Mathematics **2015**, *3*, 190-257; doi:10.3390/math3020190

OPEN ACCESS

mathematics

ISSN 2227-7390

www.mdpi.com/journal/mathematics

Article

Maxwell–Lorentz Electrodynamics Revisited via the Lagrangian Formalism and Feynman Proper Time Paradigm

Nikolai N. Bogolubov, Jr. [1,2], Anatolij K. Prykarpatski [2,3,]* and Denis Blackmore [4]

[1] Mathematical Institute of RAS, Moscow, Russian Federation;
E-Mail: nikolai_bogolubov@hotmail.com

[2] The Abdus Salam International Centre of Theoretical Physics, Trieste, Italy

[3] The Department of Applied Mathematics at AGH University of Science and Technology, Krakow 30059, Poland

[4] Department of Mathematical Sciences and Center for Applied Mathematics and Statistics, New Jersey Institute of Technology, Newark, NJ 07102-1982 USA; E-Mail: deblac@m.njit.edu

* Author to whom correspondence should be addressed; E-Mail: pryk.anat@ua.fm;
Tel.:+48-605-940-710; Fax:+48-126-173-165.

Academic Editor: Palle Jorgensen

Received: 25 January 2015 / Accepted: 27 March 2015 / Published: 17 April 2015

Abstract: We review new electrodynamics models of interacting charged point particles and related fundamental physical aspects, motivated by the classical A.M. Ampère magnetic and H. Lorentz force laws electromagnetic field expressions. Based on the Feynman proper time paradigm and a recently devised vacuum field theory approach to the Lagrangian and Hamiltonian, the formulations of alternative classical electrodynamics models are analyzed in detail and their Dirac type quantization is suggested. Problems closely related to the radiation reaction force and electron mass inertia are analyzed. The validity of the Abraham-Lorentz electromagnetic electron mass origin hypothesis is argued. The related electromagnetic Dirac–Fock–Podolsky problem and symplectic properties of the Maxwell and Yang–Mills type dynamical systems are analyzed. The crucial importance of the remaining reference systems, with respect to which the dynamics of charged point particles is framed, is explained and emphasized.

Keywords: Ampère's law; Lorentz force; Lorenz constraint; Maxwell electromagnetic equations; Lagrangian and Hamiltonian formalisms; radiation theory; vacuum field theory approach

PACS classifications: 11.10.Ef, 11.15.Kc, 11.10.-z; 11.15.-q, 11.10.Wx, 05.30.-d

1. Classical Relativistic Electrodynamics Models Revisited: Lagrangian and Hamiltonian Analysis

1.1. Introductory Setting

Classical electrodynamics is nowadays considered [1–4] as the most fundamental physical theory, largely owing to the depth of its theoretical foundations and wealth of experimental verifications. In this work we describe a new approach to the classical Maxwell theory, based on a vacuum field medium model, and reanalyze some of the modern classical electrodynamics problems related with the description of a charged point particle dynamics under an external electromagnetic field. We remark here that by *"a charged point particle"* we as usual understand an elementary material charged particle whose internal spatial structure is assumed to be unimportant and is not taken into account, if the contrary is not specified.

We shall use the least action principle to discuss, for various charged point particle dynamics the important physical principles characterizing the related electrodynamical vacuum field structure. In particular, the main classical relativistic relationships characterizing the charge point particle dynamics are obtained using the least action principle within Feynman's approach to the Maxwell electromagnetic equations and the Lorentz type force derivation. Moreover, for each least action principle constructed, we describe the corresponding Hamiltonian pictures and present the related energy conservation laws. Making use of this modified least action approach, a classical hadronic string model is analyzed in detail.

The classical Lorentz force expression with respect to an arbitrary inertial reference frame has engendered many theoretical and experimental controversies, such as the relativistic potential energy impact on the charged point particle mass, the Aharonov–Bohm effect [5–7] and the Abraham–Lorentz–Dirac radiation force [1,2,8] expression. In an effort to explain this, R. Feynman [9] in his "Lectures on Physics" wrote:

"Now we would like to state the law that for quantum mechanics replaces the law $F = q\mathbf{v}\times\mathbf{B}$*. It will be the law that determines the behavior of quantum mechanical particles in an electromagnetic field. Since what happens is determined by amplitudes, the law must tell us how the magnetic influences affect the amplitudes; we are no longer dealing with the acceleration of the particle. The law is the following: the phase of the amplitude to arrive via any trajectory is changed by the presence of a magnetic field by an amount equal to the integral of the vector potential along the whole trajectory times the charge of the particle over Planck's constant. That is,*

$$\text{Magnetic change in phase } = -\frac{q}{\hbar}\int \mathbf{A}\cdot d\mathbf{s} \qquad (1)$$

If there were no magnetic field there would be a certain phase of arrival. If there is a magnetic field anywhere, the phase of the arriving wave is increased by the integral in Eq. (1). Although we will *not need to use it for our present discussion, let us mention that the effect of an electrostatic field is to produce a phase change given by the negative of the time integral of the scalar potential :*

$$\text{Electric change in phase } = -\frac{q}{\hbar}\int \phi \cdot dt$$

These two expressions are correct not only for static fields, but together give the correct result for any electromagnetic field, static or dynamic. This is the law that replaces $F = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$*."*

Consequently, the analysis of the Lorentz force subject to the assumed vacuum field medium is a very interesting and important problem, which was discussed by E. Fermi, G. Schott, R. Feynman, F. Dyson [9–14] and many other physicists. To describe the essence of the electrodynamic problems related to the description of charged point particle dynamics under external electromagnetic field, let us begin with the classical Lorentz force expression

$$dp/dt = F_\xi := \xi E + \xi u \times B \tag{1.1}$$

Here $\xi \in \mathbb{R}$ is a particle electric charge, $u \in T(\mathbb{R}^3)$ is its velocity [15,16] vector, expressed here in the light speed c units,

$$E := -\partial A/\partial t - \nabla\varphi \tag{1.2}$$

is the associated external electric field and

$$B := \nabla \times A \tag{1.3}$$

is the corresponding external magnetic field, acting on the charged particle, expressed in terms of suitable vector $A : M^4 \to \mathbb{E}^3$ and scalar $\varphi : M^4 \to \mathbb{R}$ potentials. Moreover, "∇" is the standard gradient operator with respect to the spatial variable $r \in \mathbb{E}^3$ and $\times$ is the usual vector product in three-dimensional Euclidean space $\mathbb{E}^3 := (\mathbb{R}^3, < \cdot, \cdot >)$, which is naturally endowed with the classical scalar product $< \cdot, \cdot >$. These potentials are defined on the Minkowski space $M^4 \simeq \mathbb{R} \times \mathbb{E}^3$, which models a chosen laboratory reference frame $\mathcal{K}_t$. Now, it is a well-known fact [2,3,9,17] that the force expression (1.1) does not take into account the dual influence of the charged particle on the electromagnetic field and should be considered valid only if the particle charge $\xi \to 0$. This also means that expression (1.1) cannot be used for studying the interaction between two different moving charged point particles, as was pedagogically demonstrated in the classical manuals [2,9]. The classical Lorentz force expression (1.1) is a natural consequence of the interaction of a charged point particle with an ambient electromagnetic field, and its derivation based on the general principles of dynamics was analyzed in detail by R. Feynman and F. Dyson [9–11].

Taking this into account, it is natural to reanalyze this problem from the classical perspective, using only the Maxwell-Faraday wave theory aspect to specifying the corresponding vacuum field medium. Other questionable inferences from the classical electrodynamics theory, which strongly motivated the analysis in this work, are related both to an alternative interpretation of the well-known *Lorenz condition*, imposed on the four-vector of electromagnetic observable potentials $(\varphi, A) : M^4 \to T^*(M^4)$ and the classical Lagrangian formulation [2] of charged particle dynamics under external electromagnetic field. The Lagrangian is strongly dependent on an important Einsteinian notion of the rest reference frame $\mathcal{K}_\tau$

and the related least action principle, so before explaining it in more detail, we first analyze the classical Maxwell electromagnetic theory from a strictly dynamical point of view.

Let us consider with respect to a laboratory reference frame $\mathcal{K}_t$ the additional *Lorenz condition*

$$\partial\varphi/\partial t + <\nabla, A> = 0 \tag{1.4}$$

a priori assuming the Lorentz invariant wave scalar field equation

$$\partial^2\varphi/\partial t^2 - \nabla^2\varphi = \rho \tag{1.5}$$

and the charge continuity equation

$$\partial\rho/\partial t + <\nabla, J> = 0 \tag{1.6}$$

where $\rho : M^4 \to \mathbb{R}$ and $J : M^4 \to \mathbb{E}^3$ are, respectively, the charge and current densities of the ambient matter. Then one can show [18,19] that the Lorentz invariant wave equation

$$\partial^2 A/\partial t^2 - \nabla^2 A = J \tag{1.7}$$

and the classical electromagnetic Maxwell field equations [1–3,9,17]

$$\begin{aligned} \nabla \times E + \partial B/\partial t = 0, \quad & <\nabla, E> = \rho \\ \nabla \times B - \partial E/\partial t = J, \quad & <\nabla, B> = 0 \end{aligned} \tag{1.8}$$

hold for all $(t, r) \in M^4$ with respect to the chosen laboratory reference frame $\mathcal{K}_t$.

Notice here that, inversely, Maxwell's equations (1.8) do not directly reduce, via definitions (1.2) and (1.3), to the wave field Equations (1.5) and (1.7) without the Lorenz condition (1.4). This fact is very important and suggests that when it comes to a choice of governing equations, it may be reasonable to replace Maxwell's equation (1.8) with the Lorenz condition (1.4) and the charge continuity Equation (1.6). To make the equivalence statement above more transparent, we formulate it as the following proposition.

Proposition 1.1. *The Lorentz invariant wave equation (1.5) together with the Lorenz condition (1.4) for the observable potentials* $(\varphi, A) : M^4 \to T^*(M^4)$ *and the charge continuity relationship (1.6) are completely equivalent to the Maxwell field equation (1.8).*

Proof. Substituting (1.4), into (1.5), one easily obtains

$$\partial^2\varphi/\partial t^2 = - <\nabla, \partial A/\partial t> = <\nabla, \nabla\varphi> + \rho \tag{1.9}$$

which implies the gradient expression

$$<\nabla, -\partial A/\partial t - \nabla\varphi> = \rho \tag{1.10}$$

Taking into account the electric field definition (1.2), expression (1.10) reduces to

$$<\nabla, E> = \rho \tag{1.11}$$

which is the second of the first pair of Maxwell's equations (1.8).

Now upon applying $\nabla\times$ to definition (1.2), we find using the definition (1.3), that

$$\nabla \times E + \partial B/\partial t = 0 \tag{1.12}$$

which is the first pair of the Maxwell equations (1.8). Upon differentiating Equation (1.5) with respect to the time $t \in \mathbb{R}$ and taking into account the charge continuity Equation (1.6), one finds that

$$< \nabla, \partial^2 A/\partial t^2 - \nabla^2 A - J >= 0 \tag{1.13}$$

This is equivalent to the wave Equation (1.7) if one observes that the current vector $J : M^4 \to \mathbb{E}^3$ is defined by means of the charge continuity equation (1.6) up to a vector function $\nabla \times S : M^4 \to \mathbb{E}^3$. Now applying operation $\nabla\times$ to the definition (1.3), it follows from the wave Equation (1.7) that

$$\begin{aligned}\nabla \times B &= \nabla \times (\nabla \times A) = \nabla < \nabla, A > -\nabla^2 A = \\ &= -\nabla(\partial\varphi/\partial t) - \partial^2 A/\partial t^2 + (\partial^2 A/\partial t^2 - \nabla^2 A) = \\ &= \frac{\partial}{\partial t}(-\nabla\varphi - \partial A/\partial t) + J = \partial E/\partial t + J,\end{aligned} \tag{1.14}$$

which leads directly to

$$\nabla \times B = \partial E/\partial t + J,$$

which is the first of the second pair of the Maxwell equations (1.8). The final "*no magnetic charge*"equation

$$< \nabla, B >=< \nabla, \nabla \times A >= 0,$$

in (1.8) follows directly from the elementary identity $< \nabla, \nabla\times >= 0$, thereby completing the proof. □

This proposition allows to consider the observable potential functions $(\varphi, A) : M^4 \to T^*(M^4)$ as fundamental ingredients of the ambient *vacuum field medium*, by means of which we can try to describe the related physical behavior of charged point particles imbedded in space-time M^4. The following observation provides strong support for this approach:

Observation. *The Lorenz condition (1.4) actually means the scalar potential field $\varphi : M^4 \to \mathbb{R}$ continuity relationship, whose origin lies in some new field conservation law, characterizing the deep intrinsic structure of the vacuum field medium.*

To make this observation more transparent and precise, let us recall the definition [2,3,9,17] of the electric current $J : M^4 \to \mathbb{E}^3$ in the dynamical form

$$J := \rho u, \tag{1.15}$$

where the vector $u \in T(\mathbb{R}^3)$ is the corresponding charge velocity. Thus, the following continuity relationship

$$\partial\rho/\partial t + < \nabla, \rho u >= 0 \tag{1.16}$$

holds, which can easily be rewritten [20] as the integral conservation law

$$\frac{d}{dt}\int_{\Omega_t} \rho(t, r) d^3 r = 0 \tag{1.17}$$

for a charge inside any bounded domain $\Omega_t \subset \mathbb{E}^3$, moving in the space-time M^4 with respect to the natural evolution equation

$$dr\ /dt := u. \tag{1.18}$$

The above reasoning leads to the following result.

Proposition 1.2. *The Lorenz condition (1.4) is equivalent to the integral conservation law*

$$\frac{d}{dt}\int_{\Omega_t}\varphi(t,r)d^3r = 0, \tag{1.19}$$

where $\Omega_t \subset \mathbb{E}^3$ *is any bounded domain, moving with respect to the charged point particle* ξ *evolution equation*

$$dr/dt = u(t,r) \tag{1.20}$$

which represents the velocity vector of related local potential field changes propagating in the Minkowski space-time M^4. *Moreover, for a particle with the distributed charge density* $\rho : M^4 \to \mathbb{R}$, *the following Umov type local energy conservation relationship*

$$\frac{d}{dt}\int_{\Omega_t}\frac{\rho(t,r)\varphi(t,r)}{(1-|u(t,r)|^2)^{1/2}}d^3r = 0 \tag{1.21}$$

holds for any $t \in \mathbb{R}$.

Proof. Consider first the corresponding solutions to potential field Equation (1.5), taking into account condition (1.15). Owing to the standard results from [2,9], one finds that

$$A = \varphi u \tag{1.22}$$

which gives rise to the following form of the Lorenz condition (1.4):

$$\partial\varphi/\partial t + <\nabla, \varphi u> = 0 \tag{1.23}$$

This obviously can be rewritten [20] as the integral conservation law (1.19), so the expression (1.19) is verified.

To prove the local energy conservation relationship (1.21) it is necessary to combine the conditions (1.16), (1.23) and find that

$$\partial(\rho\varphi)/\partial t + <u, \nabla(\rho\varphi)> + 2\rho\varphi <\nabla, u> = 0 \tag{1.24}$$

Recall now that the infinitesimal volume transformation $d^3r = \chi(t,r)d^3r_0$, where the Jacobian $\chi(t,r) := |\partial r(t;r_0)/\partial r_0|$ of the corresponding transformation $r : \Omega_{t_0} \to \Omega_t$, induced by the Cauchy problem for the differential relationship (1.20) for any $t \in \mathbb{R}$, satisfies the evolution equation

$$d\chi/dt = <\nabla, u> \chi \tag{1.25}$$

easily following from (1.20). Then applying the operator $\int_{\Omega_{t_0}}(...)\chi^2 d^3r_0$ to the equality (1.24) , one obtains

$$\begin{aligned} 0 = \int_{\Omega_{t_0}}\tfrac{d}{dt}(\rho\varphi\chi^2)d^3r_0 = \tfrac{d}{dt}\int_{\Omega_{t_0}}(\rho\varphi\chi)\chi d^3r_0 = \\ = \tfrac{d}{dt}\int_{\Omega_t}(\rho\varphi\chi\ )d^3r\ := \ \tfrac{d}{dt}\mathcal{E}(\xi;\Omega_t) \end{aligned} \tag{1.26}$$

Here we denoted the conserved charge $\xi := \int_{\Omega_t} \rho(t,r)d^3r$ and the local energy conservation quantity $\mathcal{E}(\xi;\Omega_t) := \int_{\Omega_t}(\rho\varphi\chi\ )d^3r$. The latter quantity can be simplified, owing to the infinitesimal Lorentz invariance four-volume measure relationship $d^3r(t,r_0) \wedge dt = d^3r_0 \wedge dt_0$. Here the variables $(t,r) \in \mathbb{R}_t \times \Omega_t \subset M^4$ are, in the present context, taken with respect to the moving reference frame $\mathcal{K}_t$ related to the infinitesimal charge quantity $d\xi(t,r) := \rho(t,r)d^3r$. The variables $(t_0,r_0) \in \mathbb{R}_{t_0} \times \Omega_{t_0} \subset M^4$ are taken with respect to the laboratory reference frame $\mathcal{K}_{t_0}$, related to the infinitesimal charge quantity $d\xi(t_0,r_0) = \rho(t_0,r_0)d^3r_0$, satisfying the charge conservation invariance $d\xi(t,r) = d\xi(t_0,r_0)$. The above mentioned infinitesimal Lorentz invariance relationships make it possible to calculate the local energy conservation quantity $\mathcal{E}(\xi;\Omega_0)$ as

$$\begin{aligned}\mathcal{E}(\xi;\Omega_0) &= \int_{\Omega_t}(\rho\varphi\chi\ )d^3r = \int_{\Omega_t}(\rho\varphi\frac{d^3r}{d^3r_0}\ )d^3r = \\ &= \int_{\Omega_t}(\rho\varphi\frac{d^3r\wedge dt}{d^3r_0\wedge dt}\ )d^3r = \int_{\Omega_t}(\rho\varphi\frac{d^3r_0\wedge dt_0}{d^3r_0\wedge dt}\ )d^3r = \\ &= \int_{\Omega_t}(\rho\varphi\frac{dt_0}{dt})\ d^3r = \int_{\Omega_t}\frac{\rho\varphi\ d^3r}{(1-|u|^2)^{1/2}},\end{aligned} \tag{1.27}$$

where we took into account that $dt = dt_0(1-|u|^2)^{1/2}$. Thus, owing to (1.26) and (1.27), the local energy conservation relationship (1.21) is satisfied, proving the proposition. $\square$

The local energy conservation quantity (1.27) can be rewritten as

$$\mathcal{E}(\xi;\Omega_t) = \int_{\Omega_t}\frac{d\xi(t,r)\varphi(t,r)}{(1-|u|^2)^{1/2}} := \int_{\Omega_t} d\mathcal{E}(t,r) \tag{1.28}$$

where $d\mathcal{E}(t,r) = d\xi(t,r)\varphi(t,r)(1-|u|^2)^{-1/2}$ is the distributed in vacuum electromagnetic field energy density, related to the electric charge $d\xi(t,r)$ located at the point $(t,r) \in M^4$.

The above proposition suggests a physically motivated interpretation of electrodynamic phenomena in terms of what should naturally be called *the vacuum potential field*, which determines the observable interactions between charged point particles. More precisely, we can *a priori* endow the ambient vacuum medium with a scalar potential energy field density function $W := \xi\varphi : M^4 \to \mathbb{R}$, where $\xi \in \mathbb{R}_+$ is the value of an elementary charge quantity, satisfying the governing *vacuum field equations*

$$\begin{gathered}\partial^2W/\partial t^2 - \nabla^2 W = \rho\xi, \quad \partial W/\partial t + <\nabla,\hat{A}>= 0, \\ \partial^2\hat{A}/\partial t^2 - \nabla^2\hat{A} = \xi\rho v, \quad \hat{A} = Wv\end{gathered} \tag{1.29}$$

taking into account the external charged sources, which possess a virtual capability for disturbing the vacuum field medium. Moreover, this vacuum potential field function $W : M^4 \to \mathbb{R}$ allows the natural potential energy interpretation, whose origin should be assigned not only to the charged interacting medium, but also to any other medium possessing interaction capabilities, including for instance, material particles, interacting due to gravity.

This leads naturally to the next important step, consisting in deriving the equation governing the corresponding potential field $\bar{W} : M^4 \to \mathbb{R}$, assigned to a charged point particle moving in the vacuum field medium with velocity $u \in T(\mathbb{R}^3)$ and located at point $r(t) = R(t) \in \mathbb{E}^3$ at time $t \in \mathbb{R}$. As can be readily shown [18,19,21], the corresponding evolution equation governing the related potential field function $\bar{W} : M^4 \to \mathbb{R}$, assigned to a moving in the space $\mathbb{E}^3$ charged particle ξ under the stationary distributed field sources, has the form

$$\frac{d}{dt}(-\bar{W}u) = -\nabla \bar{W} \tag{1.30}$$

where $\bar{W} := W(t, r)|_{r\to R(t)}$, $u(t) := dR(t)/dt$ at point particle location $(t, R(t)) \in M^4$.

Similarly, if there are two interacting charged point particles, located at points $r(t) = R(t)$ and $r_f(t) = R_f(t) \in \mathbb{E}^3$ at time $t \in \mathbb{R}$ and moving, respectively, with velocities $u := dR(t)/dt$ and $u_f := dR_f(t)/dt$, the corresponding potential field function $\bar{W}' : M^4 \to \mathbb{R}$, considered with respect to the reference frame $\mathcal{K}'$ specified by Euclidean coordinates $(t', r - r_f) \in \mathbb{E}^4$ and moving with the velocity $u_f \in T(\mathbb{R}^3)$ subject to the laboratory reference frame $\mathcal{K}_t$, should satisfy [19,22] with respect to the reference frame $\mathcal{K}'$ the dynamical equality

$$\frac{d}{dt'}[-\bar{W}'(u' - u'_f)] = -\nabla \bar{W}' \tag{1.31}$$

where we have denoted the velocity vectors $u' := dr/dt'$, $u'_f := dr_f/dt' \in T(\mathbb{R}^3)$. The latter comes with respect to the laboratory reference frame $\mathcal{K}_t$ about the dynamical equality

$$\frac{d}{dt}[-\bar{W}\ (u - u_f)] = -\nabla \bar{W}(1 - |u_f|^2) \tag{1.32}$$

The dynamical potential field Equations (1.30) and (1.31) appear to have important properties and can be used as means for representing classical electrodynamic phenomena. Consequently, we shall proceed to investigate their physical properties in more detail and compare them with classical results for Lorentz type forces arising in the electrodynamics of a moving charged point particles in an external electromagnetic field.

In this investigation, we were in part inspired by works [23–27] and studies [28,29] devoted to solving the classical problem of reconciling gravitational and electrodynamic charges in the Mach–Einstein ether paradigm. First, we shall revisit the classical Mach–Einstein relativistic electrodynamics of a moving charged point particle, and second, we study the resulting electrodynamic theories associated with our vacuum potential field dynamical Equations (1.30) and (1.31), making use of the fundamental Lagrangian and Hamiltonian formalisms which were devised in [18,30].

1.2. Classical Relativistic Electrodynamics Revisited

The classical relativistic electrodynamics of a freely moving charged point particle in the Minkowski space-time $M^4 := \mathbb{R} \times \mathbb{E}^3$ is based on the Lagrangian approach [2,3,9,17] with Lagrangian function

$$\mathcal{L} := -m_0(1 - |u|^2)^{1/2} \tag{1.33}$$

where $m_0 \in \mathbb{R}_+$ is the so-called particle rest mass and $u \in T(\mathbb{R}^3)$ is its spatial velocity in the Euclidean space $\mathbb{E}^3$, expressed here and in the sequel in light speed units (with light speed c). The least action principle in the form

$$\delta S = 0, \quad S := -m_0 \int_{t_1}^{t_2} (1 - |u|^2)^{1/2} dt \tag{1.34}$$

for any fixed temporal interval $[t_1, t_2] \subset \mathbb{R}$ gives rise to the well-known relativistic relationships for the mass of the particle

$$m = m_0(1 - |u|^2)^{-1/2} \tag{1.35}$$

the momentum of the particle

$$p := mu = m_0 u(1 - |u|^2)^{-1/2} \tag{1.36}$$

and the energy of the particle

$$\mathcal{E}_0 = m = m_0(1 - |u|^2)^{-1/2} \tag{1.37}$$

It follows from [2,3], that the origin of the Lagrangian (1.33) can be extracted from the action

$$S := -m_0 \int_{t_1}^{t_2} (1 - |u|^2)^{1/2} dt = -m_0 \int_{\tau_1}^{\tau_2} d\tau \tag{1.38}$$

on the suitable temporal interval $[\tau_1, \tau_2] \subset \mathbb{R}$, where, by definition,

$$d\tau := dt(1 - |u|^2)^{1/2} \tag{1.39}$$

and $\tau \in \mathbb{R}$ is the so-called, proper temporal parameter assigned to a freely moving particle with respect to the rest reference frame $\mathcal{K}_r$. The action (1.38) is rather questionable from the dynamical point of view, since it is physically defined with respect to the rest reference frame $\mathcal{K}_\tau$, giving rise to the constant action $S = -m_0(\tau_2 - \tau_1)$, as the limits of integrations $\tau_1 < \tau_2 \in \mathbb{R}$ were taken to be fixed from the very beginning. Moreover, considering this particle to have charge $\xi \in \mathbb{R}$ and be moving in the Minkowski space-time M^4 under action of an electromagnetic field $(\varphi, A) \in \mathbb{R} \times \mathbb{E}^3$, the corresponding classical (relativistic) action functional is chosen (see [2,3,9,17,18,30]) as follows:

$$S := \int_{\tau_1}^{\tau_2} [-m_0 d\tau + \xi < A, \dot{r} > d\tau - \xi\varphi(1 - |u|^2)^{-1/2} d\tau] \tag{1.40}$$

with respect to the *rest reference system*, parameterized by the Euclidean space-time variables $(\tau, r) \in \mathbb{E}^4$, where we have denoted $\dot{r} := dr/d\tau$ in contrast to the definition $u := dr/dt$. The action (1.40) can be rewritten with respect to the laboratory reference frame $\mathcal{K}_t$ the moving with velocity vector $u \in \mathbb{E}^3$ as

$$S = \int_{t_1}^{t_2} \mathcal{L} dt, \quad \mathcal{L} := -m_0(1 - |u|^2)^{1/2} + \xi < A, u > -\xi\varphi \tag{1.41}$$

on the suitable temporal interval $[t_1, t_2] \subset \mathbb{R}$, which gives rise to the following [2,3,9,17] dynamical expressions

$$P = p + \xi A, \quad p = mu, \quad m = m_0(1 - |u|^2)^{-1/2} \tag{1.42}$$

for the particle momentum and

$$\mathcal{E}_0 = (m_0^2 + |P - \xi A|^2)^{1/2} + \xi\varphi \tag{1.43}$$

for the charged particle ξ energy, where, by definition, $P \in \mathbb{E}^3$ is the common momentum of the particle and the ambient electromagnetic field at a space-time point $(t, r) \in M^4$.

The expression (1.43) for the particle energy $\mathcal{E}_0$ also is open to question, since the potential energy $\xi\varphi$, entering additively, has no affect on the particle mass $m = m_0(1 - |u|^2)^{-1/2}$. This was noticed by L. Brillouin [31], who remarked that the fact that the potential energy has no affect on the particle mass tells us that "... any possibility of existence of a particle mass related with an external potential energy, is completely excluded." Moreover, it is necessary to stress here that the least action principle (1.41), formulated with respect to the laboratory reference frame $\mathcal{K}_t$ time parameter $t \in \mathbb{R}$, appears logically inadequate, for there is a strong physical inconsistency with other time parameters of the Lorentz equivalent reference frames. This was first mentioned by R. Feynman in [32], in his efforts to rewrite the Lorentz force expression with respect to the rest reference frame $\mathcal{K}_\tau$. This and other special relativity theory and electrodynamics problems stimulated many prominent physicists of the past [3,31–34] and present [21,24,26,35–44] to try to develop alternative relativity theories based on completely different space-time and matter structure principles.

There also is another controversial inference from the action expression (1.41). As one can easily show [2,3,9,17], the corresponding dynamical equation for the Lorentz force is given as

$$dp/dt = F_\xi := \xi E + \xi u \times B \tag{1.44}$$

We have defined here, as before,

$$E := -\partial A/\partial t - \nabla\varphi \tag{1.45}$$

for the corresponding electric field and

$$B := \nabla \times A \tag{1.46}$$

for the related magnetic field, acting on the charged point particle ξ. The expression (1.44) means, in particular, that the Lorentz force F depends linearly on the particle velocity vector $u \in T(\mathbb{R}^3)$, and so there is a strong dependence on the reference frame with respect to which the charged particle ξ moves. Attempts to reconcile this and some related controversies [21,31,32,45] forced Einstein to devise his special relativity theory and proceed further to creating his general relativity theory trying to explain the gravity by means of geometrization of space-time and matter in the Universe. Here, we must mention that the classical Lagrangian function $\mathcal{L}$ in (1.41) is written in terms of a combination of terms expressed by means of both the Euclidean rest reference frame variables $(\tau, r) \in \mathbb{E}^4$ and arbitrarily chosen Minkowski reference frame variables $(t, r) \in M^4$.

These problems were recently analyzed using a completely different "no-geometry" approach [19,21,22], where new dynamical equations were derived, which were free of the controversial elements mentioned above. Moreover, this approach avoided the introduction of the well known Lorentz transformations of the space-time reference frames with respect to which the action functional (1.41) is invariant. From this point of view, there are interesting for discussion conclusions in [37,46–48], in which some electrodynamic models, possessing intrinsic Galilean and Poincaré–Lorentz symmetries, are reanalyzed from diverse geometrical points of view. Subject to a possible geometric space-type structure and the related vacuum field background, exerting the decisive influence on the particle dynamics, we need to mention the recent works [49,50] and the closely related classical articles [51,52]. Next, we shall revisit the results obtained in [18,19] from the classical Lagrangian and Hamiltonian formalisms [30] in order to shed new light on the physical underpinnings of the vacuum field theory approach to the study of combined electromagnetic and gravitational effects.

1.3. Ampère's Law in Electrodynamics–The Classical and Modified Lorentz Forces Derivations

Ampère's ingenious classical analysis of magnetically interacting to each other two electric currents in thin conductors, as is well known, was based [2,3,9,17] on the following experimental fact: the force between two electric currents depends on the distance between conductors, their mutual spatial orientation and the quantitative values of currents. Under the assumption of the infinitesimal superposition principle, A.M. Ampère derived a general analytical expression for the force between two infinitesimal elements of currents:

$$df(r, r') = I\, I' \frac{(r - r')}{|r - r'|^2} \alpha(s, s'; n) dl dl' \tag{1.47}$$

where vectors $r, r' \in \mathbb{E}^3$ point at infinitesimal currents $dr = s dl$, $dr' = s' dl'$ with normalized orientation vectors $s, s' \in \mathbb{E}^3$ of two closed conductors l and l' carrying currents $I \in \mathbb{R}$ and $I' \in \mathbb{R}$, respectively, and the unit vector $n := (r - r')/|r - r'|$, the spatial orientations of these infinitesimal elements are fixed, and the function $\alpha : (\mathbb{S}^2)^2 \times \mathbb{S}^2 \to \mathbb{R}$ is a real-valued smooth mapping. Then taking into account the mutual symmetry between the infinitesimal elements of currents dl and dl', belonging respectively to these two electric conductors, the infinitesimal force (1.47) was assumed by Ampère *to locally satisfy Newton's third law*:

$$df(r, r') = -df(r', r) \tag{1.48}$$

with the mapping

$$\alpha(s, s'; n) = \frac{\mu_0}{4\pi}(3k_1 < s, n >< s', n > +k_2 < s, s' >) \tag{1.49}$$

where $< \cdot, \cdot >$ is the natural scalar product in $\mathbb{E}^3$ and $k_1, k_2 \in \mathbb{R}$ are some still undetermined real and dimensionless parameters. The assumption (1.48) is apparently very restrictive and can be considered as reasonable only subject to a stationary system of conductors under regard, when the mutual action at a distance principle [2,9] can be applied. As J.C. Maxwell [53] observed "... we may draw the conclusions, first, that action and reaction are not always equal and opposite, and second, that apparatus may be constructed to generate any amount of work from its own resources. For let two oppositely electrified bodies A and B travel along the line joining them with equal velocities in the direction AB, then if either the potential or the attraction of the bodies at a given time is that due to their position at some former time (as these authors suppose), B, the foremost body, will attract A forwards more than B attracts A backwards. Now let A and B be kept apart by a rigid rod. The combined system, if set in motion in the direction AB, will pull in that direction with a force which may either continually augment the velocity, or may be used as an inexhaustible source of energy."

Based on the fact that there is no possibility to measure the force between two infinitesimal current elements, A.M. Ampère took into account (1.48), (1.49) and calculated the corresponding force exerted by the whole conductor l' on an infinitesimal current element of other conductor under regard:

$$\begin{aligned} dF(r) &:= \oint_{l'} df(r,r') = \\ &= \tfrac{I\,I'\mu_0}{4\pi}\oint_{l'}\tfrac{(r-r')}{|r-r'|^2}(3k_1 < dr, \tfrac{r-r'}{|r-r'|} >< dr', \tfrac{r-r'}{|r-r'|} > + k_2\tfrac{r-r'}{|r-r'|} < dr, dr' >) = \\ &= \tfrac{I\,I'\mu_0}{4\pi}\oint_{l'}\nabla_{r'}\left(\tfrac{1}{|r-r'|}\right)(3k_1 < dr, r-r' >< dr', r-r' > + k_2 < dr, dr' >) \end{aligned} \tag{1.50}$$

which can be equivalently transformed as

$$\begin{aligned} dF(r) &= \tfrac{I\,I'\mu_0}{4\pi}\oint_{l'}\nabla_{r'}\left(\tfrac{1}{|r-r'|}\right)(3k_1 < dr, r-r' >< dr', r-r' > + k_2 < dr, dr' >) = \\ &= \tfrac{I\,I'\mu_0}{4\pi}\oint_{l'}\nabla_{r'}\left(\tfrac{1}{|r-r'|}\right)[k_1(3 < dr, r-r' >< dr', r-r' > - \\ &- < dr, dr' >) + (k_1+k_2) < dr, dr' >] = \\ &= -k_1\tfrac{\mu_0 I}{4\pi} < dr, \nabla\oint_{l'}\left(\tfrac{I'dr'}{|r-r'|}\right) > -(k_1+k_2) < \nabla, \oint_{l'} < dr, \tfrac{I'dr'}{|r-r'|} > \end{aligned} \tag{1.51}$$

owing to the integral identity

$$\oint_{l'}\nabla_{r'}\left(\tfrac{1}{|r-r'|}\right)(3 < dr, r-r' >< dr', r-r' > - < dr, dr' >) =< dr, \nabla > \oint_{l'}\tfrac{dr'}{|r-r'|} \tag{1.52}$$

which can be easily checked by means of integration by parts. Introducing the vector potential

$$A(r) := \frac{\mu_0 I'}{4\pi}\oint_{l'}\frac{dr'}{|r-r'|} \tag{1.53}$$

generated by the conductor l' at point $r \in \mathbb{E}^3$, belonging to the infinitesimal element dl of the conductor l, the resulting infinitesimal force (1.50) gives rise to the following expression:

$$\begin{aligned} dF(r) &= k_1(-I < dr, \nabla)A(r) + I\nabla < dr, A(r) >) - (2k_1+k_2)I\nabla < dr, A(r) >= \\ &= k_1 I dr \times (\nabla \times A(r)) - (2k_1+k_2)I\nabla < dr, A(r) >= \\ &= k_1 J(r)d^3r \times B(r) - (2k_1+k_2)\nabla < Jd^3r, A(r) > \end{aligned} \tag{1.54}$$

where we have taken into account the standard magnetic field definition

$$B(r) := \nabla \times A(r) \tag{1.55}$$

and the corresponding current density relationship

$$J(r)d^3r := Idr \tag{1.56}$$

There are clearly many possible choices for the dimensionless parameters $k_1, k_2 \in \mathbb{R}$. In his analysis, Ampère chose the case when $k_1 = 1, k_2 = -2$ and obtained the now well-known *magnetic force* expression

$$dF(r) = J(r)d^3r \times B(r) \tag{1.57}$$

which easily reduces to the *classical Lorentz expression*

$$df_L(r) = \xi u \times B(r) \tag{1.58}$$

for a force exerted by an external magnetic field on a point particle moving with a constant velocity $u \in T(\mathbb{R}^3)$ with an electric charge $\xi \in \mathbb{R}$.

If one takes *an alternative choice* and sets $k_1 = 1, k_2 = -1$, the expression (1.54) yields *a modified magnetic Lorentz type force*, exerted by an external magnetic field generated by a moving charged particle with a velocity $u' \in T(\mathbb{R}^3)$ on a point particle, endowed with the electric charge $\xi \in \mathbb{R}$ and moving with a velocity $u \in T(\mathbb{R}^3)$:

$$dF_L(r) = J(r)d^3r \times B(r) - \nabla < J(r)d^3r, A(r) > \tag{1.59}$$

which was briefly discussed in [21,54,55] and recently obtained and analyzed in detail from the Lagrangian point of view in [18,19,22,56] in the following equivalent to (1.32) infinitesimal form:

$$\delta f_L(r) = \xi u \times (\nabla \times \xi \delta A(r)) - \xi \nabla < u - u_f, \delta A(r) > \tag{1.60}$$

where $\delta A(r) \in T^*(\mathbb{R}^3)$ denotes the magnetic potential generated by an external charged point particle moving with velocity $u_f \in T(\mathbb{R}^3)$ and exerting the magnetic force $\delta f_L(r)$ on the charged particle located at point $r \in \mathbb{R}^3$ and moving with velocity $u \in T(\mathbb{R}^3)$ with respect to a common reference system $\mathcal{K}_t$. We also need to mention here that the modified Lorentz force expression (1.59) does not take into account the resulting pure electric force as the conductors l and l' are considered to be electrically neutral. Simultaneously, we see that the magnetic potential has a physical significance in its own right [8,21,54,56] and has meaning in a way that extends beyond the calculation of force fields.

To obtain the Lorentz force (1.59) exerted by the external magnetic field generated by *the whole conductor* l' on an infinitesimal current element dl of the conductor l, it is necessary to integrate the expression (1.60) along this conductor loop l' :

$$
\begin{aligned}
&dF_L(r) := \oint_{l'} \delta f_L(r) = J(r)dr \times (\nabla \times \oint_{l'} \delta A(r)) - \nabla < J(r)dr, \oint_{l'} \delta A(r) > + \\
&+\nabla \oint_{l'} < u', \xi\delta A(r) >= J(r)dr \times (\nabla \times A(r)) - \nabla < J(r)dr, \oint_{l'} \delta A(r) > + \\
&+\nabla \oint_{l'} < dr', \xi\delta A(r)/dt >= J(r)dr \times B(r) - \nabla < J(r)dr, \oint_{l'} \delta A(r) > + \\
&+\nabla \int_{S(l')} < dS(l'), \nabla \times \xi\delta A(r)/dt >= J(r)dr \times B(r) - \nabla < J(r)dr, \oint_{l'} \delta A(r) > + \\
&+\nabla \oint_{l'} < dS(l'), \xi\delta B(r)/dt >= J(r)dr \times B(r) - \nabla < J(r)dr, \oint_{l'} \delta A(r) > + \\
&+\xi\nabla(d\Phi(r)/dt) = J(r)dr \times B(r) - \nabla < J(r)dr, A(r) > -\rho(r)d^3r\nabla\overline{W} = \\
&= J(r)dr \times B(r) - \nabla < J(r)dr, \oint_{l'} \delta A(r) > +\rho(r)d^3r(-\nabla\overline{W} - \partial A(r)/\partial t) = \\
&= J(r)dr \times B(r) - \nabla < J(r)dr, \oint_{l'} \delta A(r) > +\rho(r)d^3rE(r)
\end{aligned}
\tag{1.61}
$$

that is the equality

$$
dF(r) = \rho(r)d^3rE(r) + J(r)d^3r \times B(r) - \nabla < J(r)d^3r, A(r) > \tag{1.62}
$$

where, by the electric field $E(r) := -\nabla\overline{W} - \partial A(r)/\partial t$. Now one can easily derive from (1.62) the desired *Lorentz force* expression (6.24), if one takes into account that the whole electric field $E(r) \simeq 0$ owing to the neutrality of the conductors. Concerning the latter it is worth mentioning the following remark of D. Kastler [57] :

"It is true that Ampere's formula is no more admissible today, because it is based on the Newtonian idea of instantaneous action at a distance and it leads notably to the strange consequence that two consecutive elements of the same current should repel each other. Ampere presumed to have demonstrated experimentally this repulsion force, but on this point he was wrong. The modern method, the more rational in order to establish the existence of electrodynamics forces and to determine their value consists in starting from the electrostatic interaction law of Coulomb between two charges (two electrons), whose one of them is at rest in the adopted frame of reference and studying how the interaction forces transform when one goes, thanks to the Lorentz-Einstein relations, to a system of coordinates in which both charges are in motion. One sees the appearance of additional forces proportional to e^2/c^2, e being the electrostatic charge and c the light velocity, hence one sees that not only the spin but also the magnetic moment of the electron are of relativistic origin - as Dirac has shown - but that the whole of electromagnetic forces has such an origin."

Thereby, the above analysis of Ampère's derivation of the magnetic force expression (1.54), as well as its consequences (1.59) and (1.60) make it possible to suppose that the missed modified Lorentz type force expression (1.61) could also be embedded into the classical relativistic Lagrangian and related Hamiltonian formalisms, eventually giving rise to new aspects and interpretations of many "strange" experimental phenomena observed during the past few centuries.

2. Vacuum Field Theory Electrodynamics Equations: Lagrangian Analysis

2.1. A Point Particle Moving in Vacuo—An Alternative Electrodynamic Model

In the vacuum field theory approach to combining electromagnetism and gravity, devised in [18,19], the main vacuum potential field function $\bar{W} : M^4 \to \mathbb{R}$, related to a charged point particle ξ under the external stationary distributed field sources, satisfies the dynamical Equation (1.29), namely

$$\frac{d}{dt}(-\bar{W}u) = -\nabla\bar{W} \tag{2.1}$$

in the case when the external charged particles are at rest, where, as above, $u := dr/dt$ is the particle velocity with respect to some reference system.

To analyze the dynamical Equation (2.1) from the Lagrangian point of view, we write the corresponding action functional as

$$S := -\int_{t_1}^{t_2} \bar{W}dt = -\int_{\tau_1}^{\tau_2} \bar{W}(1 + |\dot{r}|^2)^{1/2}\, d\tau \tag{2.2}$$

expressed with respect to the rest reference frame $\mathcal{K}_\tau$. Fixing the proper temporal parameters $\tau_1 < \tau_2 \in \mathbb{R}$, one finds from the least action principle ($\delta S = 0$) that

$$\begin{aligned} p &:= \partial\mathcal{L}/\partial\dot{r} = -\bar{W}\dot{r}(1 + |\dot{r}|^2)^{-1/2} = -\bar{W}u, \\ \dot{p} &:= dp/d\tau = \partial\mathcal{L}/\partial r = -\nabla\bar{W}(1 + |\dot{r}|^2)^{1/2} \end{aligned} \tag{2.3}$$

where, owing to (2.2), the corresponding Lagrangian function is

$$\mathcal{L} := -\bar{W}(1 + |\dot{r}|^2)^{1/2} \tag{2.4}$$

Recalling now the definition of the particle mass

$$m := -\bar{W} \tag{2.5}$$

and the relationships

$$d\tau = dt(1 - |u|^2)^{1/2}, \; \dot{r}d\tau = udt \tag{2.6}$$

from (2.3) we easily obtain exactly the dynamical Equation (2.1). Moreover, one now readily find that the dynamical mass, defined by means of expression (2.5), is given as

$$m = m_0(1 - |u|^2)^{-1/2}$$

which coincides with the Equation (1.35) of the preceding section. Now one can formulate the following proposition using the above results

Proposition 2.1. *The alternative freely moving point particle electrodynamic model (2.1) allows the least action formulation (2.2) with respect to the "rest" reference frame variables, where the Lagrangian function is given by expression (2.4). Its electrodynamics is completely equivalent to that of a classical relativistic freely moving point particle, described in Subsection 1.2.*

2.2. A Moving Two Charge System in a Vacuum—An Alternative Electrodynamic Model

We proceed now to the case when our charged point particle ξ moves in the space-time with velocity vector $u \in T(\mathbb{R}^3)$ and interacts with another external charged point particle ξ_f, moving with velocity vector $u_f \in T(\mathbb{R}^3)$ with respect to a common reference frame $\mathcal{K}_t$. As was shown in [18,19], the respectively modified dynamical equation for the vacuum potential field function $\bar{W}' : M^4 \to \mathbb{R}$ subject to the moving reference frame $\mathcal{K}'$ is given by equality (1.31), or

$$\frac{d}{dt'}[-\bar{W}'(u' - u'_f)] = -\nabla\bar{W}' \tag{2.7}$$

where, as before, the velocity vectors $u' := dr/dt', u'_f := dr_f/dt' \in T(\mathbb{R}^3)$. Since the external charged particle ξ_f moves in the space-time M^4, it generates the related magnetic field $B := \nabla \times A$, whose magnetic vector potentials $A : M^4 \to \mathbb{E}^3$ and $A' : M^4 \to \mathbb{E}^3$ are defined, owing to the results of [18,19,21], as

$$\xi A := \bar{W}u_f, \quad \xi A' := \bar{W}'u'_f \tag{2.8}$$

Whence, taking into account that the field potential

$$\bar{W} = \bar{W}'(1 - |u_f|^2)^{-1/2} \tag{2.9}$$

and the particle momentum $p' = -\bar{W}'u' = -\bar{W}u$, equality (2.7) becomes equivalent to

$$\frac{d}{dt'}(p' + \xi A') = -\nabla\bar{W}' \tag{2.10}$$

if considered with respect to the moving reference frame $\mathcal{K}'$, or to the Lorentz type force equality

$$\frac{d}{dt}(p + \xi A) = -\nabla\bar{W}(1 - |u_f|^2) \tag{2.11}$$

if considered with respect to the laboratory reference frame $\mathcal{K}_t$, owing to the classical Lorentz invariance relationship (2.9), as the corresponding magnetic vector potential, generated by the external charged point test particle ξ_f with respect to the reference frame $\mathcal{K}'$, is identically equal to zero. To imbed the dynamical Equation (2.11) into the classical Lagrangian formalism, we start from the following action functional, which naturally generalizes the functional (2.2):

$$S := -\int_{\tau_1}^{\tau_2} \bar{W}'(1 + |\dot{r} - \dot{r}_f|^2)^{1/2}\, d\tau \tag{2.12}$$

Here, as before, $\bar{W}'$ is the respectively calculated vacuum field potential $\bar{W}$ subject to the moving reference frame $\mathcal{K}'$, $\dot{r} = u'dt'/d\tau, \dot{r}_f = u'_f dt'/d\tau$, $d\tau = dt'(1 - |u' - u'_f|^2)^{1/2}$, which take into account the relative velocity of the charged point particle ξ subject to the reference frame $\mathcal{K}'$, specified by the Euclidean coordinates $(t', r - r_f) \in \mathbb{R}^4$, and moving simultaneously with velocity vector $u_f \in T(\mathbb{R}^3)$ with respect to the laboratory reference frame $\mathcal{K}_t$, specified by the Minkowski coordinates $(t, r) \in M^4$ and related to those of the reference frame $\mathcal{K}'$ and $\mathcal{K}_\tau$ by means of the following infinitesimal relationships:

$$dt^2 = (dt')^2 + |dr_f|^2, \quad (dt')^2 = d\tau^2 + |dr - dr_f|^2 \tag{2.13}$$

So, it is clear in this case that our charged point particle ξ moves with the velocity vector $u' - u'_f \in T(\mathbb{R}^3)$ with respect to the reference frame $\mathcal{K}'$ in which the external charged particle ξ_f is at rest. Thereby, we have reduced the problem of deriving the charged point particle ξ dynamical equation solved in Subsection 2.1.

Now we can compute the least action variational condition $\delta S = 0$, taking into account that, owing to (2.12), the corresponding Lagrangian function with respect to the rest reference frame $\mathcal{K}_\tau$ is given as

$$\mathcal{L} := -\bar{W}'(1 + |\dot{r} - \dot{r}_f|^2)^{1/2} \tag{2.14}$$

As a result of simple calculations, the generalized momentum of the charged particle ξ equals

$$\begin{aligned} P &:= \partial\mathcal{L}/\partial\dot{r} = -\bar{W}'(\dot{r} - \dot{r}_f)(1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} = \\ &= -\bar{W}'\dot{r}(1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} + \bar{W}'\dot{r}_f(1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} = \\ &= m'u' + \xi A' := p' + \xi A' = p + \xi A \end{aligned} \tag{2.15}$$

where, owing to (2.9) the vectors $p' := -\bar{W}'u' = -\bar{W}u = p \in \mathbb{E}^3$, $A' = \bar{W}'u'_f = \bar{W}u_f = A \in \mathbb{E}^3$, and giving rise to the dynamical equality

$$\frac{d}{d\tau}(p' + \xi A') = -\nabla\bar{W}'(1 + |\dot{r} - \dot{r}_f|^2)^{1/2} \tag{2.16}$$

with respect to the rest reference frame $\mathcal{K}_\tau$. As $dt' = d\tau(1 + |\dot{r} - \dot{r}_f|^2)^{1/2}$ and $(1 + |\dot{r} - \dot{r}_f|^2)^{1/2} = (1 - |u' - u'_f|^2)^{-1/2}$, we obtain from (2.16) the equality

$$\frac{d}{dt'}(p' + \xi A') = -\nabla\bar{W}' \tag{2.17}$$

exactly coinciding with equality (2.10) subject to the moving reference frame $\mathcal{K}'$. Now, making use of expressions (2.13) and (2.9), one can rewrite (2.17) as that with respect to the laboratory reference frame $\mathcal{K}_t$:

$$\begin{aligned} &\tfrac{d}{dt'}(p' + \xi A') = -\nabla\bar{W}' \Rightarrow \\ &\Rightarrow \tfrac{d}{dt'}\left(\tfrac{-\bar{W}u'}{(1+|u'_f|^2)^{1/2}} + \tfrac{\xi\bar{W}u'_f}{(1+|u'_f|^2)^{1/2}}\right) = -\tfrac{\nabla\bar{W}}{(1+|u'_f|^2)^{1/2}} \Rightarrow \\ &\Rightarrow \tfrac{d}{dt'}\left(\tfrac{-\bar{W}dr}{(1+|u'_f|^2)^{1/2}dt'} + \tfrac{\xi\bar{W}dr_f/}{(1+|u'_f|^2)^{1/2}}\right) = -\tfrac{\nabla\bar{W}}{(1+|u'_f|^2)^{1/2}} \Rightarrow \\ &\Rightarrow \tfrac{d}{dt}\left(-\bar{W}\tfrac{dr}{dt} + \xi\bar{W}\tfrac{dr_f}{dt}\right) = -\nabla\bar{W}(1 - |u_f|^2) \end{aligned} \tag{2.18}$$

exactly coinciding with (2.11):

$$\frac{d}{dt}(p + \xi A) = -\nabla\bar{W}(1 - |u_f|^2) \tag{2.19}$$

Remark 2.2. The Equation (2.19) allows to infer the following important and physically reasonable phenomenon: if the test charged point particle velocity $u_f \in T(\mathbb{R}^3)$ tends to the light velocity $c = 1$, the corresponding acceleration force $F_{ac} := -\nabla\bar{W}(1 - |u_f|^2)$ is vanishing. Thereby, the electromagnetic fields, generated by such rapidly moving charged point particles, have no influence on the dynamics of charged objects if observed with respect to an arbitrarily chosen laboratory reference frame $\mathcal{K}_t$.

The latter Equation (2.19) can be rewritten as

$$
\begin{aligned}
dp/dt &= -\nabla\bar{W} - \xi dA/dt + \nabla\bar{W}|u_f|^2 = \\
&= \xi(-\xi^{-1}\nabla\bar{W} \ - \ \partial A/\partial t) - \xi < u, \nabla > A + \xi\nabla < A, u_f >
\end{aligned} \tag{2.20}
$$

or, using the well-known [2] identity

$$
\nabla < a, b >=< a, \nabla > b+ < b, \nabla > a + b \times (\nabla \times a) + a \times (\nabla \times b) \tag{2.21}
$$

where $a, b \in \mathbb{E}^3$ are arbitrary vector functions, in the standard Lorentz type form

$$
dp/dt = \xi E + \xi u \times B - \ \nabla < \xi A, u - u_f > \tag{2.22}
$$

The result (2.22), being before found and written down with respect to the moving reference frame $\mathcal{K}'$ in [18,19,21] and in [58] yet with some inconsistency, makes it possible to formulate the next important proposition.

Proposition 2.3. *The alternative classical relativistic electrodynamic model (2.10) allows the least action formulation based on the action functional (2.12) with respect to the rest reference frame $\mathcal{K}_\tau$, where the Lagrangian function is given by expression (2.14). The resulting Lorentz type force expression equals (2.22) modified by the additional force component $F_c := -\nabla < \xi A, u - u_f >$, important for explanation [5–7] of the well known Aharonov–Bohm effect.*

2.3. A moving charged point particle formulation dual to the classical alternative electrodynamic model

It is easy to see that the action functional (2.12) is written utilizes the classical Galilean transformations. If we now consider the action functional (2.2) for a charged point particle moving with respect to a reference frame $\mathcal{K}_\tau$, and take into account its interaction with an external magnetic field generated by the vector potential $A : M^4 \to \mathbb{E}^3$, it can be naturally generalized as

$$
S := \int_{t_1}^{t_2} (-\bar{W}dt + \xi < A, dr >) = \int_{\tau_1}^{\tau_2} [-\bar{W}(1 + |\dot{r}|^2)^{1/2} \ + \xi < A, \dot{r} >] d\tau \tag{2.23}
$$

where $d\tau = dt(1 - |u|^2)^{1/2}$.

Thus, the corresponding common particle-field momentum takes the form

$$
\begin{aligned}
P &:= \partial\mathcal{L}/\partial\dot{r} = -\bar{W}\dot{r}(1 + |\dot{r}|^2)^{-1/2} + \xi A = \\
&= mu + \xi A := p + \xi A
\end{aligned} \tag{2.24}
$$

and satisfies

$$
\begin{aligned}
\dot{P} &:= dP/d\tau = \partial\mathcal{L}/\partial r = -\nabla\bar{W}(1 + |\dot{r}|^2)^{1/2} \ + \xi\nabla < A, \dot{r} >= \\
&= -\nabla\bar{W}(1 - |u|^2)^{-1/2} + \xi\nabla < A, u > (1 - |u|^2)^{-1/2}
\end{aligned} \tag{2.25}
$$

where

$$\mathcal{L} := -\bar{W}(1+|\dot{r}|^2)^{1/2} \ + \xi < A, \dot{r} > \tag{2.26}$$

is the corresponding Lagrangian function. Since $d\tau = dt(1-|u|^2)^{1/2}$, one easily finds from (2.25) that

$$dP/dt = -\nabla\bar{W} + \xi\nabla < A, u > \tag{2.27}$$

Upon substituting (2.24) into (2.27) and making use of the identity (2.21), we obtain the classical expression for the Lorentz force F acting on the moving charged point particle ξ :

$$dp/dt := F = \xi E + \xi u \times B \tag{2.28}$$

where,

$$E := -\xi^{-1}\nabla\bar{W} - \partial A/\partial t \tag{2.29}$$

is its associated electric field and

$$B := \nabla \times A \tag{2.30}$$

is the corresponding magnetic field. This result can be summarized as follows:

Proposition 2.4. *The classical relativistic Lorentz force (2.28) allows the least action formulation (2.23) with respect to the rest reference frame variables, where the Lagrangian function is given by formula (2.26). Yet its electrodynamics, described by the Lorentz force (2.28), is not equivalent to the classical relativistic moving point particle electrodynamics, described by means of the Lorentz force (1.44), as the inertial mass expression $m = -\bar{W}$ does not coincide with that of (1.35).*

Expressions (2.28) and (2.22) are equal to the gradient term $F_c := -\xi\nabla < A, u - u_f >$, which reconciles the Lorentz forces acting on a charged moving particle ξ with respect to different reference frames. This fact is important for our vacuum field theory approach since it uses no special geometry and makes it possible to analyze both electromagnetic and gravitational fields simultaneously by employing the new definition of the dynamical mass by means of expression (2.5).

2.4. Vacuum Field Theory Electrodynamics Equations: Hamiltonian Analysis

Any Lagrangian theory has an equivalent canonical Hamiltonian representation via the classical Legendre transformation [15,17,59–61]. As we have already formulated our vacuum field theory of a moving charged particle ξ in Lagrangian form, we proceed now to its Hamiltonian analysis making use of the action functionals (2.2), (2.14) and (2.23).

Take, first, the Lagrangian function (2.4) and the momentum expression (2.3) for defining the corresponding Hamiltonian function with respect to the moving reference frame $\mathcal{K}_\tau$:

$$\begin{aligned} H :=& < p, \dot{r} > -\mathcal{L} = \\ &= - < p, p > \bar{W}^{-1}(1-|p|^2/\bar{W}^2)^{-1/2} + \bar{W}(1-|p|^2/\bar{W}^2)^{-1/2} = \\ &= -|p|^2\bar{W}^{-1}(1-|p|^2/\bar{W}^2)^{-1/2} + \bar{W}^2\bar{W}^{-1}(1-|p|^2/\bar{W}^2)^{-1/2} = \\ &= -(\bar{W}^2-|p|^2)(\bar{W}^2-|p|^2)^{-1/2} = -(\bar{W}^2-|p|^2)^{1/2} \end{aligned} \tag{2.31}$$

It is easy to show [15–17,59,61] that the Hamiltonian function (2.31) is a conservation law of the dynamical field Equation (2.1), that is for all $\tau, t \in \mathbb{R}$

$$dH/d\tau = dH/dt = 0 \tag{2.32}$$

which naturally leads to an energy interpretation of H. Thus, we can represent the particle energy as

$$\mathcal{E} = (\bar{W}^2 - |p|^2)^{1/2} \tag{2.33}$$

Accordingly the Hamiltonian equivalent to the vacuum field Equation (2.1) can be written as

$$\begin{aligned} \dot{r} &:= dr/d\tau = \partial H/\partial p = p(\bar{W}^2 - |p|^2)^{-1/2} \\ \dot{p} &:= dp/d\tau = -\partial H/\partial r = \bar{W}\nabla\bar{W}(\bar{W}^2 - |p|^2)^{-1/2} \end{aligned} \tag{2.34}$$

and we have the following result.

Proposition 2.5. *The alternative freely moving point particle electrodynamic model (2.1) allows the canonical Hamiltonian formulation (2.34) with respect to the "rest" reference frame variables, where the Hamiltonian function is given by expression (2.31). Its electrodynamics is completely equivalent to the classical relativistic freely moving point particle electrodynamics described in Subsection 2.1.*

Analogously, one can now use the Lagrangian (2.14) to construct the Hamiltonian function for the dynamical field Equation (4.14), describing the motion of charged particle ξ in an external electromagnetic field in the canonical Hamiltonian form:

$$\dot{r} := dr/d\tau = \partial H/\partial P, \qquad \dot{P} := dP/d\tau = -\partial H/\partial r \tag{2.35}$$

where

$$\begin{aligned} H &:=< P, \dot{r} > -\mathcal{L} = \\ &=< P, \dot{r}_f - P\bar{W}^{\prime,-1}(1 - |P|^2/\bar{W}^{\prime,2})^{-1/2} > +\bar{W}'[\bar{W}^{\prime,2}(\bar{W}^{\prime,2} - |P|^2)^{-1}]^{1/2} = \\ &=< P, \dot{r}_f > +|P|^2(\bar{W}^{\prime,2} - |P|^2)^{-1/2} - \bar{W}^{\prime,2}(\bar{W}^{\prime,2} - |P|^2)^{-1/2} = \\ &= -(\bar{W}^{\prime,2} - |P|^2)(\bar{W}^{\prime,2} - |P|^2)^{-1/2} + < P, \dot{r}_f >= \\ &= -(\bar{W}^{\prime,2} - |P|^2)^{1/2} - \xi < A', P > (\bar{W}^{\prime,2} - |P|^2)^{-1/2} = \\ &= -(\bar{W}^2 - |\xi A|^2 - |P|^2)^{1/2} - \xi < A, P > (\bar{W}^2 - |\xi A|^2 - |P|^2)^{-1/2} \end{aligned} \tag{2.36}$$

with respect to the laboratory reference frame $\mathcal{K}_t$. Here we took into account that, owing to definitions (2.8), (2.9) and (4.17),

$$\begin{aligned} \xi A' &:= \bar{W}'u_f' = \bar{W}'dr_f/dt' = \xi A = \\ &= \bar{W}'\frac{dr_f}{d\tau} \cdot \frac{d\tau}{dt'} = \bar{W}'\dot{r}_f(1 - |u - u_f|)^{1/2} = \\ &= \bar{W}'\dot{r}_f(1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} = \\ &= -\bar{W}'\dot{r}_f(\bar{W}^{\prime,2} - |P|^2)^{1/2}\bar{W}^{\prime,-1} = -\dot{r}_f(\bar{W}^{\prime,2} - |P|^2)^{1/2} \end{aligned} \tag{2.37}$$

and, in particular,

$$\dot{r}_f = -\xi A(\bar{W}'^{,2} - |P|^2)^{-1/2}, \ \ \bar{W} = \bar{W}'(1 - |u_f|^2)^{-1/2} \tag{2.38}$$

where $A : M^4 \to \mathbb{R}^3$ is the related magnetic vector potential generated by the moving external charged particle ξ_f. Equations (2.35) can be rewritten with respect to the laboratory reference frame $\mathcal{K}_t$ in the form

$$dr/dt = u, \quad dp/dt = \xi E + \xi u \times B - \xi \nabla < A, u - u_f > \tag{2.39}$$

which coincides with the result (2.22).

Whence, we see that the Hamiltonian function (2.36) satisfies the energy conservation conditions

$$dH/d\tau = dH/dt' = dH/dt = 0 \tag{2.40}$$

for all τ, t' and $t \in \mathbb{R}$, and that the suitable energy expression is

$$\mathcal{E} = (\bar{W}^2 \ - \xi^2|A|^2 - |P|^2)^{1/2} + \xi < A, P > (\bar{W}^2 \ - \xi^2|A|^2 - |P|^2)^{-1/2} \tag{2.41}$$

where the generalized momentum $P = p + \xi A$. The result (2.41) differs essentially from that obtained in [2], which makes use of the Einsteinian Lagrangian for a moving charged point particle ξ in an external electromagnetic field. Thus, we obtain the following result:

Proposition 2.6. *The alternative classical relativistic electrodynamic model (2.39), which is intrinsically compatible with the classical Maxwell Equations (1.6), allows the Hamiltonian formulation (2.35) with respect to the rest reference frame variables, where the Hamiltonian function is given by expression (2.36).*

The inference above is a natural candidate for experimental validation of our theory. It is strongly motivated by the following remark.

Remark 2.7. It is necessary to mention here that the Lorentz force expression (2.39) uses the particle momentum $p = mu$, where the dynamical "mass" $m := -\bar{W}$ satisfies condition (2.41). This gives rise to the following crucial relationship between the particle energy $\mathcal{E}_0$ and its rest mass $m_0 = -\bar{W}_0$ (for the initial velocity $u \ = 0$ at time $t = 0$) :

$$\mathcal{E}_0 = m_0 \frac{(1 - |\xi A_0/m_0|^2)}{(1 - 2|\xi A_0/m_0|^2)^{1/2}} \tag{2.42}$$

or, equivalently, under the condition $|\xi A_0/m_0|^2 < 1/2$

$$m_0 = \mathcal{E}_0 \left(\frac{1}{2} + |\xi A_0/\mathcal{E}_0|^2 \pm \frac{1}{2}\sqrt{1 - 4|\xi A_0/\mathcal{E}_0|^2} \right)^{1/2} \tag{2.43}$$

where $A_0 := A|_{t=0} \in \mathbb{E}^3$, which differs markedly from the classical expression $m_0 = \mathcal{E}_0 - \xi\varphi_0$, following from (1.43) and does not a priori depend on the external potential energy $\xi\varphi_0$. As the quantity $|\xi A_0/\mathcal{E}_0| \to 0$ if the energy modulus $|\mathcal{E}_0| \to \infty$, the following asymptotic mass values follow from (2.43):

$$\bar{m}_0 \simeq \ \mathcal{E}_0, \qquad m_0^{(\pm)} \simeq \ \pm\sqrt{2}|\xi A_0|. \tag{2.44}$$

The first mass value $\bar{m}_0 \simeq \ \mathcal{E}_0$ looks like the relativistic physics standard, yet the second mass values $m_0^{(\pm)} \simeq \pm\sqrt{2}|\xi A_0|$ give rise to the existence at large enough energies of charged particle excitations of the vacuo with both positive and negative mass values.

To make this difference more clear, we now analyze the Lorentz force (2.28) from the Hamiltonian point of view based on the Lagrangian function (2.26). Thus, we obtain that the corresponding Hamiltonian function

$$\begin{aligned} H :=&< P, \dot{r} > -\mathcal{L} =< P, \dot{r} > +\bar{W}(1+|\dot{r}|^2)^{1/2} - \xi < A, \dot{r} >= \\ &=< P - \xi A, \dot{r} > +\bar{W}(1+|\dot{r}|^2)^{1/2} = \\ &= - < p, p > \bar{W}^{-1}(1-|p|^2/\bar{W}^2)^{-1/2} + \bar{W}(1-|p|^2/\bar{W}^2)^{-1/2} = \\ &= -(\bar{W}^2 - |p|^2)(\bar{W}^2-|p|^2)^{-1/2} = -(\bar{W}^2-|p|^2)^{1/2} \end{aligned} \tag{2.45}$$

Since $p = P - \xi A$, expression (2.45) assumes the final "*no interaction*" [2,3,62,63] form

$$H = -(\bar{W}^2 - |P - \xi A|^2)^{1/2} \tag{2.46}$$

which is conserved with respect to the evolution Equations (2.24) and (2.25), that is

$$dH/d\tau = dH/dt = 0 \tag{2.47}$$

for all $\tau, t \in \mathbb{R}$. These equations are equivalent to the following Hamiltonian system

$$\begin{aligned} \dot{r} &= \partial H/\partial P = (P - \xi A)(\bar{W}^2 - |P-\xi A|^2)^{-1/2}, \\ \dot{P} &= -\partial H/\partial r = (\bar{W}\nabla\bar{W} - \nabla < \xi A, (P-\xi A) >)(\bar{W}^2 - |P-\xi A|^2)^{-1/2} \end{aligned} \tag{2.48}$$

as one can readily check by direct calculations. Actually, the first equation

$$\begin{aligned} \dot{r} &= (P - \xi A)(\bar{W}^2 - |P-\xi A|^2)^{-1/2} = p(\bar{W}^2 - |p|^2)^{-1/2} = \\ &= mu(\bar{W}^2 - |p|^2)^{-1/2} = -\bar{W}u(\bar{W}^2-|p|^2)^{-1/2} = u(1-|u|^2)^{-1/2} \end{aligned} \tag{2.49}$$

holds, owing to the condition $d\tau = dt(1-|u|^2)^{1/2}$ and definitions $p := mu$, $m = -\bar{W}$, postulated from the very beginning. Similarly we obtain that

$$\begin{aligned} \dot{P} &= -\nabla\bar{W}(1-|p|^2/\bar{W}^2)^{-1/2} + \nabla < \xi A, u > (1-|p|^2/\bar{W}^2)^{-1/2} = \\ &= -\nabla\bar{W}(1-|u|^2)^{-1/2} + \nabla < \xi A, u > (1-|u|^2)^{-1/2} \end{aligned} \tag{2.50}$$

coincides with Equation (2.27) in the evolution parameter $t \in \mathbb{R}$. This can be formulated as the next result.

Proposition 2.8. *The dual to the classical relativistic electrodynamic model (2.28) allows the canonical Hamiltonian formulation (2.48) with respect to the rest reference frame variables, where the Hamiltonian function is given by expression (2.46). Moreover, this formulation circumvents the "mass-potential energy" controversy surrounding the classical electrodynamical model (1.41).*

The modified Lorentz force expression (2.28) and the related rest energy relationship are characterized by the following remark.

Remark 2.9. If we make use of the modified relativistic Lorentz force expression (2.28) as an alternative to the classical one of (1.44), the corresponding charged particle ξ energy expression (2.46) also gives rise to a true physically reasonable energy expression (at the velocity $u := 0 \in \mathbb{E}^3$ at the initial time moment $t = 0$); namely, $\mathcal{E}_0 = m_0$ instead of the physically controversial classical expression $\mathcal{E}_0 = m_0 + \xi\varphi_0$, where $\varphi_0 := \varphi|_{t=0}$, corresponding to the case (1.43).

2.5. Quantization of Electrodynamics Models via the Vacuum Field Theory Approach

2.5.1. The Problem Setting

Recently [18,19] we devised a new regular no-geometry approach to deriving the electrodynamics of a moving charged point particle ξ in an external electromagnetic field from first principles. This approach has, in part, reconciled the mass-energy controversy [31] in classical relativistic electrodynamics. Using the vacuum field theory approach initially proposed in [18,19,21], we reanalyzed this problem above both from the Lagrangian and Hamiltonian perspective and derived key expressions for the corresponding energy functions and Lorentz type forces acting on a moving charged point particle ξ.

Since all of our electrodynamics models were represented here in canonical Hamiltonian form, they are well suited to the application of Dirac quantization [64–69] and the corresponding derivation of related Schrödinger type evolution equations. We describe these procedures in this section.

2.5.2. Free Point Particle Electrodynamics Model and Its Quantization

The charged point particle electrodynamics models, discussed in Sections 1.2 and 1.3, were also considered in [18] from the dynamical point of view, where a Dirac quantization of the corresponding conserved energy expressions was attempted. However, from the canonical point of view, the true quantization procedure should be based on the relevant canonical Hamiltonian formulation of the models given in (2.34), (2.35) and (2.48).

In particular, consider a free charged point particle electrodynamics model characterized by (2.34) and having the Hamiltonian equations

$$\begin{aligned} dr/d\tau &:= \partial H/\partial p = -p(\bar{W}^2 - |p|^2)^{-1/2} \\ dp/d\tau &:= -\partial H/\partial r = -\bar{W}\nabla\bar{W}(\bar{W}^2 - |p|^2)^{-1/2} \end{aligned} \tag{2.51}$$

where $\bar{W} : M^4 \to \mathbb{R}$ defined in the preceding sections is the corresponding vacuum field potential characterizing the medium field structure, $(r,p) \in T^*(\mathbb{R}^3) \simeq \mathbb{E}^3 \times \mathbb{E}^3$ are the standard canonical coordinate-momentum variables on the cotangent space $T^*(\mathbb{R}^3)$, $\tau \in \mathbb{R}$, is the proper rest reference frame $\mathcal{K}_\tau$ time parameter of the moving particle, and $H : T^*(\mathbb{R}^3) \to \mathbb{R}$ is the Hamiltonian function

$$H := -(\bar{W}^2 - |p|^2)^{1/2} \tag{2.52}$$

expressed here and hereafter in light speed units. The rest reference frame $\mathcal{K}_\tau$, parameterized by variables $(\tau, r) \in \mathbb{E}^4$, is related to any other reference frame $\mathcal{K}_t$ in which our charged point particle ξ moves with velocity vector $u \in \mathbb{E}^3$. The frame $\mathcal{K}_t$ is parameterized by variables $(t, r) \in M^4$ via the Euclidean infinitesimal relationship

$$dt^2 = d\tau^2 + |dr|^2 \tag{2.53}$$

which is equivalent to the Minkowskian infinitesimal relationship

$$d\tau^2 = dt^2 - |dr|^2. \tag{2.54}$$

The Hamiltonian function (2.52) clearly satisfies the energy conservation conditions

$$dH/d\tau = dH/dt = 0 \tag{2.55}$$

for all $t, \tau \in \mathbb{R}$. This means that the energy

$$\mathcal{E} = (\bar{W}^2 - |p|^2)^{1/2} \tag{2.56}$$

can be treated by means of the Dirac quantization scheme [64,67] to obtain, as $\hbar \to 0$, (or the light speed $c \to \infty$) the governing Schrödinger type dynamical equation. To do this following the approach in [18,67], we need to make canonical operator replacements $\mathcal{E} \to \hat{\mathcal{E}} := -\frac{\hbar}{i}\frac{\partial}{\partial\tau}$, $p \to \hat{p} := \frac{\hbar}{i}\nabla$, as $\hbar \to 0$, in the following energy expression:

$$\mathcal{E}^2 := (\hat{\mathcal{E}}\psi, \hat{\mathcal{E}}\psi) = (\psi, \hat{\mathcal{E}}^2\psi) = (\psi, \hat{H}^+\hat{H}\psi) \tag{2.57}$$

where $(\cdot,\cdot)$ is the standard L_2 - inner product. It follows from (2.56) that

$$\hat{\mathcal{E}}^2 = \bar{W}^2 - |p|^2 = \hat{H}^+\hat{H} \tag{2.58}$$

is a suitable operator factorization in the Hilbert space $\mathcal{H} := L_2(\mathbb{R}^3;\mathbb{C})$ and $\psi \in \mathcal{H}$ is the corresponding normalized quantum vector state. Since the following elementary identity

$$\bar{W}^2 - |p|^2 = \bar{W}(1 - \bar{W}^{-1}|p|^2\bar{W}^{-1})^{1/2}(1 - \bar{W}^{-1}|p|^2\bar{W}^{-1})^{1/2}\bar{W} \tag{2.59}$$

holds, we can use (2.58) and (2.59) to define the operator

$$\hat{H} := (1 - \bar{W}^{-1}|p|^2\bar{W}^{-1})^{1/2}\bar{W} \tag{2.60}$$

Upon calculating the operator expression (2.60) as $\hbar \to 0$ up to operator accuracy $O(\hbar^4)$, it is easy see that

$$\hat{H} = \frac{|p|^2}{2m(u)} + \bar{W} := -\frac{\hbar^2}{2m(u)}\nabla^2 + \bar{W}, \tag{2.61}$$

where we have taken into account the dynamical mass definition $m(u) := -\bar{W}$ (in the light speed units). Consequently, using (2.57) and (2.61), we obtain up to operator accuracy $O(\hbar^4)$ the following Schrödinger type evolution equation

$$i\hbar\frac{\partial\psi}{\partial\tau} := \hat{\mathcal{E}}\psi = \hat{H}\psi = -\frac{\hbar^2}{2m(u)}\nabla^2\psi + \bar{W}\psi \tag{2.62}$$

with respect to the rest reference frame $\mathcal{K}_\tau$ evolution parameter $\tau \in \mathbb{R}$. For a related evolution parameter $t \in \mathbb{R}$ parameterizing a reference frame $\mathcal{K}_t$, the Equation (2.62) takes the form

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2 m_0}{2m(u)^2}\nabla^2\psi - m_0\psi \tag{2.63}$$

Here we used the fact that it follows from (2.56) that the classical mass relationship

$$m(u) = m_0(1 - |u|^2)^{-1/2} \tag{2.64}$$

holds, where $m_0 \in \mathbb{R}_+$ is the corresponding rest mass of our point particle ξ.

The linear Schrödinger Equation (2.63) for the case $\hbar/c \to 0$ actually coincides with the well-known expression [2,9,67,68] from classical quantum mechanics.

2.5.3. Classical Charged Point Particle Electrodynamics Model and Its Quantization

We start here from the first vacuum field theory reformulation of the classical charged point particle electrodynamics (introduced in Subsection 2.1) and based on the conserved Hamiltonian function (2.46)

$$H := -(\bar{W}^2 - |P - \xi A|^2)^{1/2} \tag{2.65}$$

where $\xi \in \mathbb{R}$ is the particle charge, $(\bar{W}, A) \in \mathbb{R} \times \mathbb{E}^3$ is the corresponding representation of the electromagnetic field potentials and $P \in \mathbb{E}^3$ is the common generalized particle-field momentum

$$P := p + \xi A, \qquad p := mu \tag{2.66}$$

which satisfies the classical Lorentz force equation. Here $m := -\bar{W}$ is the observable dynamical mass of our charged particle, and $u \in \mathbb{E}^3$ is its velocity vector with respect to a chosen reference frame $\mathcal{K}_t$, all expressed in light speed units.

Our electrodynamics based on (2.65) is canonically Hamiltonian, so the Dirac quantization scheme

$$P \to \hat{P} := \frac{\hbar}{i}\nabla, \qquad \mathcal{E} \to \hat{\mathcal{E}} := -\frac{\hbar}{i}\frac{\partial}{\partial\tau} \tag{2.67}$$

should be applied to the energy expression

$$\mathcal{E} := (\bar{W}^2 - |P - \xi A|^2)^{1/2} \tag{2.68}$$

following from the conservation conditions

$$dH/dt = 0 = dH/d\tau \tag{2.69}$$

satisfied for all $\tau, t \in \mathbb{R}$.

Proceeding as above, we can factorize the operator $\hat{\mathcal{E}}^2$ as

$$\begin{aligned} \bar{W}^2 - |\hat{P} - \xi A|^2 = \bar{W}(1 - \bar{W}^{-1}|\hat{P} - \xi A|^2\bar{W})^{1/2} \times \\ \times(1 - \bar{W}^{-1}|\hat{P} - \xi A|^2\bar{W}^{-1})^{1/2}\bar{W} := \hat{H}^+\hat{H} \end{aligned}$$

where (as $\hbar/c \to 0$, $\hbar c = const$)

$$\hat{H} := \frac{1}{2m(u)}|\frac{\hbar}{i}\nabla - \xi A|^2 + \bar{W} \tag{2.70}$$

up to operator accuracy $O(\hbar^4)$. Hence, the related Schrödinger evolution equation in the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^3; \mathbb{C})$ is

$$i\hbar\frac{\partial\psi}{\partial\tau} := \hat{\mathcal{E}}\psi = \hat{H}\psi = \frac{1}{2m(u)}|\frac{\hbar}{i}\nabla - \xi A|^2\psi + \bar{W}\psi \tag{2.71}$$

with respect to the rest reference frame $\mathcal{K}_\tau$ evolution parameter $\tau \in \mathbb{R}$, and corresponding Schrödinger type evolution equation with respect to the evolution parameter $t \in \mathbb{R}$ takes the form

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{m_0}{2m(u)^2}|\frac{\hbar}{i}\nabla - \xi A|^2\psi - m_0\psi \tag{2.72}$$

The Schrödinger Equation (2.71) (as $\hbar/c \to 0$) coincides [67,70] with the classical quantum mechanics version.

2.5.4. Modified Charged Point Particle Electrodynamics Model and Its Quantization

From the canonical viewpoint, we now turn to the true quantization procedure for the electrodynamics model, characterized by (2.16) and having the Hamiltonian function (2.36)

$$H := -(\bar{W}^2 - \xi^2|A|^2 - |P|^2)^{1/2} - \xi < A, P > (\bar{W}^2 - \xi^2|A|^2 - |P|^2)^{-1/2} \tag{2.73}$$

Accordingly the suitable energy function is

$$\mathcal{E} := (\bar{W}^2 - \xi^2|A|^2 - |P|^2)^{1/2} + \xi < A, P > (\bar{W}^2 - \xi^2|A|^2 - |P|^2)^{-1/2} \tag{2.74}$$

where, as before,

$$P := p + \xi A, \quad p := mu, \;\; m := -\bar{W}, \tag{2.75}$$

is a conserved quantity for (2.16), which we shall canonically quantize via the Dirac procedure (2.67). Toward this end, let us consider the quantum condition

$$\mathcal{E}^2 := (\hat{\mathcal{E}}\psi, \hat{\mathcal{E}}\psi) = (\psi, \hat{\mathcal{E}}^2\psi), \quad (\psi, \psi) := 1, \tag{2.76}$$

where, $\hat{\mathcal{E}} := -\frac{\hbar}{i}\frac{\partial}{\partial t}$ and $\psi \in \mathcal{H} = L_2(\mathbb{R}^3; \mathbb{C})$ is a normalized quantum state vector. Making use of the energy function (2.74), one readily computes that

$$\mathcal{E}^2 = \bar{W}^2 - |P - \xi A|^2 + \xi^2 < A, P > (\bar{W}^2 - |P|^2)^{-1} < P, A >,$$

which transforms by the canonical Dirac type quantization $P \to \hat{P} := \frac{\hbar}{i}\nabla$ into the symmetrized operator expression

$$\hat{\mathcal{E}}^2 = \bar{W}^2 - |\hat{P} - \xi A|^2 + \xi^2 < A, \hat{P} > (\bar{W}^2 - |\hat{P}|^2)^{-1} < \hat{P}, A > . \tag{2.77}$$

Factorizing the operator (2.77) in the form $\hat{\mathcal{E}}^2 = \hat{H}^+\hat{H}$, and retaining only terms up to $O(\hbar^4)$ (as $\hbar/c \to 0$), we compute that

$$\hat{H} := \frac{1}{2m(u)}|\frac{\hbar}{i}\nabla - \xi A|^2 - \frac{\xi^2}{2m^3(u)} < A, \frac{\hbar}{i}\nabla >< \frac{\hbar}{i}\nabla, A >, \tag{2.78}$$

where, as before, $m(u) = -\bar{W}$ in light speed units. Thus, owing to (2.76) and (2.78), the resulting Schrödinger evolution equation is

$$i\hbar\frac{\partial\psi}{\partial\tau} := \hat{H}\psi = \frac{1}{2m(u)}|\frac{\hbar}{i}\nabla - \xi A|^2\psi - \frac{\xi^2}{2m^3(u)} < A, \frac{\hbar}{i}\nabla >< \frac{\hbar}{i}\nabla, A > \psi \tag{2.79}$$

with respect to the rest reference frame proper evolution parameter $\tau \in \mathbb{R}$, which can be recast in the form

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial\tau} = -\frac{\hbar^2}{2m(u)}\Delta\psi - \frac{1}{2m(u)} < [\frac{\hbar}{i}\nabla, \xi A]_+ > \psi - \\ -\frac{\xi^2}{2m^3(u)} < A, \frac{\hbar}{i}\nabla >< \frac{\hbar}{i}\nabla, A > \psi \end{aligned} \tag{2.80}$$

where $[\cdot,\cdot]_+$ is the formal anti-commutator of operators. Similarly one also obtains the related Schrödinger equation with respect to the time parameter $t \in \mathbb{R}$, which we shall not dwell upon here. The result (2.79) differs only slightly from the classical Schrödinger evolution Equation (2.71). Simultaneously, its form (2.80) almost completely coincides with the classical ones from [3,67,70] modulo the evolution considered with respect to the rest reference time parameter $\tau \in \mathbb{R}$. This suggests that we must more thoroughly reexamine the physical motivation of the principles underlying the classical electrodynamic models, described by the Hamiltonian functions (2.65) and (2.73) and giving rise to different Lorentz type force expressions. A more deeply considered and extended analysis of this matter is forthcoming in a paper now in preparation.

Remark 2.10. All of the dynamical field equations discussed above are canonical Hamiltonian systems with respect to the corresponding proper rest reference frames $\mathcal{K}_\tau$, parameterized by suitable time parameters $\tau \in \mathbb{R}$. Upon passing to the basic laboratory reference frame $\mathcal{K}_t$ with the time parameter $t \in \mathbb{R}$, naturally the related Hamiltonian structure is lost, giving rise to a new interpretation of the real particle motion. Namely, one that has an absolute sense only with respect to the proper reference system, and otherwise is completely relative with respect to all other reference frames. As for the Hamiltonian expressions (2.31), (2.36) and (2.46), one observes that they all depend strongly on the vacuum potential energy field function $\bar{W} : M^4 \to \mathbb{R}$, thereby avoiding the mass problem of the classical energy expression pointed out by L. Brillouin [31]. It should be noted that the canonical Dirac quantization procedure can be applied only to the corresponding dynamical field systems considered with respect to their proper rest reference frames. Some comments are in order concerning the classical relativity principle. We have obtained our results relying only on the natural notion of the rest reference frame and its suitable Lorentzian parametrization with respect to any other moving reference frame. It seems reasonable then that the true state changes of a moving charged particle ξ are exactly realized only with respect to its proper rest reference system. Then the only remaining question would be about the physical justification of the corresponding relationship between time parameters of moving and rest reference frames.

The relationship between reference frames that we have used throughout is expressed as

$$d\tau = dt(1 - |u|^2)^{1/2} \tag{2.81}$$

where $u := dr/dt \in \mathbb{E}^3$ is the velocity vector with which the rest reference frame $\mathcal{K}_\tau$ moves with respect to another arbitrarily chosen reference frame $\mathcal{K}_t$. Expression (2.81) implies, in particular, that

$$dt^2 - |dr|^2 = d\tau^2 \tag{2.82}$$

which is identical to the classical infinitesimal Lorentz invariant. This is not a coincidence, since all our dynamical vacuum field equations were derived in turn [18,19] from the governing equations of the vacuum potential field function $W : M^4 \to \mathbb{R}$ in the form

$$\partial^2 W/\partial t^2 - \nabla^2 W = \xi\rho,\ \partial W/\partial t + \nabla(vW) = 0,\ \partial\rho/\partial t + \nabla(v\rho) = 0 \tag{2.83}$$

which is *a priori* Lorentz invariant. Here $\rho \in \mathbb{R}$ is the charge density and $v := dr/dt$ the associated local velocity of the vacuum field potential evolution. Consequently, the dynamical infinitesimal Lorentz invariant (2.82) reflects this intrinsic structure of Equation (2.83). If it is rewritten in the following nonstandard Euclidean form:

$$dt^2 = d\tau^2 + |dr|^2 \tag{2.84}$$

it gives rise to a completely different relationship between the reference frames $\mathcal{K}_t$ and $\mathcal{K}_\tau$, namely

$$dt = d\tau(1 + |\dot{r}|^2)^{1/2} \tag{2.85}$$

where $\dot{r} := dr/d\tau$ is the related particle velocity with respect to the rest reference system. Thus, we observe that all our Lagrangian analysis in this section is based on the corresponding functional expressions written in these "Euclidean" space-time coordinates and with respect to which the least action principle was applied. Thus, there are two alternatives - the first is to apply the least action principle to the corresponding Lagrangian functions expressed in the Minkowski space-time variables with respect to an arbitrarily chosen reference frame $\mathcal{K}_t$, and the second is to apply the least action principle to the corresponding Lagrangian functions expressed in Euclidean space-time variables with respect to the rest reference frame $\mathcal{K}_\tau$.

This leads us to a slightly amusing but thought-provoking observation: It follows from our analysis that all of the results of classical special relativity related with the electrodynamics of charged point particles can be obtained (in a one-to-one correspondence) using our new definitions of the dynamical particle mass and the least action principle with respect to the associated Euclidean space-time variables in the rest reference system.

An additional remark concerning the quantization procedure of the proposed electrodynamics models is in order: If the dynamical vacuum field equations are expressed in canonical Hamiltonian form, as we have done in this paper, only straightforward technical details are required to quantize the equations and obtain the corresponding Schrödinger evolution equations in suitable Hilbert spaces of quantum states. There is another striking implication from our approach: the Einsteinian equivalence principle [2,3,9,32,45] is rendered superfluous for our vacuum field theory of electromagnetism and gravity.

Using the canonical Hamiltonian formalism devised here for the alternative charged point particle electrodynamics models, we found it rather easy to treat the Dirac quantization. The results obtained compared favorably with classical quantization, but it must be admitted that we still have not given a compelling physical motivation for our new models. This is something that we plan to revisit in future investigations. Another important aspect of our vacuum field theory no-geometry (geometry-free) approach to combining the electrodynamics with the gravity, is the manner in which it singles out the decisive role of the rest reference frame $\mathcal{K}_\tau$. More precisely, all of our electrodynamics models allow both Lagrangian and Hamiltonian formulations with respect to the rest reference system evolution parameter $\tau \in \mathbb{R}$, which are well suited to the canonical quantization. The physical nature of this fact remains is as yet not quite clear. In fact, as far as we know [2,3,38,39,45], there is no physically reasonable explanation of this decisive role of the rest reference system, except for that given by R. Feynman who argued in [9] that the relativistic expression for the classical Lorentz force (1.44) has physical sense only with respect to the rest reference frame variables $(\tau, r) \in \mathbb{R} \times \mathbb{E}^3$. In future research we plan to analyze the quantization scheme in more detail and begin work on formulating a vacuum quantum field theory of infinitely many particle systems.

3. The Modified Lorentz Force, Radiation Theory and the Abraham–Lorentz Electron Inertia Problem

3.1. Introductory Setting

It is well known that Maxwell equations, which are fundamental in modern physics, allow two main forms of representations: either by means of the electric and magnetic fields or by the electric and magnetic potentials. The latter were mainly considered as a mathematically motivated representation useful for different applications but having no physical significance.

That the situation is not so simple and the evidence that the magnetic potential demonstrates the physical properties was doubtless, the physics community understood when Y. Aharonov and D. Bohm [5] formulated their "paradox" concerning the measurement of magnetic field outside a separated region where it is completely vanishing. Later, similar effects were also revealed in the superconductivity theory of Josephson media. As the existence of any electromagnetic field in the ambient space can be tested only owing to its interaction with electric charges, their dynamical behavior, being of great importance, was deeply studied by M. Faraday, A. Ampère and H. Lorentz subject to its classical Newton's second law form. Namely, the classical Lorentz force

$$dp/dt = \xi E + \xi \frac{u}{c} \times B \tag{3.1}$$

was derived, where E and $B \in \mathbb{E}^3$ are, respectively, electric and magnetic fields, acting on a point charged particle $\xi \in \mathbb{R}$, possessing the momentum $p = mu$, where $m \in \mathbb{R}_+$ is the observed particle mass and $u \in T(\mathbb{R}^3)$ is its velocity, measured with respect to a suitably chosen laboratory reference frame $\mathcal{K}_t$.

That the Lorentz force (3.1) is not a completely satisfactory expression was well known by Lorentz himself, as the nonuniform Maxwell equations also describe the electromagnetic fields, radiated by any accelerated charged particle. This follows directly from well-known expressions for the Lienard–Wiechert electromagnetic four-potential $(\varphi, A) : M^4 \to T^*(M^4)$, related to the electromagnetic fields by means of the well-known [1,2,8] relationships

$$E := -\nabla\varphi - \frac{1}{c}\frac{\partial A}{\partial t}, \quad B := \nabla \times A \tag{3.2}$$

This fact had inspired many physicists to "improve" the classical Lorentz force expression (3.1) and its modification was then suggested by G.A. Schott [14] and later by M. Abraham and P.A.M. Dirac (see [1,8]), who found that the so called classical "*radiation reaction*" force, owing to the self-interaction of a charged particle with charge $\xi \in \mathbb{R}$, equals

$$dp/dt = \xi E + \xi \frac{u}{c} \times B + \frac{2\xi^2}{3c^3} d^2u/dt^2 \tag{3.3}$$

The additional self-reaction force expression

$$F_r := \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2} \tag{3.4}$$

depending on the particle acceleration immediately begged questions concerning its physical meaning, since for instance, a uniformly accelerated charged particle, owing to the expression (3.3), feels no radiation reaction, contradicting the fact that any accelerated charged particle radiates electromagnetic waves. This "paradox" was a challenging problem during the twentieth century [1,14,67,71–73] and still remains to be explained [4,58,74] . As there exist different approaches to explaining this reaction radiation phenomenon, we mention here only the most popular ones such as the Wheeler–Feynman [75] "absorber radiation" theory, based on a very sophisticated elaboration of the retarded and advanced solutions to the nonuniform Maxwell equations, the vacuum Casimir effect approach devised in [25,76], and the construction of Teitelbom [77] which extensively exploits the intrinsic structure of the electromagnetic energy tensor subject to the advanced and retarded solutions to the nonuniform Maxwell equations.

It is also worth mentioning here very the nontrivial development of the Teitelbom's theory devised recently in [78,79] and applied to the non-abelian Yang–Mills equations, which are natural generalizations of the Maxwell equations. Nonetheless, all of these explanations do not prove to be satisfactory from the modern physics of view. Taking this state of art into account, we will reanalyze the structure of the "radiative" Lorentz type force (3.3) using the vacuum field theory approach of Section 1 and find that this force allows some natural slight modification.

3.2. The Radiation Reaction Force: Vacuum Field Theory Approach

In this section, we will develop further our vacuum field theory approach, devised in [18,19], to the electromagnetic Maxwell and Lorentz electron theories and show that it is in complete agreement with the classical results and even more: it allows some nontrivial generalizations, which may have some important physical applications. It will also be shown that the closely related electron mass problem can be satisfactorily explained via the devised vacuum field theory approach and the spatial electron structure assumption.

The modified Lorentz force, acting on a particle of charge $\xi \in \mathbb{R}$ and exerted by a moving with velocity $u_f \in T(\mathbb{R}^3)$ charged particle $\xi_f \in \mathbb{R}$, was derived in Section 1 and is

$$dp/dt := F_s = \xi E + \xi \frac{u}{c} \times B - \nabla < \xi A \, , (u - u_f)/c > \tag{3.5}$$

where $(\varphi, A) \in T^*(M^4)$ is the external electromagnetic potential calculated with respect to a fixed laboratory reference frame $\mathcal{K}_t$. To take into account the self-interaction of this particle we will make use of a spatially distributed charge density $\rho : M^4 \to \mathbb{R}$, satisfying the condition

$$\xi = \int_{\mathbb{R}^3} \rho(t,r) d^3 r \tag{3.6}$$

for all $t \in \mathbb{R}$ subject to this laboratory reference frame $\mathcal{K}_t$ with coordinates $(t, r) \in M^4$. Then, owing to (3.5) and results from Section 1, the self-interacting force of this spatially structured charge $\xi \in \mathbb{R}$ can be expressed with respect to this laboratory reference frame $\mathcal{K}_t$ in the following equivalent form:

$$\begin{gathered} dp/dt = -\tfrac{1}{c} \int_{\mathbb{R}^3} d^3 r \rho(t,r) \tfrac{d}{dt} A_s(t,r) - \\ - \int_{\mathbb{R}^3} d^3 r \rho(t,r) \nabla \varphi_s(t,r) \left(1 - |u/c|^2\right) \; = \end{gathered} \tag{3.7}$$

where

$$\varphi_s(t,r) = \int_{\mathbb{R}^3} \frac{\rho(t',r')|_{ret} d^3 r'}{|r - r'|}, \quad A_s(t,r) = \frac{1}{c} \int_{\mathbb{R}^3} \frac{J(t',r')|_{ret} d^3 r'}{|r - r'|} \tag{3.8}$$

the well-known retarded Lienard–Wiechert potentials, which should be calculated at the retarded time parameter $t' := t - |r - r'|/c \in \mathbb{R}$. Taking into account the continuity relationship

$$\partial\rho/\partial t + <\nabla, J> = 0 \tag{3.9}$$

for the spatially distributed charge density $\rho : M^4 \to \mathbb{R}$ and current $J = \rho u : M^4 \to \mathbb{E}^3$ and the Taylor expansions for retarded potentials (3.8)

$$\begin{aligned} \varphi_s(t,r) &= \sum_{n\in\mathbb{Z}_+} \frac{\partial^n}{\partial t^n} \int_{\mathbb{R}^3} \frac{(-|r-r'|)^n}{c^n n!} \frac{\rho(t,r')d^3r'}{|r-r'|}, \\ A_s(t,r) &= \sum_{n\in\mathbb{Z}_+} \frac{\partial^n}{\partial t^n} \int_{\mathbb{R}^3} \frac{(-|r-r'|)^n}{c^{n+1} n!} \frac{J(t,r')d^3r'}{|r-r'|} \end{aligned} \tag{3.10}$$

from (3.7) and (3.10). assuming for brevity the spherical charge distribution is small ($|u/c| \ll 1$) and, respectively, slow acceleration, followed by calculations similar to those of [1,58], one can obtain that

$$\begin{aligned} F_s = {}& \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{n!c^n}(1-|u/c|^2) \int_{\mathbb{R}^3} d^3r \rho(t,r) \int_{\mathbb{R}^3} d^3r' \tfrac{\partial^n}{\partial t^n}\rho(t,r')\nabla|r-r'|^{n-1} + \\ & + \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{n!c^{n+2}} \int_{\mathbb{R}^3} d^3r \rho(t,r)|r-r'|^{n-1} \tfrac{\partial^{n+1}}{\partial t^{n+1}} J(t,r'] = \\ = {}& \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{n!c^{n+2}}(1-|u/c|^2) \int_{\mathbb{R}^3} d^3r \rho(t,r) \int_{\mathbb{R}^3} d^3r' \tfrac{\partial^{n=2}}{\partial t^{n+2}}\rho(t,r')\nabla|r-r'|^{n+1} + \\ & + \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{n!c^{n+2}} \int_{\mathbb{R}^3} d^3r \rho(t,r) \int_{\mathbb{R}^3} d^3r'|r-r'|^{n-1} \tfrac{\partial^{n+1}}{\partial t^{n+1}} J(t,r') \end{aligned} \tag{3.11}$$

The relationship above can be rewritten, owing to the charge continuity Equation (3.9), and gives rise to the radiation force expression

$$\begin{aligned} F_s = {}& \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^n}{n!c^{n+2}}(1-|u/c|^2) \int_{\mathbb{R}^3} d^3r \rho(t,r) \int_{\mathbb{R}^3} d^3r'|r-r'|^{n-1} \tfrac{\partial^{n+1}}{\partial t^{n+1}} \left(\tfrac{J(t,r')}{n+2} + \tfrac{n-1}{n+2} \tfrac{<r-r',J(t,r')>(r-r')}{|r-r'|^2} \right) + \\ & + \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{n!c^{n+2}} \int_{\mathbb{R}^3} d^3r \rho(t,r) \int_{\mathbb{R}^3} d^3r'|r-r'|^{n-1} \tfrac{\partial^{n+1}}{\partial t^{n+1}} J(t,r') = \\ = {}& \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{n!c^{n+2}}(1-|u/c|^2) \int_{\mathbb{R}^3} d^3r \rho(t,r) \int_{\mathbb{R}^3} d^3r'|r-r'|^{n-1} \tfrac{\partial^{n+1}}{\partial t^{n+1}} \left(\tfrac{J(t,r')}{n+2} + \tfrac{n-1}{n+2} \tfrac{|r-r',u|^2 J(t,r')}{|r-r'|^2|u|^2} \right) + \\ & + \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{n!c^{n+2}} \int_{\mathbb{R}^3} d^3r \rho(t,r) \int_{\mathbb{R}^3} d^3r'|r-r'|^{n-1} \tfrac{\partial^{n+1}}{\partial t^{n+1}} J(t,r') \end{aligned} \tag{3.12}$$

Now, having applied to (3.12) the rotational symmetry property for calculation of the internal integral, one easily obtains that

$$\begin{aligned} F_s = {}& \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^n}{n!c^{n+2}}(1-|u/c|^2) \int_{\mathbb{E}^3} d^3r \rho(t,r) \int_{\mathbb{R}^3} d^3r'|r-r'|^{n-1} \tfrac{\partial^{n+1}}{\partial t^{n+1}} \left(\tfrac{J(t,r')}{n+2} + \tfrac{(n-1)J(t,r')}{3(n+2)} \right) + \\ & + \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^{n+1}}{n!c^n} \int_{\mathbb{R}^3} d^3r \rho(t,r) \int_{\mathbb{R}^3} d^3r' \tfrac{|r-r'|^{n+1}}{c^2} \tfrac{\partial^{n+1}}{\partial t^{n+1}} J(t,r') = \\ = {}& \tfrac{d}{dt}[\sum_{n\in\mathbb{Z}_+} \tfrac{2(-1)^{n+1}}{3n!c^{n+2}} \int_{\mathbb{R}^3} d^3r \rho(t,r) \int_{\mathbb{R}^3} d^3r'|r-r'|^{n-1} \tfrac{\partial^n}{\partial t^n} J(t,r') - \\ & - \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^n}{3n!c^{n+4}} \tfrac{|u|^2}{}) \int_{\mathbb{R}^3} d^3r \rho(t,r) \int_{\mathbb{R}^3} d^3r'|r-r'|^{n-1} \tfrac{\partial^n}{\partial t^n} J(t,r')] \end{aligned} \tag{3.13}$$

where we took into account [1] that in case of the spherical charge distribution the following equalities

$$\int_{\mathbb{E}^3} d^3r \int_{\mathbb{R}^3} d^3r' \rho(t,r)\rho(t,r') \tfrac{|<r-r',u(t)>|^2}{|r-r'|^2|u(t)|^2} = \tfrac{1}{3}\xi^2,$$

$$\int_{\mathbb{R}^3} d^3r \ < \nabla, J(t,r) > \int_{\mathbb{R}^3} d^3r' |r-r'|^{n-1} \tfrac{\partial^n}{\partial t^n} J(t,r') = 0, \tag{3.14}$$

$$\int_{\mathbb{R}^3} d^3r \int_{\mathbb{R}^3} d^3r \rho(t,r)\rho(t,r') \tfrac{(r-r')}{|r-r'|^3} = 0$$

hold for all $n \in \mathbb{Z}_+$. Thus, from (3.14) one easily finds up to the $O(1/c^4)$ accuracy the following radiation reaction force expression:

$$dp/dt = F_s = -\frac{d}{dt}\left(\frac{4\mathcal{E}_{es}}{3c^2}u(t)\right) - \frac{d}{dt}\left(\frac{2\mathcal{E}_{es}}{3c^2}\,|u/c|^2\,u(t)\right) + \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2} + O(1/c^4) = \tag{3.15}$$

$$= -\frac{d}{dt}\left(\frac{4}{3}m_{0,es}(1+\frac{|u/c|^2}{2})u(t)\right) + \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2} + O(1/c^4) =$$

$$= -\frac{d}{dt}\left(\frac{4}{3}\frac{m_{0,es}u(t)}{(1-|u/c|^2)^{1/2}}\right) \ + \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2} + O(1/c^4) =$$

$$= -\frac{d}{dt}\left(\frac{4}{3}m_{es}u(t)\right) + \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2} + O(1/c^4)$$

where we defined, respectively, the electrostatic self-interaction repulsive energy as

$$\mathcal{E}_{es} := \frac{1}{2}\int_{\mathbb{R}^3} d^3r \int_{\mathbb{R}^3} d^3r' \frac{\rho(t,r)\rho(t,r')}{|r-r'|} \tag{3.16}$$

the electromagnetic charged particle rest and inertial masses as

$$m_{0,es} := \frac{\mathcal{E}_{es}}{c^2}, \quad m_{es} := \frac{m_{0,es}}{(1-|u/c|^2)^{1/2}} \tag{3.17}$$

Now from (3.5) one obtains that

$$\frac{d}{dt}\left[(m_g + \frac{4}{3}m_{es})u\right] = \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2} + O(1/c^4) \tag{3.18}$$

where we made use of the inertial mass definition

$$m_g := -\bar{W}_g/c^2, \quad \nabla\bar{W}_g \simeq 0 \tag{3.19}$$

following from the vacuum field theory approach, where the $m_g \in \mathbb{R}$ is the corresponding gravitational mass of the charged particle ξ, generated by the vacuum field potential $\bar{W}_g$. The corresponding radiation force

$$F_r = \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2} + O(1/c^4) \tag{3.20}$$

coincides exactly with the classical Abraham–Lorentz–Dirac results. From (3.18) it follows that the observable physical charged particle mass $m_{ph} \simeq m_g + \frac{4}{3}m_{es}$ consists of two impacts: the electromagnetic and gravitational components, giving rise to the final force expression

$$\frac{d}{dt}(m_{ph}u) = \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2} + O(1/c^4) \tag{3.21}$$

This means, in particular, that the real physically observed "inertial" mass m_{ph} of an electron strongly depends on the external physical interaction with the ambient vacuum medium, as was recently demonstrated using completely different approaches in [25,76], based on the vacuum Casimir effect considerations. Moreover, the assumed above boundedness of the electrostatic self-energy $\mathcal{E}_{es}$ appears to be completely equivalent to the existence of so-called intrinsic Poincaré type "*tensions*", analyzed in [71,76], and to the existence of a special compensating Coulomb "*pressure*", suggested in [25], guaranteeing the observable electron stability.

3.3. Comments

The charged particle radiation problem, revisited in this section, allows to conceive the following explanation of the point charged particle mass as that of a compact and stable object which should possess the vacuum interaction potential $\bar{W} \in \mathbb{R}^3$ of negative sign as follows from (3.19). The latter can be satisfied iff the equality (3.19) holds, thereby imposing on the intrinsic charged particle structure [74] some nontrivial geometrical constraints. Moreover, as follows from the physically observed particle mass expressions (3.19) the electrostatic potential energy, being of repulsive force origin, does contribute to the full mass as its main component.

There exist different relativistic generalizations of the force expression (3.18), which suffer the same common physical inconsistency related with the no radiation effect of a charged point particle at uniform motion.

Another problem closely related to the radiation reaction force analyzed above is the search for an explanation to the Wheeler and Feynman reaction radiation mechanisms, called the absorption radiation theory, based on the Mach type interaction of a charged point particle with the ambient vacuum electromagnetic medium. Concerning this problem, one can also observe some of its relationships with the one devised here via the vacuum field theory approach, but this question needs a more detailed and extended analysis.

4. Electron Inertia via the Feynman Proper Time Paradigm and Vacuum Field Theory Approach

4.1. Introduction

As was reported by F. Dyson [10], the original Feynman approach derivation of the electromagnetic Maxwell equations was based on an *a priori* general form of the classical Newton type force, acting on a charged point particle moving in three-dimensional space $\mathbb{R}^3$ endowed with the canonical Poisson brackets on the phase variables, defined on the associated tangent space $T(\mathbb{R}^3)$. As a result of this approach there only the first part of the Maxwell equations were derived, as the second part, owing to F.

Dyson [10], is related with the charged matter nature, which appeared to be hidden. Trying to complete this Feynman approach to the derivation of Maxwell's equations more systematically we have observed [80] that the original Feynman's calculations, based on Poisson brackets analysis, were performed on the *tangent space* $T(\mathbb{R}^3)$ which is, subject to the problem posed, not physically proper. The true Poisson brackets can be correctly defined only on the *coadjoint phase space* $T^*(\mathbb{R}^3)$, as seen from the classical Lagrangian equations and the related Legendre transformation [15,16,59,81] from $T(\mathbb{R}^3)$ to $T^*(\mathbb{R}^3)$. Moreover, within this observation, the corresponding dynamical Lorentz type equation for a charged point particle should be written for the particle momentum, not for the particle velocity,

whose value is well defined only with respect to the proper relativistic reference frame, associated with the charged point particle owing to the fact that the Maxwell equations are Lorentz invariant.

Thus, from the very beginning, we shall reanalyze the structure of the Lorentz force exerted on a moving charged point particle with a charge $\xi \in \mathbb{R}$ by another point charged particle with a charge $\xi_f \in \mathbb{R}$, making use of the classical Lagrangian approach, and rederive the corresponding electromagnetic Maxwell equations. The latter appears to be strongly related to the charged point mass structure of the electromagnetic origin as was suggested by R. Feynman and F. Dyson.

4.2. Feynman Proper Time Paradigm Analysis

Consider a charged point particle moving in an electromagnetic field. For its description, it is convenient to introduce a trivial fiber bundle structure π: $\mathcal{M} \to \mathbb{R}^3, \mathcal{M} = \mathbb{R}^3 \times G$, with the abelian structure group $G := \mathbb{R}\backslash\{0\}$, equivariantly acting on the canonically symplectic coadjoint space $T^*(\mathcal{M})$ endowed both with the canonical symplectic structure

$$\begin{aligned}\omega^{(2)}(p,y;r,g) &:= d\,pr^*\alpha^{(1)}(r,g) =< dp, \wedge dr > + \\ &+ < dy, \wedge g^{-1}dg >_{\mathcal{G}} + < ydg^{-1}, \wedge dg >_{\mathcal{G}}\end{aligned} \tag{4.1}$$

for all $(p,y;r,g) \in T^*(\mathcal{M})$, where $\alpha^{(1)}(r,g) :=< p, dr > + < y, g^{-1}dg >_{\mathcal{G}} \in T^*(\mathcal{M})$ is the corresponding Liouville form on $\mathcal{M}$, and with a connection one-form $\mathcal{A} : M \to T^*(M) \times \mathcal{G}$ as

$$\mathcal{A}(r,g) := g^{-1} < \xi A(r), dr > g + g^{-1}dg \tag{4.2}$$

with $\xi \in \mathcal{G}^*, (r,g) \in \mathbb{R}^3 \times G, < \cdot,\cdot >$ being the scalar product in $\mathbb{E}^3$. The corresponding curvature 2-form $\Sigma^{(2)} \in \Lambda^2(\mathbb{R}^3) \otimes \mathcal{G}$ is

$$\Sigma^{(2)}(r) := d\mathcal{A}(r,g) + \mathcal{A}(r,g) \wedge \mathcal{A}(r,g) = \xi \sum_{i,j=1}^{3} F_{ij}(r) dr^i \wedge dr^j \tag{4.3}$$

where

$$F_{ij}(r) := \frac{\partial A_j}{\partial r^i} - \frac{\partial A_i}{\partial r^j} \tag{4.4}$$

for $i,j = \overline{1,3}$ with respect to the reference frame $\mathcal{K}_t$, characterized by the phase space coordinates $(r,p) \in T^*(\mathbb{R}^3)$. As an element $\xi \in \mathcal{G}^*$ is still not fixed, it is natural to apply the standard [15,16,59] invariant Marsden–Weinstein–Meyer reduction to the orbit factor space $\tilde{P}_\xi := P_\xi / G_\xi$ subject to the related momentum mapping $l : T^*(\mathcal{M}) \to \mathcal{G}^*$, constructed with respect to the canonical symplectic structure (4.1) on $T^*(\mathcal{M})$, where, by definition, $\xi \in \mathcal{G}^*$ is constant, $P_\xi := l^{-1}(\xi) \subset T^*(\mathcal{M})$ and $G_\xi = \{g \in G : Ad_G^*\xi\}$ is the isotropy group of the element $\xi \in \mathcal{G}^*$.

As a result of the Marsden–Weinstein–Meyer reduction, one finds that $G_\xi \simeq G$, the factor-space $\tilde{P}_\xi \simeq T^*(\mathbb{R}^3)$ is endowed with a suitably reduced symplectic structure $\bar{\omega}_\xi^{(2)} \in T^*(\tilde{P}_\xi)$ and the corresponding Poisson brackets on the reduced manifold $\tilde{P}_\xi$ are

$$\begin{aligned} &\{r^i, r^j\}_\xi = 0,\ \{p_j, r^i\}_\xi = \delta_j^i, \\ &\{p_i, p_j\}_\xi = \xi F_{ij}(r) \end{aligned} \tag{4.5}$$

for $i, j = \overline{1,3}$, considered with respect to the reference frame $\mathcal{K}_t$. Introducing a new momentum variable

$$\tilde{\pi} := p + \xi A(r) \tag{4.6}$$

on $\tilde{P}_\xi$, it is easy to verify that $\bar{\omega}_\xi^{(2)} \to \tilde{\omega}_\xi^{(2)} :=< d\tilde{\pi}, \wedge dr >$, giving rise to the following *"minimal interaction"* canonical Poisson brackets:

$$\{r^i, r^j\}_{\tilde{\omega}_\xi^{(2)}} = 0,\ \{\tilde{\pi}_j, r^i\}_{\tilde{\omega}_\xi^{(2)}} = \delta_j^i,\ \{\tilde{\pi}_i, \tilde{\pi}_j\}_{\tilde{\omega}_\xi^{(2)}} = 0 \tag{4.7}$$

for $i, j = \overline{1,3}$ with respect to some new reference frame $\tilde{\mathcal{K}}_{t'}$, characterized by the phase space coordinates $(r, \tilde{\pi}) \in \tilde{P}_\xi$ and an evolution parameter $t' \in \mathbb{R}$ if and only if the Maxwell field equations

$$\partial F_{ij}/\partial r_k + \partial F_{jk}/\partial r_i + \partial F_{ki}/\partial r_j = 0 \tag{4.8}$$

are satisfied on $\mathbb{R}^3$ for all $i, j, k = \overline{1,3}$ with the curvature tensor $F_{ij}(r) := \partial A_j/\partial r^i - \partial A_i/\partial r^j$, $i, j = \overline{1,3}$, $r \in \mathbb{R}^3$.

Now we proceed to a dynamic description of the interaction between two moving charged point particles ξ and ξ_f, moving respectively, with the velocities $u := dr/dt$ and $u_f := dr_f/df$ subject to the reference frame $\mathcal{K}_t$. Unfortunately, there is a fundamental problem in correctly formulating a physically suitable action functional and the related least action condition. There are clearly possibilities such as

$$S_p^{(t)} := \int_{t_1}^{t_2} dt \mathcal{L}_p^{(t)}(r; dr/dt) \tag{4.9}$$

on a temporal interval $[t_1, t_2] \subset \mathbb{R}$ with respect to the laboratory reference frame $\mathcal{K}_t$,

$$S_p^{(t')} := \int_{t_1'}^{t_2'} dt' \mathcal{L}_p^{(t')}(r; dr/dt') \tag{4.10}$$

on a temporal interval $[t_1', t_2'] \subset \mathbb{R}$ with respect to the moving reference frame $\mathcal{K}_{t'}$ and

$$S_p^{(\tau)} := \int_{\tau_1}^{\tau_2} dt \mathcal{L}_p^{(\tau)}(r; dr/d\tau) \tag{4.11}$$

on a temporal interval $[\tau_1, \tau_2] \subset \mathbb{R}$ with respect to the proper time reference frame $\mathcal{K}_\tau$, naturally related to the moving charged point particle ξ.

It was first observed by Poincaré and by Minkowski [3,82,83] that the temporal differentials dt and dt' are not closed differential one-forms, which physically means that a particle can traverse many different paths in space $\mathbb{R}^3$ during any given proper time interval $d\tau$, naturally related to its motion. This fact was stressed [3,84–87] by Einstein, Minkowski and Poincaré, and later exhaustively analyzed by R. Feynman, who argued [9] that the dynamical equation of a moving point charged particle is physically sensible only with respect to its proper time reference frame. This is Feynman's proper time reference frame paradigm, which was recently further elaborated and applied both to the electromagnetic Maxwell equations in [82,83] and to the Lorentz type equation for a moving charged point particle under external electromagnetic field in [18,65,80,88]. As it was there argued from a physical point of view, the least action principle should be applied only to the expression (4.11) written with respect to the proper time reference frame $\mathcal{K}_\tau$, whose temporal parameter $\tau \in \mathbb{R}$ is independent of an observer and is a closed differential one-form. Consequently, this action functional is also mathematically sensible, which in part reflects the Poincaré's and Minkowski's observation that the infinitesimal quadratic interval

$$d\tau^2 = (dt')^2 - |dr - dr_f|^2 \tag{4.12}$$

relating the reference frames $\mathcal{K}_{t'}$ and $\mathcal{K}_\tau$, can be invariantly used for the four-dimensional relativistic geometry. The most natural way to contend with this problem is to first consider the quasi-relativistic dynamics of the charged point particle ξ with respect to the moving reference frame $\mathcal{K}_{t'}$ subject to which the charged point particle ξ_f is at rest. Therefore, it possible to write down a suitable action functional (4.10), up to $O(1/c^2)$, as the light velocity $c \to \infty$, where the Lagrangian function $\mathcal{L}_p^{(t')}(r; dr/dt')$ can be naturally chosen as

$$\mathcal{L}_p^{(t')}(r; dr/dt') := m'(r)\left|dr/dt' - dr_f/dt'\right|^2/2 - \xi\varphi'(r) \tag{4.13}$$

where $m'(r) \in \mathbb{R}_+$ is the charged particle ξ mass parameter and $\varphi'(r)$ is the potential function generated by the charged particle ξ_f at a point $r \in \mathbb{R}^3$ with respect to the reference frame $\mathcal{K}_{t'}$. Since the standard temporal relationships between reference frames $\mathcal{K}_t$ and $\mathcal{K}_{t'}$:

$$dt' = dt(1 - |dr_f/dt'|^2)^{1/2} \tag{4.14}$$

as well as between the reference frames $\mathcal{K}_{t'}$ and $\mathcal{K}_\tau$:

$$dt = dt'(1 - |dr/dt' - dr_f/dt'|^2)^{1/2} \tag{4.15}$$

give rise, up to $O(1/c^2)$, as $c \to \infty$, to $dt' \simeq dt$ and $d\tau \simeq dt'$, respectively, it is easy to verify that the least action condition $\delta S_p^{(t')} = 0$ is equivalent to the dynamical equation

$$d\pi/dt = \nabla\mathcal{L}_p^{(t')}(r; dr/dt) = \nabla m(\frac{1}{2}\left|dr/dt - dr_f/dt\right|^2) - \xi\nabla\varphi(r) \tag{4.16}$$

where we have defined the generalized canonical momentum as

$$\pi := \partial\mathcal{L}_p^{(t')}(r; dr/dt)/\partial(dr/dt) = m(dr/dt - dr_f/dt) \tag{4.17}$$

with the dash signs dropped and denoted by "∇" the usual gradient operator in $\mathbb{E}^3$. Equating the canonical momentum expression (4.17) with respect to the reference frame $\mathcal{K}_{t'}$ to that of (4.6) with respect to the reference frame $\tilde{\mathcal{K}}_{t'}$, and identifying the reference frame $\tilde{\mathcal{K}}_t$ with $\mathcal{K}_{t'}$, one obtains the important particle mass determining expression

$$m = -\xi\varphi(r) \tag{4.18}$$

which follows from the relationship

$$\varphi(r)dr_f/dt = A(r) \tag{4.19}$$

This is well known in the classical electromagnetic theory [1] for potentials $(\varphi, A) \in T^*(M^4)$ satisfying the Lorentz condition

$$\partial\varphi(r)/\partial t + <\nabla, A(r)> = 0 \tag{4.20}$$

yet the expression (4.18) looks very nontrivial in relating the *"inertial"* mass of the charged point particle ξ to the electric potential, both generated by the ambient charged point particles ξ_f. As was argued in articles [56,65,80], the above mass phenomenon is closely related and from a physical perspective shows its deep relationship to the classical electromagnetic mass problem.

Before further analysis of the completely relativistic the charge ξ motion under consideration, we substitute the mass expression (4.18) into the quasi-relativistic action functional (4.10) with the Lagrangian (4.13). As a result, we obtain two possible action functional expressions, taking into account two main temporal parameters choices:

$$S_p^{(t')} = -\int_{t_1'}^{t_2'} \xi\varphi'(r)(1 + \frac{1}{2}\,|dr/dt' - dr_f/dt'|^2)dt' \tag{4.21}$$

on an interval $[t_1', t_2'] \subset \mathbb{R}$, or

$$S_p^{(\tau)} = -\int_{\tau_1}^{\tau_2} \xi\varphi'(r)(1 + \frac{1}{2}\,|dr/d\tau - dr_f/d\tau|^2)d\tau \tag{4.22}$$

on an $[\tau_1, \tau_2] \subset \mathbb{R}$. It is easy to see that the first expression (4.16) is unsatisfactory upon transforming to the proper time relativistic representation form the suitable quasi-relativistic limit for the Lagrangian function (4.13). On the other hand, the direct relativistic generalization of (4.22) follows:

$$\begin{aligned} S_p^{(\tau)} &= -\int_{\tau_1}^{\tau_2} \xi\varphi'(r)(1 + \frac{1}{2}\,|dr/d\tau - dr_f/d\tau|^2)d\tau \simeq \\ &\simeq -\int_{\tau_1}^{\tau_2} \xi\varphi'(r)(1 + |dr/d\tau - dr_f/d\tau|^2)^{1/2}d\tau = \\ &= -\int_{\tau_1}^{\tau_2} \xi\varphi'(r)(1 - |dr/dt' - dr_f/dt'|)^{-1/2}d\tau = -\int_{t_1'}^{t_2'} \xi\varphi'(r)dt' \end{aligned} \tag{4.23}$$

giving rise to the correct (from the physical point of view) relativistic action functional form (4.10), suitably transformed to the proper time reference frame representation (4.11) via the Feynman proper time paradigm. Thus, we have shown that the true action functional procedure consists in a physically motivated choice of either the action functional expression form (4.9) or (4.10). Then, it is transformed to the proper time action functional representation form (4.11) in the Feynman paradigm, and the least action principle is applied.

Concerning the above problem of describing the motion of a charged point particle ξ in the electromagnetic field generated by another moving charged point particle ξ, it must be mentioned that we have chosen the quasi-relativistic functional expression (4.13) in the form (4.10) with respect to the moving reference frame $\mathcal{K}_{t'}$, because its form is more physically acceptable, since the charged point particle ξ_f is then at rest.

Based on the above relativistic action functional expression

$$S_p^{(\tau)} := -\int_{\tau_1}^{\tau_2} \xi\varphi'(r)(1+|dr/d\tau - dr_f/d\tau|^2)^{1/2}d\tau \tag{4.24}$$

written with respect to the proper reference from $\mathcal{K}(\tau; r-r_f)$, one finds the following evolution equation:

$$d\pi_p/d\tau = -\xi\nabla\varphi'(r)(1+|dr/d\tau - dr_f/d\tau|^2)^{1/2} \tag{4.25}$$

where the generalized momentum is given by the relationship (4.17):

$$\pi_p = m(dr/dt - dr_f/dt) \tag{4.26}$$

Making use of the relativistic transformation (4.14) and the next one (4.15), the Equation (4.25) is easily transformed to

$$\frac{d}{dt}(p+\xi A) = -\nabla\varphi(r)(1-|u_f|^2) \tag{4.27}$$

where we took into account the definitions: (4.18) for the charged particle ξ mass, (4.19) for the magnetic vector potential and $\varphi(r) = \varphi'(r)/(1-|u_f|^2)^{1/2}$ for the scalar electric potential with respect to the laboratory reference frame $\mathcal{K}_t$. Equation (4.27) can be further transformed, using elementary vector algebra, to the classical Lorentz type form:

$$dp/dt = \xi E + \xi u \times B - \xi\nabla < u - u_f, A > \tag{4.28}$$

where

$$E := -\partial A/\partial t - \nabla\varphi \tag{4.29}$$

is the related electric field and

$$B := \nabla \times A \tag{4.30}$$

is the related magnetic field, exerted by the moving charged point particle ξ_f on the charged point particle ξ with respect to the laboratory reference frame $\mathcal{K}_t$. The Lorentz type force Equation (4.28) was obtained in [18,80] in terms of the moving reference frame $\mathcal{K}_{t'}$, and recently reanalyzed in [88,89]. The reanalysis was derived in part from Ampère's classical works [90,91] on constructing the magnetic force between two neutral conductors with stationary currents.

For the Lorentz force Equation (4.28) it is a natural problem to analyze its form in the case of many external charged point particles $\xi_j \in \mathbb{R}$, $j \in \mathbb{Z}_+$, moving with velocities $dr_j/dt, j \in \mathbb{Z}_+$, with respect to the laboratory reference frame $\mathcal{K}_t$. In this case there is no possibility of choosing a common moving reference frame $\mathcal{K}_{t'}$ with respect to which all of the charged particles $\xi_j, j \in \mathbb{Z}_+$, are at rest. However, we do have the unique proper time parameter $\tau \in \mathbb{R}$, related to each charged point particle $\xi_j, j \in \mathbb{Z}_+$, via the infinitesimal relativistic transformation expressions

$$dt'_j = d\tau(1-|dr/dt'_j - dr_f/dt'_j|^2)^{-1/2} \tag{4.31}$$

to the moving reference frames $\mathcal{K}_{t'_j}, j \in \mathbb{Z}_+$, fixing the τ-clock for all the charged particles. Thus, making use of the same scheme as demonstrated above, we can express together with the superposition principle, the net Lorentz type force expression for the charged point particle ξ as

$$dp/dt = \xi\bar{E} + \xi u \times \bar{B} - \xi\nabla\sum_{j\in\mathbb{Z}_+} < u - u_j, A_j > \tag{4.32}$$

where

$$\bar{E} := \sum_{j\in\mathbb{Z}_+} E_j,\ \bar{B} = \sum_{j\in\mathbb{Z}_+} B_j \tag{4.33}$$

and $A_j \in T^*(\mathbb{R}^3), j \in \mathbb{Z}_+$, are magnetic vector potentials generated by the set of distant charged point particles $\xi_j, j \in \mathbb{Z}_+$. As this system of external charges is on average neutral, that is $\sum_{j\in\mathbb{Z}_+} \xi_j \simeq 0$, and their spatial distribution is on average symmetric with respect to the charge signs and velocities, one obtains from (4.32) that

$$dp/dt = \xi\ \bar{E} + \xi u \times \bar{B} \tag{4.34}$$

which the classical Lorentz type expression for the charged point particle ξ moving under the influence of an external electromagnetic field with respect to the laboratory reference frame $\mathcal{K}_t$.

Equation (4.34) can naturally be physically interpreted as the Lorentz type force exerted by a virtual net charge $\bar{\xi}$ at rest and located at the centroid of the charges with respect to $\mathcal{K}_t$. Consequently, one can write the corresponding effective relativistic invariant action functional in the form

$$\bar{S}_p^{(t)} := \int_{t_1}^{t_2} dt(m_\xi + < \bar{A}, dr/dt > -\xi\ \bar{\varphi}) \tag{4.35}$$

on an interval $[t_1, t_2] \subset \mathbb{R}$ with respect to $\mathcal{K}_t$. Here $m_\xi \in \mathbb{R}$ is a possible internal charged particle mass energy value and as before, $\bar{\varphi} := \sum_{j\in\mathbb{Z}_+} \varphi_j$, $\bar{A} := \sum_{j\in\mathbb{Z}_+} A_j$, and we also took into account took the suitable relativistic electric potentials transformations from the moving reference frames $\mathcal{K}_{t'_j}, j \in \mathbb{Z}_+$, to the laboratory reference frame $\mathcal{K}_t$ with respect to which the averaged set of charges $\bar{\xi}$ is assumed to be virtually at rest so that

$$-\ \varphi'_j dt'_j = \varphi_j dt + < A_j, dr > \tag{4.36}$$

holds for all $j \in \mathbb{Z}_+$ and gives rise, upon summing over $j \in \mathbb{Z}_+$, to

$$-\sum_{j\in\mathbb{Z}_+} \varphi'_j dt'_j = -\ \bar{\varphi} dt + < \bar{A}, dr > \tag{4.37}$$

used for construction of the action functional (4.35). As this is considered to be written for the averaged set of charges $\bar{\xi}$, whose virtual location is now assumed to be at rest, we can apply to this action functional (4.35) the Feynman proper time paradigm and construct the corresponding physically reasonable action functional

$$\bar{S}_p^{(\tau)} = \int_{\tau_1}^{\tau_2} d\tau(-\xi\ \bar{\varphi} + \xi < \bar{A}, dr/d\tau >)(1 + |dr/d\tau|^2)^{1/2} \tag{4.38}$$

defined on an independent time interval $[\tau_1, \tau_2] \subset \mathbb{R}$ with respect to the proper time reference frame $\mathcal{K}_\tau$, whose time parameter $\tau \in \mathbb{R}$ is infinitesimally related to the laboratory time parameter $t \in \mathbb{R}$ as

$$d\tau = dt(1 - |dr/dt|^2)^{-1/2} \tag{4.39}$$

Applying the least action principle to the functional (4.38) one easily obtains the evolution equation

$$\frac{d}{dt}(p + \xi\bar{A}) = -\xi\nabla\ \bar{\varphi} + \xi\nabla < \bar{A}, u >, \tag{4.40}$$

where, as before, the charged particle ξ momentum is defined classically as

$$p := m\, dr/dt \tag{4.41}$$

and its mass parameter is defined as

$$m\ := -\xi\ \bar{\varphi}(r). \tag{4.42}$$

As the four-vector potentials $(\varphi_j, A_j) \in T^*(M^4), j \in \mathbb{Z}_+$, where $M^4 := \mathbb{R} \times \mathbb{E}^3$ is the standard Minkowski pseudo-Euclidean metric space, satisfy the Lorentz conditions

$$\partial\varphi_j/\partial t + <\nabla, A_j> = 0 \tag{4.43}$$

for any $j \in \mathbb{Z}_+$, it is evident that the same condition

$$\partial\ \bar{\varphi}/\partial t + <\nabla, \bar{A}> = 0 \tag{4.44}$$

holds also for the averaged potentials $(\ \bar{\varphi}, \bar{A}) \in T^*(M^4)$. The same standard calculations applied to the expression (4.40) yield the (same as (4.34)) Lorentz force equation

$$dp/dt = \xi\bar{E} + \xi u \times \bar{B}, \tag{4.45}$$

thereby demonstrating the mathematical agreement between two physically different approaches to its derivation, based on the classical averaging procedure and the superposition principle.

4.3. Analysis of the Maxwell and Lorentz Force Equations

4.3.1. The Maxwell Equations

As a moving charged particle ξ_f generates the suitable electric field (4.29) and magnetic field (4.30) via their electromagnetic potential $(\varphi, A) \in T^*(M^4)$ with respect a laboratory reference frame $\mathcal{K}_t$, we will supplement them naturally by means of the external material equations describing the relativistic charge conservation law:

$$\partial\rho/\partial t + <\nabla, J> = 0 \tag{4.46}$$

where $(\rho, J) \in T^*(M^4)$ is a related four-vector for the charge and current distribution in the space $\mathbb{R}^3$. Moreover, one can augment the Equation (4.46) with the experimentally well established the Gauss law

$$<\nabla, E> = \rho \tag{4.47}$$

to calculate the quantity $\Delta\varphi := <\nabla, \nabla\varphi>$ from the expression (4.29):

$$\Delta\varphi = -\frac{\partial}{\partial t} <\nabla, A> - <\nabla, E> \tag{4.48}$$

Having taken into account the relativistic Lorentz condition (4.20) and the expression (4.47) one easily finds that the wave equation

$$\partial^2\varphi/\partial t^2 - \Delta\varphi = \rho \tag{4.49}$$

holds with respect to the laboratory reference frame $\mathcal{K}_t$. Applying the rot-operation "$\nabla\times$" to the expression (4.29) we obtain, owing to the expression (4.30), the equation

$$\nabla \times E + \partial B/\partial t = 0 \tag{4.50}$$

giving rise, together with (4.47), to the first pair of the classical Maxwell equations. To obtain the second pair of the Maxwell equations, it is first necessary to apply the rot-operation "$\nabla\times$"to the expression (4.30):

$$\nabla \times B = \partial E/\partial t + (\partial^2 A/\partial t^2 - \Delta A) \tag{4.51}$$

and then apply $-\partial/\partial t$ to the wave Equation (4.49) to obtain

$$\begin{gathered} -\frac{\partial^2}{\partial t^2}(\frac{\partial \varphi}{\partial t}) + < \nabla, \nabla\frac{\partial \varphi}{\partial t} >= \frac{\partial^2}{\partial t^2} < \nabla, A > - \\ - < \nabla, \nabla < \nabla, A >>=< \nabla, \frac{\partial^2 A}{\partial t^2} - \nabla \times (\nabla \times A) - \Delta A >= \\ =< \nabla, \frac{\partial^2 A}{\partial t^2} - \Delta A >=< \nabla, J > \end{gathered} \tag{4.52}$$

The result (4.52) leads to the relationship

$$\partial^2 A/\partial t^2 - \Delta A = J \tag{4.53}$$

if we take into account that both the vector potential $A \in \mathbb{E}^3$ and the vector of current $J \in \mathbb{E}$ are determined up to a rot-vector expression $\nabla \times S$ for some smooth vector-function $S : M^4 \to \mathbb{E}^3$. Inserting the relationship (4.53) into (4.51), we obtain (4.50) and the second pair of the Maxwell equations:

$$\nabla \times B = \partial E/\partial t + J, \;\; \nabla \times E = \partial B/\partial t \tag{4.54}$$

It is important that the system of Equation (4.54) can be represented by means of the least action principle $\delta S^{(t)}_{f-p} = 0$, where the action functional

$$S^{(t)}_{f-p} := \int_{t_1}^{t_2} dt \mathcal{L}^{(t)}_{f-p} \tag{4.55}$$

is defined on an interval $[t_1, t_2] \subset \mathbb{R}$ by the Lagrangian function

$$\mathcal{L}^{(t)}_{f-p} = \int_{\mathbb{R}^3} d^3 r((|E|^2 - |B|^2)/2+ < J, A > -\rho\varphi) \tag{4.56}$$

with respect to the laboratory reference frame $\mathcal{K}_t$. From (4.56) we deduce that the generalized field momentum

$$\pi_f := \partial\mathcal{L}^{(t)}_{f-p}/\partial(\partial A/\partial t) = -E \tag{4.57}$$

and its evolution is given as

$$\partial \pi_f/\partial t := \delta\mathcal{L}^{(t)}_{f-p}/\delta A \; = J - \nabla \times B, \tag{4.58}$$

which is equivalent to the first Maxwell equation of (4.54). As the Maxwell equations allow the least action representation, it is easy to derive [15,16,18,59,65] their dual Hamiltonian formulation with the Hamiltonian function

$$H_{f-p} := \int_{\mathbb{R}^3} d^3 r < \pi_f, \partial A/\partial t > -\mathcal{L}^{(t)}_{f-p} = \int_{\mathbb{R}^3} d^3 r((|E|^2 - |B|^2)/2- < J, A >) \tag{4.59}$$

satisfying the invariant condition

$$dH_{f-p}/dt = 0 \tag{4.60}$$

for all $t \in \mathbb{R}$.

It is worth noting here that the Maxwell equations were derived under the important condition that the charged system $(\rho, J) \in T^*(M^4)$ exerts no influence on the ambient electromagnetic field potentials $(\varphi, A) \in T^*(M^4)$. As this is not actually the case owing to the damping radiation reaction on accelerated charged particles, one can try to describe this self-interacting influence by means of the modified least action

principle, making use of the Lagrangian expression (4.56) in the case of a separate charged particle ξ. Following the well-known approach from [2] this Lagrangian can be recast (in the Gauss units) as

$$\begin{gathered}
\mathcal{L}_{f-p}^{(t)} = \int_{\mathbb{R}^3} d^3r(\tfrac{1}{2} < -\nabla\varphi - \tfrac{1}{c}\partial A/\partial t, -\nabla\varphi - \tfrac{1}{c}\partial A/\partial t > -\tfrac{1}{2} < \nabla \times (\nabla \times A), A >)+ \\
+ \int_{\mathbb{R}^3} d^3r(\tfrac{1}{c} < J, A > -\rho\varphi)- < k(t), dr/dt >= \\
= \int_{\mathbb{R}^3} d^3r(\tfrac{1}{2} < -\nabla\varphi, E > -\tfrac{1}{2c} < \partial A/\partial t, E > -\tfrac{1}{2} < A, \nabla \times B > + \\
+\tfrac{1}{c} < J, A > -\rho\varphi)+ < k(t), dr/dt >= \\
= \int_{\mathbb{R}^3} d^3r(\tfrac{1}{2}\varphi < \nabla, E > +\tfrac{1}{2c} < A, \partial E/\partial t > -\tfrac{1}{2c} < A,\ J + \partial E/\partial t > +\tfrac{1}{c} < J, A > -\rho\varphi)- \\
-\tfrac{1}{2c}\tfrac{d}{dt}\int_{\mathbb{R}^3} d^3r < A, E > -\tfrac{1}{2}\lim_{r\to\infty}\int_{\mathbb{S}_r^2} < \varphi E+ < A \times B, dS_r^2 > - < k(t), dr/dt > = \\
= \tfrac{1}{2}\int_{\mathbb{R}^3} d^3r(\tfrac{1}{c} < J, A > -\rho\varphi) - \tfrac{1}{2c}\tfrac{d}{dt}\int_{\mathbb{R}^3} d^3r < A, E > - \\
-\tfrac{1}{2}\lim_{r\to\infty}\int_{\mathbb{S}_r^2} < \varphi E+ < A \times B, dS_r^2 > - < k(t), dr/dt > -
\end{gathered} \quad , \tag{4.61}$$

where we have introduced an as yet undetermined internal charged particle stability energy impact $m_\xi c^2$ and radiation damping force $k(t) \in \mathbb{E}^3$, as well as a two-dimensional sphere $\mathbb{S}_r^2$ of radius $r \to \infty$. If we also assume that the radiated charged particle energy at infinity is negligible, the Lagrangian function (4.61) becomes equivalent to

$$\mathcal{L}_{f-p}^{(t)} = \tfrac{1}{2}\int_{\mathbb{R}^3} d^3r(\tfrac{1}{c} < J, A > -\rho\varphi)- < k(t), dr/dt > \tag{4.62}$$

which we now need to calculate taking into account that the electromagnetic potentials $(\varphi, A) \in T^*(M^4)$ are retarded and given as $1/c \to 0$ in the following Lienard–Wiechert form:

$$\begin{gathered}
\varphi = \int_{\mathbb{R}^3} d^3r' \frac{\rho(t', r')}{|r - r'|}\bigg|_{t'=t-|r-r'|/c} = \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^3} d^3r' \frac{\rho(t-\varepsilon, r')}{|r - r'|} + \\
+ \frac{1}{2c^2}\int_{\mathbb{R}^3} d^3r'|r - r'|\partial^2\rho(t, r')/\partial t^2 + \frac{1}{6c^3}\int_{\mathbb{R}^3} d^3r'|r - r'|^2\partial\rho(t, r')/\partial t + O(1/c^4),
\end{gathered}$$

$$A=\frac{1}{c}\int_{\mathbb{R}^3}d^3r'\frac{J(t',r')}{|r-r'|}\bigg|_{t'=t-|r-r'|/c}=\lim_{\varepsilon\downarrow 0}\frac{1}{c}\int_{\mathbb{R}^3}d^3r'\frac{J(t-\varepsilon,r')}{|r-r'|}- \tag{4.63}$$

$$-\frac{1}{c^2}\int_{\mathbb{R}^3}d^3r'\partial J(t,r')/\partial t+\frac{1}{2c^3}\int_{\mathbb{R}^3}d^3r'|r-r'|\partial^2J(t,r')/\partial t^2+O(1/c^4).$$

Here the current density $J(t,r)=\rho(t,r)dr(t)/dt$ for all $t\in\mathbb{R}$, $r\in\Omega(\xi):=supp\ \rho(t;r)\subset\mathbb{R}^3$ is the compact support of the charged particle density distribution. Moreover, the limit as $\varepsilon\to+0$ takes into account that the potentials (4.63) are both retarded and singular at the charged particle ξ center, moving in space with the velocity $u\in T(\mathbb{R}^3)$ with respect to the laboratory reference frame $\mathcal{K}_t$. As a result of simple calculations of the kind in [1] and the suitable regularization procedure one finds that, up to $O(1/c^4)$, the electric potential integral in (4.62), equals

$$\lim\nolimits_{\varepsilon\downarrow 0}\int_{\mathbb{R}^3}d^3r\rho(t,r)\int_{\mathbb{R}^3}d^3r'\tfrac{\rho(t-\varepsilon,r')}{|r-r'|}=$$

$$=\int_{\mathbb{R}^3}d^3r\rho(t,r)\int_{\mathbb{R}^3}d^3r'\tfrac{\rho(t,r')}{|r'-r|}-\lim\nolimits_{\varepsilon\downarrow 0}\int_{\mathbb{R}^3}d^3r\rho(t,r)\int_{\mathbb{R}^3}d^3r'\tfrac{\varepsilon\partial\rho(t,r')/\partial t}{|r-r'|}=$$

$$=\int_{\mathbb{R}^3}d^3r\rho(t,r)\int_{\mathbb{R}^3}d^3r'\tfrac{\rho(t,r')}{|r-r'|}+\lim\nolimits_{\varepsilon\downarrow 0}\int_{\mathbb{R}^3}d^3r\rho(t,r)\int_{\mathbb{R}^3}d^3r'\tfrac{\varepsilon<\nabla',J(t;r')>}{|r'-r|}=\int_{\mathbb{R}^3}d^3r\rho(t,r)\int_{\mathbb{R}^3}d^3r'\tfrac{\rho(t,r')}{|r-r'|}$$

$$-\lim\nolimits_{\varepsilon\downarrow 0}\int_{\mathbb{R}^3}d^3r\rho(t,r)\int_{\mathbb{R}^3}d^3r'<\tfrac{\varepsilon u}{|r'-r|},\tfrac{r'-r}{|r-r'|^2}>\rho(t;r'):=2\mathcal{E}_{es}-m_\xi|u|^2, \tag{4.64}$$

where we denoted the averaged, as $\varepsilon\downarrow 0$, limiting integral expression by

$$\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^3}d^3r\rho(t,r)\int_{\mathbb{R}^3}d^3r'<\frac{\varepsilon u}{|r'-r|},\frac{r'-r}{|r-r'|^2}>\rho(t;r'):=m_\xi|u|^2. \tag{4.65}$$

This expression depends strongly on the internal electron structure, thus ensuring its stability. The same regularization scheme applied to the expression $\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^3}d^3r'\partial J(t-\varepsilon,r')/\partial t$ does not change its value.

Thus, making use of the expressions (4.63), (4.64), the Lagrangian function (4.62) yields

$$\mathcal{L}^{(t)}_{f-p}=\frac{\mathcal{E}^{es}}{6c^2}|dr/dt|^2-<k(t),dr/dt>-\mathcal{E}_{es}(1-|u|^2/c^2)-m_\xi|u|^2/2, \tag{4.66}$$

where

$$\mathcal{E}_{es}:=\frac{1}{2}\int_{\mathbb{R}^3}d^3r\int_{\mathbb{R}^3}d^3r'\frac{\rho(t,r')\rho(t,r')}{|r-r'|} \tag{4.67}$$

is the electrostatic energy of the charged particle ξ.

To obtain the corresponding evolution equation for our charged particle ξ we need, within the Feynman proper time paradigm, to transform the Lagrangian function (4.66) to one with respect to the charged particle proper time reference frame $\mathcal{K}_\tau$:

$$\mathcal{L}^{(\tau)}_{f-p}=(m_{es}/6)|\dot r|^2(1+|\dot r|^2/c^2)^{-1/2}-m_{es}c^2(1+|\dot r|^2/c^2)^{-1/2}- \tag{4.68}$$

$$-<k(t),\dot r>-m_\xi|\dot r|^2/2(1+|\dot r|^2/c^2)^{-1/2}$$

where $\dot{r} := dr/d\tau$ is the charged particle ξ velocity with respect to the proper reference frame $\mathcal{K}_\tau$ and by $m_{es} := \mathcal{E}_{es}/c^2$ is its so called electrostatic mass.

As a result, the generalized charged particle ξ momentum up to $O(1/c^4)$ equals

$$\pi_p := \partial \mathcal{L}_{f-p}^{(\tau)}/\partial r = \tfrac{1}{3}\tfrac{m_{es}\dot{r}}{(1+|\dot{r}|^2/c^2)^{1/2}} - \tfrac{m_{es}|\dot{r}|^2\dot{r}}{6c^2(1+|\dot{r}|^2/c^2)^{3/2}} + \tfrac{m_{es}\dot{r}}{(1+|\dot{r}|^2/c^2)^{1/2}} -$$
$$-k(t) - \tfrac{m_\xi \dot{r}}{(1+|\dot{r}|^2/c^2)^{1/2}} + \tfrac{m_\xi |\dot{r}|^2\dot{r}}{2(1+|\dot{r}|^2/c^2)^{1/2}} \simeq \tfrac{1}{3}m_{es}u(1-|u|^2/c^2)^{1/2} + \tag{4.69}$$
$$+m_{es}u(1-|u|^2/c^2)^{1/2} - k(t) - m_\xi u \simeq (-m_\xi + \tfrac{4}{3}m_{es})u - k(t),$$

where $u := dr/dt$ is the charged particle ξ velocity with respect to the laboratory reference frame $\mathcal{K}_t$. The generalized momentum (4.69) satisfies the following evolution equation with respect to $\mathcal{K}_\tau$

$$d\pi_p/d\tau := \partial \mathcal{L}_{f-p}^{(\tau)}/\partial r = 0, \tag{4.70}$$

which is equivalent to, with respect to the laboratory reference frame $\mathcal{K}_t$, the Lorentz type equation

$$\frac{d}{dt}(-m_\xi u + \frac{4}{3}m_{es}u) = -dk(t)/dt. \tag{4.71}$$

The evolution Equation (4.70) allows the corresponding canonical Hamiltonian formulation on the phase space $T^*(\mathbb{R}^3)$ with the Hamiltonian function

$$H_{f-p} := < \pi_p, \dot{r} > - \mathcal{L}_{f-p}^{(\tau)} \simeq < \tfrac{1}{3}\tfrac{m_{es}\dot{r}}{(1+|\dot{r}|^2/c^2)^{1/2}} + \tfrac{m_{es}\dot{r}}{(1+|\dot{r}|^2/c^2)^{1/2}} -$$
$$-k(t) - \tfrac{m_\xi \dot{r}}{(1+|\dot{r}|^2/c^2)^{1/2}}, \dot{r} > -(m_{es}/6)|\dot{r}|^2(1+|\dot{r}|^2/c^2)^{-1/2} +$$
$$+m_{es}c^2(1+|\dot{r}|^2/c^2)^{-1/2} + < k(t), \dot{r} > +(m_\xi/2)|\dot{r}|^2(1+|\dot{r}|^2/c^2)^{-1/2} =$$
$$= \tfrac{1}{3}m_{es}|\dot{r}|^2(1+|\dot{r}|^2/c^2)^{-1/2} + \ m_{es}|\dot{r}|^2(1+|\dot{r}|^2/c^2)^{-1/2} - < k(t), \dot{r} > - \tag{4.72}$$
$$-m_\xi|\dot{r}|^2(1+|\dot{r}|^2/c^2)^{-1/2} - (m_{es}/6)|\dot{r}|^2(1+|\dot{r}|^2/c^2)^{-1/2} +$$
$$+m_{es}c^2(1+|\dot{r}|^2/c^2)^{-1/2} + < k(t), \dot{r} > +(m_\xi/2)|\dot{r}|^2(1+|\dot{r}|^2/c^2)^{-1/2}$$
$$\simeq [\ \tfrac{(-m_\xi + m_{es}/3)|\pi_p + k(t)|^2}{2(-m_\xi + 4m_{es}/3)^2} + m_{es}c^2](1 - \tfrac{|\pi_p + k(t)|^2}{(-m_\xi + 4m_{es}/3)^2 c^2}\ )^{-1/2}$$

satisfying for all τ,$t \in \mathbb{R}$ the conservation conditions

$$\frac{d}{d\tau}H_{f-p} = 0 = \frac{d}{dt}H_{f-p}\,. \tag{4.73}$$

To determine the damping radiation force $k(t) \in \mathbb{E}^3$, we can make use of the Lorentz type force expression (4.28) in the proper case $u = u_f \in T(\mathbb{R}^3)$ and obtain, as in [1], up to $O(1/c^4)$ accuracy, the resulting Abraham–Lorentz force as

$$\frac{d}{dt}(-m_\xi u + \frac{4}{3}m_{es}u) = \frac{2\xi^2}{3c^3}d^2u/dt^2. \tag{4.74}$$

Comparing the force expressions (4.71) and (4.74), one finds that, up to $O(1/c^4)$ accuracy,

$$k(t) = \frac{2\xi^2}{3c^3} du/dt, \tag{4.75}$$

which should be understood as a smooth function of the temporal parameter $t \in \mathbb{R}$. Moreover, looking at the Equation (4.74) one can define the physical observable charged particle ξ mass parameter as

$$m_{ph} := -m_\xi + \frac{4}{3} m_{es}. \tag{4.76}$$

For the mass parameter $m_\xi \in \mathbb{R}$ in the expression (4.76) to be determined, we need to analyze in detail the charged particle ξ stability condition and try to understand its relationship to the additional momentum production. Before proceeding to this analysis, we review some important results devoted to the stability problem of a charged particle such as an electron and try to determine a related additional momentum generation mechanism.

Remark 4.1. Some years ago in [58] a "solution" to the above "$4/3$-electron mass" problem was suggested that was expressed by the physical mass relationship (4.76) and formulated more than one hundred years ago by H. Lorentz and M. Abraham. Unfortunately, the "solution" appeared to be erroneous as one can easily see from the incorrect assumptions on which the work in [58] was based. The first one concerns the particle-field momentum conservation, taken there in the form

$$\frac{d}{dt}(p + \xi A) = 0 \tag{4.77}$$

and the second one is a speculation about the $1/2$-coefficient imbedded into the calculation of the Lorentz type self-interaction force

$$F := -\frac{1}{2c} \int_{\mathbb{R}^3} d^3 r \rho(t;r) \partial A(t;r)/\partial t. \tag{4.78}$$

There it was incorrectly argued by the reasoning that the expression (4.78) represents "... the interaction of a given element of charge with all other parts, otherwise we count twice that reciprocal action" (cited from [58], page 2710). This claim is fallacious as it was not taken into account the important fact that the interaction in the integral (4.78) is, in reality, retarded and it should be considered as that calculated for two virtually different charged particles, as in the classical works of H. Lorentz and M. Abraham. As for the first assumption (4.77), it suffices to recall that a vector of the field momentum $\xi A \in \mathbb{E}^3$ is not independent and is, in the charged particle model considered, strongly related to the local flow of the electromagnetic energy in the Lorentz constraint form:

$$\partial(\xi\varphi)/\partial t + < \nabla, c\xi A >= 0. \tag{4.79}$$

The constraint implies the validity of the Lienard–Wiechert expressions (4.62) for potentials for calculation of the integral (4.78), which was exploited in [58]. Thus, the Equation (4.77), following the classical Newton second law, should be replaced by

$$\frac{d}{dt'}(p + \xi A) = -\nabla(\xi\varphi') \tag{4.80}$$

written with respect to the reference frame $\mathcal{K}_{t'}$ subject to which the charged particle ξ is at rest. Taking into account that the relativistic relationships $dt = dt'(1 - |u|^2/c^2)^{1/2}$ and $\varphi' = \varphi(1 - |u|^2/c^2)^{1/2}$ hold with respect to the laboratory reference frame $\mathcal{K}_t$, it follows from (4.80) that

$$\begin{aligned} &\tfrac{d}{dt}(p + \xi A) = -\xi\nabla\varphi(1 - |u|^2/c^2) = \\ &= -\xi\nabla\varphi + \tfrac{\xi}{c}\nabla < u, u\varphi/c >= -\xi\nabla\varphi + \tfrac{\xi}{c}\nabla < u, A > . \end{aligned} \tag{4.81}$$

Here we made use of the well-known relationship $A = u\varphi/c$ for the vector potential generated by this charged particle ξ moving in space with the velocity $u \in T(\mathbb{R}^3)$ with respect to $\mathcal{K}_t$. Now from the Equation (4.81) one can derive the final expression for the evolution of the charged particle ξ momentum:

$$\begin{aligned} dp/dt &= -\xi\nabla\varphi - \frac{\xi}{c}dA/dt + \frac{\xi}{c}\nabla < u, A >= \\ &= -\xi\nabla\varphi - \frac{\xi}{c}\partial A/\partial t - \frac{\xi}{c} < u, \nabla > A + \frac{\xi}{c}\nabla < u, A >= \\ &= \xi E + \frac{\xi}{c}u \times (\nabla \times A) = \xi E + \frac{\xi}{c}u \times B, \end{aligned} \tag{4.82}$$

which is exactly the well-known Lorentz force expression, used in the works of H. Lorentz and M. Abraham.

Recently, there has been other interesting research devoted to this “4/3-electron mass” problem, amongst which we would like to mention [25,76], whose arguments are based on the charged shell electron model and are quite similar - each assumes a virtual interaction of the electron with the ambient “dark” radiation energy. This interaction was first clearly demonstrated in [25], where a suitable compensation mechanism for the related singular electrostatic Coulomb electron energy and the wide band vacuum electromagnetic radiation energy fluctuations deficit inside the electron shell was shown to be harmonically realized as the electron shell radius $a \to 0$. Moreover, this compensation occurs when the induced outward directed electrostatic Coulomb pressure on the whole electron coincides, up to the sign, with that induced by the above vacuum electromagnetic energy fluctuations outside the electron shell, as was manifested by their absence inside the electron shell.

Actually, the outward directed electrostatic spatial Coulomb pressure on the electron is

$$\eta_{coul} := \lim_{a\to 0} \frac{\varepsilon_0 |E|^2}{2}\bigg|_{r=a} = \lim_{a\to 0} \frac{\xi^2}{32\varepsilon_0\pi^2 a^4}, \tag{4.83}$$

where $E = \frac{\xi r}{4\pi\varepsilon_0 |r|^3} \in \mathbb{E}^3$ is the electrostatic field at point $r \in \mathbb{R}$ with respect to the electron center at the point $r = 0 \in \mathbb{R}$. The related inward directed vacuum electromagnetic fluctuations spatial pressure is

$$\eta_{vac} := \lim_{\Omega\to\infty} \frac{1}{3}\int_0^{\Omega} d\mathcal{E}(\omega), \tag{4.84}$$

where $d\mathcal{E}(\omega)$ is the electromagnetic energy fluctuations density for a frequency $\omega \in \mathbb{R}$, and $\Omega \in \mathbb{R}$ is the corresponding electromagnetic frequency cutoff. The integral (4.84) can be calculated by taking into account the quantum statistical recipe [32,35,92] that

$$d\mathcal{E}(\omega) := \hbar\omega \frac{d^3p(\omega)}{h^3}, \tag{4.85}$$

where the Planck constant $h := 2\pi\hbar$ and the electromagnetic wave momentum $p(\omega) \in \mathbb{E}^3$ satisfy the relativistic relationship

$$|p(\omega)| = \hbar\omega/c. \tag{4.86}$$

Whence, by substituting (4.86) into (4.85) one obtains

$$d\mathcal{E}(\omega) = \frac{\hbar\omega^3}{2\pi^2c^3}d\omega, \tag{4.87}$$

which implies, in view of (4.84), the following vacuum electromagnetic energy fluctuations spatial pressure

$$\eta_{vac} = \lim_{\Omega\to\infty} \frac{\hbar\Omega^4}{24\pi^2c^3}. \tag{4.88}$$

For the charged electron shell model to be stable at rest it is necessary to equate the inward (4.88) and outward (4.83) spatial pressures:

$$\lim_{\Omega\to\infty} \frac{\hbar\Omega^4}{24\pi^2c^3} = \lim_{a\to 0} \frac{\xi^2}{32\varepsilon_0\pi^2a^4}, \tag{4.89}$$

giving rise to the balance electron shell radius $a_b \to 0$ limiting condition:

$$a_b = \lim_{\Omega\to\infty} \left[\Omega^{-1} \left(\frac{3\xi^2c^2}{2\hbar} \right)^{1/4} \right]. \tag{4.90}$$

Simultaneously we can calculate the corresponding Coulomb and electromagnetic fluctuations energy deficit inside the electron shell:

$$\Delta W_b := \frac{1}{2}\int_{a_b}^{\infty} \varepsilon_0|E|^2d^3r - \int_0^{a_b} d^3r \int_0^{\Omega} d\mathcal{E}(\omega) = \frac{\xi^2}{8\pi\varepsilon_0a_b} - \frac{\hbar\Omega^4a_b^3}{6\pi c^3} = 0, \tag{4.91}$$

additionally ensuring the electron shell model stability.

Another important consequence of this pressure-energy compensation mechanism can be derived concerning the electron mass component $m_\xi \in \mathbb{R}$, entering the momentum expression (4.69) in the case of the electron movement. Namely, following the reasoning in [76], one can observe that during the electron movement there arises an additional hidden and not compensated for, velocity $u \in T(\mathbb{R}^3)$ directed electrostatic Coulomb surface self-pressure acting only on the *front half part* of the electron shell and equal to

$$\eta_{surf} := \frac{|E\xi|}{4\pi a_b^2}\frac{1}{2} = \frac{\xi^2}{32\pi\varepsilon_0a_b^4}, \tag{4.92}$$

apparently coinciding with the already compensated for outward directed electrostatic Coulomb spatial pressure (4.83). As it is evident that during the electron motion in space its surface electric current energy flow does not vanish [76], it follows that the electron momentum gains an additional mechanical impact, which can be expressed as

$$\pi_\xi := -\eta_{surf}\frac{4\pi a_b^3}{3c^2}u = -\frac{1}{3}\frac{\xi^2}{8\pi\varepsilon_0 a_b c^2}u = -\frac{1}{3}m_{es}u, \tag{4.93}$$

where we took into account that in this electron shell model the corresponding electrostatic electron mass equals its electrostatic energy

$$m_{es} = \frac{\xi^2}{8\pi\varepsilon_0 a_b c^2}. \tag{4.94}$$

Thus, one can claim that, owing to the structural stability of the electron shell model, its generalized self-interaction momentum $\pi_p \in T^*(\mathbb{R}^3)$ gains during the movement with velocity $u = dr/dt \in T(\mathbb{R}^3)$ the additional backward directed hidden impact (4.93), which can be identified with the momentum component

$$\pi_\xi = -m_\xi u, \tag{4.95}$$

entering the momentum expression (4.69). Owing to (4.75), this becomes

$$\begin{aligned} \pi_p &= (-m_\xi + \frac{4}{3}m_{es})u - \frac{2\xi^2}{3c^3}d^2u/dt^2 = \\ &= (-\frac{1}{3}m_{es} + \frac{4}{3}m_{es})u - \frac{2\xi^2}{3c^3}d^2u/dt^2 = \\ &= m_{es}u - \frac{2\xi^2}{3c^3}d^2u/dt^2, \end{aligned} \tag{4.96}$$

which strongly supports the electromagnetic origin of the electron mass that was first conceived by H. Lorentz and M. Abraham.

The result above makes it possible to reanalyze the calculation of the Lagrangian function (4.66), based on the averaged limiting integral expression (4.65), taking into account the electron shell model and its dynamical stability. In particular, the averaged limiting integral expression (4.65) can be calculated in the above dynamically stable electron shell model as follows:

$$\begin{aligned} &\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^3} d^3r\rho(t;r)\int_{\mathbb{R}^3} d^3r' < \tfrac{\varepsilon u}{|r'-r|}, \tfrac{r'-r}{|r-r'|^2} > \rho(t;r') \simeq \\ &\simeq \tfrac{1}{2}\lim_{\varepsilon\downarrow 0}\tfrac{1}{3}\int_{\mathbb{R}^3} d^3r\rho(t;r)\int_{\mathbb{R}^3} d^3r' < \tfrac{\varepsilon u}{|r'-r|}, \tfrac{\varepsilon u}{\varepsilon^2 c^2} > \rho(t;r') = \\ &= \tfrac{1}{2}\cdot\tfrac{2\mathcal{E}_{es}}{3}|u|^2 = \tfrac{1}{3}m_{es}|u|^2 := m_\xi|u|^2. \end{aligned} \tag{4.97}$$

Here, we took into account that, owing to the retarded electron self-interaction, only one half of the charged electron shell, separated by the distance $|r' - r| = \varepsilon c$, generates an additional impact in the Lagrangian function (4.66), as the second half is shadowed by the electron shell interior with the absent electric field. Thus, upon substituting $m_\xi = \frac{1}{3}m_{es}$ into the final electron physical mass expression (4.76), one obtains

$$m_{ph} := -\frac{1}{3}m_{es} + \frac{4}{3}m_{es} = m_{es}, \tag{4.98}$$

which also supports the Abraham–Lorentz suggestion about the origin of the electromagnetic electron mass.

4.3.2. Comments

The electromagnetic mass origin problem was reanalyzed in details within the Feynman proper time paradigm and related vacuum field theory approach by means of the fundamental least action principle and the Lagrangian and Hamiltonian formalisms. The resulting electron inertia appeared to coincide in part, in the quasi-relativistic limit, with the momentum expression obtained more than one hundred years ago by M. Abraham and H. Lorentz [93–96], yet it proved to contain an additional hidden impact owing to the imposed electron stability constraint, which was taken into account in the original action functional as some preliminarily undetermined constant component. As it was demonstrated in [25,76], this stability constraint can be successfully realized within the charged shell model of electron at rest, if to take into account the existing ambient electromagnetic "dark" energy fluctuations, whose inward directed spatial pressure on the electron shell is compensated by the related outward directed electrostatic Coulomb spatial pressure as the electron shell radius satisfies some limiting compatibility condition. The latter also allows to compensate simultaneously the corresponding electromagnetic energy fluctuations deficit inside the electron shell, thereby forbidding the external energy to flow into the electron. In contrary to the lack of energy flow inside the electron shell, during the electron movement the corresponding internal momentum flow is not vanishing owing to the non vanishing hidden electron momentum flow caused by the surface pressure flow and compensated by the suitably generated surface electric current flow. As it was shown, this backward directed hidden momentum flow makes it possible to justify the corresponding self-interaction electron mass expression and to state, within the electron shell model, the fully electromagnetic electron mass origin, as it has been conceived by H. Lorentz and M. Abraham and strongly supported by R. Feynman in his Lectures [9]. This consequence is also independently supported by means of the least action approach, based on the Feynman proper time paradigm and the suitably calculated regularized retarded electric potential impact into the charged particle Lagrangian function.

The charged particle radiation problem, revisited in this Section, allowed to conceive the explanation of the charged particle mass as that of a compact and stable object which should be exerted by a vacuum field interaction energy potential $\bar{W}: \ M^4 \to \mathbb{R}$ of negative sign as follows from (3.19). The latter can be satisfied iff the expression (3.18) holds, thereby imposing on the intrinsic charged particle structure [74] some nontrivial geometrical constraints. Moreover, as follows from the physically observed particle mass expressions (3.19) the electrostatic potential energy, having its origin in the repulsive force, does contribute to the full mass as its main energy component.

There exist different relativistic generalizations of the force expression (3.18), which suffer the same common physical inconsistency related to the no radiation effect of a charged particle in uniform motion.

Another deeply related problem to the radiation reaction force analyzed above is the search for an explanation to the Wheeler and Feynman reaction radiation mechanism, called the absorption radiation theory, strongly based on the Mach type interaction of a charged particle with the ambient vacuum electromagnetic medium. Concerning this problem, one can also observe some of its relationships with the one devised here within the vacuum field theory approach, but this question needs a more detailed and extended analysis.

5. Charged Point Particle Dynamics and a Hadronic String Model Analysis

5.1. Classical Relativistic Electrodynamics Foundations: A Charged Point Particle Analysis

It is well known [2,3,9,97] that the relativistic least action principle for a point charged particle ξ in the Minkowski space-time $M^4 \simeq \mathbb{E}^3 \times \mathbb{R}$ can be formulated on a time interval $[t_1, t_2] \subset \mathbb{R}$ (in the light speed units) as

$$\delta S^{(t)} = 0, \quad S^{(t)} := \int_{\tau(t_1)}^{\tau(t_2)} (-m_0 d\tau - \xi < \mathcal{A}, dx >_{M^4}) =$$
$$= \int_{s(t_1)}^{s(t_2)} (-m_0 < \dot{x}, \dot{x} >_{M^4}^{1/2} - \xi < \mathcal{A}, \dot{x} >_{M^4}) ds. \tag{5.1}$$

Here $\delta x(s(t_1)) = 0 = \delta x(s(t_2))$ are the boundary constraints, $m_0 \in \mathbb{R}_+$ is the so called particle rest mass, the four-vector $x := (r, t) \in M^4$ is the particle location in M^4 and $\dot{x} := dx/ds \in T(M^4)$ is the particle Euclidean "four-vector" velocity with respect to a laboratory reference frame $\mathcal{K}_t$, parameterized by means of the Minkowski space-time parameters $(r, s(t)) \in M^4$ and related to each other vis the infinitesimal Lorentz interval relationship

$$d\tau :=< dx, dx >_{M^4}^{1/2} := ds < \dot{x}, \dot{x} >_{M^4}^{1/2}, \tag{5.2}$$

$\mathcal{A} \in T^*(M^4)$ is an external electromagnetic four-vector potential, satisfying the classical Maxwell equations [2,3,9], $< \cdot, \cdot >_{\mathcal{H}}$ is the corresponding scalar product in a finite-dimensional vector space $\mathcal{H}$ and $T(M^4), T^*(M^4)$ are, respectively, the tangent and cotangent spaces [15,17,59,60,98] to the Minkowski space M^4. In particular, $< x, x >_{M^4} := t^2 - < r, r >_{\mathbb{E}^3}$ for any $x := (r, t) \in M^4$.

The subintegral expression in (5.1)

$$\mathcal{L}^{(t)} := -m_0 < \dot{x}, \dot{x} >_{M^4}^{1/2} - \xi < \mathcal{A}, \dot{x} >_{M^4}, \tag{5.3}$$

is the related Lagrangian function, whose first part is proportional to the particle world line length with respect to the proper rest reference frame $\mathcal{K}_\tau$ and the second part is proportional to the pure electromagnetic particle-field interaction with respect to the Minkowski laboratory reference frame $\mathcal{K}_t$. Moreover, the positive rest mass parameter $m_0 \in \mathbb{R}_+$ is introduced into (5.3) as an external physical ingredient, also describing the point particle with respect to the proper rest reference frame $\mathcal{K}_\tau$. The electromagnetic four-vector potential $\mathcal{A} \in T^*(M^4)$ is at the same time expressed as a four-vector, constructed and measured with respect to the Minkowski laboratory reference frame $\mathcal{K}_t$ that appears to be rather controversial from physical point of view, since the action functional (5.1) is forced to be extremal with respect to the laboratory reference frame $\mathcal{K}_t$. This, in particular, means that the actual physical motion of our charged point particle, realized with respect to the proper rest reference frame $\mathcal{K}_\tau$, somehow depends on external observation data [9,31,38,39,99] with respect to the chosen laboratory reference frame $\mathcal{K}_t$. This aspect was never discussed in the physical literature except by R. Feynman in [9], who argued that the relativistic expression for the classical Lorentz force has a physical sense only with respect to the Euclidean rest reference frame $\mathcal{K}_\tau$ variables $(r, \tau) \in \mathbb{E}^4$ related to the Minkowski laboratory reference frame $\mathcal{K}_t$ parameters $(r, t) \in M^4$ by means of the infinitesimal relationship

$$d\tau :=< dx, dx >_{M^4}^{1/2} = dt(1 - |u|^2)^{1/2}, \tag{5.4}$$

where $u := dr/dt \in T(\mathbb{E}^3)$ is the point particle velocity with respect to the reference frame $\mathcal{K}_t$.

It should be pointed out here that to be correct, it would be necessary to include in the action functional the additional part describing the electromagnetic field itself. But this part is not taken into account, since it is generally assumed [29,34,35,41–43,45,71,100] that the charged particle influence on the electromagnetic field is negligible. This is true if the particle charge value ξ is very small and the support $supp\mathcal{A} \subset M^4$ of the electromagnetic four-vector potential is compact. It is clear that in the case of two interacting charged particles, the above assumption is invalid, as it is necessary to take into account the relative motion of two particles and the varying interaction energy. This aspect of the action functional selection problem appears to be very important when one tries to analyze the related Lorentz type forces exerted on each other by charged particles. We shall return to this problem in the sequel.

Having calculated the least action condition (5.1), we easily obtain from (5.3) the classical relativistic dynamical equations

$$dP/ds := -\partial \mathcal{L}^{(t)}/\partial x = -\xi \nabla_x < \mathcal{A}, \dot{x} >_{M^4}, \qquad P := -\partial \mathcal{L}^{(t)}/\partial \dot{x} = m_0 \dot{x} < \dot{x}, \dot{x} >_{M^4}^{-1/2} + \xi \mathcal{A}, \tag{5.5}$$

where $P \in T^*(M^4)$ is the common particle-field momentum of the interacting system.

Now at $s = t \in \mathbb{R}$, by means of the standard infinitesimal change of variables (5.4) we can easily obtain from (5.5) the classical Lorentz force expression

$$dp/dt = \xi E + \xi u \times B \tag{5.6}$$

with the relativistic particle momentum and mass

$$p := mu, \quad m := m_0(1 - |u|^2)^{-1/2}, \tag{5.7}$$

the electric field

$$E := -\partial A/\partial t - \nabla \varphi \tag{5.8}$$

and the magnetic field

$$B := \nabla \times A, \tag{5.9}$$

where we have expressed the electromagnetic four-vector potential as $\mathcal{A} := (A, \varphi) \in T^*(M^4)$.

The Lorentz force (5.6), owing to our preceding assumption, is the force exerted by the external electromagnetic field on our charged point particle, whose charge ξ is so negligible that it does not exert any influence on the field. This fact becomes very important if we try to make use of the Lorentz force expression (5.6) for the case of two interacting charged particles, since then one cannot assume that ξ exerts negligible influence on other charged particle. Thus, the corresponding Lorentz force between two charged particles should be suitably altered. Nevertheless, modern physics [1,2,8,24,40,47,64,67] did not make this necessary Lorentz force modification, and the classical expression (5.6) is used just about everywhere. This situation was observed and analyzed concerning the related physical aspects in [21], where it was shown that the electromagnetic Lorentz force between two moving charged particles can be modified in such a way that it ceases to be dependent on their relative motion, which is contrary to the classical relativistic case.

Unfortunately, the least action principle approach to analyzing the Lorentz force structure in was completely ignored in [21], thereby giving rise to some incorrect and physically unmotivated conclusions concerning the mathematical physics foundations of modern electrodynamics. To make the problem more transparent, we will analyze it in the next section from the vacuum field theory approach recently devised in [18,19,30].

5.2. Least Action Principle Analysis

Consider the least action principle (5.1) and observe that the extremality condition

$$\delta S^{(t)} = 0, \qquad \delta x(s(t_1)) = 0 = \delta x(s(t_2)) \tag{5.10}$$

is calculated with respect to the laboratory reference frame $\mathcal{K}_t$, whose point particle coordinates $(r,t) \in M^4$ are parameterized by means of an arbitrary parameter $s \in \mathbb{R}$ owing to expression (5.2). Recalling now the definition of the invariant proper rest reference frame $\mathcal{K}_\tau$ time parameter (5.4), we compute that at the critical parameter value $s = \tau \in \mathbb{R}$ the action functional (5.1) on the fixed interval $[\tau_1, \tau_2] \subset \mathbb{R}$ is

$$S^{(t)} = \int_{\tau_1}^{\tau_2} (-m_0 - \xi < \mathcal{A}, \dot{x} >_{M^4}) d\tau \tag{5.11}$$

under the additional constraint

$$< \dot{x}, \dot{x} >_{M^4}^{1/2} = 1, \tag{5.12}$$

where $\dot{x} := dx/d\tau$, $\tau \in \mathbb{R}$.

The expressions (5.11) and (5.12) require comment since the Lagrangian function

$$\mathcal{L}^{(t)} := -m_0 - \xi < \mathcal{A}, \dot{x} >_{M^4} \tag{5.13}$$

corresponding to (5.11) depends only virtually on the unobservable rest mass parameter $m_0 \in \mathbb{R}$ and, evidently, it has no direct impact on the resulting particle dynamical equations following from the condition $\delta S^{(t)} = 0$. However, the rest mass springs up as a suitable Lagrangian multiplier owing to the imposed constraint (5.12). To demonstrate this, consider the extended Lagrangian function (5.13) in the form

$$\mathcal{L}_\lambda^{(t)} := -m_0 - \xi < \mathcal{A}, \dot{x} >_{M^4} - \lambda(< \dot{x}, \dot{x} >_{M^4}^{1/2} - 1), \tag{5.14}$$

where $\lambda \in \mathbb{R}$ is a suitable Lagrangian multiplier. The resulting Euler equations are

$$\begin{gathered} P_r := \partial \mathcal{L}_\lambda^{(t)} / \partial \dot{r} = \xi A + \lambda \dot{r}, \;\; P_t := \partial \mathcal{L}_\lambda^{(t)} / \partial \dot{t} = -\xi \varphi - \lambda \dot{t}, \\ \partial \mathcal{L}_\lambda^{(t)} / \partial \lambda = < \dot{x}, \dot{x} >_{M^4}^{1/2} - 1 = 0, \;\; dP_r/d\tau = \xi \nabla_r < A, \dot{r} >_{\mathbb{E}^3} - \xi \dot{t} \nabla_r \varphi, \\ dP_t/d\tau = \xi < \partial A/\partial t, \dot{r} >_{\mathbb{E}^3} - \xi \dot{t} \partial \varphi / \partial t, \end{gathered} \tag{5.15}$$

giving rise, owing to relationship (5.4), to the following dynamical equations:

$$\frac{d}{dt}(\lambda u \dot{t}) = \xi E + \xi u \times B, \;\; \frac{d}{dt}(\lambda \dot{t}) = \xi < E, u >_{\mathbb{E}^3} . \tag{5.16}$$

Here

$$E := -\partial A/\partial t - \nabla\varphi, \;\; B = \nabla \times A \tag{5.17}$$

are the corresponding electric and magnetic fields. As a simple consequence of (5.16), one obtains

$$\frac{d}{dt}\ln(\lambda\dot{t}) + \frac{d}{dt}\ln(1-|u|^2)^{1/2} = 0, \tag{5.18}$$

which is equivalent for all $t \in \mathbb{R}$, to the relationship

$$\lambda\dot{t}(1-|u|^2)^{1/2} = \lambda := m_0 \tag{5.19}$$

in virtue of the relationship (5.4), where $m_0 \in \mathbb{R}_+$ is a constant, which could be interpreted as the rest mass of our charged point particle ξ. Actually, the first equation of (5.16) can be rewritten as

$$dp/dt = \xi E + \xi u \times B, \tag{5.20}$$

where

$$p := mu, \; m := \lambda\dot{t} = m_0(1-|u|^2)^{-1/2}, \tag{5.21}$$

coinciding exactly with that of (5.4).

Thus, we have retrieved all of the results obtained in section above, making use of the action functional (5.11), represented with respect to the rest reference frame $\mathcal{K}_\tau$ under the constraint (5.12). During these derivations, we faced a very delicate inconsistency property in the definition of the action functional $S^{(t)}$, defined with respect to the rest reference frame $\mathcal{K}_\tau$ and depending on the external electromagnetic potential function $\mathcal{A} : M^4 \to T^*(M^4)$ constructed with respect to the laboratory reference frame $\mathcal{K}_t$. Namely, this potential function, as a physically observable quantity, is defined and measurable only with respect to the fixed laboratory reference frame $\mathcal{K}_t$. This, in particular, means that a physically reasonable action functional should be constructed by means of an expression depending strongly on the laboratory reference frame $\mathcal{K}_t$ by means of coordinates $(r,t) \in M^4$ and subsequently transformed with respect to the rest reference frame $\mathcal{K}_\tau$ coordinates $(r,\tau) \in \mathbb{E}^4$, to obtain the charged point particle ξ motion. Thus, the corresponding action functional should initially be cast in the form

$$S^{(\tau)} = \int_{t(\tau_1)}^{t(\tau_2)} (-\xi < \mathcal{A}, \dot{x} >_{\mathbb{E}^3})dt, \tag{5.22}$$

where $\dot{x} := dx/dt$, $t \in \mathbb{R}$ is calculated on a time interval $[t(\tau_1), t(\tau_2)] \subset \mathbb{R}$, suitably related to the proper motion of the charged point particle ξ on the true time interval $[\tau_1, \tau_2] \subset \mathbb{R}$ with respect to the rest reference frame $\mathcal{K}_\tau$ and whose charge value is assumed to be so negligible that it exerts no influence on the external electromagnetic field. The problem now arises: how does one correctly compute the variation $\delta S^{(\tau)} = 0$ of the action functional (5.22)?

To reply to this question, we turn to Feynman [9], where he argued, when deriving the relativistic Lorentz force expression, that the real charged particle dynamics can be physically and unambiguously determined only with respect to the rest reference frame time parameter. In particular, Feynman wrote: "...we calculate a growth Δx for a small time interval Δt. But in the other reference frame the interval

Δt may correspond to changing both t' and x', thereby at the change of the only t' the suitable change of x will be other... Making use of the quantity $d\tau$ one can determine a good differential operator $d/d\tau$, as it is invariant with respect to the Lorentz reference frames transformations". This means that if our charged particle ξ moves in the Minkowski space M^4 during the time interval $[t_1, t_2] \subset \mathbb{R}$ with respect to the laboratory reference frame $\mathcal{K}_t$, its proper real and invariant time of motion with respect to the rest reference frame $\mathcal{K}_r$ will be $[\tau_1, \tau_2] \subset \mathbb{R}$.

As a corollary of Feynman's arguments, we arrive at the necessity to rewrite the action functional (5.22) as

$$S^{(\tau)} = \int_{\tau_1}^{\tau_2} (-\xi < \mathcal{A}, \dot{x} >_{M^4}) d\tau, \quad \delta x(\tau_1) = 0 = \delta x(\tau_2), \tag{5.23}$$

where $\dot{x} := dx/d\tau$, $\tau \in \mathbb{R}$, under the additional constraint

$$< \dot{x}, \dot{x} >_{M^4}^{1/2} = 1, \tag{5.24}$$

which is equivalent to the infinitesimal transformation (5.4). Simultaneously the proper time interval $[\tau_1, \tau_2] \subset \mathbb{R}$ is mapped on the time interval $[t_1, t_2] \subset \mathbb{R}$ by means of the infinitesimal transformation

$$dt = d\tau(1 + |\dot{r}|^2)^{1/2}, \tag{5.25}$$

where $\dot{r} := dr/d\tau$, $\tau \in \mathbb{R}$. Thus, we can now pose the true least action problem equivalent to (5.23) as

$$\delta S^{(\tau)} = 0, \qquad \delta r(\tau_1) = 0 = \delta r(\tau_2), \tag{5.26}$$

where the functional

$$S^{(\tau)} = \int_{\tau_1}^{\tau_2} [-\bar{W}(1 + |\dot{r}|^2)^{1/2} + \xi < A, \dot{r} >_{\mathbb{E}^3}] d\tau \tag{5.27}$$

is characterized by the Lagrangian function

$$\mathcal{L}^{(\tau)} := -\bar{W}(1 + |\dot{r}|^2)^{1/2} + \xi < A, \dot{r} >_{\mathbb{E}^3} \tag{5.28}$$

Here, $\bar{W} := \xi\varphi$, which is a suitable vacuum field [18,19,21,30] potential function. The resulting Euler equation gives rise to the following relationships

$$\begin{aligned} P &:= \partial \mathcal{L}^{(\tau)}/\partial \dot{r} = -\bar{W}\dot{r}(1 + |\dot{r}|^2)^{-1/2} + \xi A, \\ dP/d\tau &:= \partial \mathcal{L}^{(\tau)}/\partial r = -\nabla \bar{W}(1 + |\dot{r}|^2)^{1/2} + \xi \nabla < A, \dot{r} >_{\mathbb{E}^3}. \end{aligned} \tag{5.29}$$

Making now use once more of the infinitesimal transformation (5.25) and the crucial dynamical particle mass definition [18,19,21] (in the light speed units)

$$m := -\bar{W} \tag{5.30}$$

we can easily rewrite Equations (5.29) with respect to the parameter $t \in \mathbb{R}$ as the classical relativistic Lorentz force:

$$dp/dt = \xi E + \xi u \times B, \tag{5.31}$$

where

$$p := mu, \qquad u := dr/dt, \tag{5.32}$$
$$B := \nabla \times A, \quad E := -\xi^{-1}\nabla \bar{W} - \partial A/\partial t.$$

Thus, we have again obtained the relativistic Lorentz force expression (5.31), which is slightly different from (5.6), since the classical relativistic momentum expression of (5.7) does not completely coincide with our modified relativistic momentum expression

$$p = -\bar{W}u \tag{5.33}$$

that is strongly on the scalar vacuum field potential function $\bar{W} : M^4 \to \mathbb{R}$. But if we recall that our action functional (5.23) was written under the assumption that the particle charge value ξ is negligible and does not influence on the electromagnetic field source, we can make use of the result in [19,21,30] that the vacuum field potential function $\bar{W} : M^4 \to \mathbb{R}$, owing to (5.31)-(5.33), satisfies as $\xi \to 0$ the dynamical equation

$$d(-\bar{W}u)/dt = -\nabla \bar{W}, \tag{5.34}$$

having the solution

$$-\bar{W} = m_0(1 - |u|^2)^{-1/2}, \; m_0 = -\left.\bar{W}\right|_{u=0}. \tag{5.35}$$

Accordingly we have arrived, owing to (5.35) and (5.33), at an almost complete agreement of our result (5.31) for the relativistic Lorentz force with that of (5.6) under the condition $\xi \to 0$. Moreover, we have also proved the following result.

Proposition 5.1. *Under the assumption of negligible influence of a charged point particle ξ on an external electromagnetic field source, a true physically reasonable action functional can be given by expression (5.22), which is equivalently defined with respect to the rest reference frame $\mathcal{K}_\tau$ in form (5.23),(5.24). The resulting relativistic Lorentz force (5.31) coincides almost exactly with that of (5.6), obtained from the classical Einstein type action functional (5.1), but the momentum expression (5.33) differs from the classical expression (5.7), which takes into account the related vacuum field potential interaction energy impact.*

The next result then follows directly.

Corollary 5.2. *The Lorentz force expression (5.31) must in due course corrected in the case when the weak charge ξ influence assumption is not valid.*

Remark 5.3. The infinitesimal relationship (5.25) reflects the Euclidean nature of transformations $\mathbb{R} \ni t \rightleftharpoons \tau \in \mathbb{R}$.

In spite of the results obtained above by means of two different least action principles (5.1) and (5.23), we admit that the first one has some logical gaps, which may give rise to unpredictable, unexplainable and even nonphysical effects. Among these gaps we mention: $i)$ the definition of Lagrangian function (5.3) depends on the external and undefined rest mass parameter with respect to the rest reference frame $\mathcal{K}_r$ time $\tau \in \mathbb{R}$, and serving as a variational integrand with respect to the laboratory reference frame $\mathcal{K}_t$

time $t \in \mathbb{R}$; $ii)$ the least action condition (5.1) is calculated with respect to fixed boundary conditions at the ends of a time interval $[t_1, t_2] \subset \mathbb{R}$, so the resulting dynamics are strongly dependent on the chosen "laboratory reference frame $\mathcal{K}_t$, and following the Feynman argument [9,32], is physically unreasonable; $iii)$ the resulting relativistic particle mass and its energy depend only on the particle velocity in the laboratory reference frame $\mathcal{K}_t$ and do not take into account the vacuum field potential energy, which exerts a significant action on the particle motion; $iv)$ the negligible influence assumption for a charged point particle is also physically inconsistent.

6. The Dirac–Fock–Podolsky Problem and Symplectic Properties of the Maxwell and Yang–Mills Dynamical Systems

6.1. Introduction

When investigating different dynamical systems on canonical symplectic manifolds, invariant under the action of symmetry groups, additional mathematical structures often appear, the analysis of which shows their importance for understanding many related problems. For example there is the Cartan connection on an associated principal fiber bundle, which enables a more detailed description of the properties of a dynamical system in the case of its reduction on certain associated invariant submanifolds and quotient spaces.

Problems related to the investigation of properties of reduced dynamical systems on symplectic manifolds were studied, e.g., in [15,60–62,89], where the relationship between a symplectic structure on the reduced space and the available connection on a principal fiber bundle was formulated in explicit form. Other aspects of dynamical systems related to properties of reduced symplectic structures were studied in [63,101], where, in particular, the reduced symplectic structure was explicitly described in the framework of the classical Dirac scheme, and several applications to nonlinear (including celestial) dynamics were given.

It is well known [3,17,19,30,64,67] that the Hamiltonian theory of Maxwell's equations faces a very important classical problem; namely, introducing into its unique formalism the well-known Lorentz conditions, ensuring both the wave structure of propagating quanta and the positivity of energy. Unfortunately, in spite of classical studies on this problem by Dirac, Fock and Podolsky [69], the problem remains open, and the Lorentz condition is included in modern electrodynamics as an external constraint - and not one that follows from the initial Hamiltonian (or Lagrangian) theory. Moreover, when trying to quantize the electromagnetic theory, as was shown by Pauli, Dirac, Bogolubov and Shirkov and others [3,64,67], the quantum Lorentz condition could not be satisfied in the context of the existing quantum approaches, except in the average sense, since it is incompatible with the related quantum dynamics. This problem motivated us to seek a solution from the so-called symplectic reduction theory [62,89,101], which allows the systematic introduction into the Hamiltonian formalism the external charge and current conditions, giving rise to a partial solution to the Lorentz condition problem. Some applications of the method to Yang–Mills type equations interacting with a point charged particle, are presented in detail. In particular, based on the analysis of reduced geometric structures on fibered manifolds, invariant under the action of a symmetry group, we construct the symplectic structures associated with connection forms on suitable principal fiber bundles. We present mathematical preliminaries of the related Poissonian structures on the corresponding reduced symplectic manifolds, which are often used [15,63,101] in various problems of dynamics in modern mathematical physics. Then we apply them to study the non-standard Hamiltonian properties of the Maxwell and Yang–Mills type dynamical systems. In addition, we formulate a symplectic analysis of the important Lorentz type constraints, which describe the electrodynamic vacuum properties.

We formulate a symplectic reduction theory of the classical Maxwell electromagnetic field equations and prove [18] that the important Lorentz condition, ensuring the existence of electromagnetic waves [2,9,64], can be naturally included in the Hamiltonian picture, thereby solving the Dirac, Fock and Podolsky problem [69] mentioned above. Moreover, we use symplectic reduction theory to study the Poissonian structures and the classical minimal interaction principle related to Yang–Mills equations.

6.2. Hamiltonian Analysis of the Maxwell Dynamical Systems

We take the Maxwell electromagnetic equations to be

$$\begin{gathered} \partial E/\partial t = \nabla \times B - J, \quad \partial B/\partial t = -\nabla \times E, \\ < \nabla, E >= \rho, \qquad < \nabla, B >= 0 \end{gathered} \tag{6.1}$$

on the cotangent phase space $T^*(N)$ to $N \subset T(D;\mathbb{E}^3)$, which is the smooth manifold of smooth vector fields on an open domain $D \subset \mathbb{R}^3$, all expressed in light speed units. Here $(E, B) \in T^*(N)$ is a vector of electric and magnetic fields, $\rho : D \to \mathbb{R}$ and $J : D \to \mathbb{E}^3$ are, respectively, fixed charge and current densities in the domain D, satisfying the equation of continuity

$$\partial\rho/\partial t + < \nabla, J >= 0 \tag{6.2}$$

holding for all $t \in \mathbb{R}$, where the symbol "∇" is the gradient operation with respect to a variable $x \in D$, and $\times$ is the usual vector product in $\mathbb{E}^3 := (\mathbb{R}^3, < \cdot, \cdot >)$, which is the standard three-dimensional Euclidean vector space $\mathbb{R}^3$ endowed with the usual scalar product $< \cdot, \cdot >$.

Aiming to represent Equations (6.1) as those on reduced symplectic space, we define an appropriate configuration space $M \subset \mathcal{T}(D;\mathbb{E}^3)$ with a vector potential field coordinate $A \in M$. The cotangent space $T^*(M)$ may be identified with pairs $(A;Y) \in T^*(M)$, where $Y \in \mathcal{T}^*(D;\mathbb{E}^3)$ is a suitable vector field density in D. On the space $T^*(M)$ there exists the canonical symplectic form $\omega^{(2)} \in \Lambda^2(T^*(M))$, allowing, owing to the definition of the Liouville from

$$\lambda(\alpha^{(1)})(A;Y) = \int_D d^3x (< Y, dA >:= (Y, dA) \tag{6.3}$$

the canonical expression

$$\omega^{(2)} := d\lambda(\alpha^{(1)}) = (dY, \wedge dA). \tag{6.4}$$

Here $\wedge$ denotes exterior differentiation, d^3x, $x \in D$, is the Lebesgue measure in the domain D and $pr : T^*(M) \to M$ is the standard projection on the base space M. Now, we define a Hamiltonian function $\tilde{H} \in \mathcal{D}(T^*(M))$ as

$$H(A, Y) = 1/2[(Y, Y) + (\nabla \times A, \nabla \times A) + (< \nabla, A >, < \nabla, A >)] \tag{6.5}$$

describing the well-known vacuum Maxwell equations when the densities $\rho = 0$ and $J = 0$. Actually, owing to (6.4), one easily obtains from (6.5) the equations

$$\begin{aligned} \partial A/\partial t &:= \delta H/\delta Y = Y, \\ \partial Y/\partial t &:= -\delta H/\delta A = -\nabla \times B + \nabla < \nabla, A >, \end{aligned} \tag{6.6}$$

which are the true vacuum wave equations, where

$$B := \nabla \times A \tag{6.7}$$

is the corresponding magnetic field. Now defining

$$E := -Y - \nabla W \tag{6.8}$$

for some function $W : M \to \mathbb{R}$ as the corresponding electric field, the system of Equation (6.6) becomes, in view of definition (6.7),

$$\partial B/\partial t = -\nabla \times E, \ \ \partial E/\partial t = \nabla \times B \tag{6.9}$$

exactly coinciding with the Maxwell equations in a vacuum when the Lorentz condition

$$\partial W/\partial t + < \nabla, A >= 0 \tag{6.10}$$

is included.

Since definition (6.8) was essentially imposed rather than arising naturally from the Hamiltonian approach and our equations are valid only for a vacuum, we shall try to improve upon these matters by employing the reduction approach devised in Section 2. We start with the Hamiltonian (6.5) and observe that it is invariant with respect to the following abelian symmetry group $G := \exp \mathcal{G}$, where $\mathcal{G} \simeq C^{(1)}(D; \mathbb{R})$, acts on the base manifold M naturally lifted to $T^*(M)$; namely, for any $\psi \in \mathcal{G}$ and $(A, Y) \in T^*(M)$

$$\varphi_\psi(A) := A + \nabla\psi, \quad \varphi_\psi(Y) = Y. \tag{6.11}$$

The 1-form (6.3) is also invariant under the transformation (6.11) since

$$\begin{aligned} \varphi_\psi^* \lambda(\alpha^{(1)})(A, Y) = (Y, dA + \nabla d\psi) = \\ = (Y, dA) - (< \nabla, Y >, d\psi) = \lambda(\alpha^{(1)})(A, Y) \end{aligned} \tag{6.12}$$

where we used the condition $d\psi \simeq 0$ in $\Lambda^1(T^*(M))$ for any $\psi \in \mathcal{G}$. Thus, the corresponding momentum mapping (6.11) is given as

$$l(A, Y) = - < \nabla, Y > \tag{6.13}$$

for all $(A, Y) \in T^*(M)$. If $\rho \in \mathcal{G}^*$ is fixed, one can define the reduced phase space $\bar{\mathcal{M}}_\rho := l^{-1}(\rho)/G$, since the isotropy group $G_\rho = G$, owing to its commutativity and the condition (6.11). Consider now a principal fiber bundle $p : M \to N$ with the abelian structure group G and a base manifold N taken as

$$N := \{B \in \mathcal{T}(D; \mathbb{E}^3) : \quad < \nabla, \ B >= 0, \quad < \nabla, E(S) >= \rho\}, \tag{6.14}$$

where

$$p(A) := B = \nabla \times A. \tag{6.15}$$

We can construct a connection 1-form $\mathcal{A} \in \Lambda^1(M) \otimes \mathcal{G}$ on this bundle, where for all $A \in M$

$$\mathcal{A}(A) \cdot \hat{A}_*(l) = 1, \quad d < \rho, \mathcal{A}(A) >_{\mathcal{G}} = \Omega_\rho^{(2)}(A) \in H^2(M; \mathbb{Z}), \tag{6.16}$$

where $\mathcal{A}(A) \in \Lambda^1(M)$ is a differential 1-form, which we choose in the following form:

$$\mathcal{A}(A) := -(W, d < \nabla, A >) \tag{6.17}$$

where $W \in C^{(1)}(D; \mathbb{R})$ is a scalar function, not yet defined. As a result, the Liouville form (6.3) transforms into

$$\lambda(\tilde{\alpha}_\rho^{(1)}) := (Y, dA) - (W, d < \nabla, A >) = (Y + \nabla W, dA) := (\tilde{Y}, \ dA), \ \tilde{Y} := Y + \nabla W \tag{6.18}$$

giving rise to the corresponding canonical symplectic structure on $T^*(M)$ as

$$\tilde{\omega}_\rho^{(2)} := d\lambda(\tilde{\alpha}_\rho^{(1)}) = (d\tilde{Y}, \wedge dA). \tag{6.19}$$

Then the Hamiltonian function (6.5), as a function on $T^*(M)$, transforms into

$$\tilde{H}_\rho(A, \tilde{Y}) = 1/2[(\tilde{Y}, \tilde{Y}) + (\nabla \times A, \nabla \times A) + (< \nabla, A >, < \nabla, A >)] \tag{6.20}$$

coinciding with the well-known Dirac–Fock–Podolsky [64,69] Hamiltonian expression. The corresponding Hamiltonian equations on the cotangent space $T^*(M)$,

$$\begin{aligned} &\partial A/\partial t := \delta \tilde{H}/\delta \tilde{Y} = \tilde{Y}, \quad \tilde{Y} := -E - \nabla W, \\ &\partial \tilde{Y}/\partial t := -\delta \tilde{H}/\delta A = -\nabla \times (\nabla \times A) + \nabla < \nabla, A > \end{aligned}$$

describe true wave processes related to the Maxwell equations in a vacuum, which do not take into account boundary charge and current density conditions. Actually, from (6.20) we obtain

$$\partial^2 A/\partial t^2 - \nabla^2 A = 0 \Longrightarrow \partial E/\partial t + \nabla(\partial W/\partial t \ + \ < \nabla, A >) = -\nabla \times B \tag{6.21}$$

giving rise to the true vector potential wave equation, but the electromagnetic Faraday induction law is only satisfied if the Lorentz condition (6.10) is imposed.

To remedy this situation, we will apply the symplectic space reduction technique devised in Section 2. Namely, owing to the standard reduction theorem [15,59,61,101], the constructed cotangent manifold $T^*(N)$ is symplectomorphic to the corresponding reduced phase space $\bar{\mathcal{M}}_\rho$, that is

$$\bar{\mathcal{M}}_\rho \simeq \{(B; S) \in T^*(N) : < \nabla, E(S) >= \rho, \quad < \nabla, B >= 0\} \tag{6.22}$$

with the reduced canonical symplectic 2-form

$$\omega_\rho^{(2)}(B, S) = (dB, \wedge dS = d\lambda(\alpha_\rho^{(1)})(B, S), \quad \lambda(\alpha_\rho^{(1)})(B, S) := -(S, dB), \tag{6.23}$$

where

$$\nabla \times S + F + \nabla W = -\tilde{Y} := E + \nabla W, \quad < \nabla, F >:= \rho \tag{6.24}$$

for some fixed vector mapping $F \in C^{(1)}(D;\mathbb{E}^3)$, depending on the imposed boundary conditions. The result (6.23) follows directly from substituting the expression for the electric field $E = \nabla \times S + F$ into the symplectic structure (6.19), and taking into account that $dF = 0$ in $\Lambda^1(M)$. Then the Hamiltonian function (6.20) reduces to the following symbolic form:

$$H_\rho(B,S) = 1/2[(B,B) + (\nabla \times S + F + \nabla W, \nabla \times S + F + \nabla W) + \\ +(< \nabla, (\nabla\times)^{-1}B >, < \nabla, (\nabla\times)^{-1}B >)], \tag{6.25}$$

where $(\nabla\times)^{-1}$ denotes the corresponding inverse curl-operation, mapping [101] the divergence-free subspace $C^{(1)}_{div}(D;\mathbb{E}^3) \subset C^{(1)}(D;\mathbb{E}^3)$ into itself. Then, it follows from (6.25) that the Maxwell Equation (6.1) become a canonical Hamiltonian system on the reduced phase space $T^*(N)$, endowed with the canonical symplectic structure (6.23) and the modified Hamiltonian function (6.25). In fact, one easily calculates that

$$\partial S/\partial t := \delta H/\delta B = B - (\nabla\times)^{-1}\nabla < \nabla, (\nabla\times)^{-1}B >, \tag{6.26}$$
$$\partial B/\partial t := -\delta H/\delta S = -\nabla \times (\nabla \times S + F + \nabla W) := -\nabla \times E$$

where we made use of the definition $E = \nabla \times S + F$ and the elementary identity $\nabla \times \nabla = 0$. Thus, the second equation of (6.26) coincides with the second Maxwell equation of (6.1) in the classical form

$$\partial B/\partial t = -\nabla \times E.$$

Moreover, from (6.24), owing to (6.26), one finds via differentiation with respect to $t \in \mathbb{R}$ that

$$\partial E/\partial t = \partial F/\partial t + \nabla \times \partial S/\partial t = \\ = \partial F/\partial t + \nabla \times B \tag{6.27}$$

as well as, in virtue of (6.2),

$$< \nabla, \partial F/\partial t >= \partial\rho/\partial t = - < \nabla, J > \tag{6.28}$$

So, we can find from (6.28) that, up to the non-essential curl-terms $\nabla \times (\cdot)$, the following relationship

$$\partial F/\partial t = -J \tag{6.29}$$

holds. Actually, it follows from the equation of continuity (6.2) that the current density vector $J \in C^{(1)}(D;\mathbb{E}^3)$ is defined up to curl-terms $\nabla \times (\cdot)$ which can be included in the right-hand side of (6.29). Now, substituting (6.29) into (6.27), we obtain exactly the first Maxwell equation of (6.1):

$$\partial E/\partial t = \nabla \times B - J, \tag{6.30}$$

which is supplemented, naturally, with the external boundary constraint conditions

$$< \nabla, B >= 0, \quad < \nabla, E >= \rho, \\ \partial\rho/\partial t + < \nabla, J >= 0, \tag{6.31}$$

owing to the continuity relationship (6.2) and definition (6.22).

Concerning the wave equations related to the Hamiltonian system (6.26), it can readily be shown that the electric field E is recovered from the second equation as

$$E := -\partial A/\partial t - \nabla W, \tag{6.32}$$

where $W \in C^{(1)}(D;\mathbb{R})$ is a smooth function depending on the vector field $A \in M$. To determine this dependence, we substitute (6.29) into Equation (6.30), taking into account that $B = \nabla \times A$ which yields

$$\partial^2 A/\partial t^2 - \nabla(\partial W/\partial t + < \nabla, A >) = \nabla^2 A + J \tag{6.33}$$

With the above, if we now impose the Lorentz condition (6.10), we obtain from (6.33) the corresponding true wave equations in space-time, taking into account the external charge and current density conditions (6.31).

Notwithstanding our progress so far, the problem of naturally fulfilling the Lorentz constraint (6.10) in the canonical Hamiltonian formalism still remains to be completely solved. To this end, we are compelled to analyze the structure of the Liouville 1-form (6.18) for Maxwell equations in vacuo on a slightly extended functional manifold $M \times L$. As a first step, we rewrite the 1-form (6.18) as

$$\begin{aligned} \lambda(\tilde{\alpha}^{(1)}_\rho) := (\tilde{Y}, dA) = (Y + \nabla W, dA) = (Y, dA) + \\ +(W, -d < \nabla, A >) := (Y, dA) + (W, d\chi), \end{aligned} \tag{6.34}$$

where

$$\chi := - < \nabla, A > . \tag{6.35}$$

Considering now the elements $(Y, A; \chi, W) \in T^*(M \times L)$ as new canonical variables on the extended cotangent phase space $T^*(M \times L)$, where $L := C^{(1)}(D;\mathbb{R})$, we can rewrite the symplectic structure (6.19) in the following canonical form

$$\tilde{\omega}^{(2)}_\rho := d\lambda(\tilde{\alpha}^{(1)}_\rho) = (dY, \wedge dA) + (dW, \wedge d\chi) \tag{6.36}$$

Subject to the Hamiltonian function (6.20), we obtain the expression

$$H(A, Y; \chi, W) = 1/2[(Y - \nabla W, Y - \nabla W) + (\nabla \times A, \nabla \times A) + (\chi, \chi)] \tag{6.37}$$

with respect to which the corresponding Hamiltonian equations take the form:

$$\begin{aligned} \partial A/\partial t &:= \delta H/\delta Y = Y - \nabla W, \quad Y := -E, \\ \partial Y/\partial t &:= -\delta H/\delta A = -\nabla \times (\nabla \times A), \\ \partial \chi/\partial t &:= \delta H/\delta W =< \nabla, Y - \nabla W >, \\ \partial W/\partial t &:= -\delta H/\delta \chi = -\chi. \end{aligned} \tag{6.38}$$

From (6.38), we obtain the following using the external boundary conditions (6.31):

$$\begin{aligned} &\partial B/\partial t + \nabla \times E = 0, \quad \partial^2 W/\partial t^2 - \nabla^2 W = \rho, \\ &\partial E/\partial t - \nabla \times B = 0, \quad \partial^2 A/\partial t^2 - \nabla^2 A = -\nabla(\partial W/\partial t + < \nabla, A >). \end{aligned} \tag{6.39}$$

Clearly, these equations describe electromagnetic Maxwell equations in a vacuum, but without the Lorentz condition (6.10). Consequently, as above, we shall apply the reduction technique devised in Section 2 to the symplectic structure (6.36) . We find that under transformations (6.24) the corresponding reduced manifold $\bar{\mathcal{M}}_\rho$ becomes endowed with the symplectic structure

$$\bar{\omega}_\rho^{(2)} := (dB, \wedge dS) + (dW, \wedge d\chi) \tag{6.40}$$

and the Hamiltonian (6.37) assumes the form

$$H(S, B; \chi, W) = 1/2[(\nabla \times S + F + \nabla W, \nabla \times S + F + \nabla W) + (B, B) + (\chi, \chi)] \tag{6.41}$$

having Hamiltonian equations

$$\begin{aligned} \partial S/\partial t &:= \delta H/\delta B = B, \qquad \partial W/\partial t := -\delta H/\delta \chi = -\chi \\ \partial B/\partial t &:= -\delta H/\delta S = -\nabla \times (\nabla \times S + F + \nabla W) = -\nabla \times E \\ \partial \chi/\partial t &:= \delta H/\delta W = - < \nabla, \nabla \times S + F + \nabla W >= - < \nabla, E > -\Delta W \end{aligned} \tag{6.42}$$

coinciding with the Maxwell equation (6.1) under conditions (6.24). Thus, they describe true wave processes in a vacuum, as well as the electromagnetic Maxwell equations, taking into account *a priori* both the imposed external boundary conditions (6.31) and the Lorentz condition (6.10), and solving the problem mentioned in [64,69]. Actually, it follows readily from (6.42) that

$$\begin{gathered} \partial^2 W/\partial t^2 - \Delta W = \rho, \qquad \partial W/\partial t + < \nabla, A >= 0 \\ \nabla \times B = J + \partial E/\partial t, \qquad \partial B/\partial t = -\nabla \times E \end{gathered} \tag{6.43}$$

Now from (6.43) and (6.31) one can easily calculate [19,30] the magnetic wave equation

$$\partial^2 A/\partial t^2 - \Delta A = J \tag{6.44}$$

supplementing the suitable wave equation on the scalar potential $W \in L$ and finishing the calculations. Thus, we have proved the following result.

Proposition 6.1. *The electromagnetic Maxwell Equation (6.1) together with the Lorentz condition (6.10) are equivalent to the Hamiltonian system (6.42) with respect to the canonical symplectic structure (6.40) and Hamiltonian function (6.41), which reduce to the electromagnetic Equation (6.43) and (6.44) under the external boundary conditions (6.31).*

The above result can (eventually) be used to develop an alternative quantization procedure for Maxwell's electromagnetic equations that is free of some of the quantum operator problems discussed in detail in [64]. We hope to consider this aspect of the quantization problem in an investigation planned for the near future.

7. Conclusions

In this section we have demonstrated the complete legacy of the Feynman's approach to the Lorentz force based derivation of the Maxwell electromagnetic field equations. Moreover, we have succeeded in finding the exact relationship between Feynman's approach and the vacuum field approach of Sections 1–3, and introduced in [18,19]. The results obtained provide strong arguments for the deep physical foundations residing in the vacuum field theory approach, based on which one can describe the physical phenomena of electromagnetism and gravity, and (perhaps) eventually do both simultaneously. For gravity, the approach is physically based on the particle "inertial" mass expression (4.98), which follows naturally both from Feynman's proper time paradigm applied to the Lorentz force derivation, and from the vacuum field theory.

Acknowledgments

The authors are very thankful to Harold E. Puthoff for his correspondence, important remarks and suggestions, which markedly improved the exposition of our main results during the final preparation of this paper.

Author Contributions

This work was carried out in collaboration between all authors. Author Nikolai N. Bogolubov, Jr. designed the study, performed the analysis of the state-of-the-art in the field within the devised vacuum field theory approach to investigating electrodynamics models and wrote the first draft of the manuscript. Author Denis Blackmore and Author Anatolij K. Prykarpatski jointly developed the Lagrangian and Hamiltonian aspects of the vacuum field theory approach and analyzed in detail related physical properties of diverse electrodynamics models. All of the authors read and approved the final manuscript.

The authors declare no conflict of interest

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