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Riemann Solitons and Ricci Bi-Conformal Vector Fields on 4-Dimensional Oscillator Group

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Abstract

We consider Riemann soliton vector fields and Ricci bi-conformal vector fields on the oscillator group. We prove that the oscillator group admits Riemann solitons. Subsequently, we provide a complete classification of all Ricci bi-conformal vector fields admitted by the oscillator group and identify those that belong to specific categories, namely gradient-type vector fields, Killing vector fields, and Ricci collineation using the partial differential equations.

Keywords: oscillator group; Riemann solitons; partial differential equations; conformal vector fields

MSC: 53C21; 53C44

1. Introduction

Let (M^n, g) be a pseudo-Riemannian manifold. A Riemann flow on (M^n, g) with the Riemann curvature tensor R was introduced by Udrişte [1,2] as

$$\frac{\partial}{\partial t} G(t) = -2R(g(t)),$$

where $G = \frac{1}{2} g \odot g$ and for two $(0,2)$-tensors ω and θ, the product $\odot$ is defined by

$$\begin{aligned}(\omega \odot \theta)(U_1, U_2, U_3, U_4) \quad = \quad & \omega(U_1, U_4)\theta(U_2, U_3) + \omega(U_2, U_3)\theta(U_1, U_4) \\ & -\omega(U_1, U_3)\theta(U_2, U_4) - \omega(U_2, U_4)\theta(U_1, U_3),\end{aligned}$$

for all vector fields U_1, U_2, U_3, U_4. Manifold (M^n, g) is said to be a Riemann soliton [3] and denoted by (M^n, g, μ, V) if

$$2R + \mu g \odot g + g \odot \mathcal{L}_V g = 0, \tag{1}$$

for some vector field V and constant μ, where $\mathcal{L}_V$ denotes the Lie derivative along V. The Riemann soliton is called expanding, shrinking, or steady depending on whether μ is

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positive, negative, or zero, respectively. If for some smooth function f we have $V = \mathrm{grad} f$, then the Riemann soliton is called a gradient Riemann soliton, and (1) becomes

$$2R + \mu g \odot g + 2g \odot \nabla^2 f = 0.$$

A Riemann soliton is referred to as an almost-Riemann soliton if the parameter μ is taken to be a smooth function. Moreover, the Riemann soliton represents a fixed point of the associated Riemann flow.

Riemann solitons have been extensively studied by researchers across various geometric structures, including almost-co-Kähler manifolds [4], within the framework of contact geometry [5,6], and on para-Sasakian manifolds [7].

A vector field X defined on a pseudo-Riemannian manifold (M^n, g) is termed a conformal vector field [8,9] if it satisfies the condition $\mathcal{L}_X g = 2\psi g$ for some smooth scalar function ψ. In the special case when $\psi = 0$, the vector field X reduces to a Killing vector field. In recent years, several extensions of the concept of conformal vector fields have attracted considerable attention.

One such generalization is the notion of a generalized Kerr–Schild vector field, characterized by the relations

$$\mathcal{L}_X g = \alpha g + \beta\, l \otimes l, \qquad \mathcal{L}_X l = \gamma l,$$

where α, β, and γ are smooth functions, and l is a null 1-form. When $\alpha = 0$, this framework reduces to the classical Kerr–Schild vector field structure [10]. These vector fields have found important applications in constructing exact solutions to the vacuum Einstein field equations and have also been investigated theoretically.

Furthermore, a symmetric $(0,2)$-tensor h on (M^n, g) is said to be a square root of the metric g if it satisfies the identity $h_{ik}h^k_j = g_{ij}$. Another meaningful extension is the bi-conformal vector field [11], which is defined through the pair of conditions

$$\mathcal{L}_X g = \alpha g + \beta h, \qquad \mathcal{L}_X h = \alpha h + \beta g,$$

where α and β are smooth scalar functions and h denotes a symmetric square root of the metric tensor g. Building upon this concept, De et al. [12] introduced the class of Ricci bi-conformal vector fields. A vector field Y is called a Ricci bi-conformal vector field [12] if the following conditions are satisfied:

$$\mathcal{L}_Y g = \alpha g + \beta S, \qquad \mathcal{L}_Y S = \alpha S + \beta g, \tag{2}$$

for some smooth functions α and β, where S denotes the Ricci tensor corresponding to the metric g. For further developments and related studies, see [13–17].

In this work, we provide a comprehensive classification of all Riemann solitons and Ricci bi-conformal vector fields defined on the oscillator group and determine which among them are of gradient soliton type. Furthermore, we demonstrate that every Ricci bi-conformal vector field on the oscillator group is simultaneously a Killing vector field as well as a Ricci collineation.

2. Preliminaries

The oscillator group G_λ is defined by $\mathbb{R} \times \mathbb{C} \times \mathbb{R}$ with the action [18]

$$(x, z, y).(u, w, v) = (x + u + \frac{1}{2} Im(\bar{z} e^{i\lambda y} w), z + e^{i\lambda y} w, y + v),$$

which is a connected, simply connected Lie group with Lie algebra $\mathfrak{g}_\lambda =< \{U, V, X, Y\} >$ having nonzero brackets

$$[V, X] = U, \qquad [Y, V] = \lambda X, \qquad [Y, X] = -\lambda V,$$

where $\lambda > 0$ is a constant. It is endowed with left-invariant Lorentzian metrics $g = \langle , \rangle_a$ with $-1 < a < 1$. It has non-zero scalar products on g of the element U, V, X, Y given by

$$\langle U, U\rangle_a = \langle Y, Y\rangle_a = a, \quad \langle U, Y\rangle_a = \langle V, V\rangle_a = \langle X, X\rangle_a = 1.$$

If $a = 0$ and $\lambda = 1$, then g is also right-invariant [19]. If $a \neq 0$ then g is not bi-invariant. The oscillator group can be considered a subgroup $GL(4, \mathbb{R})$ of the form

$$G_\lambda = \{M_\lambda(u, v, x, y) \in GL(4, \mathbb{R}) | u, v, x, y \in \mathbb{R}\},$$

where

$$M_\lambda(u, v, x, y) = exp(uU)exp(vV)exp(xX)exp(yY).$$

In coordinate system (u, v, x, y), at a neighborhood of an arbitrary point, the oscillator metric is given by

$$g = adu^2 + 2axdudv + 2dudy + (1 + ax^2)dv^2 + 2xdvdy + dx^2 + ady^2, \tag{3}$$

where a is a constant [20]. The oscillator group is named after Streater [21] because its Lie algebra is related to the harmonic oscillator problem. This group has very useful properties in both geometry and mathematical physics. The following topics of oscillator geometry are explored: Ricci solitons [22], Yang–Baxter [23], Ricci collineations [24], Einstein–Yang–Mills [25], geodesic orbit space [26], homogeneous structures [27], adjoint orbits [18], spherical analysis [28], and spectra of compact quotients [29].

Let ∇ be the Levi–Civita connection of the metric g. The curvature tensor R of g is then determined by

$$R(V, W) = \nabla_{[V,W]} - [\nabla_V, \nabla_W],$$

and the Ricci tensor S of g is described by $S(V, W) = tr(Z \to R(V, Z)W)$. Let the partial differential equations

$$\partial_1 = \frac{\partial}{\partial u}, \quad \partial_2 = \frac{\partial}{\partial v}, \quad \partial_3 = \frac{\partial}{\partial x}, \quad \partial_4 = \frac{\partial}{\partial y}.$$

Then, the non-zero components of ∇ in the term of partial differential equations are given by

$$\begin{aligned}
\nabla_{\partial_1}\partial_2 &= -\frac{1}{2}a\partial_3, & \nabla_{\partial_1}\partial_3 &= -\frac{1}{2}ax\partial_1 + \frac{1}{2}a\partial_2,\\
\nabla_{\partial_2}\partial_2 &= -ax\partial_3, & \nabla_{\partial_2}\partial_3 &= \frac{1}{2}(1 - ax^2)\partial_1 + \frac{1}{2}ax\partial_2,\\
\nabla_{\partial_2}\partial_4 &= -\frac{1}{2}\partial_3, & \nabla_{\partial_3}\partial_4 &= -\frac{1}{2}x\partial_1 + \frac{1}{2}\partial_2,
\end{aligned}$$

and the only non-zero components of the Riemannian curvature tensor associated with (3) are given as follows:

$$\begin{aligned}
&R_{1212} = \frac{1}{4}a^2, \quad R_{1242} = \frac{1}{4}a, \quad R_{2323} = \frac{1}{4}a^2x^2 - \frac{3}{4}a, \quad R_{2424} = \frac{1}{4}, \quad R_{1334} = -\frac{1}{4}a,\\
&R_{1313} = \frac{1}{4}a^2, \quad R_{3434} = \frac{1}{4}, \quad R_{2334} = -\frac{1}{4}ax, \quad R_{1314} = -\frac{1}{4}a, \quad R_{1323} = \frac{1}{4}a^2x.
\end{aligned} \tag{4}$$

The Ricci tensor of the oscillator metric is determined by

$$S = \begin{pmatrix} \frac{1}{2}a^2 & \frac{1}{2}a^2x & 0 & \frac{1}{2}a \\ \frac{1}{2}a^2x & \frac{1}{2}a(ax^2-1) & 0 & \frac{1}{2}ax \\ 0 & 0 & -\frac{1}{2}a & 0 \\ \frac{1}{2}a & \frac{1}{2}ax & 0 & \frac{1}{2} \end{pmatrix} \tag{5}$$

with respect to the basis $\{\partial_1, \partial_2, \partial_3, \partial_4\}$.

We consider an arbitrary vector field $Y = Y^i\partial_i$ on the oscillator group, where $Y^i = Y_i(u, v, x, y)$, $i = 1, 2, 3, 4$ are smooth functions. We obtain

$$\begin{aligned}
(\mathcal{L}_Y g)_{11} &= 2a\partial_1 Y^1 + 2ax\partial_1 Y^2 + 2\partial_1 Y^4, \\
(\mathcal{L}_Y g)_{12} &= ax\partial_1 Y^1 + a\partial_2 Y^1 + \partial_1 Y^2 + ax^2\partial_1 Y^2 + ax\partial_2 Y^2 + aY^3 + x\partial_1 Y^4 + \partial_2 Y^4, \\
(\mathcal{L}_Y g)_{13} &= a\partial_3 Y^1 + ax\partial_3 Y^2 + \partial_1 Y^3 + \partial_3 Y^4, \\
(\mathcal{L}_Y g)_{14} &= \partial_1 Y^1 + a\partial_4 Y^1 + x\partial_1 Y^2 + ax\partial_4 Y^2 + a\partial_1 Y^4 + \partial_4 Y^4, \\
(\mathcal{L}_Y g)_{22} &= 2ax\partial_2 Y^1 + 2\partial_2 Y^2 + 2ax^2\partial_2 Y^2 + 2axY^3 + 2x\partial_2 Y^4, \\
(\mathcal{L}_Y g)_{23} &= ax\partial_3 Y^1 + \partial_3 Y^2 + ax^2\partial_3 Y^2 + \partial_2 Y^3 + x\partial_3 Y^4, \\
(\mathcal{L}_Y g)_{24} &= \partial_2 Y^1 + ax\partial_4 Y^1 + x\partial_2 Y^2 + \partial_4 Y^2 + ax^2\partial_4 Y^2 + Y^3 + a\partial_2 Y^4 + x\partial_4 Y^4, \\
(\mathcal{L}_Y g)_{33} &= 2\partial_3 Y^3, \\
(\mathcal{L}_Y g)_{34} &= \partial_3 Y^1 + x\partial_3 Y^2 + \partial_4 Y^3 + a\partial_3 Y^4, \\
(\mathcal{L}_Y g)_{44} &= 2\partial_4 Y^1 + 2x\partial_4 Y^2 + 2a\partial_4 Y^4,
\end{aligned} \tag{6}$$

where $(\mathcal{L}_Y g)_{ij} = (\mathcal{L}_Y g)(\partial_i, \partial_j)$ for $i = 1, 2, 3, 4$. The Lie derivative of the Ricci tensor S along Y is given by

$$\begin{aligned}
(\mathcal{L}_Y S)_{11} &= a\Big(\partial_1 Y^4 + ax\partial_1 Y^2 + a\partial_1 Y^1\Big), \\
(\mathcal{L}_Y S)_{12} &= \frac{1}{2}a\Big(\partial_2 Y^4 + x\partial_1 Y^4 + ax\partial_2 Y^2 + ax^2\partial_1 Y^2 - \partial_1 Y^2 + a\partial_2 Y^1 + ax\partial_1 Y^1 + aY^3\Big), \\
(\mathcal{L}_Y S)_{13} &= \frac{1}{2}a\Big(\partial_3 Y^4 - \partial_1 Y^3 + ax\partial_3 Y^2 + a\partial_3 Y^1\Big), \\
(\mathcal{L}_Y S)_{14} &= \frac{1}{2}\Big(a\partial_4 Y^4 + \partial_1 Y^4 + a^2x\partial_4 Y^2 + ax\partial_1 Y^2 + a^2\partial_4 Y^1 + a\partial_1 Y^1\Big), \\
(\mathcal{L}_Y S)_{22} &= a\Big(x\partial_2 Y^4 + ax^2\partial_2 Y^2 - \partial_2 Y^2 + ax\partial_2 Y^1 + axY^3\Big), \\
(\mathcal{L}_Y S)_{23} &= \frac{1}{2}a\Big(x\partial_3 Y^4 - \partial_2 Y^3 + ax^2\partial_3 Y^2 - \partial_3 Y^2 + ax\partial_3 Y^1\Big), \\
(\mathcal{L}_Y S)_{24} &= \frac{1}{2}\Big(ax\partial_4 Y^4 + \partial_2 Y^4 + a^2x^2\partial_4 Y^2 - a\partial_4 Y^2 + ax\partial_2 Y^2 + a^2x\partial_4 Y^1 + a\partial_2 Y^1 + aY^3\Big), \\
(\mathcal{L}_Y S)_{33} &= -a\partial_3 Y^3, \\
(\mathcal{L}_Y S)_{34} &= \frac{1}{2}\partial_3 Y^4 - \frac{1}{2}a\partial_4 Y^3 + \frac{1}{2}ax\partial_3 Y^2 + \frac{1}{2}a\partial_3 Y^1, \\
(\mathcal{L}_Y S)_{44} &= \partial_4 Y^4 + ax\partial_4 Y^2 + a\partial_4 Y^1,
\end{aligned} \tag{7}$$

where $(\mathcal{L}_Y S)_{ij} = (\mathcal{L}_Y S)(\partial_i, \partial_j)$ for $i = 1, 2, 3, 4$.

3. Riemann Solitons on the Oscillator Group

In this section, we investigate the existence of Riemann solitons on the oscillator group. From (1), we have

$$2R(U_1, U_2, U_3, U_4) = -2\mu[g(U_1, U_4)g(U_2, U_3) - g(U_1, U_3)g(U_2, U_4)]$$

$$-[g(U_1,U_4)\mathcal{L}_Vg(U_2,U_3)+g(U_2,U_3)\mathcal{L}_Vg(U_1,U_4)] \quad (8)$$
$$+[g(U_1,U_3)\mathcal{L}_Vg(U_2,U_4)+g(U_2,U_4)\mathcal{L}_Vg(U_1,U_3)],$$

for any vector fields U_1, U_2, U_3, U_4.

By substituting ∂_i instead of U_i, we examine all possible cases. For example, if we substitute $U_1=\partial_1,\ U_2=\partial_2,\ U_3=\partial_1,\ U_4=\partial_2$ in the above formula, then we have

$$2R_{1212}=-2\mu(g_{12}^2-g_{11}g_{22})-2g_{12}(\mathcal{L}_Vg)_{12}+g_{11}(\mathcal{L}_Vg)_{22}+g_{22}(\mathcal{L}_Vg)_{11}.$$

Therefore, by using (8), the oscillator group is a Riemann soliton with potential vector field V if and only if

$$\begin{aligned}
2R_{1212}&=-2\mu(g_{12}^2-g_{11}g_{22})-2g_{12}(\mathcal{L}_Vg)_{12}+g_{11}(\mathcal{L}_Vg)_{22}+g_{22}(\mathcal{L}_Vg)_{11},\\
2R_{1313}&=2\mu g_{11}g_{33}+g_{11}(\mathcal{L}_Vg)_{33}+g_{33}(\mathcal{L}_Vg)_{11},\\
2R_{1414}&=-2\mu(g_{14}^2-g_{11}g_{44})-2g_{14}(\mathcal{L}_Vg)_{14}+g_{11}(\mathcal{L}_Vg)_{44}+g_{44}(\mathcal{L}_Vg)_{11},\\
2R_{1242}&=-2\mu(g_{12}g_{24}-g_{14}g_{22})-g_{12}(\mathcal{L}_Vg)_{24}-g_{24}(\mathcal{L}_Vg)_{12}+g_{14}(\mathcal{L}_Vg)_{22}+g_{22}(\mathcal{L}_Vg)_{14},\\
2R_{2323}&=2\mu g_{22}g_{33}+g_{22}(\mathcal{L}_Vg)_{33}+g_{33}(\mathcal{L}_Vg)_{22},\\
2R_{1314}&=-g_{14}(\mathcal{L}_Vg)_{13}+g_{11}(\mathcal{L}_Vg)_{34},\\
2R_{1334}&=-2\mu g_{14}g_{33}-g_{14}(\mathcal{L}_Vg)_{33}-g_{33}(\mathcal{L}_Vg)_{14},\\
2R_{4334}&=-2\mu g_{33}g_{44}-g_{44}(\mathcal{L}_Vg)_{33}-g_{33}(\mathcal{L}_Vg)_{44},\\
2R_{2334}&=-2\mu g_{24}g_{33}-g_{24}(\mathcal{L}_Vg)_{33}-g_{33}(\mathcal{L}_Vg)_{24},\\
2R_{2424}&=-2\mu(g_{24}^2-g_{22}g_{44})-2g_{24}(\mathcal{L}_Vg)_{24}+g_{22}(\mathcal{L}_Vg)_{44}+g_{44}(\mathcal{L}_Vg)_{22},\\
2R_{1323}&=2\mu g_{12}g_{33}+g_{12}(\mathcal{L}_Vg)_{33}+g_{33}(\mathcal{L}_Vg)_{12},\\
2R_{1213}&=-g_{12}(\mathcal{L}_Vg)_{13}+g_{11}(\mathcal{L}_Vg)_{23},\\
2R_{1223}&=-g_{22}(\mathcal{L}_Vg)_{13}+g_{12}(\mathcal{L}_Vg)_{23},\\
2R_{1324}&=-g_{14}(\mathcal{L}_Vg)_{23}+g_{12}(\mathcal{L}_Vg)_{34},\\
2R_{1224}&=-2\mu(g_{14}g_{12}-g_{11}g_{24})-g_{14}(\mathcal{L}_Vg)_{12}-g_{12}(\mathcal{L}_Vg)_{14}+g_{11}(\mathcal{L}_Vg)_{24}+g_{24}(\mathcal{L}_Vg)_{11},\\
2R_{1434}&=-g_{14}(\mathcal{L}_Vg)_{34}+g_{44}(\mathcal{L}_Vg)_{13}.
\end{aligned}$$

Applying the components of the metric (3) and Riemann curvature tensor (4) in the above equations, we have

$$\frac{1}{2}a^2=2\mu a-2ax(\mathcal{L}_Vg)_{12}+a(\mathcal{L}_Vg)_{22}+(1+ax^2)(\mathcal{L}_Vg)_{11}, \quad (9)$$
$$\frac{1}{2}a^2=2\mu a+a(\mathcal{L}_Vg)_{33}+(\mathcal{L}_Vg)_{11}, \quad (10)$$
$$0=-2\mu(1-a^2)-2(\mathcal{L}_Vg)_{14}+a(\mathcal{L}_Vg)_{44}+a(\mathcal{L}_Vg)_{11}, \quad (11)$$
$$\frac{1}{2}a=2\mu-ax(\mathcal{L}_Vg)_{24}-x(\mathcal{L}_Vg)_{12}+(\mathcal{L}_Vg)_{22}+(1+ax^2)(\mathcal{L}_Vg)_{14}, \quad (12)$$
$$\frac{1}{2}a^2x^2-\frac{3}{2}a=2\mu(1+ax^2)+(1+ax^2)(\mathcal{L}_Vg)_{33}+(\mathcal{L}_Vg)_{22}, \quad (13)$$
$$-\frac{1}{2}a=-(\mathcal{L}_Vg)_{13}+a(\mathcal{L}_Vg)_{34}, \quad (14)$$
$$-\frac{1}{2}a=-2\mu-(\mathcal{L}_Vg)_{33}-(\mathcal{L}_Vg)_{14}, \quad (15)$$
$$-\frac{1}{2}=-2\mu a-a(\mathcal{L}_Vg)_{33}-(\mathcal{L}_Vg)_{44}, \quad (16)$$
$$-\frac{1}{2}ax=-2\mu x-x(\mathcal{L}_Vg)_{33}-(\mathcal{L}_Vg)_{24}, \quad (17)$$

$$\frac{1}{2} = -2\mu(x^2 - a(1+ax^2)) - 2x(\mathcal{L}_V g)_{24} + (1+ax^2)(\mathcal{L}_V g)_{44} + a(\mathcal{L}_V g)_{22}, \quad (18)$$

$$\frac{1}{2}a^2 x = 2\mu a x + ax(\mathcal{L}_V g)_{33} + (\mathcal{L}_V g)_{12}, \quad (19)$$

$$0 = -ax(\mathcal{L}_V g)_{13} + a(\mathcal{L}_V g)_{23}, \quad (20)$$

$$0 = -(1+ax^2)(\mathcal{L}_V g)_{13} + ax(\mathcal{L}_V g)_{23}, \quad (21)$$

$$0 = -(\mathcal{L}_V g)_{23} + ax(\mathcal{L}_V g)_{34}, \quad (22)$$

$$0 = -(\mathcal{L}_V g)_{12} - ax(\mathcal{L}_V g)_{14} + a(\mathcal{L}_V g)_{24} + x(\mathcal{L}_V g)_{11}, \quad (23)$$

$$0 = -(\mathcal{L}_V g)_{34} + a(\mathcal{L}_V g)_{13}. \quad (24)$$

Consider the above set of equations. From Equations (20) and (21), it can be concluded that $(\mathcal{L}_V g)_{13} = a(\mathcal{L}_V g)_{23} = 0$. Equation (22) gives $a^2(\mathcal{L}_V g)_{34} = 0$. Applying it in Equation (14), we deduce $a = 0$. Thus, $(\mathcal{L}_V g)_{23} = 0$. Equation (9) implies that $(\mathcal{L}_V g)_{11} = 0$ and Equation (11) leads to $(\mathcal{L}_V g)_{14} = -\mu$. Equation (23) yields $(\mathcal{L}_V g)_{12} = 0$. Using Equation (16), one gets $(\mathcal{L}_V g)_{44} = \frac{1}{2}$. From Equations (17) and (18), we get $(\mathcal{L}_V g)_{24} = -\mu x$ and $(\mathcal{L}_V g)_{33} = -\mu$. On the other hand, from Equations (12) and (13), we have $(\mathcal{L}_V g)_{22} = -\mu$. Equation (24) leads to $(\mathcal{L}_V g)_{34} = 0$. Now, using (6), we have the following system of equations:

$$\partial_1 V^4 = 0, \quad (25)$$

$$\partial_1 V^2 + x\partial_1 V^4 + \partial_2 V^4 = 0, \quad (26)$$

$$\partial_1 V^3 + \partial_3 V^4 = 0, \quad (27)$$

$$\partial_1 V^1 + x\partial_1 V^2 + \partial_4 V^4 = -\mu, \quad (28)$$

$$2\partial_2 V^2 + 2x\partial_2 V^4 = -\mu, \quad (29)$$

$$\partial_3 V^2 + \partial_2 V^3 + x\partial_3 V^4 = 0, \quad (30)$$

$$\partial_2 V^1 + x\partial_2 V^2 + \partial_4 V^2 + V^3 + x\partial_4 V^4 = -\mu x, \quad (31)$$

$$2\partial_3 V^3 = -\mu, \quad (32)$$

$$\partial_3 V^1 + x\partial_3 V^2 + \partial_4 V^3 = 0, \quad (33)$$

$$2\partial_4 V^1 + 2x\partial_4 V^2 = \frac{1}{2}. \quad (34)$$

We solve the above equations to determine whether the oscillator with the metric g admits a Riemann soliton. Equation (25) yields $V^4 = K_4(v,x,y)$ for some smooth function K_4. Substituting this into Equations (26)–(28), and then integrating with respect to u, we have

$$V^2 = K_2(v,x,y) - u\partial_2 K_4, \quad V^3 = K_3(v,x,y) - u\partial_3 K_4, \quad (35)$$

$$V^1 = K_1(v,x,y) - \mu u - xK_2 + xu\partial_2 K_4 - u\partial_4 K_2, \quad (36)$$

for some smooth functions K_1, K_2, and K_3. Using (32) we get $\partial_3 K_3 - u\partial_{33}^2 K_4 = -\frac{1}{2}\mu$. Since this is a polynomial with respect to u, it follows that $\partial_{33}^2 K_4 = 0$ and $\partial_3 K_3 = -\frac{1}{2}\mu$. Therefore,

$$K_3 = -\frac{1}{2}\mu x + K_5(v,y), \quad K_4 = K_6(v,y)x + K_7(v,y) \quad (37)$$

for some smooth functions K_5, K_6, and K_7. Substituting (35) and (37) into Equation (29) gives $\partial_2 K_2 - u(\partial_{22}^2 K_6 x + \partial_{22}^2 K_7) + x(\partial_2 K_6 x + \partial_2 K_7) = -\frac{1}{2}\mu$. As this is a polynomial in u, we arrive at $\partial_{22}^2 K_6 = \partial_{22}^2 K_7 = 0$ and $\partial_2 K_2 + x(\partial_2 K_6 x + \partial_2 K_7) = -\frac{1}{2}\mu$. Thus,

$$K_6 = K_8(y)v + K_9(y), \quad K_7 = K_{10}(y)v + K_{11}(y), \quad (38)$$

$$K_2 = -\frac{1}{2}\mu v - (K_8(y)v + K_9(y))x^2 - (K_{10}(y)v + K_{11}(y))x + K_{12}(x,y) \quad (39)$$

for some smooth functions $K_8, \cdots, K_{12}$. Substituting (35)–(39) into (30), we deduce

$$-(K_8 v + K_9)x - (K_{10} v + K_{11}) + \partial_3 K_{12} - 2uK_8 + \partial_2 K_5 = 0,$$

which is a polynomial with respect to u. Hence, we find $K_8 = 0$, and

$$-(K_8 v + K_9)x - (K_{10} v + K_{11}) + \partial_3 K_{12} + \partial_2 K_5 = 0. \tag{40}$$

Applying (34), we conclude that

$$\partial_4 K_1 - u\left(K_9'' x + K_{10}'' v + K_{11}''\right) = \frac{1}{4},$$

which implies $K_9'' = K_{10}'' = K_{11}'' = 0$ and $\partial_4 K_1 = \partial_4 K_1$. Therefore,

$$K_1 = \frac{1}{4} y + K_{13}(v, x), \quad K_9 = b_1 y + b_2, \quad K_{10} = b_3 y + b_4, \quad K_{11} = b_5 y + b_6, \tag{41}$$

for some smooth function K_{13} and some constants $b_1, \cdots, b_6$. Using Equation (33), we deduce that

$$\partial_3 K_{13} + \frac{1}{2}\mu v + (b_1 y + b_2)x^2 + ((b_3 y + b_4)v + b_5 y + b_6)x - K_{12} + \partial_4 K_5 + u(b_3 y + b_4 - 2b_1) = 0$$

which is a polynomial with respect to u. Hence, we find $b_3 = 0$, $b_4 = 2b_1$, and

$$\partial_3 K_{13} + \frac{1}{2}\mu v + (b_1 y + b_2)x^2 + (2b_1 v + b_5 y + b_6)x - K_{12} + \partial_4 K_5 = 0. \tag{42}$$

Also, Equation (31) leads to

$$\partial_2 K_{13} - \frac{1}{2}\mu x + \partial_4 K_{12} + K_5 - (b_1 y + b_2)u = -\mu x$$

which is a polynomial with respect to u. Hence, we find $b_1 = b_2 = 0$, and

$$\partial_2 K_{13} - \frac{1}{2}\mu x + \partial_4 K_{12} + K_5 = -\mu x. \tag{43}$$

By differentiating (40) with respect to v, we have $\partial_{22}^2 K_5 = 0$. Thus, $K_5 = K_{14}(y)v + K_{15}(y)$ for some smooth functions K_{14} and K_{15}. Also, by differentiating (40) with respect to x, we have $\partial_{33}^2 K_{12} = 0$. Thus, $K_{12} = K_{16}(y)x + K_{17}(y)$ for some smooth functions K_{16} and K_{17}. Equation (40) becomes

$$-b_5 y - b_6 + K_{16} + K_{14} = 0. \tag{44}$$

By differentiating (42) and (43) with respect to y, we obtain

$$b_5 x - K_{16}' x - K_{17}' + K_{14}'' v + K_{15}'' = 0, \quad K_{16}'' x + K_{17}'' + K_{14}' v + K_{15}' = 0,$$

which imply that $K_{16}' = b_5$, $K_{14}' = 0$, $K_{15}' + K_{17}'' = 0$, and $K_{15}'' - K_{17}' = 0$. Therefore,

$$K_5 = b_7 v + b_9 + b_{10}\cos y + b_{11}\sin y, \tag{45}$$

$$K_{12} = (b_5 y + b_6 - b_7)x + b_{12} + b_{11}\cos y - b_{10}\sin y, \tag{46}$$

for some constants $b_7, \cdots, b_{12}$. Substituting (45) and (46) in (42) and (43) gives $b_5 = 0$ and

$$K_{13} = -\frac{1}{2}\mu v x - \frac{1}{2}b_7 x^2 + b_{12}x - \frac{1}{2}b_7 v^2 - b_9 v + b_{13}$$

for some constant b_{13}. Therefore,

$$\begin{cases} V^1 = -\mu u + \frac{1}{4}y + \frac{1}{2}b_7x^2 - \frac{1}{2}b_7v^2 - b_9v + b_{13} - (b_{11}\cos y - b_{10}\sin y)x, \\ V^2 = -\frac{1}{2}\mu v - b_7x + b_{12} + b_{11}\cos y - b_{10}\sin y, \\ V^3 = -\frac{1}{2}\mu x + b_7v + b_9 + b_{10}\cos y + b_{11}\sin y, \\ V^4 = b_6. \end{cases} \tag{47}$$

Theorem 1. *An oscillator group with metric (3) is a steady, shrinking, expanding Riemann soliton, and potential vector field V satisfies (47).*

Now, suppose that $V = \nabla h$ for some smooth function h with respect to the metric g. Then, for $a = 0$, we have

$$\nabla h = (x^2\partial_1 h - x\partial_2 h + \partial_4 h)\partial_1 + (\partial_2 h - x\partial_1 h)\partial_2 + \partial_3 h\partial_3 + \partial_1 h\partial_4.$$

Therefore,

$$x^2\partial_1 h - x\partial_2 h + \partial_4 h = V^1,\ \ \partial_2 h - x\partial_1 h = V^2,\ \ \partial_3 h = V^3,\ \ \partial_1 h = V^4, \tag{48}$$

and

$$\partial_1 h = b_6, \tag{49}$$

$$\partial_2 h = b_6x - \frac{1}{2}\mu v - b_7x + b_{12} + b_{11}\cos y - b_{10}\sin y, \tag{50}$$

$$\partial_3 h = -\frac{1}{2}\mu x + b_7v + b_9 + b_{10}\cos y + b_{11}\sin y, \tag{51}$$

$$\partial_4 h = -\mu u + \frac{1}{4}y - \frac{1}{2}b_7x^2 - \frac{1}{2}b_7v^2 - b_9v + b_{13} + (-\frac{1}{2}\mu v + b_{12})x. \tag{52}$$

Condition $\partial^2_{14}h = \partial^2_{41}h$ leads to $\mu = 0$. Equation $\partial^2_{23}h = \partial^2_{32}h$ implies that $b_6 = 2b_7$. Relation $\partial^2_{24}h = \partial^2_{42}h$ yields $b_7 = b_9 = b_{10} = b_{11} = 0$. Also, $\partial^2_{34}h = \partial^2_{43}h$ implies that $b_{12} = 0$. Hence,

$$\partial_1 h = 0, \quad \partial_2 h = 0, \partial_3 h = 0, \quad \partial_4 h = \frac{1}{4}y + b_{13},$$

and by integrating with respect to the variables, the potential function is $h = \frac{1}{8}y^2 + b_{13}y + k$ for some constant k.

Corollary 1. *A Riemann soliton on an oscillator group with metric (3) is a steady gradient Riemann soliton with potential function $h = \frac{1}{8}y^2 + ay + b$ for some constants a, b.*

4. Ricci Bi-Conformal Vector Fields on the Oscillator Group

In this section, we investigate the presence of Ricci bi-conformal vector fields on the oscillator group. By inserting (3), (5), (6), and (7) in (2), we get

$$2a\partial_1 Y^1 + 2ax\partial_1 Y^2 + 2\partial_1 Y^4 = a\alpha + \frac{1}{2}a^2\beta, \tag{53}$$

$$ax\partial_1 Y^1 + a\partial_2 Y^1 + \partial_1 Y^2 + ax^2\partial_1 Y^2 + ax\partial_2 Y^2 + aY^3 + x\partial_1 Y^4 + \partial_2 Y^4 = ax\alpha + \frac{1}{2}a^2x\beta, \tag{54}$$

$$a\partial_3 Y^1 + ax\partial_3 Y^2 + \partial_1 Y^3 + \partial_3 Y^4 = 0, \tag{55}$$

$$\partial_1 Y^1 + a\partial_4 Y^1 + x\partial_1 Y^2 + ax\partial_4 Y^2 + a\partial_1 Y^4 + \partial_4 Y^4 = \alpha + \frac{1}{2}a\beta, \tag{56}$$

$$2ax\partial_2 Y^1 + 2\partial_2 Y^2 + 2ax^2\partial_2 Y^2 + 2axY^3 + 2x\partial_2 Y^4 = (1 + ax^2)\alpha + \frac{1}{2}a(ax^2 - 1)\beta, \tag{57}$$

$$ax\partial_3 Y^1 + \partial_3 Y^2 + ax^2\partial_3 Y^2 + \partial_2 Y^3 + x\partial_3 Y^4 = 0, \tag{58}$$

$$\partial_2 Y^1 + ax\partial_4 Y^1 + x\partial_2 Y^2 + \partial_4 Y^2 + ax^2\partial_4 Y^2 + Y^3 + a\partial_2 Y^4 + x\partial_4 Y^4 = x\alpha + \frac{1}{2}ax\beta, \tag{59}$$

$$2\partial_3 Y^3 = \alpha - \frac{1}{2}a\beta, \tag{60}$$

$$\partial_3 Y^1 + x\partial_3 Y^2 + \partial_4 Y^3 + a\partial_3 Y^4 = 0, \tag{61}$$

$$2\partial_4 Y^1 + 2x\partial_4 Y^2 + 2a\partial_4 Y^4 = a\alpha + \frac{1}{2}\beta, \tag{62}$$

$$a\left(\partial_1 Y^4 + ax\partial_1 Y^2 + a\partial_1 Y^1\right) = \frac{1}{2}a^2\alpha + a\beta, \tag{63}$$

$$\frac{1}{2}a\left(\partial_2 Y^4 + x\partial_1 Y^4 + ax\partial_2 Y^2 + ax^2\partial_1 Y^2 - \partial_1 Y^2 + a\partial_2 Y^1 + ax\partial_1 Y^1 + aY^3\right) = \frac{1}{2}a^2x\alpha + ax\beta, \tag{64}$$

$$\frac{1}{2}a\left(\partial_3 Y^4 - \partial_1 Y^3 + ax\partial_3 Y^2 + a\partial_3 Y^1\right) = 0, \tag{65}$$

$$\frac{1}{2}\left(a\partial_4 Y^4 + \partial_1 Y^4 + a^2x\partial_4 Y^2 + ax\partial_1 Y^2 + a^2\partial_4 Y^1 + a\partial_1 Y^1\right) = \frac{1}{2}a\alpha + \beta, \tag{66}$$

$$a\left(x\partial_2 Y^4 + ax^2\partial_2 Y^2 - \partial_2 Y^2 + ax\partial_2 Y^1 + axY^3\right) = \frac{1}{2}a(ax^2 - 1)\alpha + (1 + ax^2)\beta, \tag{67}$$

$$\frac{1}{2}a\left(x\partial_3 Y^4 - \partial_2 Y^3 + ax^2\partial_3 Y^2 - \partial_3 Y^2 + ax\partial_3 Y^1\right) = 0, \tag{68}$$

and

$$\frac{1}{2}\left(ax\partial_4 Y^4 + \partial_2 Y^4 + a^2x^2\partial_4 Y^2 - a\partial_4 Y^2 + ax\partial_2 Y^2 + a^2x\partial_4 Y^1 + a\partial_2 Y^1 + aY^3\right) = \frac{1}{2}ax\alpha + x\beta, \tag{69}$$

$$-a\partial_3 Y^3 = -\frac{1}{2}a\alpha + \beta, \tag{70}$$

$$\frac{1}{2}\partial_3 Y^4 - \frac{1}{2}a\partial_4 Y^3 + \frac{1}{2}ax\partial_3 Y^2 + \frac{1}{2}a\partial_3 Y^1 = 0, \tag{71}$$

$$\partial_4 Y^4 + ax\partial_4 Y^2 + a\partial_4 Y^1 = \frac{1}{2}\alpha + a\beta, \tag{72}$$

Equation (65) yields $a = 0$ or $\partial_3 Y^4 - \partial_1 Y^3 + ax\partial_3 Y^3 + a\partial_3 Y^1 = 0$. If $a = 0$, Equation (70) gives $\beta = 0$. Equations (63)–(72) lead to the following:

$$\partial_1 Y^4 = \partial_2 Y^4 = \partial_3 Y^4 = 0, \quad \partial_4 Y^4 = \frac{1}{2}\alpha.$$

Hence, $\alpha = \alpha(y)$ and

$$Y^4 = \int \alpha dy + a_1,$$

for some constant a_1. From (54) and (57), we infer $\partial_1 Y^2 = 0$ and $\partial_2 Y^2 = \frac{1}{2}\alpha$. Therefore,

$$Y^2 = \frac{1}{2}\alpha v + F(x, y)$$

for some smooth function F. Using Equations (55) and (60), we get $\partial_1 Y^3 = 0$ and $\partial_3 Y^3 = \frac{1}{2}\alpha$. Therefore,

$$Y^3 = \frac{1}{2}\alpha x + G(v, y),$$

for some smooth function G. Equation (56) yields $\partial_1 Y^1 = \frac{1}{2}\alpha$. Then,

$$Y^1 = \frac{1}{2}\alpha u + H(v, x, y),$$

for some smooth function H. Applying Equation (58), it follows that $\partial_3 F + \partial_2 G = 0$. Then $\partial_{33}^2 F = 0$ and $G = -\partial_3 F v + G_1(y)$ for some smooth function G_1. Thus,

$$F = F_1(y)x + F_2(y),$$

$$G = -F_1(y)v + G_1(y)$$

for some smooth functions F_1 and F_2. Therefore, Y^2 and Y^3 become

$$Y^2 = \frac{1}{2}\alpha v + F_1(y)x + F_2(y),$$

$$Y^3 = \frac{1}{2}\alpha x - F_1(y)v + G_1(y).$$

Equation (62) leads to

$$\alpha'(y)u + \partial_4 H + x\alpha'(y)v + 2x^2 F_1'(y) + 2xF_2'(y) = 0.$$

Since the final equation is not a polynomial in the variables u and v, it follows that $\alpha'(y) = 0$, which implies that α must be a constant function. Also, $\partial_4 H + 2x^2 F_1'(y) + 2xF_2'(y) = 0$, which implies that

$$H = -2x^2 F_1(y) - 2xF_2(y) + H_1(v, x),$$

for some smooth function H_1. Equation (59) leads to

$$\partial_2 H_1 + F_1'(y)x + F_2'(y) + \frac{1}{2}\alpha x - F_1(y)v + G_1(y) = 0. \tag{73}$$

Differentiating the last equation concerning v, we conclude that $\partial_{22}^2 H_1 - F_1(y) = 0$. Therefore, $\partial_{22}^2 H_1 = F_1(y) = a_2$ for some constants a_2. Thus,

$$H_1 = \frac{1}{2}a_2 v^2 + H_2(x)v + H_3(x),$$

for some smooth functions H_2 and H_3. Using Equation (61), we get

$$-3xa_2 - 2F_2(y) + H_2'(x)v + H_3'(x) + G_1'(y) = 0, \tag{74}$$

a polynomial with respect to v, then $H_2'(x) = 0$. Thus, $H_2(x) = a_3$ for some constant a_3. Equation (73) yields $\alpha = 0$ and $G_1(y) = -a_3 - F_2'(y)$. Equation (74) leads to

$$H_3 = \frac{3}{2}a_2 x^2 + a_4 x + a_5, \quad F_2(y) = a_6 \sin\sqrt{2}y + a_7 \cos\sqrt{2}y + \frac{1}{2}a_4,$$

$$G_1(y) = -a_3 - a_6\sqrt{2}\cos\sqrt{2}y + a_7\sqrt{2}\sin\sqrt{2}y$$

for some constants a_4, a_5, a_6, and a_7. Therefore,

$$\begin{cases} Y^1 = -2a_2x^2 - 2x\left(a_6 \sin\sqrt{2}y + a_7\cos\sqrt{2}y\right) + \frac{1}{2}a_2v^2 + a_3v + 2a_2x^2 + a_5, \\ Y^2 = a_2x + a_6\sin\sqrt{2}y + a_7\cos\sqrt{2}y + \frac{1}{2}a_4, \\ Y^3 = -a_2v - a_3 - a_6\sqrt{2}\cos\sqrt{2}y + a_7\sqrt{2}\sin\sqrt{2}y, \\ Y^4 = a_1. \end{cases} \tag{75}$$

Now, we assume that $a \neq 0$. Using (60) and (70), we infer $(a^2-4)\beta = 0$. Also, Equations (53) and (63) give $(a-4)\beta = 0$. Then, $\beta = 0$. By definition of g, it follows that $a - \frac{1}{a} \neq 0$. From Equations (62) and (72), we obtain the partial differential equations

$$\partial_4 Y^4 = \frac{1}{2}\alpha, \qquad \partial_4(Y^1 + xY^2) = 0. \tag{76}$$

Also, Equations (55) and (65) yield

$$\partial_1 Y^3 = 0, \qquad \partial_3(aY^1 + axY^2 + Y^4) = aY^2. \tag{77}$$

Differentiating (60) with respect to u and using (77), it follows that $\partial_1 \alpha = 0$. Using Equations (58) and (68), we arrive at

$$\partial_3 Y^2 + \partial_2 Y^3 = 0. \tag{78}$$

Using (53) and comparing Equations (54) and (64), we conclude that

$$\partial_1 Y^2 = \frac{1}{2}\alpha, \qquad \partial_2(aY^1 + axY^2 + Y^4) = \frac{1}{2}a\alpha x - aY^3. \tag{79}$$

By taking the derivative from (53) and (79) with respect to v and u, respectively, we get $\partial_2 \alpha = 0$. Applying (56) and (72), one gets

$$\partial_1(Y^1 + xY^2 + aY^4) = \frac{1}{2}\alpha. \tag{80}$$

Also, from (66) and (72), we have

$$\partial_1(Y^1 + xY^2 + \frac{1}{a}Y^4) = \frac{1}{2}\alpha. \tag{81}$$

Therefore, Equations (79)–(81) imply that

$$\partial_1 Y^4 = 0, \qquad \partial_1 Y^1 = \frac{1}{2}\alpha(1-x). \tag{82}$$

Differentiating (76) and (80) with respect to u and y, respectively, we obtain $\partial_4 \alpha = 0$. From Equations (57) and (67), we arrive at

$$\partial_2 Y^2 = \frac{1}{2}\alpha, \qquad \partial_2(aY^1 + Y^4) = -aY^3. \tag{83}$$

By differentiating Equations (77) and (79) with respect to v and x, we obtain the relation

$$x\partial_3 \alpha = \alpha.$$

Similarly, differentiating (77) and (80) with respect to u and x, respectively, yields

$$\partial_3 \alpha = a\alpha.$$

Thus, $xa\alpha = \alpha$ implies that $\alpha = 0$. Inserting (76) and (83) in (69), we find $\partial_4 Y^2 = 0$, and (76) becomes $\partial_4 Y^1 = 0$. Substituting (77) in (71), we deduce

$$\partial_4 Y^3 = 0.$$

Also, substituting (77) in (61), we get

$$\partial_3 Y^4 = 0.$$

By virtue of $\partial_1 Y^1 = \partial_4 Y^1 = 0$, we obtain

$$Y^1 = A(v, x)$$

for some smooth function A. Using $\partial_1 Y^2 = \partial_2 Y^2 = \partial_4 Y^2 = 0$, we provide

$$Y^2 = B(x)$$

for some smooth function B. Since the partial differential equations $\partial_1 Y^3 = \partial_3 Y^3 = \partial_4 Y^3 = 0$, we get

$$Y^3 = C(v)$$

for some smooth function C. Since the partial differential equations $\partial_1 Y^4 = \partial_3 Y^4 = \partial_4 Y^4 = 0$, we can write

$$Y^4 = D(v)$$

for some smooth function D. From (78), we have $B'(x) + C'(v) = 0$. Then,

$$B(x) = c_1 x + c_2,$$

$$C(v) = -c_1 v + c_3$$

for some constants c_1, c_2 and c_3. Now, Equation (77) gives $\partial_3 A + x c_1 = 0$, and then $A(v, x) = -\frac{1}{2} c_1 x^2 + E(v)$ for some smooth function E. Equation (79) yields $D'(v) = -aE'(v) - aC(v)$. Using Equations (59) and (69), we have $D'(v) = 0$ and $C = -E'(v)$. Therefore, $D(v) = c_4$ and $E(v) = \frac{1}{2} c_1 v^2 - c_2 v + c_5$ for some constants c_4 and c_5. We prove that $\alpha = \beta = 0$ and

$$\begin{cases} Y^1 = -\frac{1}{2} c_1 x^2 + \frac{1}{} \\ 2c_1 v^2 - c_2 v + c_5, \\ Y^2 = c_1 x + c_2, \\ Y^3 = -c_1 v + c_3, \\ Y^4 = c_4. \end{cases} \tag{84}$$

Theorem 2. *The oscillator group (M^n, g) has some Ricci bi-conformal vector fields $Y = Y^i \partial_i$ if and only if $\alpha = \beta = 0$ and*

(i) *$Y^1, \cdots, Y^4$ admit (75) for $a = 0$.*
(ii) *$Y^1, \cdots, Y^4$ admit (84) for $a \neq 0$.*

Now, suppose that $Y = \nabla f$ for some smooth function f with respect to the metric g. Then, for $a = 0$, we have

$$\nabla f = (x^2 \partial_1 f - x \partial_2 f + \partial_4 f)\partial_1 + (\partial_2 f - x \partial_1 f)\partial_2 + \partial_3 f \partial_3 + \partial_1 f \partial_4.$$

From Theorem 2, the Ricci bi-conformal vector field Y with respect to g is a gradient vector field ∇f if and only if

$$\partial_1 f = a_1, \tag{85}$$

$$\partial_2 f = a_1 x + a_2 x + a_6 \sin\sqrt{2}y + a_7 \cos\sqrt{2}y + \frac{1}{2}a_4, \tag{86}$$

$$\partial_3 f = -a_2 v - a_3 - a_6\sqrt{2}\cos\sqrt{2}y + a_7\sqrt{2}\sin\sqrt{2}y, \tag{87}$$

$$\begin{aligned}\partial_4 f = {} & -a_1 x^2 - 2a_2 x^2 - 2x\left(a_6 \sin\sqrt{2}y + a_7\cos\sqrt{2}y\right) + \frac{1}{2}a_2 v^2 \\ & + a_3 v + 2a_2 x^2 + a_5 + x(a_1 x + a_2 x + a_6 \sin\sqrt{2}y + a_7\cos\sqrt{2}y + \frac{1}{2}a_4).\end{aligned} \tag{88}$$

From the equation $a_1 + a_2 = \partial_3\partial_2 f = \partial_2\partial_3 f = -a_2$, we get $a_1 = -2a_2$. Also, from the equation $a_6\sqrt{2}\cos\sqrt{2}y - a_7\sqrt{2}\sin\sqrt{2}y = \partial_4\partial_2 f = \partial_2\partial_4 f = a_2 v + a_3$, we obtain $a_2 = a_3 = a_6 = a_7 = 0$ and $a_1 = 0$. Using the equation $0 = \partial_4\partial_3 f = \partial_3\partial_4 f = \frac{1}{2}a_4$, we get $a_4 = 0$. Hence, with direct integration, we obtain

$$f = a_5 y + a_6.$$

Now, if $a \neq 0$, then

$$\begin{aligned}\nabla f = {} & \frac{1}{a^2-1}\Big[(a^2x^2 - x^2 + a)\partial_1 f + (x - a^2 x)\partial_2 f - \partial_4 f\Big]\partial_1 + (\partial_2 f - x\partial_1 f)\partial_2 \\ & + \partial_3 f\partial_3 + \frac{1}{a^2-1}[-\partial_1 f + a\partial_4 f]\partial_4.\end{aligned}$$

From Theorem 2, the Ricci bi-conformal vector field Y with respect to g on the oscillator group is a gradient vector field noted as ∇f if and only if $\alpha \neq 1$, and

$$\begin{aligned}\frac{1}{a^2-1}\Big[(a^2x^2 - x^2 + a)\partial_1 f + (x - a^2x)\partial_2 f - \partial_4 f\Big] &= -\frac{1}{2}c_1 x^2 + \frac{1}{2}c_1 v^2 - c_2 v + c_5, \\ \partial_2 f - x\partial_1 f &= c_1 x + c_2, \\ \partial_3 f &= -c_1 v + c_3, \\ \frac{1}{a^2-1}[-\partial_1 f + a\partial_4 f] &= c_4,\end{aligned}$$

or equivalently

$$\begin{aligned}\partial_1 f &= a\left(\frac{1}{2}c_1 x^2 + c_2 x + \frac{1}{2}c_1 v^2 - c_2 v + c_5\right) - c_4(a^2-1), \\ \partial_2 f &= c_1 x + c_2 + ax\left(c_1 x^2 + c_2 x - \frac{1}{2}c_1 x^2 + \frac{1}{2}c_1 v^2 - c_2 v + c_5\right) - c_4(a^2-1)x, \\ \partial_3 f &= -c_1 v + c_3, \\ \partial_4 f &= \frac{1}{2}c_1 x^2 + c_2 x + \frac{1}{2}c_1 v^2 - c_2 v + c_5.\end{aligned}$$

From the equation $a(c_1 x + c_2) = \partial_3\partial_1 f = \partial_1\partial_3 f = 0$, we get

$$c_1 = c_2 = 0.$$

Also, from the equation $ac_5 - c_4(a^2-1) = \partial_3\partial_2 f = \partial_2\partial_3 f = 0$, we obtain

$$ac_5 = c_4(a^2-1).$$

Therefore,

$$\partial_1 f = 0, \qquad \partial_2 f = 0, \qquad \partial_3 f = c_3, \qquad \partial_4 f = c_5.$$

Thus, with direct integration, we get

$$f = c_3x + c_5y + c_6.$$

Corollary 2. *Any Ricci bi-conformal vector field Y concerning the metric g on the oscillator group is a gradient vector field with the potential function $f = by + c$ for $a = 0$ and $f = bx + cy + d$ for $a \neq 0$, where b, c, and d are constants.*

Remark 1. *As a consequence of Theorem 2, we deduce that every Ricci bi-conformal vector field defined on the oscillator group is necessarily a Killing vector field. This indicates strong geometric rigidity, as the only conformal transformations compatible with the Ricci tensor are isometries.*

Remark 2. *A vector field Y on an n-dimensional pseudo-Riemannian manifold (M^n, g) is called a Ricci collineation vector field if $\mathcal{L}_Y S = 0$. From Theorem 2, any Ricci bi-conformal vector field on the oscillator group is a Ricci collineation vector field. This highlights the close relationship between these symmetries in this geometric structure.*

5. Conclusions

We have shown that the oscillator group admits Riemann solitons and we classified all of them. We also fully classified its Ricci bi-conformal vector fields. These are always gradient fields with linear potentials and are simultaneously Killing vector fields and Ricci collineations, indicating significant geometric rigidity.

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References

1. Udrişte, C. Riemann flow and Riemann wave. *Ann. Univ. Vest Timiș. Ser. Mat. Inf.* **2010**, *48*, 256–274.
2. Udrişte, C. Riemann flow and Riemann wave via bialternate product Riemannian metric. *arXiv* **2012**, arXiv:1112.4279. [CrossRef]
3. Hirică, I.E.; Udrişte, C. Ricci and Riemann solitons. *Balkan J. Geom. Appl.* **2016**, *21*, 35–44.
4. Biswas, G.G.; Chen, X.; De, U.C. Riemann solitons on almost co-Kähler manifolds. *Filomat* **2022**, *36*, 1403–1413. [CrossRef]
5. Devaraja, M.N.; Kumara, H.A.; Venkatesha, V. Riemannian soliton within the framework of contact geometry. *Quaest. Math.* **2021**, *44*, 637–651. [CrossRef]
6. Venkatesha, V.; Kumara, H.A.; Devaraja, M.N. Riemann solitons and almost Riemann solitons on almost Kenmotsu manifolds. *Int. J. Geom. Methods Mod. Phys.* **2020**, *17*, 2050105. [CrossRef]
7. De, K.; De, U.C. Riemann solitons on para-Sasakian geometry. *Carpathian Math. Publ.* **2022**, *14*, 395–405. [CrossRef]
8. Deshmukh, S. Geometry of Conformal Vector Fields. *Arab. J. Math.* **2017**, *23*, 44–73. [CrossRef]
9. Yano, K. *The Theory of Lie Derivatives and Its Applications*; Dover Publications: Garden City, NY, USA , 2020.
10. Coll, B.; Hldebrondt, S.R.; Senovilla, J.M.M. Kerr-Schild symmetries. *Class. Quantum Gravity* **2001**, *33*, 649–670. [CrossRef]

11. Garcia-Parrado, A.; Senovilla, J.M.M. Bi-conformal vector fields and their applications. *Class. Quantum Gravity* **2004**, *21*, 2153–2177. [CrossRef]
12. De, U.C.; Sardar, A.; Sarkar, A. Some conformal vector fields and conformal Ricci solitons on $N(k)$-contact metric manifolds. *AUT J. Math. Com.* **2021**, *2*, 61–71. [CrossRef]
13. Azami, S.; De, U.C. Ricci bi-conformal vector fields on Lorentzian four-dimensional generalized symmetric spaces. *Filomat* **2024**, *38*, 3157–3164. [CrossRef]
14. Azami, S.; De, U.C. Ricci bi-conformal vector fields on Lorentzian five-dimensional two-step nilpotent Lie groups. *Hacet. J. Math. Stat.* **2024**, *53*, 1118–1129. [CrossRef]
15. Azami, S.; Fasihi-Ramandi, G. Ricci bi-conformal vector fields on Siklos spacetimes. *Math. Interdiscip. Res.* **2024**, *9*, 45–76. [CrossRef]
16. Azami, S.; Jafari, M. Ricci bi-conformal vector fields on homogeneous Godel-type spacetimes. *J. Nonlinear Math. Phys.* **2023**, *30*, 1700–1718. [CrossRef]
17. Sohrabpour, M.; Azami, S. Ricci bi-conformal vector fields on Lorentzian Walker manifolds of low dimension. *Lobachevskii J. Math.* **2023**, *44*, 5437–5443. [CrossRef]
18. Biggs, R.; Remsing, C.C. Some remarks on the oscillator group. *Differ. Geom. Its Appl.* **2014**, *35*, 199–209. [CrossRef]
19. Levitchev, A.V. Chronogeometry of an electromagnetic wave given by a bi-invariant metric on the oscillator group. *Sib. Math. J.* **1986**, *27*, 237–245. [CrossRef]
20. Galaev, A.S. Conformally flat Lorentzian manifolds with special holonomy groups. *Sb. Math.* **2018**, *204*, 29–50. [CrossRef]
21. Streater, R.F. The representations of the oscillator group. *Comm. Math. Phys.* **1967**, *4*, 217–236. [CrossRef]
22. Calvaruso, G. Oscillator spacetimes are Ricci solitons. *Nonlinear Anal.* **2016**, *140*, 254–269. [CrossRef]
23. Boucetta, M.; Medina, A. Solutions of the Yang–Baxter equations on quadratic Lie groups: The case of oscillator groups. *J. Geom. Phys.* **2011**, *61*, 2309–2320. [CrossRef]
24. Calvaruso, G.; Zaeim, A. On the symmetries of the Lorentzian oscillator group. *Collect. Math.* **2017**, *68*, 51–67. [CrossRef]
25. Diaz, R.D.; Gadea, P.M.; Oubina, J.A. Reductive decompositions and Einstein–Yang–Mills equations associated to the oscillator group. *J. Math. Phys.* **1999**, *40*, 3490–3498. [CrossRef]
26. Batat, W.; Gadea, P.M.; Oubiña, J.A. The Lorentzian oscillator group as a geodesic orbit space. *J. Math. Phys.* **2012**, *53*, 103510. [CrossRef]
27. Gadea, P.M.; Oubiña, J.A. Homogeneous Lorentzian structures on the oscillator groups. *Arch. Math.* **1999**, *73*, 311–320. [CrossRef]
28. Amghar, W.; Fahloui, S. Spherical analysis on the $(2n+2)$-dimensional oscillator group. *J. Anal.* **2024**, *32*, 2383–2400. [CrossRef]
29. Fischer, M.; Kath, I. Spectra of compact quotients of the oscillator group. *SIGMA* **2021**, *17*, 051. [CrossRef]

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