Mathematics **2015**, *3*, 171-189; doi:10.3390/math3020171

OPEN ACCESS

mathematics

ISSN 2227-7390

www.mdpi.com/journal/mathematics

Article

Asymptotic Expansions of Fractional Derivatives and Their Applications

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Academic Editor: Hari M. Srivastava

Received: 03 March 2015 / Accepted: 07 April 2015 / Published: 15 April 2015

Abstract: We compare the Riemann–Liouville fractional integral (fI) of a function $f(z)$ with the Liouville fI of the same function and show that there are cases in which the asymptotic expansion of the former is obtained from those of the latter and the difference of the two fIs. When this happens, this fact occurs also for the fractional derivative (fD). This method is applied to the derivation of the asymptotic expansion of the confluent hypergeometric function, which is a solution of Kummer's differential equation. In the present paper, the solutions of the equation in the forms of the Riemann–Liouville fI or fD and the Liouville fI or fD are obtained by using the method, which Nishimoto used in solving the hypergeometric differential equation in terms of the Liouville fD.

Keywords: fractional derivative; asymptotic expansion; Kummer's differential equation; confluent hypergeometric function

1. Introduction

We study the asymptotic expansion of the Riemann–Liouville fractional integral (fI) and fractional derivative (fD) of a function $f(z)$, by using their relations with the corresponding Liouville fI and fD, respectively. We then present a method of deriving the asymptotic expansion of a function, when this is expressed by the Riemann–Liouville fI or fD of a function, which is analytic in a domain in the complex plane. As an example, we take up the confluent hypergeometric function.

We adopt the following definition of the Riemann–Liouville fI [1,2] (Section 2.3.2).

Definition 1. *Let $c \in \mathbb{C}$, $z \in \mathbb{C}$, $f(\zeta) \in \mathcal{L}^1(P(c,z))$ and $f(\zeta)$ be continuous in a neighborhood of $\zeta = z$. Then, the Riemann–Liouville fI of order $\lambda \in {}_+\mathbb{C}$ is defined by:*

$$ {}_R D_c^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_c^z (z-\zeta)^{\lambda-1} f(\zeta)\, d\zeta = \frac{1}{\Gamma(\lambda)} \int_0^{z-c} \eta^{\lambda-1} f(z-\eta)\, d\eta \tag{1} $$

Here, $P(c,z)$ is the path of integration from c to z, $f(\zeta) \in \mathcal{L}^1(P(c,z))$ denotes that the function $f(\zeta)$ is integrable on $P(c,z)$ and $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Z}$ represent the sets of all complex numbers, of all real numbers and of all integers, respectively. We also use notations ${}_+\mathbb{C} := \{z \in \mathbb{C} | \mathrm{Re}\, z > 0\}$, $\mathbb{Z}_{>a} := \{n \in \mathbb{Z} | n > a\}$, $\mathbb{Z}_{<b} := \{n \in \mathbb{Z} | n < b\}$, $\mathbb{Z}_{[a,b]} := \{n \in \mathbb{Z} | a \le n \le b\}$ for $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ and $\mathbb{R}_{>0} := \{x \in \mathbb{R} | x > 0\}$.

In [1], the ϕ-dependent (ϕ-dept) Liouville fI : ${}_L D_\phi^{-\lambda} f(z)$ of order $\lambda \in {}_+\mathbb{C}$ is defined for $\phi \in \mathbb{R}$. It was mentioned that it is equal to (1) if c is chosen to be $z + \infty \cdot e^{i\phi}$, in [3]. We now consider the paths of integration $P_\phi(z)$ and $P_\phi(c)$, which are from z to $z + \infty \cdot e^{i\phi}$ and from c to $c + \infty \cdot e^{i\phi}$, respectively, as shown in Figure 1. Here, we consider the cases in which $f(z)$ satisfies one of the following conditions.

Condition A. (a) $f(\zeta)$ is analytic on neighborhoods of the paths $P(c,z]$, $P_\phi[z]$ and $P_\phi(c)$, and (b) it is analytic also in the region enclosed by them.

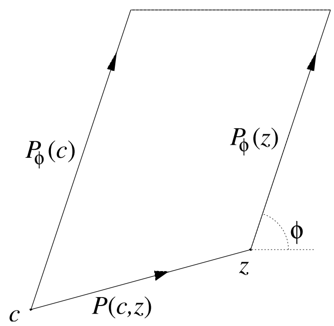

Figure 1. The paths of integration $P(c,z)$, $P_\phi(z)$ and $P_\phi(c)$.

Condition B. $f(\zeta)$ is expressed as $f(\zeta) = f_\gamma(\zeta) := (\zeta - c)^\gamma \cdot f_0(\zeta)$, where $\gamma \in \mathbb{C} \backslash \mathbb{Z}$ and $f_0(\zeta)$ satisfies Condition A with $P(c,z]$ replaced by $P[c,z]$.

Here, notations $P(c,z]$, $P_\phi[z]$ and $P_\phi(c)$ are used to denote that $f(\zeta)$ is analytic at the point $\zeta = z$, but not so at $\zeta = c$, and $P[c,z]$ and $P_\phi[c]$ are used when $f(\zeta)$ is analytic also at the point $\zeta = c$.

If the ϕ-dept Liouville fI exists, the difference of the Riemann–Liouville and ϕ-dept Liouville fIs is expressed by the path integral along the path $P_\phi(c)$. When the asymptotic expansions of the ϕ-dept Liouville fI and the last path integral are given, the asymptotic expansion of the Riemann–Liouville fI is obtained.

In [1], the fD corresponding to an fI is defined in the form of a contour integral, for a function $f(z)$, which is analytic on a neighborhood of the path of integration. They are analytic continuations of the corresponding fI. As a consequence, the relation between the Riemann–Liouville fI and the corresponding ϕ-dept Liouville fI is analytically continued to the relation between the corresponding fDs. It follows that the same recipe is useful in obtaining the asymptotic expansion of the Riemann–Liouville fD.

In Section 2, we first recall the expressions of the Riemann–Liouville fD, which are expressed by a contour integral. As examples of functions that are expressed by the Riemann–Liouville fD, we consider the incomplete gamma function $\gamma(\lambda, z)$ and the confluent hypergeometric function ${}_1F_1(a; b; z)$, in Sections 2.3 and 3, respectively.

In Section 4, we recall the expressions of the ϕ-dept Liouville fD, which are expressed by a contour integral. We then present the ϕ-dept Liouville fD of the functions of which the Riemann–Liouville fD is studied, in Sections 4.4 and 5.

We show the list of fI and fD, which we use in the present paper, with the places where they are defined.

Table 1. List of the fractional integral (fI) and fractional derivative (fD) and the places of their definitions.

	Riemann–Liouville fI and fD			**ϕ-dept Liouville fI and fD**		
fI	${}_RD_c^{-\lambda}f(z)$	Definition 1	Section 1	${}_LD_\phi^{-\lambda}f(z)$	Definition 6	Section 4.1
fD	${}_RD_c^{\nu}f(z)$	Definition 2	Section 2.1	${}_LD_\phi^{\nu}f(z)$	Definition 8	Section 4.2
	${}_CD_c^{\nu}f(z)$	Definition 3	Section 2.2	${}_HD_\phi^{\nu}f(z)$	Definition 9	Section 4.3
	${}_PD_c^{\nu}f_\gamma(z)$	Definition 4	Section 2.2	${}_HD_\phi^{\nu}f_\gamma(z)$	Definition 10	Section 4.3

In Section 6, we deform the path or contour of integration, which appears in the Riemann–Liouville fD, and show that the Riemann–Liouville fD of a function $f(z)$ is expressed as a sum of the ϕ-dept Liouville fD of $f(z)$ and a path or contour integral of $f(z)$. By writing the asymptotic expansions of the ϕ-dept Liouville fD, as well as of the path or contour integral that appears, we obtain the asymptotic expansions of the Riemann–Liouville fD under consideration. We confirm that we can obtain the asymptotic expansions of $\gamma(\lambda, z)$ and ${}_1F_1(a; b; z)$, by this procedure in Sections 6.1∼7.

The function ${}_1F_1(a; b; z)$ treated in Sections 3 and 7 is a solution of Kummer's differential equation (DE), and is expressed by a Riemann–Liouville fD. In Sections 5 and 7, we treat Kummer's function $U(a, b, z)$, which is another solution of Kummer's DE and is expressed by ϕ-dept Liouville fD. In Sections 3.1 and 5, we show that the expressions of these functions in the form of fD are obtained by the method, which Nishimoto [4] (Chapter 5, Section 2) used in obtaining the solution of the hypergeometric DE, that is the hypergeometric function, in the form of ϕ-dept Liouville fI. Concluding remarks are given in Section 8.

2. Riemann–Liouville fI and Its Analytic Continuations

2.1. Definition of Riemann–Liouville fD and the Index Law of Riemann–Liouville fI

Definition 2. *The Riemann–Liouville fD of order $\nu \in \mathbb{C}$ satisfying $\mathrm{Re}\,\nu \geq 0$ is defined by:*

$$_R D_c^{\nu} f(z) = D^m[{}_R D_c^{\nu-m} f(z)] \tag{2}$$

when the right-hand side (rhs) exists, where $m = \lfloor \mathrm{Re}\,\nu \rfloor + 1$, and $D^m f(z) = \frac{d^m}{dz^m} f(z) = f^{(m)}(z)$ for $m \in \mathbb{Z}_{>-1}$.

Here, $\lfloor x \rfloor$ for $x \in \mathbb{R}$ denotes the greatest integer not exceeding x.

Lemma 1. *Let $z = c + \zeta$ and $g(\zeta) := f(c + \zeta) = f(z)$ for $c \neq 0$. Then, ${}_R D_c^{\nu} f(z) = {}_R D_0^{\nu} g(z - c)$.*

Proof When $\nu = -\lambda$, (1) is expressed as:

$$_R D_c^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^{z-c} \eta^{\lambda-1} f(z-\eta)\, d\eta = \frac{1}{\Gamma(\lambda)} \int_0^{\zeta} \eta^{\lambda-1} g(\zeta-\eta)\, d\eta = {}_R D_0^{-\lambda} g(\zeta)$$

whose rhs represents ${}_R D_0^{-\lambda} g(z - c)$. ■

We use the following index law and Leibniz's rule, in Section 3.2.

Lemma 2. *Let $\lambda \in {}_+\mathbb{C}$, $\nu \in \mathbb{C}$ satisfy $\mathrm{Re}\,\nu \leq \mathrm{Re}\,\lambda$ and ${}_R D_c^{-\lambda} f(z)$ exist. Then:*

$$_R D_c^{\nu}[{}_R D_c^{-\lambda} f(z)] = {}_R D_c^{\nu-\lambda} f(z), \quad {}_R D_c^{\lambda}[{}_R D_c^{-\lambda} f(z)] = f(z) \tag{3}$$

A proof of this lemma for $\lambda \in \mathbb{R}_{>0}$ and $\nu \in \mathbb{R}$ is found in [2] (Section 2.6.6).

Remark 1. *In [5,6], the distribution theory in the space $\mathcal{D}'_R$ was developed, in which the index law $D^{\nu} D^{\mu} h(t) = D^{\nu+\mu} h(t)$ is always valid for $\nu \in \mathbb{C}$, $\mu \in \mathbb{C}$ and $h(t) \in \mathcal{D}'_R$, where D is the operator of differentiation in this space. Noting that all of the function and its fI and fD, which appear in (3), are regarded as regular distributions in $\mathcal{D}'_R$, we confirm the equalities in (3) in this standpoint.*

Lemma 3. *Let $\lambda \in {}_+\mathbb{C}$ and ${}_R D_c^{-\lambda} f(z)$ exist. Then:*

$$_R D_c^{-\lambda}[z \cdot f(z)] = z \cdot {}_R D_c^{-\lambda} f(z) - \lambda \cdot {}_R D_c^{-\lambda-1} f(z) \tag{4}$$

Proof We see that the both sides are equal to $\frac{1}{\Gamma(\lambda)} \int_0^{z-c} \eta^{\lambda-1}(z-\eta) f(z-\eta)\, d\eta$. ■

This Leibniz's rule is given in [7] (Section 5.5).

2.2. Analytic Continuations of Riemann–Liouville fI

In [1,8,9], analytic continuations of the Riemann–Liouville fI via contour integrals are discussed. In [1], ${}_C D_c^{\nu} f(z)$ and ${}_P D_c^{\nu} f(z)$ are defined as follows.

Definition 3. *Let $f(\zeta)$ be analytic on a neighborhood of the path $P(c, z]$ and be integrable on $P(c, z)$. Then, ${}_C D_c^\nu f(z)$ is defined by:*

$$ {}_C D_c^\nu f(z) = \frac{\Gamma(\nu+1)}{2\pi i} \int_c^{(z^+)} (\zeta - z)^{-\nu-1} f(\zeta) d\zeta \tag{5} $$

for $\nu \in \mathbb{C}\backslash\mathbb{Z}_{<0}$ where the contour of integration is the Cauchy contour $C(c, z^+)$ shown in Figure 2a, which starts from c, encircles the point z counterclockwise and goes back to c, without crossing the path $P(c, z)$. When $-n \in \mathbb{Z}_{<0}$, we put ${}_C D_c^{-n} f(z) := \lim_{\nu\to -n} {}_C D_c^\nu f(z)$, where $\nu \in \mathbb{C}\backslash\mathbb{Z}$.

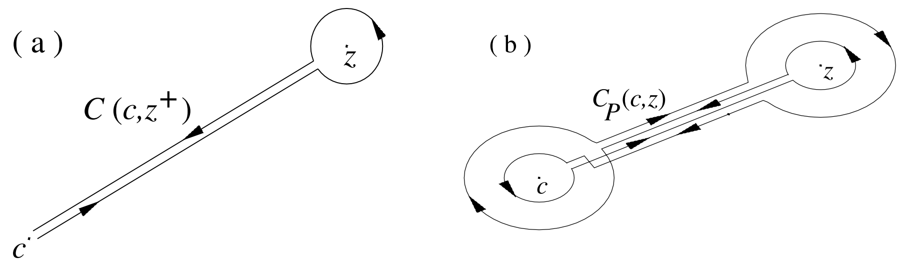

Figure 2. The contours of integration: **(a)** $C(c, z^+)$, **(b)** $C_P(c, z)$.

Definition 4. *Let $f_\gamma(\zeta) = (\zeta - c)^\gamma \cdot f_0(\zeta)$, $\gamma \in \mathbb{C}\backslash\mathbb{Z}$ and $f_0(\zeta)$ be analytic on a neighborhood of the path $P[c, z]$. Then, ${}_P D_c^\nu f_\gamma(z)$ is defined by:*

$$ {}_P D_c^\nu f_\gamma(z) = e^{-i\gamma\pi} \frac{\Gamma(\nu+1)}{4\pi \sin\gamma\pi} \int_{C_P(c,z)} (\zeta - z)^{-\nu-1} f_\gamma(\zeta) d\zeta \tag{6} $$

for $\nu \in \mathbb{C}\backslash\mathbb{Z}_{<0}$, where $C_P(c, z)$ is the Pochhammer contour shown in Figure 2b. When $-n \in \mathbb{Z}_{<0}$, we put ${}_P D_c^{-n} f_\gamma(z) := \lim_{\nu\to -n} {}_P D_c^\nu f_\gamma(z)$, where $\nu \in \mathbb{C}\backslash\mathbb{Z}$. When $n \in \mathbb{Z}_{>-1}$, we put ${}_P D_c^\nu f_n(z) := \lim_{\gamma\to n} {}_P D_c^\nu f_\gamma(z)$.

The following lemmas follow from the arguments there.

Lemma 4. *${}_C D_c^\nu f(z)$ is an analytic function of ν on $\mathbb{C}$.*

Lemma 5. *${}_P D_c^\nu f_\gamma(z)$ is analytic with respect to $\nu \in \mathbb{C}$, as well as to $\gamma \in \mathbb{C}\backslash\mathbb{Z}$.*

Lemma 6. *Let ${}_C D_c^\nu f(z)$ exist. Then, ${}_R D_c^\nu f(z)$ exists, and ${}_C D_c^\nu f(z) = {}_R D_c^\nu f(z)$.*

Lemma 7. *Let ${}_P D_c^\nu f_\gamma(z)$ exist. Then, if $\gamma + 1 \in {}_+\mathbb{C}$, ${}_C D_c^\nu f_\gamma(z)$ exists and ${}_P D_c^\nu f_\gamma(z) = {}_C D_c^\nu f_\gamma(z)$.*

In the following sections, we use ${}_C D_c^\nu f(z)$ and ${}_P D_c^\nu f_\gamma(z)$ for the Riemann–Liouville fD.

2.3. Incomplete Gamma Function

The incomplete gamma function $\gamma(\lambda, z)$, for $z \in \mathbb{C}$, is defined by:

$$ \gamma(\lambda, z) = \int_0^z \zeta^{\lambda-1} e^{-\zeta} d\zeta \tag{7} $$

when $\lambda \in {}_{+}\mathbb{C}$ and by its analytic continuation when $\mathrm{Re}\ \lambda \leq 0$ [10] (Section 12.22).

By comparing (7) and (1), we confirm the following lemma with the aid of Lemmas 6 and 4.

Lemma 8. *Let $\lambda \in \mathbb{C}\backslash\mathbb{Z}_{<1}$, $\nu \in \mathbb{C}\backslash\mathbb{Z}_{>-1}$, $z \in \mathbb{C}$ and $a \in \mathbb{C}\backslash\{0\}$. Then:*

$$\gamma(\lambda, z) = \Gamma(\lambda)e^{-z} \cdot {}_C D_0^{-\lambda} e^z, \quad {}_C D_0^{\nu} e^{az} = \frac{1}{\Gamma(-\nu)} a^{\nu} e^{az} \cdot \gamma(-\nu, az) \tag{8}$$

When $n \in \mathbb{Z}_{>-1}$, (5) shows that ${}_C D_0^n e^{az} = \frac{d^n}{dz^n} e^{az} = a^n e^{az}$.

3. Confluent Hypergeometric Function in Terms of Riemann–Liouville fD

For $a \in \mathbb{C}$, $z \in \mathbb{C}$ and $b \in \mathbb{C}\backslash\mathbb{Z}_{<1}$, the confluent hypergeometric series ${}_1F_1(a; b; z)$ is defined by:

$${}_1F_1(a; b; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{k!(b)_k} z^k \tag{9}$$

where $(z)_k$ for $z \in \mathbb{C}$ and $k \in \mathbb{Z}_{>-1}$ denotes $\prod_{l=0}^{k-1}(z + l)$ if $k \geq 1$ and one if $k = 0$. The integral representation of ${}_1F_1(a; b; z)$ is given by:

$${}_1F_1(a; b; z) = \frac{\Gamma(b)}{\Gamma(b - a)\Gamma(a)} \int_0^1 e^{zt} t^{a-1}(1 - t)^{b-a-1} dt \tag{10}$$

when $\mathrm{Re}\ a > 0$ and $\mathrm{Re}\ (b - a) > 0$, in [11] (Section 13.2.1).

This function is a solution of Kummer's DE for $a \in \mathbb{C}$, $b \in \mathbb{C}$ and $z \in \mathbb{C}$:

$$z\frac{d^2w}{dz^2} + (b - z)\frac{dw}{dz} - a \cdot w = 0 \tag{11}$$

This DE has also another solution given by:

$$z^{1-b} \cdot {}_1F_1(a - b + 1; 2 - b; z) \tag{12}$$

see [11] (Section 13.1.13).

3.1. Solution of Kummer's DE in Terms of Riemann–Liouville fD

The function ${}_1F_1(a; b; z)$ is known to be expressed in the form of (13) given below, in [8]. We now obtain the solutions of (11) expressed in terms of the Riemann–Liouville fD, by using the method that Nishimoto [4] adopted in deriving the solution of the hypergeometric DE, that is the hypergeometric function, in the form of the Liouville fD.

Proofs of the following two lemmas are presented in the following section.

Lemma 9. *There exist the following solutions of (11):*

$$w_l(z) = p_l(z)u_l(z), \quad l \in \mathbb{Z}_{[1,4]} \tag{13}$$

where:

$$p_1(z) = 1, \quad p_2(z) = e^z, \quad p_3(z) = z^{1-b}, \quad p_4(z) = z^{1-b}e^z \tag{14}$$

$$u_l(z) = \frac{\Gamma(2 - b_l)}{\Gamma(a_l - b_l + 1)} {}_P D_0^{a_l-1}[z^{a_l-b_l} e^{\delta_l \cdot z}] \tag{15}$$

The values of a_l, b_l, δ_l and their linear combinations are given in Table 2.

Table 2. Values of a_l, b_l, δ_l and their linear combinations.

l	a_l	b_l	δ_l	$a_l - b_l$	$a_l - b_l + 1$	$2 - b_l$
1	a	b	1	$a - b$	$a - b + 1$	$2 - b$
2	$b - a$	b	-1	$-a$	$-a + 1$	$2 - b$
3	$1 + a - b$	$2 - b$	1	$a - 1$	a	b
4	$1 - a$	$2 - b$	-1	$b - a - 1$	$b - a$	b

Lemma 10.

$$w_1(z) = z^{1-b} \cdot {}_1F_1(a - b + 1; 2 - b; z), \quad w_2(z) = z^{1-b} e^z \cdot {}_1F_1(1 - a; 2 - b; -z) \tag{16}$$

$$w_3(z) = {}_1F_1(a; b; z), \quad w_4(z) = e^z \cdot {}_1F_1(b - a; b; -z) \tag{17}$$

The following lemma is well known [11] (Sections 13.1.27, 13.1.28).

Remark 2. *The relations $w_1(z) = w_2(z)$ and $w_3(z) = w_4(z)$ are well known [11] (Sections 13.1.27, 13.1.28).*

3.2. Proofs of Lemmas 9 and 10

In place of (11), we now consider the solution of the equation:

$$z\frac{d^2w}{dz^2} + (b - \delta \cdot z)\frac{dw}{dz} - \delta \cdot a \cdot w = 0 \tag{18}$$

where $\delta \in \mathbb{C}$. In the proof of Lemma 9 given below, $u_l(z)$ in (13) are the solutions of (18) for $a = a_l$, $b = b_l$ and $\delta = \delta_l$, which are listed in Table 2.

The following lemma is obtained by using the method of Nishimoto [4] mentioned above.

Lemma 11. *Let* $\mathrm{Re}\ a < -1$ *and* $\mathrm{Re}\ (a - b) \geq 0$. *Then, a solution of (18) is given by:*

$$w(z) = {}_RD_0^{a-1}[z^{a-b}e^{\delta \cdot z}] \tag{19}$$

Proof We assume that a solution of (18) is expressed as $w(z) = {}_RD_0^{-\lambda}v(z)$ for $\lambda \in \mathbb{C}$ satisfying $\mathrm{Re}\ \lambda > 2$. Substituting this $w(z)$ in (18) and using Leibniz's rule given in Lemma 3, we obtain:

$${}_RD_0^{2-\lambda}[z \cdot v(z)] + {}_RD_0^{1-\lambda}[(\lambda - 2 + b - \delta \cdot z) \cdot v(z)] - \delta \cdot (\lambda - 1 + a) \cdot {}_RD_0^{-\lambda}v(z) = 0 \tag{20}$$

Putting $\lambda = 1 - a$ and hence assuming $\mathrm{Re}\ a < -1$, and applying ${}_RD_0^{\lambda-2}$ to (20), we obtain:

$$z \cdot v(z) + {}_RD_0^{-1}[(b - a - 1 - \delta \cdot z) \cdot v(z)] = 0 \tag{21}$$

with the aid of Lemma 2. This equation requires that:

$$\frac{d}{dz}[z \cdot v(z)] + (b - a - 1 - \delta \cdot z) \cdot v(z) = 0 \tag{22}$$

and $z \cdot v(z) = 0$ when $z = 0$. Now, we obtain $v(z) = z^{a-b}e^{\delta \cdot z}$. Thus, we obtain (19). ■

Proof of Lemma 9 The formula for $w_1(z)$ in (13) follows from Lemma 11 with the aid of Lemmas 5, 6 and 7. We next give a derivation of the formulas for $w_2(z)$ and $w_3(z)$ in (13). For $l = 2$ and $l = 3$, we put $w(z) = p_l(z)u_l(z)$ in (11) and obtain (18) with w, a, b and δ replaced by u_l, a_l, b_l and δ_l, respectively. We next give a derivation of the formula for $w_4(z)$ in (13). We put $w(z) = p_4(z)u_4(z) = p_3(z)p_2(z)u_4(z)$ in (11), and then, we obtain (18) with w, a, b and δ replaced by u_4, $a_4 = b_3 - a_3 = 1 - a$, $b_4 = b_3 = 2 - b$ and $\delta_4 = -\delta_3 = -1$, respectively. ■

Proof of Lemma 10 We put $q_l = \frac{\Gamma(2-b_l)}{\Gamma(a_l-b_l+1)\Gamma(-a_l+1)}$. By using Lemmas 7 and 6 and the last member of (1), (15) is expressed as:

$$u_l(z) = q_l \int_0^z \eta^{a_l-b_l}(z-\eta)^{-a_l}e^{\delta_l \cdot \eta}d\eta = q_l z^{1-b_l}\int_0^1 t^{a_l-b_l}(1-t)^{-a_l}e^{\delta_l \cdot zt}dt \tag{23}$$

when $1 + \text{Re}\,(a_l - b_l) > 0$ and $1 - \text{Re}\,a_l > 0$. This gives:

$$u_l(z) = q_l z^{1-b_l}\sum_{k=0}^{\infty}\frac{\delta_l^k \cdot z^k}{k!}\int_0^1 t^{a_l-b_l+k}(1-t)^{-a_l}dt = q_l z^{1-b_l}\Gamma(-a_l+1)\sum_{k=0}^{\infty}\frac{\Gamma(a_l-b_l+1+k)\cdot\delta_l^k}{k!\Gamma(2-b_l+k)}z^k$$

$$= z^{1-b_l}\sum_{k=0}^{\infty}\frac{(a_l-b_l+1)_k\cdot\delta_l^k}{k!(2-b_l)_k}z^k = z^{1-b_l}\cdot {}_1F_1(a_l-b_l+1;2-b_l;\delta_l\cdot z) \tag{24}$$

By using (24) in (13), we obtain (16)∼(17). ■

4. The ϕ-dept Liouville fI and Its Analytic Continuations

4.1. Definitions of ϕ-dept Liouville fI

Let $z \in \mathbb{C}$ and $\phi \in \mathbb{R}$. We denote the half line $\{z + te^{i\phi} |\ 0 < t < \infty\}$, as shown in Figure 1b, by $P_\phi(z)$. When $f(z + te^{i\phi})$ is locally integrable as a function of t in the interval $(0, \infty)$, we denote this by $f(z) \in \mathcal{L}^1_{loc}(P_\phi(z))$.

Definition 5. *Let $z \in \mathbb{C}$, $\phi \in \mathbb{R}$, $s \in \mathbb{R}$ and $f(\zeta) \in \mathcal{L}^1_{loc}(P_\phi(z))$. Let s_1 be such that $\int_1^\infty t^{-s-1}|f(z + te^{i\phi})|dt$ converges for $s > s_1$ and diverges for $s < s_1$. We then call s_1 the abscissa of convergence, and denote it by $s_1[f]$ or $s_1[f(z)]$. We then have $s_1 \in \mathbb{R}$ or $s_1 = -\infty$.*

We note that there exists such a series $\{t_l\}_{l\in\mathbb{Z}_{>0}}$ of $t_l \in \mathbb{R}_{>0}$, such that $t_l^{-s}|f(z + t_l e^{i\phi})| \to 0$ and $t_l \to \infty$ as $l \to \infty$, if $s > s_1[f]$.

Definition 6. *Let $z \in \mathbb{C}$, $\phi \in \mathbb{R}$, $f(\zeta) \in \mathcal{L}^1_{loc}(P_\phi(z))$ and $f(\zeta)$ be continuous in a neighborhood of $\zeta = z$. Let $\lambda \in {}_+\mathbb{C}$, $s_1[f] < 0$ and $\text{Re}\,\lambda < -s_1[f]$. Then, we define two types of fI, ${}_W D_\phi^{-\lambda} f(z)$ and ${}_L D_\phi^{-\lambda} f(z)$, by:*

$${}_W D_\phi^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)}\int_{P_\phi(z)}(\zeta - z)^{\lambda-1}f(\zeta)d\zeta = e^{i\phi\lambda}\frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}f(z + te^{i\phi})dt \tag{25}$$

$$
\begin{aligned}
{}_L D_\phi^{-\lambda} f(z) &= e^{i\pi\lambda} \cdot {}_W D_\phi^{-\lambda} f(z) = -\frac{1}{\Gamma(\lambda)} \int_{P_\phi(z)} (z-\zeta)^{\lambda-1} f(\zeta) d\zeta \\
&= e^{i(\phi+\pi)\lambda} \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} f(z - te^{i(\phi+\pi)}) dt \qquad (26)
\end{aligned}
$$

We call ${}_L D_\phi^{-\lambda} f(z)$ the ϕ-dept Liouville fI of $f(z)$.

4.2. Definitions of ϕ-dept Liouville fD

Let $z \in \mathbb{C}$ and $\phi \in \mathbb{R}$. When $f(z + te^{i\phi})$ is infinitely differentiable as a function of t in the interval $(0, \infty)$, we denote this by $f(z) \in \mathcal{C}^\infty(P_\phi(z))$.

Definition 7. *Let $f(z) \in \mathcal{L}^1_{loc}(P_\phi(z)) \backslash \mathcal{C}^\infty(P_\phi(z))$. We denote by $m_2[f]$, the supremum of $n \in \mathbb{Z}_{>-1}$ for which $n > s_1[f]$ and $D^n f(z)$ exists on $P_\phi(z)$. When $f(z) \in \mathcal{C}^\infty(P_\phi(z))$, we put $m_2[f] = \infty$.*

We note that if $n \in \mathbb{Z}_{>0}$ satisfies $s_1[f] < n \leq m_2[f]$, then $m_2[f^{(n)}] = m_2[f] - n$.

Definition 8. *Let $f(z) \in \mathcal{L}^1_{loc}(P_\phi(z))$. Let $\nu \in \mathbb{C}$ satisfy $\mathrm{Re}\,\nu \geq 0$ and $\mathrm{Re}\,\nu > s_1[f]$, and $m = \lfloor \mathrm{Re}\,\nu \rfloor + 1$. We then define fD, ${}_W D_\phi^\nu f(z)$, by:*

$$
\begin{aligned}
{}_W D_\phi^\nu f(z) &= {}_W D_\phi^{\nu-m} [D_W^m f(z)], \qquad \text{if } m \leq m_2[f] \qquad (27) \\
{}_W D_\phi^\nu f(z) &= D_W [{}_W D_\phi^{\nu-1} f(z)] \\
&= D_W [{}_W D_\phi^{\nu-m} [D_W^{m-1} f(z)]], \quad \text{if } m = m_2[f] + 1 \qquad (28)
\end{aligned}
$$

where $D_W^m f(z) = (-1)^m \cdot D^m f(z)$ for $m \in \mathbb{Z}_{>-1}$. Formula (28) applies when $\mathrm{Re}\,\nu - 1 > s_1[f]$ and the rhs exists. When ${}_W D_\phi^\nu f(z)$ exists, ${}_L D_\phi^\nu f(z)$ also exists and is given by:

$$
{}_L D_\phi^\nu f(z) = e^{-i\pi\nu} \cdot {}_W D_\phi^\nu f(z) \qquad (29)
$$

We call ${}_L D_\phi^\nu f(z)$ the ϕ-dept Liouville fD of $f(z)$.

4.3. Analytic Continuations of ϕ-dept Liouville fI

In [1,4,12,13], analytic continuations of ϕ-dept Liouville fI via contour integrals are discussed. The analytic continuation via Hankel's contour $C_\phi(z)$, which is shown in Figure 3, is defined as follows.

Definition 9. *Let $f(\zeta)$ be analytic on a neighborhood of $P_\phi[z]$, and $\nu \in \mathbb{C}$ satisfy $\mathrm{Re}\,\nu > s_1[f]$. Then:*

$$
\begin{aligned}
{}_H D_\phi^\nu f(z) &= \frac{\Gamma(\nu+1)}{2\pi i} \int_{z+\infty \cdot e^{i\phi}}^{(z+)} (\zeta - z)^{-\nu-1} f(\zeta) d\zeta \\
&= \frac{\Gamma(\nu+1)}{2\pi i} e^{-i\phi\nu} \int_\infty^{(0+)} t^{-\nu-1} f(z + te^{i\phi}) dt, \quad \nu \notin \mathbb{Z}_{<0} \qquad (30)
\end{aligned}
$$

$$
{}_H D_\phi^{-n} f(z) = \lim_{\nu \to -n} {}_H D_\phi^\nu f(z), \quad -n \in \mathbb{Z}_{<0} \qquad (31)
$$

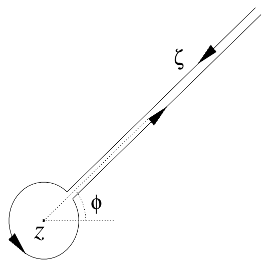

Figure 3. The contour of integration $C_\phi(z)$.

Definition 10. *Let* $f_\gamma(\zeta) = (\zeta - c)^\gamma \cdot f_0(\zeta)$ *where* $\gamma \in \mathbb{C}\backslash\mathbb{Z}$ *and* $f_0(\zeta)$ *is analytic on a neighborhood of* $P_\phi[z]$. *Then,* ${}_HD_\phi^\nu f_\gamma(z)$ *is defined by Definition 9.*

The following lemmas follow from the arguments there.

Lemma 12. ${}_HD_\phi^\nu f(z)$ *is an analytic function of* ν *on the domain* $\mathrm{Re}\ \nu > s_1[f]$.

Lemma 13. ${}_HD_\phi^\nu f_\gamma(z)$ *is analytic with respect to* $\gamma \in \mathbb{C}$, *as well as to* $\nu \in \mathbb{C}$ *on the domain* $\mathrm{Re}\ \nu > s_1[f]$.

Lemma 14. *Let* ${}_HD_\phi^\nu f(z)$ *exist. Then,* ${}_LD_\phi^\nu f(z)$ *exists, and* ${}_HD_\phi^\nu f(z) = {}_LD_\phi^\nu f(z)$ *holds.*

Lemma 15. *Let* ${}_HD_\phi^\nu f_\gamma(z)$ *exist. Then,* ${}_LD_\phi^\nu f_\gamma(z)$ *exists, and* ${}_HD_\phi^\nu f_\gamma(z) = {}_LD_\phi^\nu f_\gamma(z)$ *holds.*

We now present the index law and Leibniz's rule corresponding to Lemmas 2 and 3.

Lemma 16. *If* $\mathrm{Re}\ \nu > s_1[f]$ *and* $\mathrm{Re}\ (\mu + \nu) > s_1[f]$ *for* $\mu \in \mathbb{C}$, $\nu \in \mathbb{C}$ *and* $\phi \in \mathbb{R}$, *then:*

$$ {}_HD_\phi^\mu[{}_HD_\phi^\nu f(z)] = {}_HD_\phi^{\mu+\nu} f(z) \tag{32}$$

This index law is Theorem 4.3 in [1].

Lemma 17. *Let* $\nu \in \mathbb{C}$, $\phi \in \mathbb{R}$ *and* $f(z)$ *satisfy* $-1 + \mathrm{Re}\ \nu > s_1[f]$. *Then:*

$$ {}_HD_\phi^\nu[z \cdot f(z)] = z \cdot {}_HD_\phi^\nu f(z) + \nu \cdot {}_HD_\phi^{\nu-1} f(z) \tag{33}$$

Proof We see that both sides are equal to:

$$\frac{\Gamma(\nu+1)}{2\pi i} e^{-i\phi\nu} \int_\infty^{(0+)} t^{-\nu-1}(z + te^{i\phi}) \cdot f(z + te^{i\phi}) dt$$

■

In the following sections, we use ${}_HD_\phi^\nu f(z)$ for the Liouville fD.

4.4. The ϕ-dept Liouville fD of the Exponential Function

Lemma 18. *Let* $z \in \mathbb{C}$*,* $a \in \mathbb{C}$*,* $\nu \in \mathbb{C}$*, and* $\phi \in \mathbb{R}$ *satisfy* $\text{Re}\,(ae^{i\phi}) > 0$*. Then, the abscissa of convergence of* e^{-az} *is given by* $s_1[e^{-az}] = -\infty$*, and:*

$$ {}_W D^{\nu}_{\phi} e^{-az} = a^{\nu} e^{-az}, \quad {}_H D^{\nu}_{\phi} e^{-az} = (-a)^{\nu} e^{-az} \tag{34}$$

In particular,

$$ {}_H D^{\nu}_{\phi-\pi} e^{z} = e^{z}, \quad |\phi| < \frac{\pi}{2} \tag{35}$$

Proof When $\lambda \in {}_+\mathbb{C}$, $\text{Re}\,(ae^{i\phi}) > 0$,

$$ {}_W D^{-\lambda}_{\phi} e^{-az} = e^{i\phi\lambda} \frac{1}{\Gamma(\lambda)} \int_0^{\infty} t^{\lambda-1} e^{-a(z+te^{i\phi})} dt = a^{-\lambda} e^{-az} \tag{36}$$

by using (25). The proof of the second equation of (34) is finished by using (26) and Lemma 14. ■

We present the following lemma, but we will not use it later.

Lemma 19. *Let* $b \in \mathbb{R}\backslash\{0\}$*,* $f(z) = e^{ibz}$*. Then,* $s_1[f] = 0$ *and if* $\nu \in {}_+\mathbb{C}$ *and* $x \in \mathbb{R}_{>0}$*, there exist:*

$$ {}_H D^{\nu}_{0} e^{ibx} := \lim_{\epsilon \to 0+} {}_H D^{\nu}_{0} e^{(-\epsilon+ib)x}, \quad {}_H D^{\nu}_{-\pi} e^{ibx} := \lim_{\epsilon \to 0+} {}_H D^{\nu}_{-\pi} e^{(\epsilon+ib)x} \tag{37}$$

and:

$$ {}_H D^{\nu}_{0} e^{ibx} = (ib)^{\nu} e^{ibx}, \quad {}_H D^{\nu}_{-\pi} e^{ibx} = (ib)^{\nu} e^{ibx} \tag{38}$$

Proof We put $\phi = 0$, $f(z) = e^{(-\epsilon+ib)z}$ in (30) and take the limit $\epsilon \to 0+$. We then note that we can exchange the order of the limit and the integration, by Lebesgue's theorem [14] (p. 37), obtaining the first equation in (37). By using (34) in the rhs of that equation, we obtain the first equation in (38). ■

5. Solution of Kummer's DE in Terms of ϕ-dept Liouville fD

${}_1F_1(a; b; z)$ is a solution of Kummer's DE (11), as stated in Section 3. Kummer's function $U(a, b, z)$ is another solution of (11), which has the integral representation given by:

$$ U(a, b, z) = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-zt} t^{a-1} (1+t)^{b-a-1} dt \tag{39}$$

when $\text{Re}\, a > 0$ and $\text{Re}\, z > 0$ [11] (Section 13.2.5).

We now give a solution of (11), following the method adopted in Section 3.1 for the Riemann–Liouville fD. Proofs of the lemmas in this section are presented in the following section.

Corresponding to Lemma 9, we now have the following lemma.

Lemma 20. *There exist the following solutions of (11):*

$$ \tilde{w}_l(z) = p_l(z) \tilde{u}_l(z), \quad l \in \mathbb{Z}_{[1,4]} \tag{40}$$

where $p_l(z)$ are given in (14), and:

$$\tilde{u}_l(z) = e^{i\pi(a_l-1)} \cdot {}_H D^{a_l-1}_{\phi-(1+\delta_l)\pi/2}[z^{a_l-b_l} e^{\delta_l \cdot z}] \tag{41}$$

The values of a_l, b_l, δ_l, and their linear combinations are given in Table 2.

Lemma 21.

$$\tilde{w}_1(z) = e^{-i(1-b)\pi} z^{1-b} e^z \cdot U(1-a, 2-b, e^{-i\pi} z), \quad \tilde{w}_2(z) = z^{1-b} \cdot U(a-b+1, 2-b, z), \tag{42}$$

$$\tilde{w}_3(z) = e^z \cdot U(b-a, b, e^{-i\pi} z), \quad \tilde{w}_4(z) = U(a, b, z) \tag{43}$$

Remark 3. *We find a solution of Kummer's DE in [15] (Chapter VII, Section 8), where a solution is first obtained for the equation adjoint to Kummer's DE, by the method we adopted above, and then, a relation connecting the two solutions is used to give the above solution $\tilde{w}_4(z) = U(a, b, z)$.*

5.1. Proofs of Lemmas 20 and 21

The following lemma is obtained by using the method of Nishimoto [4], which is mentioned in Section 3.1.

Lemma 22. *A solution of (18) for $\delta = 1$ or -1 is:*

$$w(z) = {}_H D^{a-1}_{\phi-\pi(\delta+1)/2}[z^{a-b} e^{\delta \cdot z}], \quad |\phi| < \frac{\pi}{2} \tag{44}$$

Proof We assume that a solution of (18) is expressed as $w(z) = {}_H D^{\nu}_{\phi} v(z)$ for $\nu \in \mathbb{C}$. Substituting this $w(z)$ in (18) and using Leibniz's rule given in Lemma 17, we obtain (20) with λ, R and zero replaced by $-\nu$, H and ϕ, respectively. Putting $\nu = a-1$ and then applying ${}_H D_{\phi}^{-\nu-1}$, we obtain (22), with the aid of Lemma 16. If we adopt $v(z) = z^{a-b} e^{\delta \cdot z}$, then $s_1[v] = -\infty$ if $\delta = 1$ and $|\phi + \pi| < \frac{\pi}{2}$, or if $\delta = -1$ and $|\phi| < \frac{\pi}{2}$, and we obtain (44). ■

Proof of Lemma 20 Equation (40) for $l = 1$ follows from Lemma 22. For $l = 2, 3$ and 4, see the proof of Lemma 9. For $\phi - (1 + \delta_l)\pi/2$, see the last part of the proof of Lemma 22. The factors $e^{i\pi(a_l-1)}$ are so chosen that (40) gives Lemma 21. ■

Proof of Lemma 21 By using Lemmas 15 and 14 and the last member of (26), (41) is expressed as:

$$\tilde{u}_l(z) = e^{i\pi(a_l-1)} \frac{1}{\Gamma(1-a_l)} \int_0^{\infty} e^{\delta_l(z-\delta_l \cdot \eta)} \eta^{-a_l} (z - \delta_l \cdot \eta)^{a_l-b_l} d\eta \tag{45}$$

when $-\mathrm{Re}\ a_l > -1$. By putting $\eta = e^{-i\pi}\delta_l \cdot zt$, we obtain:

$$\tilde{u}_l(z) = \frac{1}{\Gamma(1-a_l)} z^{1-b_l} e^{\delta_l \cdot z} \int_0^{\infty} e^{-e^{-i\pi}\delta_l \cdot zt} t^{-a_l} (1 - e^{-i\pi} t)^{a_l-b_l} dt \tag{46}$$

when $-\mathrm{Re}\ a_l + 1 > 0$ and $-\delta_l \cdot \mathrm{Re}\ z > 0$. The integral in (39) is equal to that in (46) with a, b and z replaced by $1 - a_l$, $2 - b_l$ and $e^{-i\pi}\delta_l \cdot z$, respectively, and hence, we have:

$$\tilde{u}_l(z) = z^{1-b_l} e^{\delta_l \cdot z} \cdot U(1-a_l, 2-b_l, e^{-i\pi}\delta_l \cdot z) \tag{47}$$

By using this in (40), we obtain (42)∼(43). ■

6. Asymptotic Expansions of Riemann–Liouville fD

Comparing the third member of (26) with (1) and Figure 1, we have the following lemma.

Lemma 23. *Let Condition A in Section 1 be satisfied,* $f(\zeta) \in \mathcal{L}^1(P(c,z))$*,* $s_1[f] < 0$ *and* $s_1[f] < -\mathrm{Re}\,\lambda < 0$*. Then:*

$$ {}_R D_c^{-\lambda} f(z) = {}_L D_\phi^{-\lambda} f(z) + \frac{1}{\Gamma(\lambda)} \int_{P_\phi(c)} (z-\zeta)^{\lambda-1} f(\zeta) d\zeta \tag{48} $$

Comparing (5) and (30), Figures 2a and 4a, we have the following lemma.

Lemma 24. *Let Condition A be satisfied,* $f(\zeta) \in \mathcal{L}^1(P(c,z))$*, and* $\nu \in \mathbb{C}\backslash\mathbb{Z}_{>-1}$ *satisfy* $\mathrm{Re}\,\nu > s_1[f]$*. Then:*

$$ {}_C D_c^{\nu} f(z) = {}_H D_\phi^{\nu} f(z) + \frac{1}{\Gamma(-\nu)} \int_{P_\phi(c)} (z-\zeta)^{-\nu-1} f(\zeta) d\zeta \tag{49} $$

Remark 4. *Let Condition A be satisfied,* $f(\zeta) \in \mathcal{L}^1(P(c,z))$*, and* $n \in \mathbb{Z}_{>-1}$ *satisfy* $n > s_1[f]$*. Then,* ${}_C D_c^n f(z) = {}_H D_\phi^n f(z) = \frac{d^n}{dz^n} f(z)$*.*

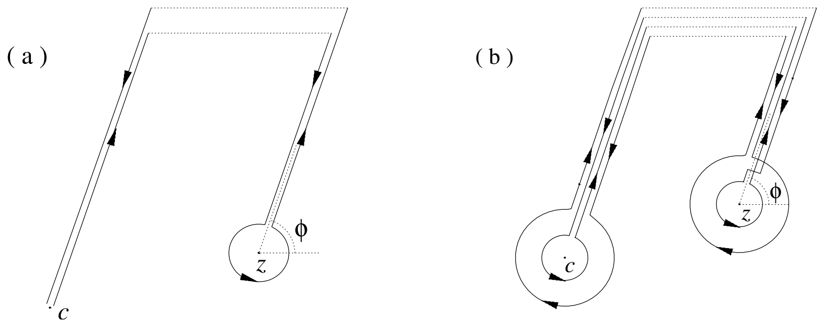

Figure 4. The contours of integration: (**a**) deformed $C(c, z^+)$; (**b**) deformed $C_P(c, z)$.

Comparing (6) and (30), Figures 2b and 4b, we have the following lemma.

Lemma 25. *Let* $f_\gamma(\zeta)$ *satisfy Condition B, and* $\nu \in \mathbb{C}\backslash\mathbb{Z}_{>-1}$ *satisfy* $\mathrm{Re}\,\nu > s_1[f_\gamma]$*. Then:*

$$ {}_P D_c^{\nu} f_\gamma(z) = {}_H D_\phi^{\nu} f_\gamma(z) + \frac{1}{\Gamma(-\nu)} \frac{e^{i\gamma\pi}}{2i\sin\gamma\pi} \int_{c+\infty\cdot e^{i\phi}}^{(c+)} (z-\zeta)^{-\nu-1} f_\gamma(\zeta) d\zeta \tag{50} $$

Remark 5. *Let* $f_\gamma(\zeta)$ *satisfy Condition B, and* $n \in \mathbb{Z}_{>-1}$ *satisfy* $n > s_1[f_\gamma]$*. Then,* ${}_P D_c^n f_\gamma(z) = {}_H D_\phi^n f_\gamma(z) = \frac{d^n}{dz^n} f_\gamma(z)$*.*

We express the integral on the rhs of (49) as:

$$ \int_{P_\phi(0)} (y-\eta)^{-\nu-1} f(c+\eta) d\eta, \tag{51} $$

where $y = z - c$ and $\eta = \zeta - c$. We now obtain the asymptotic expansion of (51) as a function of y.

We use the Taylor series given in [10] (Section 5.4):

$$(a+h)^{\mu} = \sum_{k=0}^{n-1} \binom{\mu}{k} a^{\mu-k} h^k + R_n(a,h) \tag{52}$$

for $a \in \mathbb{C}$, $h \in \mathbb{C}$, $\mu \in \mathbb{C}$ and $n \in \mathbb{Z}_{>0}$, where:

$$R_n(a,h) = \frac{h^n}{2\pi i} \int_C \frac{\xi^{\mu}}{(\xi - a - h)(\xi - a)^n} d\xi \tag{53}$$

Here, C is a contour, including a and $a+h$, but not zero.

We now choose $\alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}_{>0}$ satisfying $\alpha < 1$ and $\beta < 1$, and choose C, such that $\xi \in \mathbb{C}$ on C satisfies:

$$\alpha|a| < |\xi - a| < |h| + \beta|a|, \quad \beta|a| < |\xi - a - h|, \quad (1-\alpha)|a| < |\xi| < (1+\beta)|a| + |h| \tag{54}$$

We then choose $r \in \mathbb{R}_{>0}$, such that:

$$|\xi|^{\mathrm{Re}\,\mu} \cdot \frac{1}{r} \le |\xi^{\mu}| \le |\xi|^{\mathrm{Re}\,\mu} \cdot r, \quad \xi \in C \tag{55}$$

Remark 6. *In using the Taylor series (52), we put $a = z - c$, h being on $P_{\phi}(0)$ or $C_{\phi}(0)$, which are shown in Figures 1 and 3, respectively, and hence, $a+h$ is on $P_{\phi}(z-c)$ or $C_{\phi}(z-c)$. If we see Figure 4a with c and z replaced by zero and $z-c$, respectively, we can easily choose a contour C, as described above.*

Remark 7. *If $|\arg \xi| \le \pi$, we may choose $r = e^{\pi|\mathrm{Im}\,\mu|}$.*

We then obtain:

$$|R_n(a,h)| < R_n^*(a,h) \tag{56}$$

where:

$$R_n^*(a,h) = \begin{cases} \frac{2^{\mathrm{Re}\,\mu+1}\cdot r}{\alpha^n \beta} \frac{|h|^n}{|a|^n} \{(1+2\beta)(1+\beta)^{\mathrm{Re}\,\mu}|a|^{\mathrm{Re}\,\mu} + \frac{|h|^{\mathrm{Re}\,\mu+1}}{|a|}\}, & \mathrm{Re}\,\mu > 0 \\ \frac{2}{r\alpha(1-\alpha)^{-\mathrm{Re}\,\mu}} \frac{|h|^n}{|a|^{n-\mathrm{Re}\,\mu}} (1 + \frac{|h|}{\beta|a|}), & \mathrm{Re}\,\mu \le 0 \end{cases} \tag{57}$$

Using (52) with (56), we obtain the following lemma.

Lemma 26. *Let $g(\zeta) \in \mathcal{L}^1_{loc}(P_{\phi}(0))$, $\mathrm{Re}\,\nu > s_1[g]$, $\delta = 1$ or -1, and $n \in \mathbb{Z}_{>0}$. Then:*

$$I(1+\nu, \delta, g(\eta)) := \int_{P_{\phi}(0)} (y - \delta\cdot\eta)^{-\nu-1} g(\eta) d\eta = y^{-\nu-1} \sum_{k=0}^{n-1} \frac{(1+\nu)_k\, \delta^k A_k}{k!} \frac{1}{y^k} + \Delta_n(y) \tag{58}$$

where:

$$|\Delta_n(y)| < \Delta_n^*(y) := \left| \int_{P_{\phi}(0)} R_n^*(y,\eta) |g(\eta)| d\eta \right| \tag{59}$$

$$\Delta_n^*(y) = \begin{cases} \frac{2^{-\mathrm{Re}\,\nu}\cdot r}{\alpha^n \beta} \frac{1}{|y|^n} \{(1+2\beta)(1+\beta)^{-\mathrm{Re}\,\nu-1} A_n^* |y|^{-\mathrm{Re}\,\nu-1} + \frac{A^*_{n-\mathrm{Re}\,\nu}}{|y|}\}, & \mathrm{Re}\,\nu+1 < 0 \\ \frac{2}{r\alpha(1-\alpha)^{\mathrm{Re}\,\nu+1}} \frac{1}{|y|^{n+\mathrm{Re}\,\nu+1}} (A_n^* + \frac{A^*_{n+1}}{\beta|y|}), & \mathrm{Re}\,\nu+1 \ge 0 \end{cases} \tag{60}$$

$$A_k := \int_{P_\phi(0)} \eta^k g(\eta) d\eta, \quad A^*_\rho := \left| \int_{P_\phi(0)} |\eta|^\rho |g(\eta)| d\eta \right| \tag{61}$$

for $k \in \mathbb{Z}_{>-1}$ *and* $\rho \in \mathbb{R}_{>0}$.

Theorem 1. *Let Condition A be satisfied,* $f(\zeta) \in \mathcal{L}^1(P(c,z))$, $\nu \in \mathbb{C} \backslash \mathbb{Z}_{>-1}$ *satisfy* $\mathrm{Re}\,\nu > s_1[f]$ *and* $n \in \mathbb{Z}_{>0}$. *Then:*

$${}_C D^\nu_c f(z) = {}_H D^\nu_\phi f(z) + \frac{1}{\Gamma(-\nu)} \sum_{k=0}^{n-1} \frac{(1+\nu)_k \cdot A_k}{k!} \frac{1}{(z-c)^{\nu+k+1}} + \frac{1}{\Gamma(-\nu)} \cdot \Delta_n(z-c) \tag{62}$$

where $\Delta_n(z-c)$ *is estimated by (59) and (60), and* A_k *and* A^*_ρ *are given by (61) with* $g(\eta)$ *replaced by* $f(c+\eta)$.

Proof We express the integral on the rhs of (49) as (51). Putting $g(\eta) = f(c+\eta)$ and using (58) in (49), we obtain (62). ■

Theorem 2. *Let* $f_\gamma(\zeta)$ *satisfy Condition B,* $\nu \in \mathbb{C} \backslash \mathbb{Z}_{>-1}$ *satisfy* $\mathrm{Re}\,\nu > s_1[f_\gamma]$ *and* $n \in \mathbb{Z}_{>0}$. *Then:*

$${}_P D^\nu_c f(z) = {}_H D^\nu_\phi f(z) + \frac{1}{\Gamma(-\nu)} \sum_{k=0}^{n-1} \frac{(1+\nu)_k \cdot B_k}{k!} \frac{1}{(z-c)^{\nu+k+1}} + \frac{1}{\Gamma(-\nu)} \cdot \Delta_n(z-c) \tag{63}$$

where:

$$B_k = B_k(c) := \frac{e^{i\gamma\pi}}{2i \sin \gamma\pi} \int_{C_\phi(c)} (\zeta - c)^k f(\zeta) d\zeta = \frac{e^{i\gamma\pi}}{2i \sin \gamma\pi} \int_{C_\phi(0)} \eta^k f(c+\eta) d\eta \tag{64}$$

Remark 8. *If* $n + \gamma > -1$, $\Delta_n(z-c)$ *is estimated by (59) and (60) and if* $k + \gamma > -1$, B_k *is equal to* A_k *given by (61) with* $g(\eta)$ *replaced by* $f(c+\eta)$.

Remark 9. *When* ϕ *satisfies* $z - c = |z-c| \cdot e^{i(\phi+\pi)}$, *Theorems 1 and 2 are valid even when the Condition (b) in Condition A is not satisfied. Formula (68) given below is a trivial example.*

6.1. fD of the Exponential Function

We now apply Theorem 1 to the function $f(z) = e^{-az}$. By using (34), we obtain the following results.

Corollary 1. *Let* $a \in \mathbb{C} \backslash \{0\}$, $\nu \in \mathbb{C} \backslash \mathbb{Z}_{>-1}$ *and* $n \in \mathbb{Z}_{>0}$. *Then:*

$${}_C D^\nu_0 e^{-az} = (-a)^\nu e^{-az} + \frac{1}{\Gamma(-\nu)} \sum_{k=0}^{n-1} \frac{(1+\nu)_k}{a^{k+1}} \frac{1}{z^{\nu+k+1}} + \frac{1}{\Gamma(-\nu)} \Delta_n(z) \tag{65}$$

where:

$$|\Delta_n(z)| < \begin{cases} \frac{2n!(1+2\beta) r [2(1+\beta)|z|]^{-\mathrm{Re}\,\nu - 1} + O(|z|^{-1})}{\alpha^n \beta |a|^{n+1}} \frac{1}{|z|^n}, & \mathrm{Re}\,\nu + 1 < 0 \\ \frac{2n!(1+O(|z|^{-1}))}{\alpha^n r |a|^{n+1} [(1-\alpha)|z|]^{\mathrm{Re}\,\nu+1}} \frac{1}{|z|^n} & \mathrm{Re}\,\nu + 1 \geq 0 \end{cases} \tag{66}$$

Proof We choose ϕ, such that $ae^{i\phi} = |a|$, and obtain $A_k = e^{i\phi(k+1)} \cdot \int_0^\infty t^k e^{-|a|t} dt = \frac{k!}{a^{k+1}}$. ■

Remark 10. *In the above derivation of the second term on the rhs of (65), we use (52) in the integral* $\int_0^\infty (x+s)^{-\nu-1}e^{-as}ds$*. This derivation is the one given in [10] (Section 16.3), for Whittaker's function* $W_{k,m}(z)$*.*

When $a = -ib$, we have:

Corollary 2. *Let* $b \in \mathbb{R}\backslash\{0\}$*,* $x \in \mathbb{R}$ *and* $\nu \in \mathbb{C}\backslash\mathbb{Z}_{>-1}$*. Then, the asymptotic expansion of* ${}_C D_0^\nu e^{ibx}$ *is given by (65) with* $-a$ *and* z *replaced by* ib *and* x*, respectively.*

Remark 11. *Sakakibara [16] used the fact that the leading term of the asymptotic expansion of* ${}_C D_0^{1/2} e^{i\gamma x}$ *for* $\gamma \in \mathbb{R}$ *and* $x \gg 1$ *is* $\sqrt{i\gamma}e^{i\gamma x}$*.*

6.2. Incomplete Gamma Function

We now put $f(z) = e^z$ and $\phi = -\pi$ in (49). Then, we obtain:

$$ {}_C D_0^{-\lambda} e^z = {}_H D_{-\pi}^{-\lambda} e^z - \frac{1}{\Gamma(\lambda)} \int_0^\infty (z+t)^{\lambda-1} e^{-t} dt \tag{67} $$

By multiplying this by $\Gamma(\lambda)e^{-z}$ and comparing it with (8) and (35), we obtain a well-known formula:

$$ \gamma(\lambda, z) = \Gamma(\lambda) - \Gamma(\lambda, z) \tag{68} $$

where $\Gamma(\lambda, z) = \int_z^{z+\infty} \eta^{\lambda-1} e^{-\eta} d\eta$.

The asymptotic expansion of (67) is obtained by putting $a = -1$ and $\nu = -\lambda$ in (65) with (66). By multiplying $\Gamma(\lambda)e^{-z}$ to the result, we obtain the asymptotic expansion of (68).

7. Asymptotic Expansions of the Confluent Hypergeometric Function and Kummer's Function

Lemma 27. *Let* $a \in \mathbb{C}\backslash\mathbb{Z}_{<1}$*,* $b \in \mathbb{C}$*,* $b - a \notin \mathbb{Z}_{<1}$*,* $z \in \mathbb{C}$ *and* $|\phi| < \frac{\pi}{2}$*. Then:*

$$ {}_P D_0^{-a}[z^{b-a-1}e^{-z}] = {}_H D_\phi^{-a}[z^{b-a-1}e^{-z}] + \frac{\Gamma(b-a)}{\Gamma(a)} e^{-z} {}_H D_{\phi-\pi}^{a-b}[z^{a-1}e^z] \tag{69} $$

Proofs of the lemmas in the present section are given in the following section.

Lemma 28. *Let* $a \in \mathbb{C}\backslash\mathbb{Z}_{<1}$*,* $b \in \mathbb{C}\backslash\mathbb{Z}_{<1}$ *and* $b - a \notin \mathbb{Z}_{<1}$*. Then:*

$$ {}_1F_1(a;b;z) = e^{ia\pi} \frac{\Gamma(b)}{\Gamma(b-a)} U(a,b,z) + e^{-i(b-a)\pi} \frac{\Gamma(b)}{\Gamma(a)} e^z \cdot U(b-a, b, e^{-i\pi}z) \tag{70} $$

Remark 12. *When* $a = -m \in \mathbb{Z}_{<1}$*,* ${}_1F_1(a;b;z)$ *is a polynomial of degree* m *in* z*, by (9).*

Lemma 29. *We have the following asymptotic expansions:*

$$ U(a,b,z) = z^{-a} \sum_{k=0}^{n-1} \frac{(-1)^k (1+a-b)_k (a)_k}{k!} \frac{1}{z^k} + \epsilon_n(a,b,z) \tag{71} $$

$$U(b-a,b,e^{-i\pi}z) = e^{i(b-a)\pi} z^{a-b} \sum_{k=0}^{m-1} \frac{(1-a)_k(b-a)_k}{k!} \frac{1}{z^k} + \epsilon_m(b-a,b,z) \tag{72}$$

where $n \in \mathbb{Z}_{>0}$, $m \in \mathbb{Z}_{>0}$, *and:*

$$|\epsilon_n(a,b,z)| < \begin{cases} \frac{2(1+2\beta)[2(1+\beta)]^{\mathrm{Re}\,(b-a)-1}(a)_n r + O(|z|^{-\mathrm{Re}\,(b-a)})}{\alpha^n} \frac{1}{|z|^{n+\mathrm{Re}\,a}}, & \mathrm{Re}\,(b-a)-1>0, \\ \frac{2(a)_n(1+O(z^{-1}))}{r\alpha^n(1-\alpha)^{\mathrm{Re}\,(a-b)+1}} \frac{1}{|z|^{n+\mathrm{Re}\,a}}, & \mathrm{Re}\,(a-b)+1 \geq 0 \end{cases} \tag{73}$$

Here, α, β and r are those defined by (54) and (55).

Theorem 3. *Let the conditions in Lemma 28 be satisfied. Then, the asymptotic expansion of* ${}_1F_1(a;b;z)$ *is obtained by using (71) and (72) in the rhs of (70).*

The expansion agrees with the formula given in [11] (Section 13.5.1).

7.1. Proofs of Lemmas 27, 28 and 29

Proof of Lemma 27 We put $\nu = -a$ and $f(z) = z^{b-a-1}e^{-z}$ in (49), and then, the last term in (49) is:

$$\frac{1}{\Gamma(a)} \int_{P_\Phi(0)} (z-\zeta)^{a-1}\zeta^{b-a-1}e^{-\zeta}d\zeta = e^{i\Phi}\frac{1}{\Gamma(a)} \int_0^\infty (z-te^{i\Phi})^{a-1}(te^{i\Phi})^{b-a-1}e^{-te^{i\Phi}}dt$$

when $\mathrm{Re}\,(b-a) > 0$. By comparing this with the last member on the rhs of (26), we see that this is equal to the last term on the rhs of (69) with H replaced by L. By analytic continuation of the obtained equation, we obtain (69) with the aid of Lemmas 7 and 15. ■

Proof of Lemma 28 Multiplying (69) by $\frac{\Gamma(b)}{\Gamma(b-a)}z^{1-b}e^z$, we obtain (70), with the aid of (13) with (17) on the lhs and (43) on the rhs. ■

Proof of Lemma 29 Comparing (40) with (45) and (58), we obtain:

$$\tilde{w}_l(z) = \frac{1}{\Gamma(1-a_l)} p_l(z) e^{i(a_l-1)\pi} e^{\delta_l \cdot z} \cdot I(-a_l+b_l, \delta_l, \eta^{-a_l}e^{-\eta}) \tag{74}$$

When $l=3$ and $l=4$, this is expressed as:

$$\tilde{w}_3(z) = \frac{1}{\Gamma(b-a)} z^{1-b} e^{i(a-b)\pi} e^z \cdot I(1-a, 1, \eta^{b-a-1}e^{-\eta}) \tag{75}$$

$$\tilde{w}_4(z) = \frac{1}{\Gamma(a)} z^{1-b} e^{-ia\pi} \cdot I(1-b+a, -1, \eta^{a-1}e^{-\eta}) \tag{76}$$

We obtain (71) from (76) by using (43) on the lhs and (58) on the rhs, where we note that A_k given by (61), for $g(\eta) = e^{-\eta}\eta^{a-1}$, is:

$$A_k = \int_0^\infty \eta^k e^{-\eta}\eta^{a-1}d\eta = \Gamma(a+k) = (a)_k \cdot \Gamma(a),$$

We obtain (72) from (75) by using (43) on the lhs and (58) on the rhs, where we note that A_k given by (61), for $g(\eta) = e^{-\eta}\eta^{b-a-1}$, is $A_k = \Gamma(b-a+k) = (b-a)_k \cdot \Gamma(b-a)$. ■

8. Concluding Remarks

We presented a method of deriving the asymptotic expansion of a function, which is expressed by the Riemann–Liouville fI or fD, when these are given by contour integrals. The derivation of the formulas is done by using contour integrals, and yet, the final formulas are mostly useful for the functions of real variable.

Here, we mention [17] reviewing recent developments on fractional calculus.

Acknowledgments

The authors are grateful to the reviewers for helpful comments. With the help of them, the authors improved the descriptions in this paper.

Author Contributions

Several years ago, Ken-ichi Sato asked Tohru Morita how an asymptotic expansion given by Sakakibara in [16] is derived. by Sakakibara is derived. In the course of studying the expressions of fIs and fDs via path integrals and contour integrals in [1], the present paper emerged, where the above question is answered in Remark 11.

Conflicts of Interest

The authors declare no conflict of interest.

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