\|z_n-z\|_\infty.$$

Thus:

$$\|L(z_n)-L(z)\|_\infty \;\leq\; \frac{b^\alpha}{\Gamma(\alpha+1)}\|z_n-z\|_\infty \to 0,\ n\to\infty.$$

Furthermore:

$$\left|\int_0^t (t-s)^{-\alpha}(z_n'(s)-z'(s))ds\right| \;\leq\; \frac{t^{1-\alpha}}{1-\alpha}\|z_n'-z'\|_\infty.$$

and hence:

$$\|\bar{L}(z_n)-\bar{L}(z)\|_\infty \;\leq\; \frac{b^{1-\alpha}}{\Gamma(2-\alpha)}\|z_n'-z'\|_\infty \to 0,\ n\to\infty,$$

where:

$$\bar{L}(z) := {}^cD^\alpha z(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}z'(s), \quad t\in J.$$

Hence:

$$\begin{aligned}\|F(z_n)-F(z)\|_\infty \;\leq\; & \frac{\epsilon' b^\alpha}{\Gamma(\alpha+1)}\|f(\cdot,z_n,\bar{L}(z_n))-f(\cdot,z,\bar{L}(z))\|_\infty \\ & +\epsilon\|\bar{L}(z_n)-\bar{L}(z)\|_\infty \to 0,\ \text{as } n\to\infty.\end{aligned}$$

Step 2: F sends bounded sets in $C^1(J,\mathbb{R})$ into bounded sets in $C(J,\mathbb{R})$.
For each $t\in J$ and $z\in\overline{B}(0,r)$, we have:

$$\begin{aligned}\|(Fz)(t)\| \;\leq\; & \tfrac{\epsilon b^{1-\alpha}}{\Gamma(2-\alpha)}r+\epsilon'\left(|y_0|+\int_0^t \tfrac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}|f(s,z(s),(\bar{L}z)(s))|ds\right) \\ \leq\; & \tfrac{\epsilon b^{1-\alpha}}{\Gamma(2-\alpha)}r+\epsilon'\left(|y_0|+\int_0^t \tfrac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}(M_1|z(s)|+M_2|\bar{L}(z)(s)|+M_3)ds\right) \\ \leq\; & \tfrac{\epsilon b^{1-\alpha}}{\Gamma(2-\alpha)}r+\epsilon'|y_0|+\tfrac{\epsilon'}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\tfrac{M_2 r b^{1-\alpha}}{\Gamma(2-\alpha)}+M_1 r+M_3\right)ds.\end{aligned}$$

Then:

$$\|F(z)\|_\infty \leq \frac{\epsilon b^{1-\alpha}r}{\Gamma(2-\alpha)}+\epsilon'|y_0|+\frac{\epsilon' b^\alpha}{\Gamma(\alpha+1)}\left(\frac{M_2 r b^{1-\alpha}}{\Gamma(2-\alpha)}+M_1 r+M_3\right).$$

Step 3: F maps bounded sets into equicontinuous sets.
Let $t_1,t_2\in J$, such that $t_1<t_2$ and $z\in\overline{B}(0,r)$; we have:

$$\begin{aligned}|(Fz)(t_2)-(Fz)(t_1)| \;\leq\; & \tfrac{\epsilon r}{\Gamma(1-\alpha)}\left[\int_{t_1}^{t_2}(t_2-s)^{-\alpha}ds+\int_0^{t_1}((t_2-s)^{-\alpha}-(t_1-s)^{-\alpha})ds\right] \\ & +\tfrac{\epsilon'}{\Gamma(\alpha)}\int_0^{t_2}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}))|f(s,z(s),(\bar{L}z)(s))|ds \\ & +\tfrac{\epsilon'}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1})|f(s,z(s),(\bar{L}z)(s))|ds \\ \leq\; & \tfrac{\epsilon r}{\Gamma(2-\alpha)}(t_2-t_1)^{1-\alpha}+\tfrac{\epsilon' l}{\Gamma(\alpha+1)}(t_1(t_2^{\alpha-1}-t_1^{\alpha-1})+ \\ & \tfrac{\epsilon' l}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha+\tfrac{\epsilon r}{\Gamma(2-\alpha)}(t_1(t_2^{-\alpha}-t_1^{-\alpha}))\end{aligned}$$

where:

$$l = M_1 r+\frac{M_2 r b^{1-\alpha}}{\Gamma(2-\alpha)}+M_3.$$

Then, the right-hand side tends to zero as $t_2-t_1\to 0$. By the Arzelá-Ascoli theorem, Lemma 5, we conclude that $F: C^1(J,\mathbb{R})\to C(J,\mathbb{R})$ is a compact continuous operator. □

Now, we present a result without Lipschitz condition.

Theorem 2. *Suppose that there is a constant $M > 0$, such that for each $\lambda \in [0,1)$ and $z \in C^1(J,\mathbb{R})$ with $\|z\|_1 = M$, we have:*
(H_2) $\epsilon\, {}^cD^\alpha z + \epsilon' z \neq \lambda[\epsilon\, {}^cD^\alpha z + \epsilon'(y_0 + I^\alpha f(t, z(t), {}^cD^\alpha z))]$. *Then, the implicit Problem (1) has at least one solution. Moreover, if $0 < M_2 < 1$, then $Fix(\mathcal{L}^{-1}F)$ is compact.*

Proof. From Lemmas 6 and 7, the operators $\mathcal{L},\ F : C^1(J,\mathbb{R}) \to C(J,\mathbb{R})$ are linear, continuous, invertible and continuous and compact, respectively. Hence, $\mathcal{L}^{-1}F : C^1(J,\mathbb{R}) \to C^1(J,\mathbb{R})$ is a compact continuous operator. Set:

$$U = \{z \in C^1(J,\mathbb{R}) :\ \|z\|_1 < M\}.$$

Assume that there exists $\lambda \in [0,1)$ and $z \in \partial U$, such that

$$z = \lambda \mathcal{L}^{-1}F(z) \Rightarrow \mathcal{L}(z) = \lambda F(z).$$

Hence:

$$\epsilon\, {}^cD^\alpha z(t) + \epsilon' z(t) = \lambda[\epsilon\, {}^cD^\alpha z + \epsilon'(y_0 + I^\alpha f(t, z(t), {}^cD^\alpha z))],$$

which contradicts with (H_2). As a consequence of the nonlinear alternative of Leray–Schauder, we deduce that $\mathcal{L}^{-1}F$ has a fixed point z in U, which is a solution to Problem (1). Now, we show that $Fix(\mathcal{L}^{-1}F)$ is compact. Let $z \in Fix(\mathcal{L}^{-1}F)$, then:

$$z = (\mathcal{L}^{-1}F)(z) \Rightarrow z = y_0 + I^\alpha f(\cdot, z(\cdot), {}^cD^\alpha z) \text{ and } {}^cD^\alpha z = f(\cdot, z(\cdot), {}^cD^\alpha z).$$

Thus,

$$|z(t)| \ \leq\ |y_0| + \frac{M_3 b^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(M_2|{}^cD^\alpha z(s)| + M_1|z(s)|\right) ds$$

and:

$$|{}^cD^\alpha z(t)| \ \leq\ \frac{1}{1-M_2}(M_1|z(t)| + M_3)$$

Let:

$$\mu(t) = \sup_{\tau\in[0,t]} |z(\tau)|$$

Thus:

$$\mu(t) \ \leq\ C_1 + C_2 \int_0^t (t-s)^{\alpha-1}\mu(s)ds.$$

where:

$$C_1 = |y_0| + \frac{M_3 b^\alpha}{\Gamma(\alpha+1)} + \frac{M_3 M_2 b^\alpha}{\Gamma(\alpha+1)(1-M_2)}$$

and:

$$C_2 = \frac{M_2 M_1}{\Gamma(\alpha)(1-M_2)} + \frac{M_1}{\Gamma(\alpha)}.$$

By Lemma 2, there exists $K_3 > 0$, such that:

$$\|z\|_\infty \leq K_3.$$

Therefore,

$$\begin{aligned} \|z\|_1 &= \|(\mathcal{L}^{-1}(F(z))\|_1 \\ &\leq \|\mathcal{L}^{-1}\|\|(Fz)\|_\infty. \end{aligned}$$

Additionally, since $\|z\|_\infty \leq K_3$:

$$\Rightarrow \|^c D^\alpha z\|_\infty \leq \frac{1}{1-M_2}(M_1 K_3 + M_3)$$

and:

$$\|I^\alpha f(.,z(.),\bar{L}(z))\|_\infty \leq \frac{b^\alpha}{\Gamma(\alpha+1)}\left(\frac{M_2(M_1K_3+M_3)}{1-M_2} + M_1K_3 + M_3\right)$$

This implies that there exists $K_4 > 0$, such that:

$$\|z\|_1 < K_4.$$

Therefore, $Fix(\mathcal{L}^{-1}F)$ is bounded. Hence, $\overline{\mathcal{L}^{-1}F(Fix(\mathcal{L}^{-1}F))}$ is compact. Now, since $\mathcal{L}^{-1}F$ is a continuous operator, the set $Fix(\mathcal{L}^{-1}F)$ is a closed set. It is clear that

$$Fix(\mathcal{L}^{-1}F) \subset \overline{\mathcal{L}^{-1}F(Fix(\mathcal{L}^{-1}F))},$$

and thus, we conclude that $Fix(\mathcal{L}^{-1}F)$ is compact. $\square$

By the approximation method, we present the existence of the solution for Problem (1):

Theorem 3. *Assume that:*
(H_3) *for each $\epsilon > 0$, there exists $\delta > 0$ and $k = k(\epsilon)$, with:*

$$\underline{\lim}_{\epsilon\to 0^+} k(\epsilon) = 0$$

such that, if $|t_1 - t_2| < \delta,\ |x - \bar{x}| < \delta,\ |y - \bar{y}| < k(\epsilon)$, then we have:

$$|f(t_1, x, y) - f(t_2, \bar{x}, \bar{y})| < k(\epsilon).$$

If the approximation sequence $(z_n)_{n\in\mathbb{N}}$ defined by:

$$z_0(t) = y_0,\ t \in J,$$

$$z_n(t) = f\left(t, y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} z_{n-1}(s)ds, z_{n-1}(t)\right), \quad t \in J,\ \ n \in \mathbb{N}.$$

is bounded, then there exist at least one solution of Problem (1).

Proof. Since $(z_n)_{n\in\mathbb{N}}$ is bounded, we only need to show that $(z_n)_{n\in\mathbb{N}}$ is equicontinuous. We show this fact by induction. We consider $n = 1$; let $t_1, t_2 \in J$. Then, by (H_3) we have, for $\epsilon > 0$; we consider $r > 0$, such that $k(r) \leq \epsilon$ there exists $\delta > 0$, such that for $|t_1^\alpha - t_2^\alpha| < \eta, |t_1 - t_2| < \eta$, $\eta \leq \min\left(\frac{\delta}{2}, \frac{\delta\Gamma(\alpha+1)}{2(1+|y_0|)}, \frac{\delta\Gamma(\alpha+1)}{2M(1+|y_0|)}\right)$, we have:

$$|z_1(t_1) - z_1(t_2)| =$$
$$= \left|f\left(t_1, y_0 - \frac{t_1^\alpha}{\Gamma(\alpha+1)}y_0, y_0\right) - f\left(t_2, y_0 - \frac{t_2^\alpha}{\Gamma(\alpha+1)}y_0, y_0\right)\right| \leq$$
$$\leq k(r) \leq \epsilon.$$

We suppose that $|t_1^\alpha - t_2^\alpha| < \eta, |t_1 - t_2| < \eta$, $\eta \leq \min\left(\frac{\delta}{2}, \frac{\delta\Gamma(\alpha+1)}{2(1+|y_0|)}, \frac{\delta\Gamma(\alpha+1)}{2M(1+|y_0|)}\right)$ for each $p \leq n$. Let $p = n+1$:

$$\begin{aligned}|z_{n+1}(t_1) - z_{n+1}(t_2)| \;&=\; \left|f\left(t_1, y_0 + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}z_n(s)ds, z_n(t_1)\right)\right. \\ &\quad \left. -f\left(t_2, y_0 + \frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}z_n(s)ds, z_n(t_2)\right)\right|.\end{aligned}$$

Set:

$$F_*(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}z_n(s)ds.$$

Without loss of generality, we can assume that $t_1 \geq t_2$:

$$\begin{aligned}|F_*(t_1) - F_*(t_2)| \;&\leq\; \frac{M}{\Gamma(\alpha)}\left|\left(\int_0^{t_1}((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}))ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\right)\right| \\ &\leq\; \frac{M}{\Gamma(\alpha+1)}(t_2^\alpha - t_1^\alpha),\end{aligned}$$

where $M = \|z\|_\infty$.

By (H_3), we have:

$$\begin{aligned}|z_{n+1}(t_1) - z_{n+1}(t_2)| \;&=\; \left|f\left(t_1, y_0 + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}z_n(s)ds, z_n(t_1)\right)\right. \\ &\quad \left. -f\left(t_2, y_0 + \frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}z_n(s)ds, z_n(t_2)\right)\right| \\ &\leq\; k(r) \leq \epsilon.\end{aligned}$$

Hence, by the Arzelá-Ascoli theorem, Lemma 5, we conclude that $(z_n)_{n\in\mathbb{N}}$ is relatively compact in $C(J, \mathbb{R})$. Then, there exists a subsequence $\{z_{n_k}\}$, which converges to some limit z. Since f is continuous, this implies that:

$$z = f\left(t, y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}z(s)ds, z(s)\right).$$

Example 1. *Consider the Cauchy problem:*

$$^cD^\alpha y(t) = f(t, y) + h(^cD^\alpha y(t)),\; t \in J := [0, b], \quad y(0) = y_0 \in \mathbb{R}, \tag{11}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$*,* $h : \mathbb{R} \to \mathbb{R}$ *are continuous functions. Assume that there exist* $K_1, K_2 > 0$*, such that:*

$$|f(t,\bar{x}) - f(t,x)| \leq K_1|\bar{x} - x|, \text{ for all, } \bar{x}, x \in \mathbb{R} \text{ and } t \in J;$$

$$|h(\bar{y}) - h(y)| \leq K_2|\bar{y} - y|, \text{ for all, } \bar{y}, y \in \mathbb{R}.$$

If $\frac{b^\alpha K_1}{\alpha} + K_2 < 1$*, then, from Theorem 1, Problem (11) has a unique solution.*

For instance, take $J = [0,1]$ *and* $\alpha = \frac{1}{2}$ *with* $f(t,y) = t + \frac{1}{6}\cos y$ *and* $h({}^cD^\alpha y(t)) = \frac{1}{2}\sin({}^cD^\alpha y(t))$*,* $t \in J$ *and* $y \in \mathbb{R}$*. Then, Problem (11) becomes:*

$$ {}^cD^\alpha y(t) = t + \frac{1}{6}\cos y + \frac{1}{2}\sin({}^cD^\alpha y(t)),\ t \in J, \quad y(0) = y_0 \in \mathbb{R}, \tag{12}$$

It is clear that the functions f *and* h *in Problem (12) are continuous and:*

$$|f(t,\bar{y}) - f(t,y)| \leq \frac{1}{6}|\bar{y} - y|, \quad |h({}^cD^\alpha \bar{y}(t)) - h({}^cD^\alpha y(t))| \leq \frac{1}{2}|{}^cD^\alpha \bar{y}(t) - {}^cD^\alpha y(t)|,$$

$$\text{for all, } y, \bar{y} \in \mathbb{R}, \ \ t \in J.$$

We have:

$$\frac{b^\alpha K_1}{\alpha} + K_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6} < 1$$

This implies that Problem (12) has a unique solution.

□

4. Conclusions

We have proven an existence result for implicit fractional differential equations. In the future, we will extend the results to other fractional derivatives and boundary value problems.

Acknowledgments

The research has been partially supported by the Ministerio de Economía y Competitividad of Spain under Grants MTM2010-15314 and MTM2013–43014–P, XUNTA de Galicia, the local government, under Grant R2014/002 and co-financed by the European Community fund FEDER.

Author Contributions

Each of the authors, Juan J. Nieto, Abelghani Ouahab and V. Venktesh, contributed to each part of this study equally and read and approved the final version of this manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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