

Article

Fractional-Order Algebraic Parameter Estimation for Disturbed Differentially Flat Systems

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Abstract

Disturbances in dynamical systems pose a major challenge for parameter identification, particularly in the presence of unknown initial conditions and uncertain external influences. To address this issue, this paper proposes an algebraic parameter estimation methodology that incorporates fractional-order calculus in the Laplace domain for controlled linear engineering systems. The proposed approach eliminates the influence of unknown initial conditions and considers external disturbances that admit a local polynomial representation through Taylor series expansions over sufficiently small time intervals, while avoiding explicit numerical differentiation in the time domain. The manuscript includes analytical, numerical, and experimental validations to highlight the benefits of incorporating fractional-order differentiation in the derivation of algebraic estimators for online parameter estimation. The method is experimentally validated on two linear differentially flat electrical circuits, whose flat representations enable the proposed algebraic formulation under distinct disturbance signals. The results demonstrate that the fractional differentiation order acts as an additional tuning parameter, and that appropriately selected fractional orders can improve estimation accuracy, yielding parameter estimates consistently closer to their true values when compared with the conventional integer-order algebraic formulation.

Keywords: algebraic parameter estimation; differential flatness; fractional-order calculus; parameter identification; electrical circuits



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1. Introduction

The modeling of dynamical systems is commonly carried out using first-principle physical laws. Although this approach provides valuable insight into system behavior, obtaining accurate mathematical models in real-world applications is often challenging due to parameter uncertainties, unmodeled dynamics, and time-varying effects. Consequently, parameter identification techniques have become a fundamental tool for constructing reliable models directly from input–output data when first-principle modeling is either impractical or insufficient.

Parameter estimation methods have been successfully applied in numerous engineering domains, including power systems operation and control [1,2], fault diagnosis in DC–DC converters [3], electrical machine modeling [4,5], and robotic systems [6]. Despite their widespread adoption and effectiveness, many existing techniques exhibit significant sensitivity to measurement noise and external disturbances. In practical implementations, additional filtering stages or algorithmic modifications are often required to mitigate these effects [7]. For instance, the Extended Kalman Filter (EKF) provides robustness against process and measurement noise; however, its performance strongly depends on accurate knowledge of noise statistics, which are rarely available in real applications [8,9]. Recursive least-squares-based approaches and multi-stage estimation schemes have also been proposed to enhance robustness under disturbed conditions [10–12]. Nevertheless, these methods typically require careful tuning and may suffer from numerical sensitivity, motivating the exploration of alternative identification frameworks.

In practical engineering applications, the impact of disturbances and uncertainties becomes particularly evident in microscale systems such as MEMS and microrobotic platforms, where parameter estimation is challenged by limited sensing capabilities, multiscale dynamics, and the presence of micro-perturbations [13–15]. In these systems, disturbances often exhibit both fast transient behavior and slowly varying components, which complicates the estimation process and highlights the limitations of conventional techniques. Despite the use of advanced estimation and control strategies to compensate for these effects, the need for alternative approaches that can effectively handle such disturbance characteristics remains an open problem.

In this context, and motivated by the need for estimation techniques capable of handling disturbances and uncertainties in practical engineering systems, algebraic parameter estimation, originally introduced in [16], provides an alternative identification framework characterized by closed-form estimation expressions and favorable numerical properties. Unlike conventional identification techniques, algebraic estimators avoid explicit numerical differentiation in the time domain and instead rely on integral operators, which improve the numerical conditioning of the estimation process [17]. When disturbances are present, two main strategies have been reported in the literature: disturbance elimination [18] and disturbance estimation through local polynomial approximations [19–21]. In these approaches, disturbances are typically assumed to be smooth and slowly time-varying, which allows their local approximation by polynomial representations through Taylor series expansions over sufficiently small time intervals. In control engineering, the approximation of disturbances by means of Taylor series expansions is a widely adopted approach [22–24]. Despite these developments, most algebraic estimation approaches rely exclusively on integer-order differentiation operators, which limits the flexibility in shaping the estimator response under different disturbance characteristics and system dynamics. In particular, the use of integer-order operators restricts the ability to introduce additional tuning degrees of freedom within the algebraic framework. In this context, the incorporation of fractional-order differentiation operators represents a natural extension, as it enables a continuous adjustment of the operator order, potentially allowing a better adaptation of the estimator to the underlying system and disturbance properties. However, the systematic integration of fractional-order operators into algebraic estimation schemes for disturbed linear systems, and in particular for differentially flat systems, remains largely unexplored. Evidence supporting the relevance of fractional-order operators is provided in [25], where a modified algebraic estimator incorporating fractional-order differentiation in the complex domain is proposed. The results reported therein indicate improved estimation accuracy when compared with conventional integer-order algebraic schemes, thereby motivating further investigation into fractional-order algebraic approaches.

From a broader perspective, fractional calculus has demonstrated significant potential in the modeling and analysis of dynamical systems. Fractional-order differential equations are often employed when they provide a more accurate representation of system dynamics than their integer-order counterparts [26]. This enhanced modeling capability can be observed, for example, in the description of charge and discharge processes in supercapacitors, where fractional-order models more accurately capture the dynamic behavior of these elements [27,28]. Fractional calculus has also been successfully applied to the modeling of economic systems [29,30] and to the mathematical modeling of disease propagation phenomena [31]. Beyond continuous-time formulations, fractional operators have been incorporated into discrete-time dynamical systems with memory effects, extending classical integer-order maps [32]. Additionally, fractional calculus has been applied to the identification and control of multivariable systems, where fractional-order transfer functions and controllers provide enhanced flexibility in capturing coupled dynamics [33]. These developments suggest that fractional operators can offer structural advantages beyond pure modeling accuracy.

Complementary to these developments, differential flatness provides a structural property that enables the complete system dynamics to be parameterized in terms of a set of flat outputs and a finite number of their derivatives [34,35]. This property has been widely exploited in control design, trajectory planning, and estimation problems. Representative applications include trajectory tracking of aerial vehicles [36,37], water level control in small-scale hydroelectric plants [38], control of wheeled mobile robots [39], and control of synchronous and asynchronous electrical machines [40]. In addition, differential flatness has been applied in power electronic systems, such as in the design and control of shunt active power filters [41], in the suppression of AC transient current overshoots in dual active bridge converters [42], and in DC-DC converter control applications [43]. Owing to its inherent input–output parametrization capability, the flatness property provides a particularly convenient framework for algebraic estimation techniques.

Motivated by the challenges associated with parameter estimation in the presence of disturbances and uncertainties in practical engineering systems, and by the convergence of algebraic estimation, fractional calculus, and differential flatness, this paper proposes a fractional-order algebraic parameter estimation methodology for controlled linear engineering systems, focusing on linear differentially flat systems subject to external disturbances. Although fractional-order extensions of algebraic estimation have been previously reported, their application to systems explicitly affected by external disturbances has not been systematically addressed within a unified framework. The considered disturbance signals admit a local polynomial representation through Taylor series expansions over sufficiently small time intervals, enabling their elimination through algebraic manipulations in the Laplace domain. Integer and fractional-order differentiation in the Laplace domain are employed to eliminate the influence of unknown initial conditions and disturbance terms. Within this framework, the introduction of a fractional differentiation order provides an additional tuning degree of freedom, allowing a more flexible adjustment of the estimator without increasing its structural complexity.

The main contributions of this work can be summarized as follows:

- The development of a fractional-order algebraic parameter estimation framework for linear differentially flat systems, incorporating disturbance handling through local polynomial approximation and fractional differentiation.
- Experimental validation on two electrical circuits of increasing order and complexity subjected to voltage disturbances.

- A comparative analysis, under consistent disturbance modeling assumptions, showing that fractional-order formulations yield parameter estimates closer to the true values than those obtained using conventional integer-order estimators in the considered scenarios.

The proposed methodology is supported by analytical, numerical, and experimental validation, providing a comprehensive assessment of its performance under the considered conditions. The proposed methodology is experimentally validated on a two-mesh and a three-mesh electrical circuit. In particular, multiple disturbance scenarios and excitation conditions are considered to evaluate the behavior of the estimator under different disturbance profiles and its sensitivity to the fractional differentiation order. The results indicate that the fractional-order formulation introduces an additional tuning parameter that can improve estimation accuracy in the considered disturbed scenarios.

The proposed approach is developed under the assumption that disturbances can be locally approximated by smooth functions, which is consistent with common modeling practices in control engineering. Disturbances with strong discontinuities or high-frequency stochastic components are not explicitly modeled, although their effect is handled within the algebraic formulation through the use of integral operators.

The remainder of this paper is organized as follows. Section 2 presents a discussion of different definitions of fractional derivatives. Section 3 presents the development of the algebraic parameter estimator for linear differentially flat systems subject to disturbances and includes an illustrative example based on a second-order system. Section 4 describes the construction of the estimators and the experimental implementation for the considered case studies. Finally, conclusions and perspectives for future work are provided.

2. Fractional Calculus

Fractional calculus extends classical differentiation to non-integer orders and provides a flexible framework for modeling and analysis of dynamical systems. In the context of parameter estimation, fractional operators introduce additional degrees of freedom that can be exploited within algebraic formulations in the Laplace domain. Its applications span a wide range of engineering fields, including control systems and signal processing [44,45]. However, different definitions are not merely mathematical alternatives; they reflect distinct interpretations of the underlying physics and lead to different analytical and computational properties.

The Grünwald–Letnikov definition is closely related to finite differences, expressing the derivative as a weighted sum of past values. While this provides an intuitive interpretation in terms of system memory and facilitates numerical implementation, its dependence on the full signal history increases computational cost, making it less convenient for algebraic estimation frameworks [46].

The Riemann–Liouville definition is naturally suited for analytical developments and Laplace-domain formulations, which makes it particularly appropriate for algebraic estimation in the complex domain. Its theoretical foundations, including existence, analyticity, and integrability properties, are well established [47]. For this reason, it is adopted in the present work.

The Caputo definition allows expressing initial conditions in terms of integer-order derivatives, which is advantageous in some engineering applications. However, since in the present framework the influence of initial conditions is eliminated through algebraic manipulations in the Laplace domain, this feature is not critical.

All these definitions describe nonlocal behavior associated with memory effects, which provides modeling flexibility but also increases conceptual and computational complexity. In contrast, the proposed approach employs fractional differentiation as an algebraic operator in the Laplace domain, avoiding the explicit treatment of memory terms. Moreover,

several operational properties, including generalizations of the Leibniz rule, have been extensively studied in the literature [48].

In this context, alternative formulations such as the two-scale fractal derivative offer a different perspective by embedding multiscale effects into a rescaled time variable (e.g., $\tau = t^\alpha$), thereby avoiding explicit nonlocal operators. Although less widespread, this approach has shown potential in systems with geometric complexity or multiscale behavior, including porous media, viscoelastic materials, wave propagation, and microscale devices such as MEMS [49–51]. While such formulations provide an alternative modeling paradigm, their integration into algebraic parameter estimation frameworks remains an open research direction.

In the present work, fractional differentiation is applied to functions of the complex variable in the Laplace domain, where the use of complex powers and branch selections is well established [44,45]. This allows incorporating fractional operators without increasing the structural complexity of the estimator.

3. Algebraic Parameter Estimation in the Presence of Disturbances

Consider an n th-order linear differentially flat system described by the following input–output differential equation:

$$\sum_{i=0}^n \theta_i \frac{d^i}{dt^i} y = \sum_{j=0}^l \sigma_j \frac{d^j}{dt^j} u + d, \quad (1)$$

where y and u denote the system output and control input, respectively; θ_i and σ_j are constant system parameters; l denotes the highest-order derivative of the input, with $n \geq l$; and d represents an unknown disturbance affecting the system.

The signals y , u , and d are assumed to be piecewise continuous and of exponential order, ensuring the existence of their Laplace transforms. The disturbance d is assumed to be locally smooth and slowly time-varying, which is consistent with standard assumptions in control and system identification when algebraic estimation techniques are employed [19–21]. Under these conditions, d can be locally approximated, over a sufficiently small time interval, by a truncated Taylor series expansion, i.e.,

$$d \approx \sum_{k=0}^r \delta_k t^k, \quad (2)$$

where $\delta_k \in \mathbb{R}$ are unknown coefficients. This approximation is valid for smooth and slowly varying disturbances over sufficiently small time intervals. Disturbances with high-frequency content or stochastic components are not explicitly captured by this model. Their influence on the estimator is expected to be reduced by the smoothing effect of the integral operators [52], although this aspect is not explicitly analyzed in the present work.

To enable an algebraic manipulation of the system dynamics and facilitate the elimination of unknown terms, the Laplace transform is applied to (1), yielding

$$\sum_{i=0}^n \theta_i s^i Y(s) = \sum_{j=0}^l \sigma_j s^j U(s) + \zeta(s) + \sum_{k=0}^r \delta_k \frac{k!}{s^{k+1}}, \quad (3)$$

where $Y(s)$ and $U(s)$ denote the Laplace transforms of y and u , respectively, and

$$\zeta(s) = a_0 + a_1 s + a_2 s^2 + \cdots + a_{n-1} s^{n-1}$$

collects the terms arising from the unknown initial conditions of the system.

The objective in the estimator design is to eliminate the influence of both the initial conditions and the disturbance. Differentiation with respect to the complex variable s provides a convenient mechanism for this purpose; however, the presence of negative powers of s in (3) prevents the direct application of this operation. This difficulty is resolved by multiplying (3) by s^{r+1} , yielding

$$\sum_{i=0}^n \theta_i s^{i+r+1} Y(s) = \sum_{j=0}^l \sigma_j s^{j+r+1} U(s) + P(s), \quad (4)$$

where

$$P(s) = s^{r+1} \zeta(s) + \sum_{k=0}^r \delta_k k! s^{r-k}. \quad (5)$$

The polynomial $P(s)$ depends exclusively on nonnegative integer powers of s and has a maximum degree equal to $n + r$. Consequently, $P(s)$ can be exactly eliminated by applying a sufficient number of integer-order derivatives with respect to s , since

$$\frac{d^{n+r+1+q}}{ds^{n+r+1+q}} P(s) = 0, \quad q = 0, 1, 2, \dots$$

Once the effects of the disturbance and the initial conditions have been removed, a fractional-order differentiation with respect to s can be applied to enhance the flexibility of the estimator, following the approach in [25]. For this purpose, the Leibniz rule for the fractional derivative of a product is employed:

$$\frac{d^\gamma}{ds^\gamma} [V(s)W(s)] = \sum_{k=0}^{\infty} \binom{\gamma}{k} \left(\frac{d^{\gamma-k}}{ds^{\gamma-k}} V(s) \right) \left(\frac{d^k}{ds^k} W(s) \right), \quad (6)$$

where $\gamma \in \mathbb{R}^+$ and

$$\binom{\gamma}{k} \triangleq \frac{\Gamma(\gamma+1)}{\Gamma(k+1)\Gamma(\gamma-k+1)}. \quad (7)$$

The application of the fractional Leibniz rule is valid under appropriate regularity conditions, such as analyticity of the involved functions and suitable growth constraints, as discussed in [47,48]. In this work, the fractional derivative is defined in the sense of the Riemann–Liouville operator with respect to the complex variable s , due to its compatibility with Laplace-domain formulations [53]. Other definitions of fractional derivatives may also be considered, but their analysis is beyond the scope of this work.

For compactness, the differentiation order is defined as

$$\alpha = n + r + 1 + \beta, \quad (8)$$

where $\beta > 0$ denotes the fractional component of the derivative. Applying (6) to (4) yields

$$\begin{aligned} & \sum_{i=0}^n \left\{ \theta_i \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{d^{\alpha-k}}{ds^{\alpha-k}} Y(s) \frac{d^k}{ds^k} s^{i+r+1} \right\} \\ &= \sum_{j=0}^l \left\{ \sigma_j \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{d^{\alpha-k}}{ds^{\alpha-k}} U(s) \frac{d^k}{ds^k} s^{j+r+1} \right\}. \end{aligned} \quad (9)$$

Since the powers of s in (9) are finite, integer, and positive, successive differentiation eventually yields zero. In particular, since s^{i+r+1} is a polynomial, it follows that

$$\frac{d^k}{ds^k} s^{i+r+1} = 0, \quad \forall k > i + r + 1,$$

which ensures that the infinite series reduces exactly to a finite summation, since all higher-order terms vanish due to the polynomial nature of s^{i+r+1} . Under these considerations, (9) can be rewritten as

$$\begin{aligned} & \sum_{i=0}^n \left[\theta_{n-i} \sum_{k=0}^{o-i} \binom{\alpha}{k} F_{o,i,k} s^{o-i-k} \frac{d^{\alpha-k}}{ds^{\alpha-k}} Y(s) \right] \\ &= \sum_{j=0}^l \left[\sigma_{l-j} \sum_{k=0}^{p-j} \binom{\alpha}{k} F_{p,j,k} s^{p-j-k} \frac{d^{\alpha-k}}{ds^{\alpha-k}} U(s) \right], \end{aligned} \quad (10)$$

where

$$F_{a,b,c} = \frac{(a-b)!}{(a-b-c)!}, \quad (11)$$

and

$$\begin{aligned} o &= n + r + 1, \\ p &= l + r + 1. \end{aligned}$$

Multiplying (10) by s^{-o} ensures that no derivatives appear when mapping the expression back to the time domain:

$$\begin{aligned} & \sum_{i=0}^n \left[\theta_{n-i} \sum_{k=0}^{o-i} \binom{\alpha}{k} F_{o,i,k} s^{-i-k} \frac{d^{\alpha-k}}{ds^{\alpha-k}} Y(s) \right] \\ &= \sum_{j=0}^l \left[\sigma_{l-j} \sum_{k=0}^{p-j} \binom{\alpha}{k} F_{p,j,k} s^{l-j-k-n} \frac{d^{\alpha-k}}{ds^{\alpha-k}} U(s) \right]. \end{aligned} \quad (12)$$

Applying the inverse Laplace transform to (12) yields

$$\begin{aligned} & \sum_{i=0}^n \left[\theta_{n-i} \sum_{k=0}^{o-i} \binom{\alpha}{k} (-1)^{\alpha-k} F_{o,i,k} \int_{t_0}^{(i+k)} \Delta^{\alpha-k} y \right] \\ &= \sum_{j=0}^l \left[\sigma_{l-j} \sum_{k=0}^{p-j} \binom{\alpha}{k} (-1)^{\alpha-k} F_{p,j,k} \int_{t_0}^{(n+j+k-l)} \Delta^{\alpha-k} u \right], \end{aligned} \quad (13)$$

where

$$\int_{t_0}^{(w)} \phi(t) \triangleq \int_{t_0}^t \int_{t_0}^{\tau_1} \cdots \int_{t_0}^{\tau_{w-1}} \phi(\tau_w) d\tau_w \cdots d\tau_1, \quad (14)$$

with $w \in \mathbb{N}$ and $\Delta = t - t_0$.

It is worth noting that the term $(-1)^{\alpha-k}$ is, in general, a complex-valued quantity when $\alpha \notin \mathbb{Z}$ [54]. This expression can be interpreted using the complex exponential representation

$$(-1)^\alpha = e^{\alpha \log(-1)},$$

where $\log(\cdot)$ denotes the complex logarithm. Since the logarithm is a multivalued function, the evaluation of $(-1)^\gamma$ leads to multiple possible values associated with different branches, which can be written as

$$(-1)^\alpha = \exp[j\alpha(\pi + 2\pi k)], \quad k \in \mathbb{Z}.$$

Using Euler's formula, each branch can be expressed in rectangular form as

$$(-1)^\alpha = \cos[\alpha(\pi + 2\pi k)] + j \sin[\alpha(\pi + 2\pi k)].$$

This representation makes it explicit that fractional powers of (-1) correspond to complex numbers with unit magnitude and a phase determined by the exponent and the selected branch.

In order to obtain a consistent formulation, a specific branch must be selected. In this work, the principal branch is adopted, defined by $\arg(z) \in (-\pi, \pi]$, which yields

$$\log(-1) = j\pi,$$

and consequently

$$(-1)^\alpha = \cos(\pi\alpha) + j\sin(\pi\alpha).$$

Under this definition, the term $(-1)^{\alpha-k}$ introduces a complex phase factor of the form

$$e^{j\pi(\alpha-k)} = \cos[\pi(\alpha-k)] + j\sin[\pi(\alpha-k)],$$

which affects all terms in each expression derived from (13) in a uniform manner. Following the same procedure illustrated in the previous example, this global phase factor can be removed by multiplying (13) by $(-1)^{-\alpha}$, which corresponds to removing a global complex phase factor applied uniformly to all terms of the equation, yielding a real-valued formulation that is more convenient for parameter estimation.

It is important to emphasize that this normalization does not affect the identifiability of the parameters, since it corresponds to a scaling by a nonzero complex factor applied uniformly to all terms of the equation.

$$\begin{aligned} & \sum_{i=0}^n \left[\theta_{n-i} \sum_{k=0}^{o-i} \binom{\alpha}{k} (-1)^{-k} F_{o,i,k} \int_{t_0}^{(i+k)} \Delta^{\alpha-k} y \right] \\ &= \sum_{j=0}^l \left[\sigma_{l-j} \sum_{k=0}^{p-j} \binom{\alpha}{k} (-1)^{-k} F_{p,j,k} \int_{t_0}^{(n+j+k-l)} \Delta^{\alpha-k} u \right], \end{aligned} \quad (15)$$

The system described by the differential equation (1) involves a total of $n + l + 2$ parameters when the disturbance affecting the system is neglected. In order to ensure identifiability and enable the construction of an algebraic estimator, it is necessary to assume that at least one of these parameters is known a priori or normalized to a prescribed value. Additionally, the matrix $\mathbf{A}$ must be full rank, i.e., $\det(\mathbf{A}) \neq 0$, which depends on the excitation properties of the input signal and is required to ensure parameter identifiability. Under this assumption, the vector of unknown parameters is defined as

$$\boldsymbol{\eta} \subset \left[\boldsymbol{\theta}^\top \quad \boldsymbol{\sigma}^\top \right]^\top, \quad (16)$$

where $\boldsymbol{\eta} \in \mathbb{R}^c$ collects exclusively the unknown parameters to be estimated.

Equation (15) can then be rewritten by explicitly separating the terms associated with the unknown parameters contained in $\boldsymbol{\eta}$ from those that depend only on known quantities, yielding the following linear algebraic relation:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,c-1} & a_{1,c} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{c-1} \\ \eta_c \end{bmatrix} = b_1. \quad (17)$$

To construct a solvable system of equations from (17), an iterative integration procedure is employed. This approach allows generating additional independent equations while preserving the algebraic structure of the estimator. As a result, the following linear system is obtained:

$$\mathbf{A}\boldsymbol{\eta} = \mathbf{b}, \quad (18)$$

where the elements of the matrix $\mathbf{A} \in \mathbb{R}^{c \times c}$ and the vector $\mathbf{b} \in \mathbb{R}^c$ are recursively defined as

$$\begin{aligned} a_{g,h} &= \int_{t_0} a_{g-1,h}, & b_g &= \int_{t_0} b_{g-1}, \\ g &= 2, 3, \dots, c, & h &= 1, 2, \dots, c, \end{aligned}$$

with $a_{1,h}$ and b_1 defined by (17).

The linear system (18) can be formally solved using Cramer's rule. However, although this approach is not directly suitable for practical implementation due to numerical and structural limitations, it is useful for motivating the modified estimation law adopted in this work. Direct application of this method presents several practical difficulties, namely:

1. Indeterminacy when the determinant $\Lambda = \det(\mathbf{A})$ vanishes;
2. Unbounded growth of the integrals involved;
3. Loss of sign information in the estimated parameters.

To overcome these issues, the following modified estimation law is proposed. Let $\Lambda = \det(\mathbf{A})$ and Λ_h denote the determinant of the matrix obtained by replacing the h -th column of $\mathbf{A}$ with $\mathbf{b}$. An estimate of the parameter η_h is given by

$$\hat{\eta}_h = \frac{\text{sgn}(\Lambda_h) \int_{t_0} e^{-\mu\Delta} |\Lambda_h|}{\text{sgn}(\Lambda) \int_{t_0} e^{-\mu\Delta} |\Lambda|}, \quad (19)$$

where $\mu > 0$ is a design parameter.

The sign function is defined as

$$\text{sgn}(\kappa) = \begin{cases} 1, & \kappa \geq 0, \\ -1, & \kappa < 0. \end{cases} \quad (20)$$

The use of absolute values in (19) eliminates the indeterminacy that arises when $\Lambda = 0$, while the exponential forgetting factor $e^{-\mu\Delta}$ ensures boundedness of the integral terms over time. In particular, for any $\mu > 0$,

$$\lim_{\Delta \rightarrow \infty} \int_{t_0} e^{-\mu\Delta} |\Lambda| = C, \quad (21)$$

where C is a finite constant.

Since the use of absolute values removes sign information from the determinants, the ratio of sign functions in (19) is introduced to recover the correct sign of the estimated parameter.

The proposed formulation is consistent with the assumptions of locally smooth signals typically considered in control engineering applications. In the presence of switching or piecewise-defined inputs, the method can be applied over time intervals where differentiability holds. High-frequency disturbances and measurement noise are not explicitly captured by the polynomial disturbance model but are attenuated by the integral operators. Impulsive disturbances fall outside the theoretical assumptions, although their practical impact can be mitigated through filtering or appropriate selection of the estimation window.

Illustrative Second-Order Example

To support a clearer understanding of the estimator development, a step-by-step derivation is presented in this section. In addition, a comparison with a widely studied estimation technique, namely Recursive Least Squares, is included. The analysis is carried out through computational simulation. To begin, consider the following dynamical model described by the differential equation

$$\rho_2 \frac{d^2}{dt^2} y + \rho_1 \frac{d}{dt} y + \rho_0 y = u + d, \quad (22)$$

where y is the flat output of the system, u is the control input, and d is an external disturbance affecting the system. The parameters ρ_0 , ρ_1 , and ρ_2 are assumed to be constant. The choice of a second-order system for this example is motivated by the fact that a wide variety of electrical, mechanical, and electromechanical systems, among other engineering dynamical systems, can be represented by a differential equation of this order.

Suppose that the disturbance d present in the system can be locally approximated over a small time interval by a Taylor series expansion of order $r = 0$. Evaluating (2) yields

$$d \approx \delta_0,$$

where δ_0 is an unknown constant. It should be recalled that this approximation is valid for smooth disturbances with slow time variations. Substituting this approximation into (22) yields

$$\rho_2 \frac{d^2}{dt^2} y + \rho_1 \frac{d}{dt} y + \rho_0 y = u + \delta_0. \quad (23)$$

The next step is to apply the Laplace transform to (23), yielding

$$\rho_2 [s^2 Y(s) - y(0)s - \dot{y}(0)] + \rho_1 [sY(s) - y(0)] + \rho_0 Y(s) = U(s) + \frac{\delta_0}{s},$$

where $Y(s)$ and $U(s)$ denote the Laplace transforms of y and u , respectively. The previous expression can be rewritten as

$$\rho_2 s^2 Y(s) + \rho_1 s Y(s) + \rho_0 Y(s) = U(s) + \frac{\delta_0}{s} + \zeta(s), \quad (24)$$

with

$$\zeta(s) = \rho_2 y(0)s + \rho_2 \dot{y}(0) + \rho_1 y(0).$$

For the next step, it is desirable to eliminate negative powers of the complex variable s . Therefore, (24) is multiplied by s , yielding

$$\rho_2 s^3 Y(s) + \rho_1 s^2 Y(s) + \rho_0 s Y(s) = sU(s) + P(s), \quad (25)$$

where

$$P(s) = \rho_2 y(0)s^2 + [\rho_2 \dot{y}(0) + \rho_1 y(0)]s + \delta_0.$$

Since $P(s)$ is a polynomial in s with positive integer powers, its influence can be eliminated by differentiating (25) three times with respect to s , yielding

$$\begin{aligned} & \rho_2 \left[s^3 \frac{d^3}{ds^3} Y(s) + 9s^2 \frac{d^2}{ds^2} Y(s) + 18s \frac{d}{ds} Y(s) + 6Y(s) \right] + \rho_1 \left[s^2 \frac{d^3}{ds^3} Y(s) + 6s \frac{d^2}{ds^2} Y(s) + 6 \frac{d}{ds} Y(s) \right] \\ & + \rho_0 \left[s \frac{d^3}{ds^3} Y(s) + 3 \frac{d^2}{ds^2} Y(s) \right] = s \frac{d^3}{ds^3} U(s) + 3 \frac{d^2}{ds^2} U(s). \end{aligned} \quad (26)$$

At this point, the difference with respect to the integer-order algebraic estimator becomes apparent when the fractional-order derivative is introduced. This step relies on the Leibniz rule for the derivative of the product of two functions, which is given in (6) and repeated below:

$$\frac{d^\gamma}{ds^\gamma}[V(s)W(s)] = \sum_{k=0}^{\infty} \binom{\gamma}{k} \left(\frac{d^{\gamma-k}}{ds^{\gamma-k}} V(s) \right) \left(\frac{d^k}{ds^k} W(s) \right).$$

Care must be taken in this step when applying the derivative, since the order of the operations is important. It can be observed that the expression for the fractional derivative contains an infinite series, which must be truncated in order to make the estimator development feasible. From the expression above, it is clear that $V(s)$ is differentiated with fractional orders, whereas $W(s)$ is differentiated with integer orders. These characteristics can be exploited to truncate, or more precisely, to obtain a finite number of terms when evaluating the infinite series.

As an example, consider the term

$$s^3 \frac{d^3}{ds^3} Y(s),$$

which is associated with the parameter ρ_2 in (26). By considering

$$V(s) = \frac{d^3}{ds^3} Y(s), \quad W(s) = s^3,$$

and selecting the fractional derivative order $\beta = 3/5$, one obtains

$$\frac{d^{\frac{3}{5}}}{ds^{\frac{3}{5}}} \left[s^3 \frac{d^3}{ds^3} Y(s) \right] = \sum_{k=0}^{\infty} \binom{\frac{3}{5}}{k} \left\{ \frac{d^{\frac{3}{5}-k}}{ds^{\frac{3}{5}-k}} \left[\frac{d^3}{ds^3} Y(s) \right] \right\} \left[\frac{d^k}{ds^k} s^3 \right] \quad (27)$$

Evaluating (27) for different values of k yields:

- $k = 0$

$$\binom{\frac{3}{5}}{0} \left\{ \frac{d^{\frac{3}{5}}}{ds^{\frac{3}{5}}} \left[\frac{d^3}{ds^3} Y(s) \right] \right\} \left[\frac{d^0}{ds^0} s^3 \right] = s^3 \frac{d^{\frac{18}{5}}}{ds^{\frac{18}{5}}} Y(s).$$

- $k = 1$

$$\binom{\frac{3}{5}}{1} \left\{ \frac{d^{\frac{3}{5}-1}}{ds^{\frac{3}{5}-1}} \left[\frac{d^3}{ds^3} Y(s) \right] \right\} \left[\frac{d}{ds} s^3 \right] = \frac{9}{5} s^2 \frac{d^{\frac{13}{5}}}{ds^{\frac{13}{5}}} Y(s).$$

- $k = 2$

$$\binom{\frac{3}{5}}{2} \left\{ \frac{d^{\frac{3}{5}-2}}{ds^{\frac{3}{5}-2}} \left[\frac{d^3}{ds^3} Y(s) \right] \right\} \left[\frac{d^2}{ds^2} s^3 \right] = -\frac{18}{25} s \frac{d^{\frac{8}{5}}}{ds^{\frac{8}{5}}} Y(s).$$

- $k = 3$

$$\binom{\frac{3}{5}}{3} \left\{ \frac{d^{\frac{3}{5}-3}}{ds^{\frac{3}{5}-3}} \left[\frac{d^3}{ds^3} Y(s) \right] \right\} \left[\frac{d^3}{ds^3} s^3 \right] = \frac{42}{125} \frac{d^{\frac{3}{5}}}{ds^{\frac{3}{5}}} Y(s).$$

- $k = 4$

$$\binom{\frac{3}{5}}{4} \left\{ \frac{d^{\frac{3}{5}-4}}{ds^{\frac{3}{5}-4}} \left[\frac{d^4}{ds^4} Y(s) \right] \right\} \left[\frac{d^4}{ds^4} s^3 \right] = -\frac{21}{625} \cdot 0 \cdot \frac{d^{-\frac{2}{5}}}{ds^{-\frac{2}{5}}} Y(s) = 0.$$

As can be observed, when (27) is evaluated at $k = 4$, the result is zero, and the same result is obtained for any $k > 3$. Therefore,

$$\frac{d^{\frac{3}{5}}}{ds^{\frac{3}{5}}} \left[s^3 \frac{d^3}{ds^3} Y(s) \right] = s^3 \frac{d^{\frac{18}{5}}}{ds^{\frac{18}{5}}} Y(s) + \frac{9}{5} s^2 \frac{d^{\frac{13}{5}}}{ds^{\frac{13}{5}}} Y(s) - \frac{18}{25} s \frac{d^{\frac{8}{5}}}{ds^{\frac{8}{5}}} Y(s) + \frac{42}{125} \frac{d^{\frac{3}{5}}}{ds^{\frac{3}{5}}} Y(s).$$

It should be emphasized that this result is possible due to the particular selection of $V(s)$ and $W(s)$. If one instead considers

$$V(s) = s^3, \quad W(s) = \frac{d^3}{ds^3} Y(s),$$

the evaluation of the fractional derivative would contain an infinite number of terms, since there is no fractional derivative order for which s^3 becomes zero.

Returning to the estimator development, (26) is differentiated with a fractional-order $\beta = 3/5$, yielding

$$\begin{aligned} & \rho_2 \left[s^3 \frac{d^{\frac{18}{5}}}{ds^{\frac{18}{5}}} Y(s) + \frac{54}{5} s^2 \frac{d^{\frac{13}{5}}}{ds^{\frac{13}{5}}} Y(s) + \frac{702}{25} s \frac{d^{\frac{8}{5}}}{ds^{\frac{8}{5}}} Y(s) + \frac{1872}{125} \frac{d^{\frac{3}{5}}}{ds^{\frac{3}{5}}} Y(s) \right] \\ & + \rho_1 \left[s^2 \frac{d^{\frac{18}{5}}}{ds^{\frac{18}{5}}} Y(s) + \frac{36}{5} s \frac{d^{\frac{13}{5}}}{ds^{\frac{13}{5}}} Y(s) + \frac{234}{25} \frac{d^{\frac{8}{5}}}{ds^{\frac{8}{5}}} Y(s) \right] \\ & + \rho_0 \left[s \frac{d^{\frac{18}{5}}}{ds^{\frac{18}{5}}} Y(s) + \frac{18}{5} \frac{d^{\frac{13}{5}}}{ds^{\frac{13}{5}}} Y(s) \right] \\ & = s \frac{d^{\frac{18}{5}}}{ds^{\frac{18}{5}}} U(s) + \frac{18}{5} \frac{d^{\frac{13}{5}}}{ds^{\frac{13}{5}}} U(s). \end{aligned} \quad (28)$$

A relevant observation is that the same result as in (28) can be obtained by applying a fractional derivative of order $\alpha = n + r + 1 + \beta = 18/5$ to (25), provided that $P(s)$ is considered equal to zero.

The next step is to multiply (28) by s^{-3} so that, after returning to the time domain, the estimator does not depend on derivatives of the signals. This yields

$$\begin{aligned} & \rho_2 \left[\frac{d^{\frac{18}{5}}}{ds^{\frac{18}{5}}} Y(s) + \frac{54}{5} s^{-1} \frac{d^{\frac{13}{5}}}{ds^{\frac{13}{5}}} Y(s) + \frac{702}{25} s^{-2} \frac{d^{\frac{8}{5}}}{ds^{\frac{8}{5}}} Y(s) + \frac{1872}{125} s^{-3} \frac{d^{\frac{3}{5}}}{ds^{\frac{3}{5}}} Y(s) \right] \\ & + \rho_1 \left[s^{-1} \frac{d^{\frac{18}{5}}}{ds^{\frac{18}{5}}} Y(s) + \frac{36}{5} s^{-2} \frac{d^{\frac{13}{5}}}{ds^{\frac{13}{5}}} Y(s) + \frac{234}{25} s^{-3} \frac{d^{\frac{8}{5}}}{ds^{\frac{8}{5}}} Y(s) \right] \\ & + \rho_0 \left[s^{-2} \frac{d^{\frac{18}{5}}}{ds^{\frac{18}{5}}} Y(s) + \frac{18}{5} s^{-3} \frac{d^{\frac{13}{5}}}{ds^{\frac{13}{5}}} Y(s) \right] \\ & = s^{-2} \frac{d^{\frac{18}{5}}}{ds^{\frac{18}{5}}} U(s) + \frac{18}{5} s^{-3} \frac{d^{\frac{13}{5}}}{ds^{\frac{13}{5}}} U(s). \end{aligned} \quad (29)$$

The expression is then brought back to the time domain using the approach proposed in [25], yielding

$$\begin{aligned}
& \rho_2 \left[(-1)^{\frac{18}{5}} \Delta^{\frac{18}{5}} y + (-1)^{\frac{13}{5}} \frac{54}{5} \int_{t_0} \Delta^{\frac{13}{5}} y + (-1)^{\frac{8}{5}} \frac{702}{25} \int_{t_0}^{(2)} \Delta^{\frac{8}{5}} y + (-1)^{\frac{3}{5}} \frac{1872}{125} \int_{t_0}^{(3)} \Delta^{\frac{3}{5}} y \right] \\
& + \rho_1 \left[(-1)^{\frac{18}{5}} \int_{t_0} \Delta^{\frac{18}{5}} y + (-1)^{\frac{13}{5}} \frac{36}{5} \int_{t_0}^{(2)} \Delta^{\frac{13}{5}} y + (-1)^{\frac{8}{5}} \frac{234}{25} \int_{t_0}^{(3)} \Delta^{\frac{8}{5}} y \right] \\
& + \rho_0 \left[(-1)^{\frac{18}{5}} \int_{t_0}^{(2)} \Delta^{\frac{18}{5}} y + (-1)^{\frac{13}{5}} \frac{18}{5} \int_{t_0}^{(3)} \Delta^{\frac{13}{5}} y \right] \\
& = (-1)^{\frac{18}{5}} \int_{t_0}^{(2)} \Delta^{\frac{18}{5}} u + (-1)^{\frac{13}{5}} \int_{t_0}^{(3)} \frac{18}{5} \Delta^{\frac{13}{5}} u.
\end{aligned} \tag{30}$$

In (30), fractional powers of (-1) appear, which lead to complex values. This does not represent any difficulty for the algebraic estimator development, since these values are constants. However, for convenience in the derivation, it is desirable to simplify (30). To this end, the different branches of the fractional powers of (-1) are obtained by considering

$$\begin{aligned}
(-1)^{\frac{p}{q}} &= \exp \left[j \frac{p}{q} (\pi + 2\pi k) \right] \\
&= \cos \left[\frac{p\pi}{q} (1 + 2k) \right] + j \sin \left[\frac{p\pi}{q} (1 + 2k) \right],
\end{aligned} \tag{31}$$

with

$$p, q \in \mathbb{Z}, \quad k = 0, 1, 2, \dots, q-1.$$

The results of evaluating the different fractional powers of (-1) appearing in (30) are presented in Table 1. The principal branch is obtained by evaluating (31) at $k = 0$, which corresponds to the second column of the table.

Table 1. Rectangular representation of the distinct branches of $(-1)^{\frac{p}{q}}$ for selected fractional exponents.

γ	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$
$3/5$	$-0.309 + j0.951$	$0.809 - j0.587$	$-1 + j0$	$0.809 + j0.587$	$-0.309 - j0.951$
$8/5$	$0.309 - j0.951$	$-0.809 + j0.587$	$1 + j0$	$-0.809 - j0.587$	$0.309 + j0.951$
$13/5$	$-0.309 + j0.951$	$0.809 - j0.587$	$-1 + j0$	$0.809 + j0.587$	$-0.309 - j0.951$
$18/5$	$0.309 - j0.951$	$-0.809 + j0.587$	$1 + j0$	$-0.809 - j0.587$	$0.309 + j0.951$

The branches are computed from (31), with $k = 0, \dots, 4$, and expressed in rectangular form; the principal branch corresponds to $k = 0$, i.e., $(-1)^{\frac{p}{q}} = e^{j\pi \frac{p}{q}}$.

If the principal branch values of the fractional powers of (-1) are substituted into (30), and the resulting expression is then divided by the principal branch of $(-1)^{18/5}$, one obtains

$$\begin{aligned}
& \rho_2 \left[\Delta^{\frac{18}{5}} y - \frac{54}{5} \int_{t_0} \Delta^{\frac{13}{5}} y + \frac{702}{25} \int_{t_0}^{(2)} \Delta^{\frac{8}{5}} y - \frac{1872}{125} \int_{t_0}^{(3)} \Delta^{\frac{3}{5}} y \right] \\
& + \rho_1 \left[\int_{t_0} \Delta^{\frac{18}{5}} y - \frac{36}{5} \int_{t_0}^{(2)} \Delta^{\frac{13}{5}} y + \frac{234}{25} \int_{t_0}^{(3)} \Delta^{\frac{8}{5}} y \right] \\
& + \rho_0 \left[\int_{t_0}^{(2)} \Delta^{\frac{18}{5}} y - \frac{18}{5} \int_{t_0}^{(3)} \Delta^{\frac{13}{5}} y \right] \\
& = \int_{t_0}^{(2)} \Delta^{\frac{18}{5}} u - \frac{18}{5} \int_{t_0}^{(3)} \Delta^{\frac{13}{5}} u,
\end{aligned} \tag{32}$$

which corresponds to the evaluation of (15).

From (22), it can be observed that the control input u is not explicitly associated with a parameter, or equivalently, that its parameter is normalized to 1. With this in mind, the vector of unknown parameters can be defined as

$$\boldsymbol{\eta} = \begin{bmatrix} \rho_2 \\ \rho_1 \\ \rho_0 \end{bmatrix},$$

which allows (32) to be rewritten as

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \end{bmatrix} \begin{bmatrix} \rho_2 \\ \rho_1 \\ \rho_0 \end{bmatrix} = b_1, \quad (33)$$

with

$$\begin{aligned} a_{1,1} &= \Delta^{\frac{18}{5}} y - \frac{54}{5} \int_{t_0} \Delta^{\frac{13}{5}} y + \frac{702}{25} \int_{t_0}^{(2)} \Delta^{\frac{8}{5}} y - \frac{1872}{125} \int_{t_0}^{(3)} \Delta^{\frac{3}{5}} y, \\ a_{1,2} &= \int_{t_0} \Delta^{\frac{18}{5}} y - \frac{36}{5} \int_{t_0}^{(2)} \Delta^{\frac{13}{5}} y + \frac{234}{25} \int_{t_0}^{(3)} \Delta^{\frac{8}{5}} y, \\ a_{1,3} &= \int_{t_0}^{(2)} \Delta^{\frac{18}{5}} y - \frac{18}{5} \int_{t_0}^{(3)} \Delta^{\frac{13}{5}} y, \\ b_1 &= \int_{t_0}^{(2)} \Delta^{\frac{18}{5}} u - \frac{18}{5} \int_{t_0}^{(3)} \Delta^{\frac{13}{5}} u. \end{aligned}$$

The linear system is then constructed as

$$\mathbf{A}\boldsymbol{\eta} = \mathbf{b}, \quad (34)$$

where

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ \int_{t_0} a_{1,1} & \int_{t_0} a_{1,2} & \int_{t_0} a_{1,3} \\ \int_{t_0}^{(2)} a_{1,1} & \int_{t_0}^{(2)} a_{1,2} & \int_{t_0}^{(2)} a_{1,3} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \int_{t_0} b_1 \\ \int_{t_0}^{(2)} b_1 \end{bmatrix},$$

and the system (34) can be solved using (19).

Comparison with RLS

To evaluate the effect of the proposed fractional-order modification, a comparison is performed with a widely studied estimation technique in the literature, namely Recursive Least Squares (RLS). In order to make the comparison as consistent as possible, the variant proposed in [19] is considered, in which the disturbances are approximated through low-order Taylor series expansions. In this way, both approaches exploit the same structural assumption on the disturbance. To avoid extending this section excessively beyond its illustrative purpose, the development of the RLS estimator is omitted here, since it is based on the model presented in (23). The reader is referred to the previously cited work for further details.

The proposed comparison is carried out by means of computational simulation, based on the dynamical model presented in (22), with parameters $\rho_2 = 0.01$, $\rho_1 = 0.2$, and $\rho_0 = 0.5$. The input u is selected as a sinusoidal signal with amplitude 50 and frequency 50 Hz, while the disturbance d is taken as a constant value equal to 1.2. These signals are shown in Figure 1.

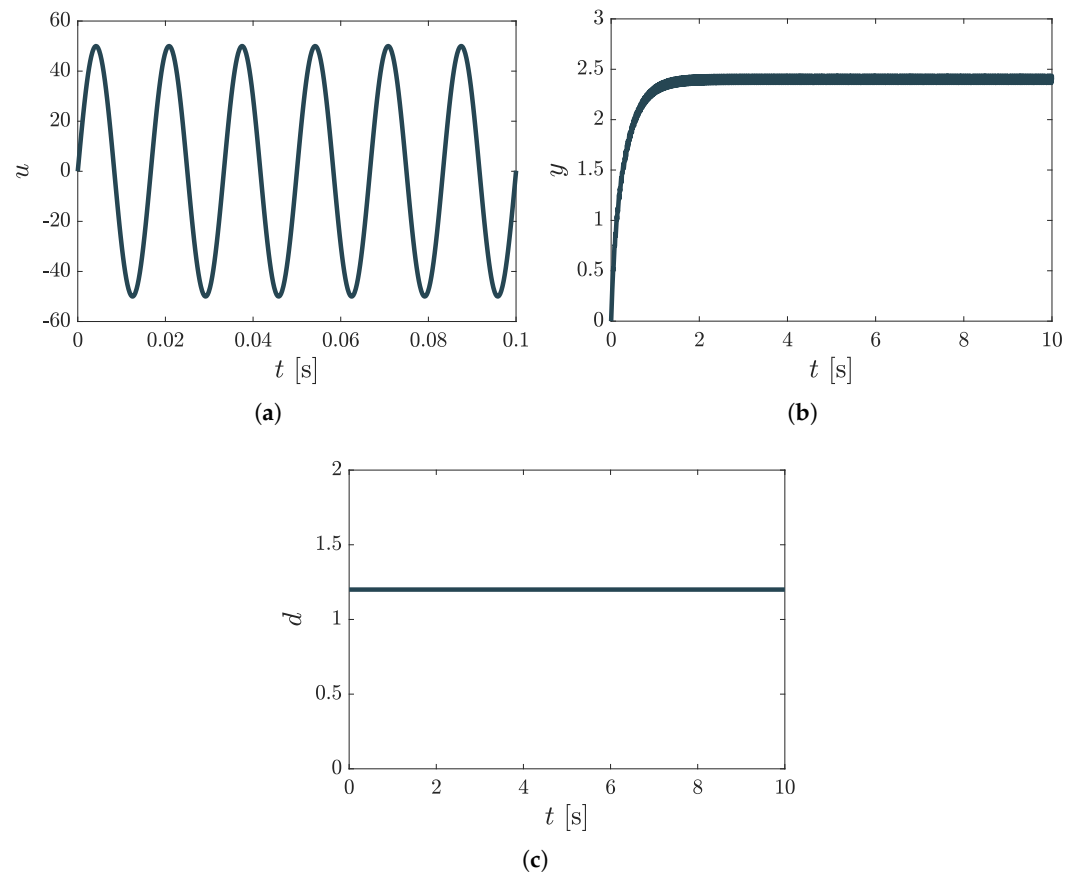


Figure 1. Simulated signals of the illustrative second-order example: (a) input signal u , (b) output signal y , and (c) disturbance signal d .

The estimation results are shown in Figure 2, where the integer-order algebraic estimator (AE), the fractional-order algebraic estimator (FAE), and the RLS method are compared. For the parameter ρ_2 , all methods provide highly accurate estimates, which makes the difference between the true value and the estimated values difficult to observe in Figure 2a. This contrasts with the estimation of ρ_1 , shown in Figure 2b, where the algebraic estimators exhibit smaller errors, while RLS presents a larger deviation. Finally, for ρ_0 , shown in Figure 2c, all methods yield estimates close to the true value, with the fractional-order algebraic estimator providing the closest result in this simulation scenario.

The relative parameter estimation errors, computed from the final values of the simulated estimates, are presented in Table 2. These results show that, for the considered example, the fractional-order algebraic estimator yields the lowest overall estimation error. It can also be observed that the integer-order algebraic estimator already provides accurate estimates, whereas the fractional-order formulation further improves the results for the considered parameters. In the case of RLS, two estimates remain close to the true parameter values; however, the estimation of ρ_1 exhibits a comparatively large error.

Table 2. Relative estimation errors (%) of the continuous-time parameters obtained with the considered estimation techniques in the illustrative second-order example.

Parameter	True Value	AE	FAE	RLS
ρ_2	0.01	0.086	0.014	−0.173
ρ_1	0.2	1.052	0.274	68.189
ρ_0	0.5	3.339	1.458	−1.572

AE: integer-order algebraic estimator; FAE: fractional-order algebraic estimator; RLS: recursive least-squares estimator.

The purpose of this illustrative example is not to establish a general ranking among estimation methods, but to provide a transparent comparison under the same disturbance modeling assumption and to highlight the effect of introducing the fractional-order parameter within the algebraic estimation framework.

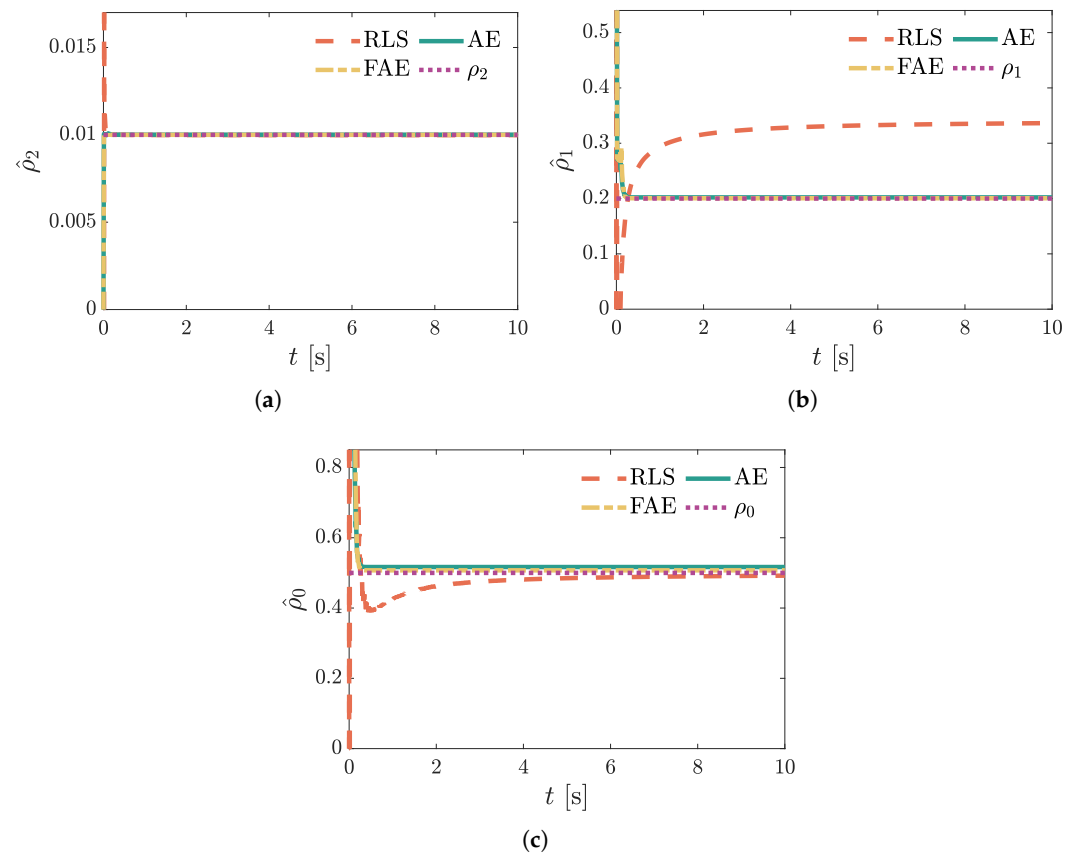


Figure 2. Parameter estimates obtained in the illustrative second-order example: (a) ρ_2 , (b) ρ_1 , and (c) ρ_0 .

4. Case Studies

To assess the performance of the proposed fractional algebraic estimator, its application to two electrically perturbed circuits is considered. The selected case studies correspond to circuits of increasing complexity, namely a two-mesh and a three-mesh configuration. These examples illustrate the applicability of the proposed methodology to practical electrical systems affected by unknown parameters and external disturbances.

In both cases, the estimation procedure follows the general framework introduced in the previous sections. Therefore, only the system models, the corresponding flat representations, and the parameters required for the estimator construction are explicitly stated, while intermediate algebraic developments are omitted for conciseness.

For both case studies, the parameter estimation is experimentally evaluated by varying the order of the fractional derivative in order to determine which value provides the best estimation performance.

For the two-mesh circuit, the values

$$\beta = \left\{ 0, \frac{1}{10}, \dots, \frac{8}{10}, \frac{9}{10} \right\}$$

are considered, whereas for the three-mesh circuit the set

$$\beta = \left\{ 0, \frac{1}{7}, \dots, \frac{5}{7}, \frac{6}{7} \right\}$$

is used. It is worth noting that when $\beta = 0$, the proposed estimator reduces to the conventional algebraic estimator.

4.1. Experimental Platform

The experimental setup used to validate the proposed methodology consists of:

1. NI USB-6363 data acquisition board;
2. Inductor bank;
3. Two-channel function generator, where the first channel supplies the main circuit voltage and the second channel provides the disturbance signal;
4. Discrete resistors and capacitors;
5. Personal computer for data acquisition and processing.

In both case studies, the sampling frequency was set to 20 kHz. Figure 3 illustrates the experimental platform employed for the validation of the proposed estimation scheme.

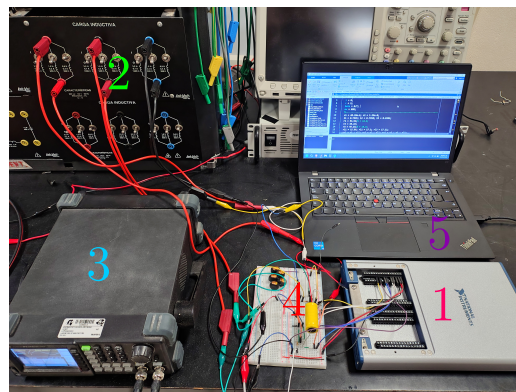


Figure 3. Experimental platform used for the implementation of the case studies.

4.2. Two-Mesh Circuit

Consider a two-mesh electrical circuit whose dynamics are described by the following set of differential equations:

$$\begin{aligned} \frac{d}{dt}i_1 &= -\frac{R_{e1}}{L_1}i_1 + \frac{R_2}{L_1}i_2 + \frac{1}{L_1}u + \frac{1}{L_1}d, \\ \frac{d}{dt}i_2 &= \frac{R_2}{L_2}i_1 - \frac{R_{e2}}{L_2}i_2 - \frac{1}{L_2}v_c, \\ \frac{d}{dt}v_c &= \frac{1}{C}i_2, \end{aligned} \quad (35)$$

with

$$\begin{aligned} R_{e1} &= R_1 + R_2 + R_{l1}, \\ R_{e2} &= R_2 + R_3 + R_{l2}, \end{aligned}$$

where i_1 and i_2 are the mesh currents, v_c is the capacitor voltage, u denotes the supply voltage, and d represents the disturbance signal. The elements L_1 and L_2 denote the inductors, whereas R_1 , R_2 , and R_3 correspond to the resistances. Finally, R_{l1} and R_{l2} represent the intrinsic resistances of the inductors.

Although the disturbance is injected into the first mesh, the proposed estimation framework is not restricted to this configuration and remains valid regardless of the disturbance location.

The system (35) is differentially flat, with flat output $y = v_c$. Using this property, the system can be rewritten as the third-order differential equation

$$\frac{d^3}{dt^3}y + \theta_2 \frac{d^2}{dt^2}y + \theta_1 \frac{d}{dt}y + \theta_0 y = \sigma_0 u + \tilde{d}, \quad (36)$$

where

$$\begin{aligned} \theta_2 &= \frac{CL_1 R_{e2} + CL_2 R_{e1}}{CL_1 L_2}, \\ \theta_1 &= \frac{L_1 + CR_{e1} R_{e2} - CR_2^2}{CL_1 L_2}, \\ \theta_0 &= \frac{R_{e1}}{CL_1 L_2}, \\ \sigma_0 &= \frac{R_2}{CL_1 L_2}. \end{aligned}$$

The disturbance representation in the flat framework differs from the original one. In this case, since the disturbance acts on the first mesh, it follows that $\tilde{d} = \sigma_0 d$.

Since (36) is a third-order system, four parameters must be estimated. By applying the estimator expression (15), the following algebraic relation is obtained:

$$\sigma_0 a_{1,1} + \theta_2 a_{1,2} + \theta_1 a_{1,3} + \theta_0 a_{1,4} = b_1, \quad (37)$$

from which a system of linear equations is constructed through successive integration. The resulting system can be written as

$$\mathbf{A}\boldsymbol{\eta} = \mathbf{b}, \quad (38)$$

where $\boldsymbol{\eta} = [\sigma_0 \ \theta_2 \ \theta_1 \ \theta_0]^\top$. The parameters are estimated using Cramer's rule with exponential weighting to improve numerical robustness.

The circuit parameters were experimentally measured in order to compare the estimated values with their true counterparts. The measured values are $C = 1.25 \ \mu\text{F}$, $L_1 = 739.5 \ \text{mH}$, $L_2 = 735.8 \ \text{mH}$, $R_1 = 54.51 \ \Omega$, $R_2 = 49.22 \ \Omega$, $R_3 = 49.19 \ \Omega$, $R_{l1} = 15.96 \ \Omega$, and $R_{l2} = 17.3 \ \Omega$.

To evaluate the performance of the estimator under different disturbance conditions, two scenarios are considered. First, a square-wave disturbance is applied, followed by a triangular disturbance. This allows assessing the robustness of the estimator under different disturbance profiles and variation rates.

4.2.1. Square-Wave Disturbance

Based on (36), the estimator parameters are selected as $n = 3$ and $l = 0$. The disturbance is locally approximated by a zero-order Taylor polynomial, yielding $r = 0$. Consequently, $o = 4$, $p = 1$, and, by choosing $\beta = \frac{1}{10}$, the total differentiation order is $\alpha = \frac{41}{10}$.

The input signal is defined as

$$u = 7.45 \sin(90 \cdot 2\pi t),$$

while the disturbance is a square-wave signal with amplitude 1 V and frequency 0.5 Hz.

The flat output y corresponds to the capacitor voltage, as shown in Figure 4. The effect of the disturbance is clearly observed as abrupt variations in the output signal.

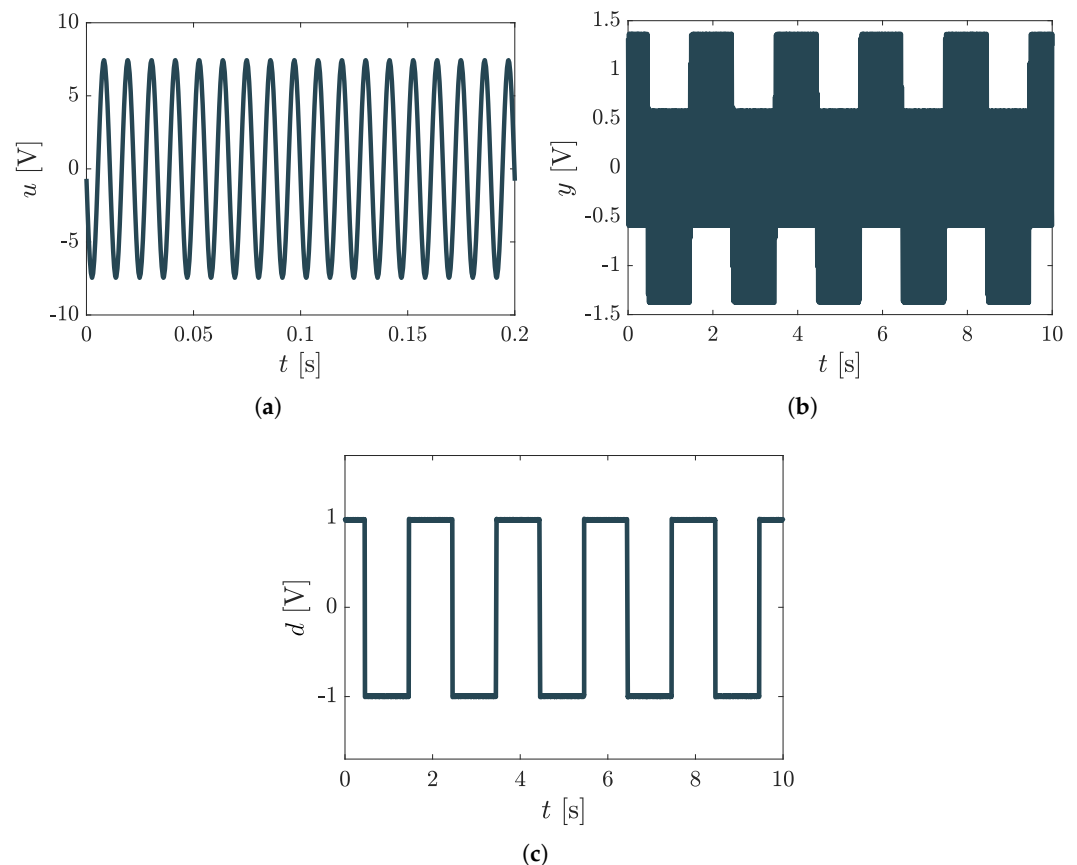


Figure 4. Measured signals of the two-mesh circuit under square-wave disturbance: (a) input voltage u , (b) flat output y , and (c) disturbance signal d .

In Figure 5, the estimated parameters for $\beta = 1/10$ are shown. An estimator reset is applied, introducing a slight time shift within the estimation window of 0.5 s. The parameters $\hat{\sigma}_0$ and $\hat{\theta}_1$ exhibit satisfactory performance, whereas $\hat{\theta}_0$ presents the largest estimation error.

Table 3 summarizes the estimation errors for different fractional orders β . When $\beta = 0$, the parameters σ_0 and θ_1 present errors below 2%, while θ_2 and θ_0 exhibit larger deviations. As β increases, distinct trends are observed. The error of σ_0 decreases monotonically, whereas the error of θ_2 increases slightly. The parameter θ_1 exhibits a significant improvement, reaching minimal error around $\beta = 8/10$. Similarly, θ_0 shows a gradual reduction in error up to $\beta = 8/10$. Overall, excluding θ_2 , the best estimation performance is achieved for $\beta \in [7/10, 8/10]$.

Table 3. Parameter estimation errors $e_{\eta_h}(\beta)$ (%) for different fractional derivative orders β in the two-mesh circuit.

$\hat{\eta}_h$	True Value	Fractional Derivative Order β									
		0	1/10	2/10	3/10	4/10	5/10	6/10	7/10	8/10	9/10
σ_0	7.236×10^7	1.056	0.995	0.935	0.875	0.817	0.760	0.706	0.655	0.607	0.563
θ_2	319.110	4.015	4.025	4.047	4.077	4.117	4.167	4.226	4.295	4.375	4.466
θ_1	1.117×10^6	0.231	0.199	0.167	0.136	0.105	0.075	0.047	0.021	−0.003	−0.024
θ_0	1.759×10^8	9.856	9.819	9.784	9.753	9.726	9.706	9.692	9.685	9.689	9.701

Note: Errors are expressed as relative percentage errors with respect to the true parameter values.

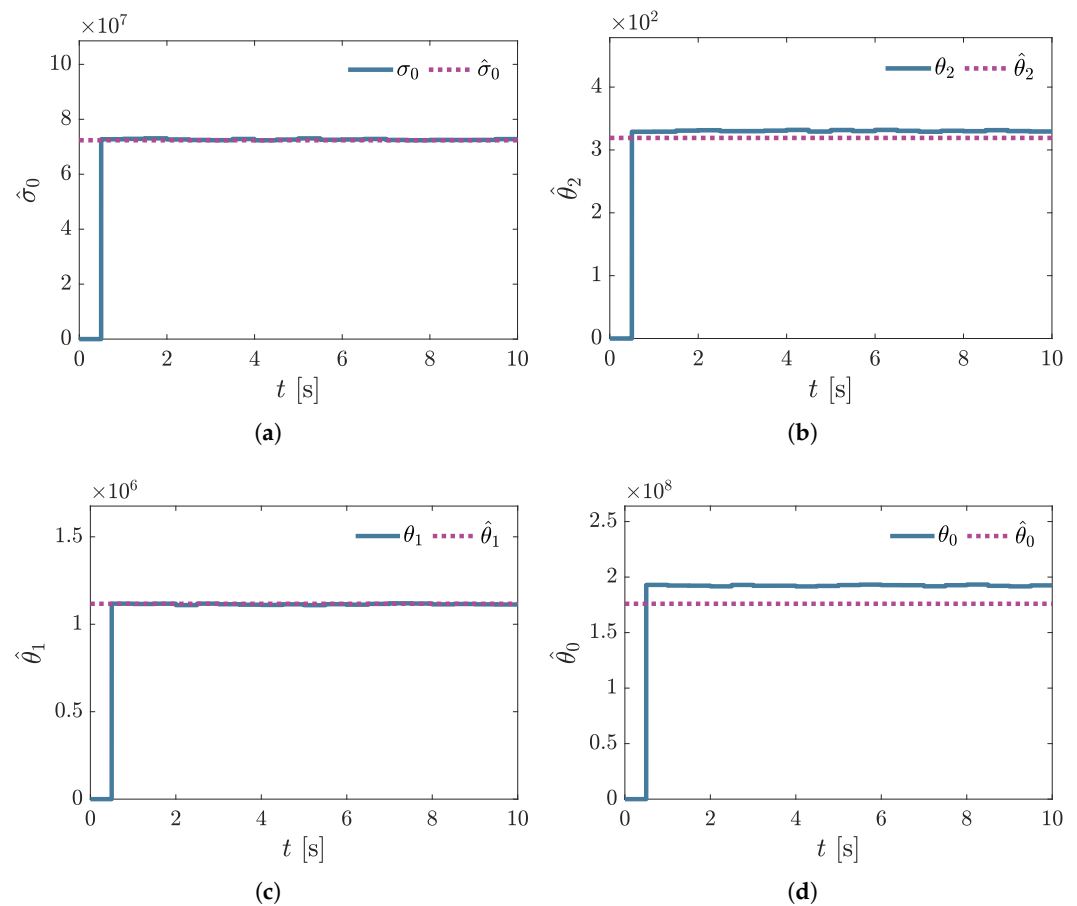


Figure 5. Time evolution of the estimated parameters of the two-mesh under square-wave disturbance: (a) σ_0 ; (b) θ_2 ; (c) θ_1 ; (d) θ_0 .

To provide a global performance measure, the following index is defined:

$$J(\beta) = \frac{1}{c} \sum_{h=1}^c |e_{\eta_h}(\beta)|, \quad (39)$$

where c is the number of estimated parameters.

This metric allows identifying the fractional order that yields the best overall estimation performance. The results, shown in Figure 6, indicate that the average estimation error decreases as β increases, reaching its minimum at $\beta = 7/10$.

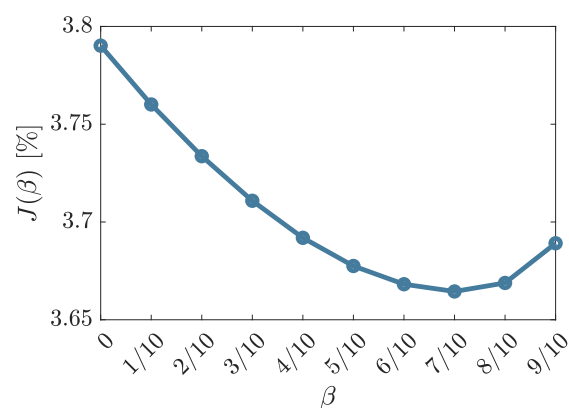


Figure 6. Average absolute estimation error index $J(\beta)$ for the two-mesh circuit as a function of the fractional differentiation order β .

The combined analysis of Table 3 and (39) provides a comprehensive assessment of the estimator performance. Although the optimal fractional order is not explicitly determined in this work, these results highlight the potential of β as a tuning parameter, motivating future research on its systematic selection.

4.2.2. Triangular-Wave Disturbance

In this scenario, the system is perturbed using a triangular waveform with amplitude 1 V and frequency 2 Hz. The input signal remains unchanged with respect to the previous case, namely a sinusoidal signal with amplitude 7.45 V and frequency 90 Hz.

For the estimator design, the parameters are selected as $n = 3$ and $l = 0$. In contrast to the previous case, the disturbance is locally approximated by a first-order Taylor polynomial, yielding $r = 1$. Consequently, $o = 5$ and $p = 2$, and by selecting $\beta = \frac{1}{10}$, the total differentiation order becomes $\alpha = \frac{51}{10}$.

The signals corresponding to this scenario are shown in Figure 7. The input signal, depicted in Figure 7a, is identical to that of the previous case. The effect of the disturbance is reflected in the flat output y , shown in Figure 7b, where more pronounced variations are observed due to the injected perturbation. The triangular disturbance signal d , with a duty cycle of 50%, is shown in Figure 7c. These signals are used for the implementation of the parameter estimation procedure.

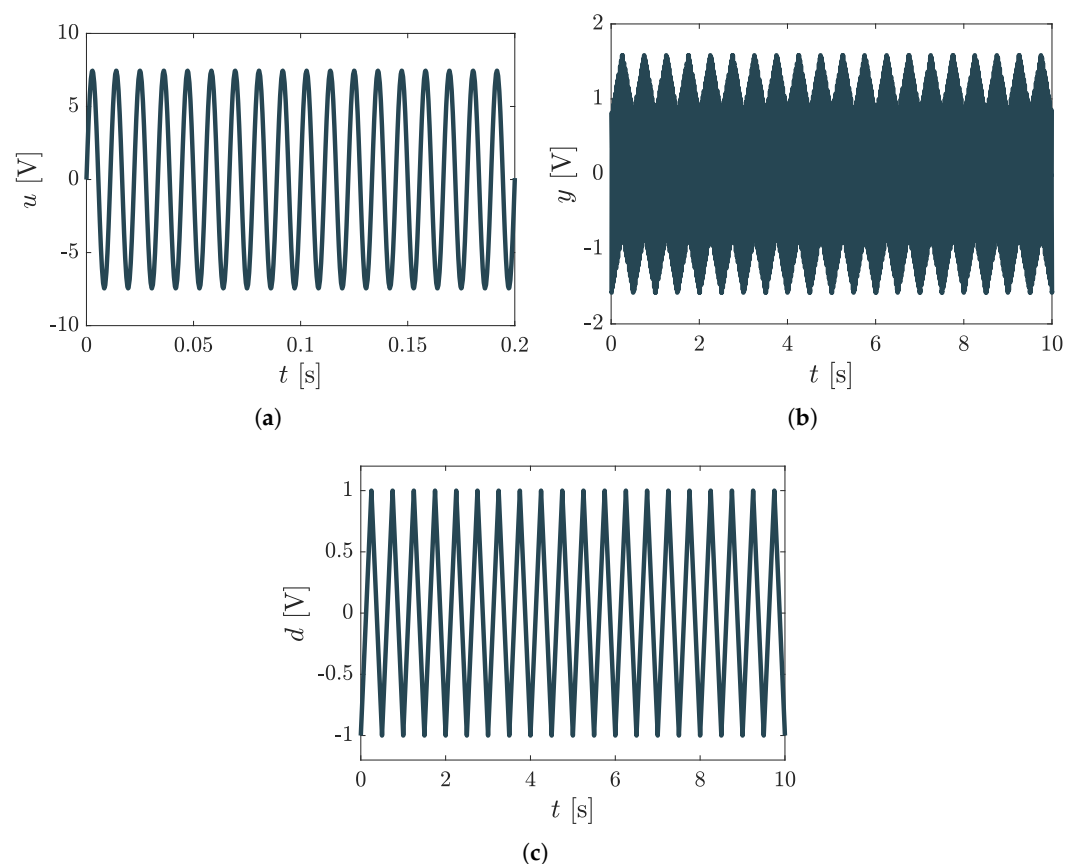


Figure 7. Measured signals of the two-mesh circuit under triangular disturbance: (a) input voltage u , (b) flat output y , and (c) disturbance signal d .

The estimation results obtained for $\beta = \frac{1}{10}$ are presented in Figure 8. The overall behavior is similar to that observed in the square-wave disturbance case. The largest estimation error is associated with the parameter θ_0 , whereas σ_0 and θ_1 exhibit the smallest deviations.

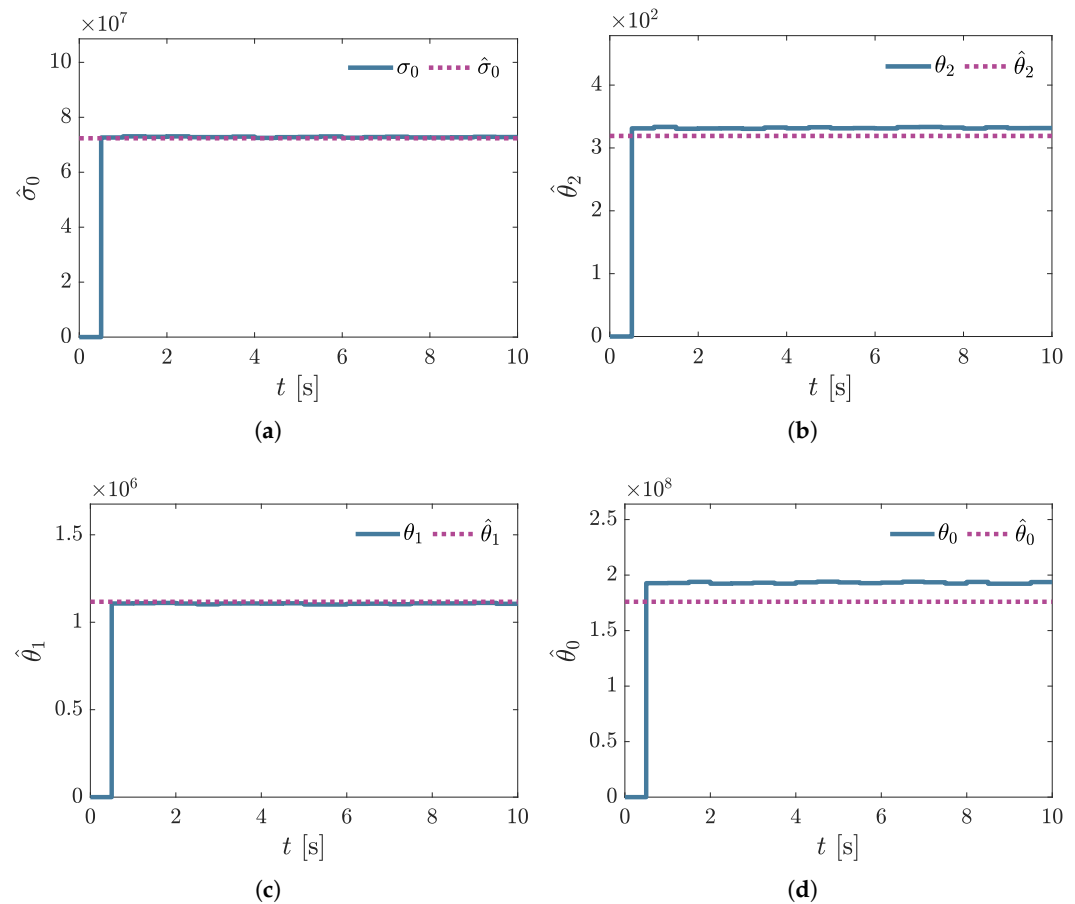


Figure 8. Estimated parameters of the two-mesh circuit under triangular-wave disturbance: (a) σ_0 ; (b) θ_2 ; (c) θ_1 ; (d) θ_0 .

Table 4 presents the estimation errors for different values of the fractional-order β . A trend similar to that observed in the previous case is identified, with the lowest errors occurring around $\beta = 7/10$. However, it is important to note that the overall error levels are higher compared with the square-wave disturbance case.

Table 4. Parameter estimation errors (%) for different fractional derivative orders β in the two-mesh circuit.

$\hat{\eta}_h$	True Value	Fractional Derivative Order β									
		0	1/10	2/10	3/10	4/10	5/10	6/10	7/10	8/10	9/10
σ_0	7.236×10^7	1.193	1.148	1.102	1.056	1.010	0.963	0.917	0.872	0.828	0.786
θ_2	319.110	4.603	4.513	4.438	4.377	4.332	4.303	4.289	4.320	4.508	4.756
θ_1	1.117×10^6	−0.478	−0.506	−0.535	−0.564	−0.593	−0.622	−0.650	−0.677	−0.703	−0.728
θ_0	1.759×10^8	10.437	10.331	10.234	10.146	10.068	10.001	9.944	9.897	9.862	9.944

Note: Errors are expressed as relative percentage errors with respect to the true parameter values.

This behavior indicates that, although the proposed estimator is capable of mitigating the effect of disturbances, the estimation accuracy is still influenced by the disturbance characteristics, particularly when the validity of the local polynomial approximation is reduced.

A global performance analysis is carried out using the index defined in (39), computed from the values reported in Table 4. The results are shown in Figure 9. It can be observed that the average estimation error is higher than in the previous scenario, which is consistent

with the more demanding disturbance profile. Nevertheless, the minimum value of $J(\beta)$ is again obtained at $\beta = 7/10$.

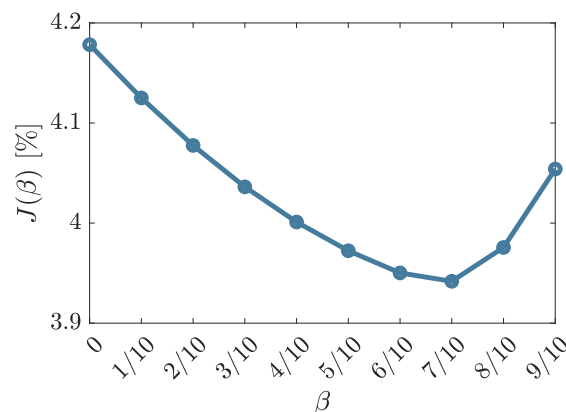


Figure 9. Average absolute estimation error of the parameters in the two-mesh circuit as a function of the fractional derivative order.

These results are particularly relevant, as both disturbance scenarios exhibit a similar trend in the behavior of $J(\beta)$, with a minimum consistently achieved at $\beta = 7/10$. This suggests that the optimal fractional order is not solely dependent on the specific disturbance waveform, but is also influenced by the intrinsic dynamics of the system and the structure of the estimator.

This observation provides valuable insight into the role of β as a tuning parameter and motivates further research toward the systematic determination of an optimal fractional differentiation order.

4.3. Three-Mesh Circuit

The second case study considers a three-mesh electrical circuit described by the dynamic model

$$\begin{aligned}
 \frac{d}{dt}i_1 &= -\frac{R_{e1}}{L_1}i_1 - \frac{1}{L_1}v_{c1} + \frac{1}{L_1}u + \frac{1}{L_1}d, \\
 \frac{d}{dt}i_2 &= -\frac{R_{e2}}{L_2}i_2 + \frac{R_2}{L_2}i_3 + \frac{1}{L_2}v_{c1}, \\
 \frac{d}{dt}i_3 &= \frac{R_2}{L_3}i_2 - \frac{R_{e3}}{L_3}i_3 - \frac{1}{L_3}v_{c2}, \\
 \frac{d}{dt}v_{c1} &= \frac{1}{C_1}i_1 - \frac{1}{C_1}i_2, \\
 \frac{d}{dt}v_{c2} &= \frac{1}{C_2}i_3,
 \end{aligned} \tag{40}$$

with

$$\begin{aligned}
 R_{e1} &= R_1 + R_{l1}, \\
 R_{e2} &= R_2 + R_{l2}, \\
 R_{e3} &= R_2 + R_3 + R_{l3},
 \end{aligned}$$

where i_1 , i_2 , and i_3 are the mesh currents, v_{c1} and v_{c2} are the capacitor voltages, u denotes the supply voltage, and d represents the disturbance signal.

This system is differentially flat, with flat output $y = v_{c2}$. The corresponding flat representation leads to the fifth-order differential equation

$$\frac{d^5}{dt^5}y + \theta_4 \frac{d^4}{dt^4}y + \theta_3 \frac{d^3}{dt^3}y + \theta_2 \frac{d^2}{dt^2}y + \theta_1 \frac{d}{dt}y + \theta_0 y = \sigma_0 u + \tilde{d}, \quad (41)$$

where the coefficients θ_i and σ_0 depend on the circuit parameters.

The component values used in this study are: $C_1 = 48.28 \mu\text{F}$, $C_2 = 1.25 \mu\text{F}$, $L_1 = 739.5 \text{ mH}$, $L_2 = 735.8 \text{ mH}$, $L_3 = 636.4 \text{ mH}$, $R_1 = 54.51 \Omega$, $R_2 = 49.22 \Omega$, $R_3 = 49.19 \Omega$, $R_{l1} = 15.96 \Omega$, $R_{l2} = 17.3 \Omega$, and $R_{l3} = 17.2 \Omega$.

As in the previous case, two disturbance scenarios are considered in order to evaluate the estimator performance under different perturbation conditions.

4.3.1. Sinusoidal Disturbance

From (41), the estimator parameters are selected as $n = 5$ and $l = 0$. The disturbance is locally approximated by a second-order Taylor polynomial, yielding $r = 2$, which results in $o = 8$ and $p = 3$. Substitution into (15) leads to a system of linear equations constructed through successive integration.

The unknown parameter vector is defined as

$$\boldsymbol{\eta} = [\sigma_0 \theta_4 \theta_3 \theta_2 \theta_1 \theta_0]^\top,$$

which is estimated using Cramer's rule with exponential weighting.

The input signal is defined as

$$u = 7.5 \sin(60 \cdot 2\pi t),$$

while the disturbance is a sinusoidal signal with amplitude 1.5 V and frequency 0.5 Hz.

Under these conditions, the disturbance can be accurately approximated by a low-order polynomial over short time intervals. The sampled signals are shown in Figure 10. The control input u is presented in Figure 10a and induces oscillations of the same frequency in the system response, depicted in Figure 10b. In contrast, the slowly varying disturbance, shown in Figure 10c, produces a noticeable deviation in the circuit response. This behavior provides a clear illustration of how disturbances influence system dynamics, highlighting the importance of explicitly accounting for their effects in parameter estimation.

Figure 11 shows the estimated parameters for $\beta = 1/7$. The results indicate good agreement with the true values, except for $\hat{\sigma}_0$, which exhibits the largest deviation.

Table 5 summarizes the estimation errors for different fractional orders β . Compared with the two-mesh circuit, the estimation performance is more sensitive to the choice of β , which can be attributed to the higher system order and increased number of parameters.

The parameter σ_0 exhibits a non-monotonic behavior, with the estimation error decreasing up to $\beta = 4/7$, followed by a significant increase. The parameter θ_4 shows a persistent negative bias that increases with β . Parameters θ_3 and θ_2 exhibit a transition from positive to negative errors, indicating a change in estimation bias as β increases. Finally, θ_1 and θ_0 achieve their lowest errors for intermediate values of β .

Overall, these results suggest that intermediate fractional orders, specifically $\beta \in [2/7, 4/7]$, provide the best trade-off between estimation accuracy and robustness.

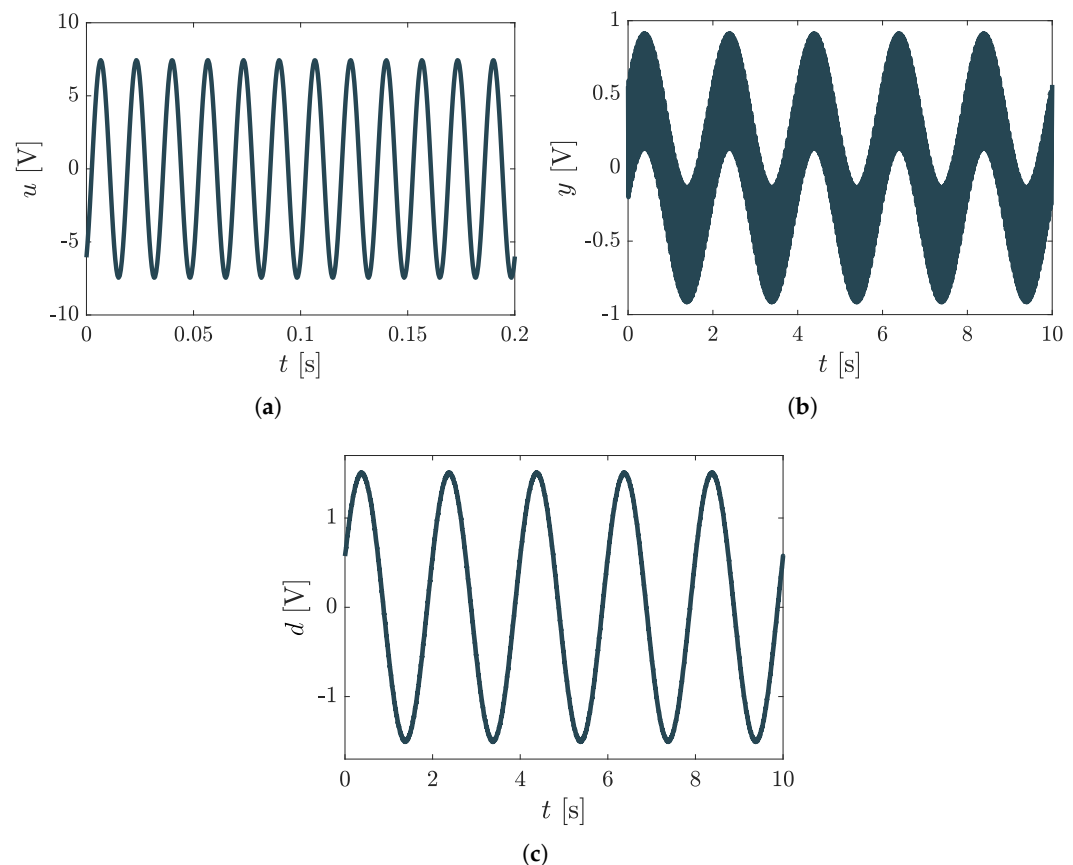


Figure 10. Signals of the three-mesh circuit: (a) input voltage u ; (b) flat output y ; (c) disturbance signal d .

Table 5. Parameter estimation errors (%) for different fractional derivative orders β in the three-mesh circuit.

$\hat{\eta}_h$	True Value	Fractional Derivative Order β						
		0	1/7	2/7	3/7	4/7	5/7	6/7
σ_0	2.355×10^{12}	7.339	6.174	5.363	4.773	4.500	5.185	7.518
θ_4	367.235	−1.471	−1.577	−1.679	−1.795	−1.939	−2.137	−2.335
θ_3	1.348×10^6	1.890	0.658	−0.433	−1.664	−3.307	−5.592	−7.720
θ_2	1.375×10^8	0.161	−1.064	−2.150	−3.373	−5.010	−7.267	−9.339
θ_1	8.222×10^{10}	2.947	1.715	0.632	−0.588	−2.240	−4.509	−6.559
θ_0	6.555×10^{12}	3.143	1.896	0.795	−0.444	−2.116	−4.410	−6.488

Note: Errors are expressed as relative percentage errors with respect to the true parameter values.

To provide a global performance measure, the index defined in (39) is evaluated with $c = 6$. The results are shown in Figure 12, where it can be observed that the minimum value of $J(\beta)$ is achieved at $\beta = 2/7$.

This result partially agrees with the trends observed in Table 5, confirming that lower fractional orders yield improved performance in this case. In contrast to the two-mesh circuit, the optimal fractional order is found below $\beta = 1/2$, highlighting the dependence of the optimal value on the system dynamics and complexity.

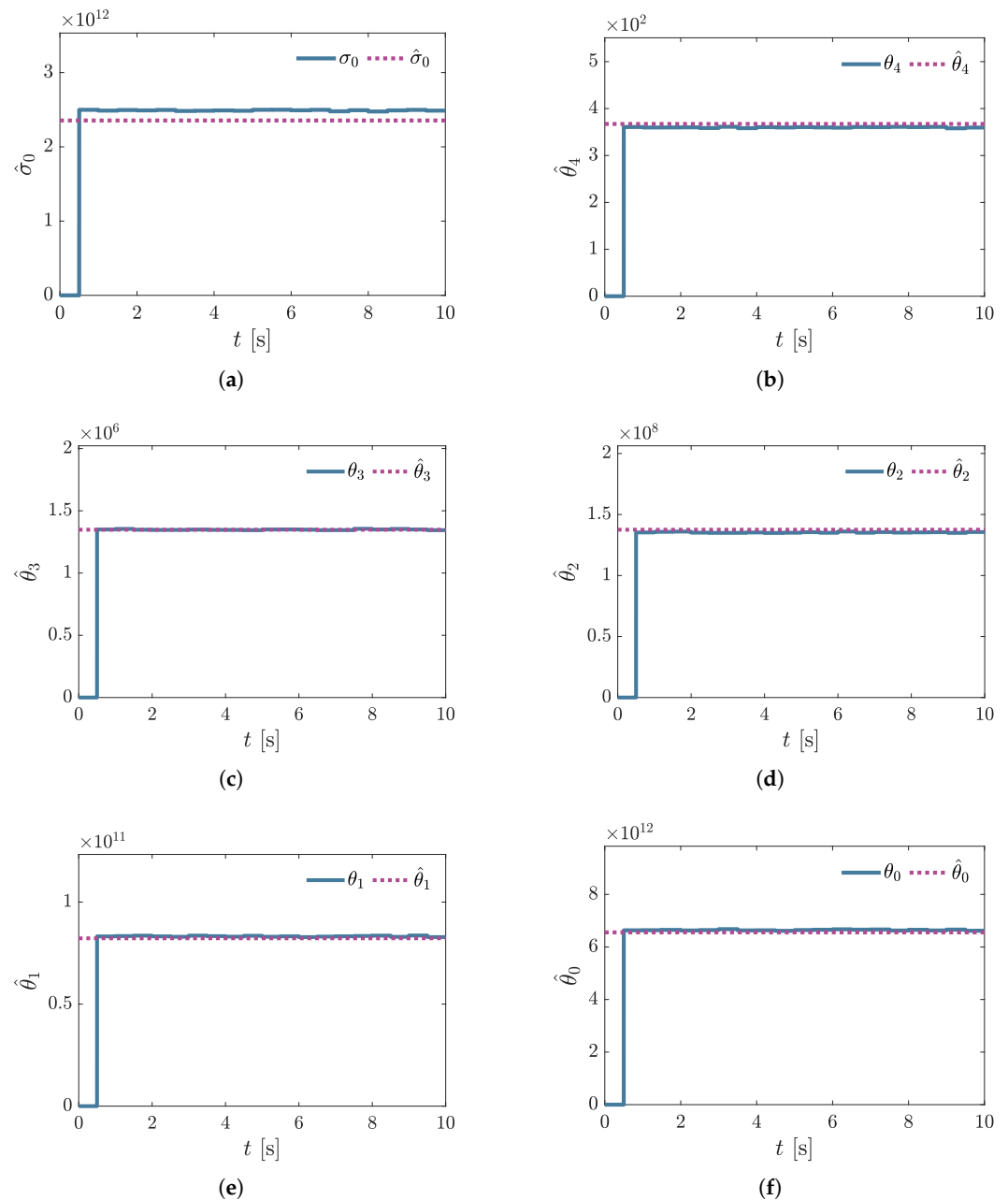


Figure 11. Estimated parameters of the three-mesh circuit under sinusoidal disturbance: (a) σ_0 ; (b) θ_4 ; (c) θ_3 ; (d) θ_2 ; (e) θ_1 ; (f) θ_0 .

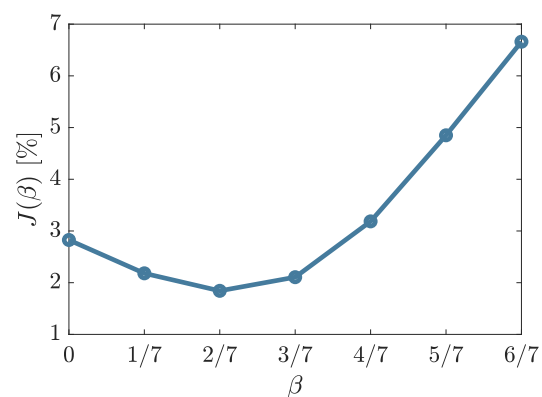


Figure 12. Average absolute estimation error $J(\beta)$ of the three-mesh circuit under sinusoidal disturbance as a function of the fractional differentiation order β .

4.3.2. Composite Disturbance

In this scenario, the input signal is defined as a sinusoidal waveform with amplitude 7.5 V and frequency 120 Hz. The disturbance d is constructed as the superposition of two signals: a triangular waveform with amplitude 1 V and frequency 1 Hz, and a sinusoidal signal with amplitude 0.1 V and frequency 1.5 Hz.

Due to the increased complexity of the excitation signals, an additional signal generator was required, since the function generator used in previous experiments was limited to two output channels.

The disturbance is locally approximated by a first-order Taylor polynomial, yielding $r = 1$. Since no derivatives of the input signal u are involved, $l = 0$, and from (41), the system order is $n = 5$. Consequently, $o = 7$ and $p = 2$. By selecting $\beta = 1/7$, the total differentiation order becomes $\alpha = 50/7$.

The signals corresponding to this case are shown in Figure 13. The increase in the excitation frequency can be observed in Figure 13a. The disturbance signal, shown in Figure 13b, exhibits the triangular waveform modulated by the superimposed sinusoidal component, resulting in small oscillatory variations. The corresponding flat output y is presented in Figure 13c, where the influence of the composite disturbance is clearly reflected in the system response.

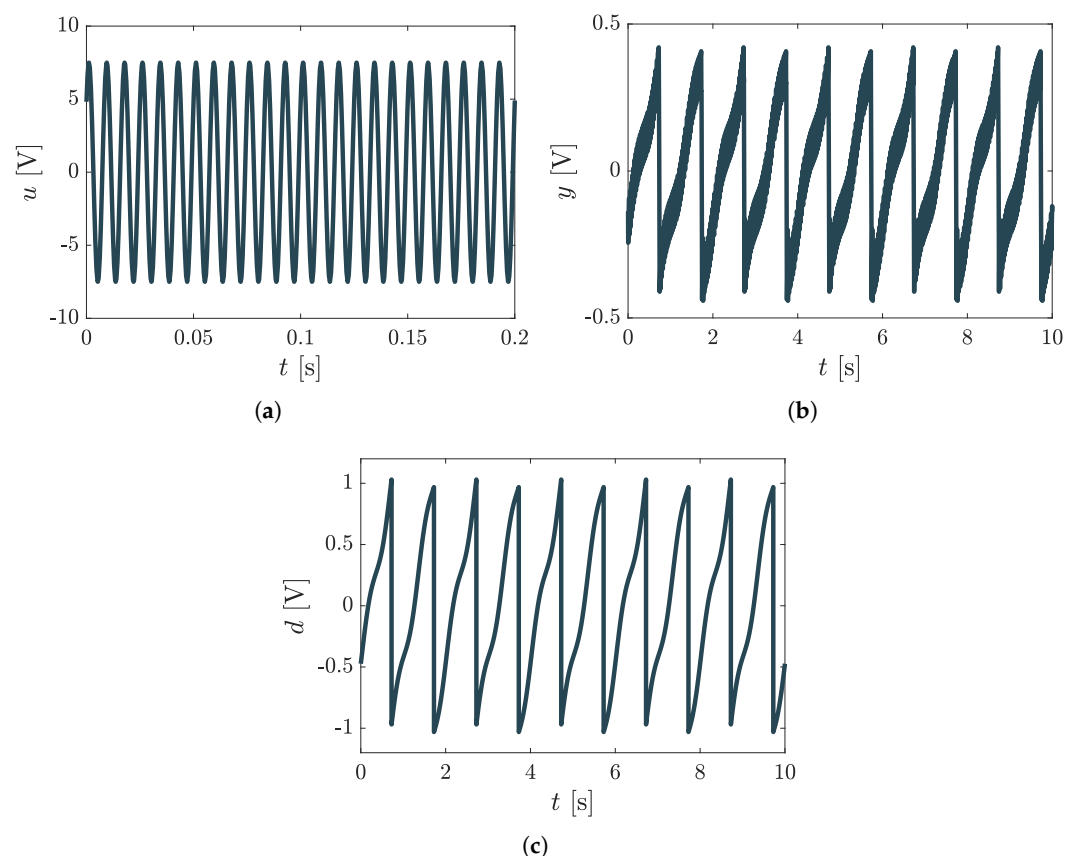


Figure 13. Signals of the three-mesh circuit under composite disturbance: (a) input voltage u ; (b) flat output y ; (c) disturbance signal d .

The estimation results obtained for $\beta = 1/7$ are shown in Figure 14. Overall, the estimator exhibits satisfactory performance under this more complex disturbance scenario. The largest estimation errors are observed in the parameters σ_0 and θ_0 , as shown in Figure 14a and Figure 14f, respectively.

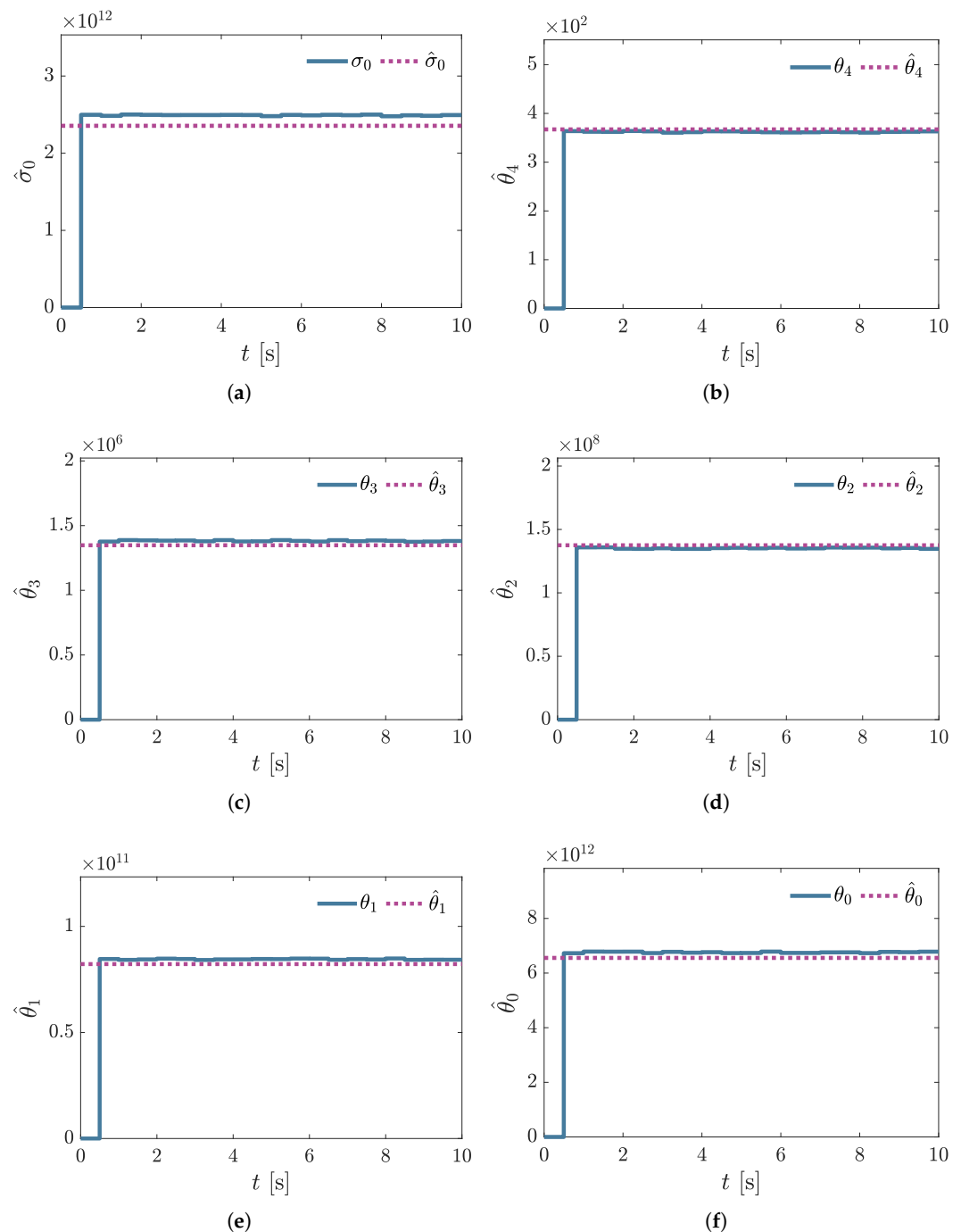


Figure 14. Estimated parameters of the three-mesh circuit under composite disturbance: (a) σ_0 ; (b) θ_4 ; (c) θ_3 ; (d) θ_2 ; (e) θ_1 ; (f) θ_0 .

The estimation errors as a function of the fractional derivative order, $e_{\eta_h}(\beta)$, are summarized in Table 6. For the parameter σ_0 , the error decreases as β increases, reaching a minimum at $\beta = 3/7$. Beyond this point, the error increases significantly, indicating the presence of an optimal fractional order.

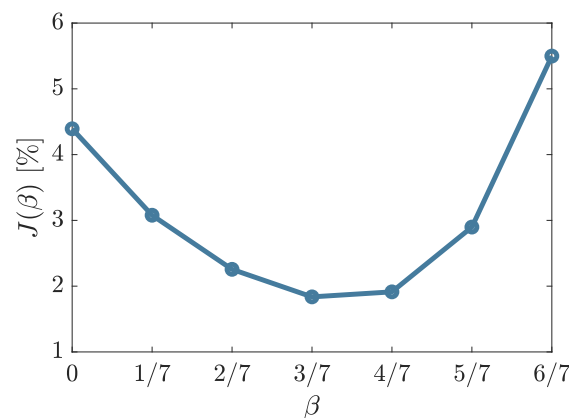
A similar behavior is observed for the remaining parameters, suggesting that the tuning of the fractional-order β is inherently nonlinear. This observation is further supported by the evaluation of the global performance index $J(\beta)$ defined in (39).

Table 6. Parameter estimation errors (%) for different fractional derivative orders β in the three-mesh circuit: estimación para perturbación de primer orden.

$\hat{\eta}_h$	True Value	Fractional Derivative Order β						
		0	1/7	2/7	3/7	4/7	5/7	6/7
σ_0	2.355×10^{12}	8.569	6.239	4.939	4.712	5.610	8.038	13.080
θ_4	367.235	−0.722	−0.861	−0.949	−1.001	−1.006	−0.901	−0.636
θ_3	1.348×10^6	4.367	3.013	2.111	1.507	1.256	2.087	4.610
θ_2	1.375×10^8	2.940	1.495	0.560	−0.025	−0.166	0.873	3.635
θ_1	8.222×10^{10}	4.742	3.293	2.350	1.752	1.592	2.626	5.405
θ_0	6.555×10^{12}	5.012	3.566	2.623	2.021	1.848	2.863	5.619

Note: Errors are expressed as relative percentage errors with respect to the true parameter values.

The results, shown in Figure 15, reveal that $J(\beta)$ exhibits a convex (parabolic-like) behavior, with a well-defined minimum. This confirms that the fractional order acts as a critical tuning parameter, whose optimal value depends on both the system dynamics and the characteristics of the disturbance.

**Figure 15.** Average absolute estimation error $J(\beta)$ of the three-mesh circuit under composite disturbance as a function of the fractional differentiation order β .

5. Discussion

The results obtained from the two-mesh and three-mesh circuits provide insight into the behavior and limitations of the proposed fractional-order algebraic estimator under different disturbance scenarios. A first important observation is that the introduction of a fractional differentiation order $\beta > 0$ leads to improved estimation accuracy in the considered cases when compared with the integer-order formulation ($\beta = 0$), confirming that the fractional operator introduces an additional degree of freedom that can be exploited to enhance estimation performance. However, the optimal value of β is not universal and depends on both the system dynamics and the disturbance characteristics. For the two-mesh circuit, the best performance is achieved around $\beta \approx 7/10$, while for the three-mesh circuit it lies in the range $\beta \in [2/7, 4/7]$, highlighting the influence of system order and complexity.

From a signal-processing perspective, β can be interpreted as a trade-off between smoothing effects and responsiveness to signal variations. Within the algebraic estimation framework, this effect is associated with the implicit integration introduced by the formulation. Higher values increase smoothing, improving the behavior under slowly varying conditions, but may introduce bias when faster variations are present. Conversely, lower values improve responsiveness but reduce smoothing, which can degrade estimation accuracy for rapidly varying signals. This behavior is further evidenced in the composite disturbance scenario, where the presence of multiple frequency components increases the

sensitivity to β and results in a more pronounced convex behavior of the performance index $J(\beta)$.

Although the proposed method mitigates the influence of disturbances, the estimation accuracy depends on the validity of the local polynomial approximation. This is consistent with the underlying assumption that disturbances are locally smooth and admit polynomial representations over sufficiently small time intervals. As the disturbance deviates from this behavior, the estimation errors increase, although the method remains stable and preserves reasonable estimation accuracy.

The proposed framework is applicable to linear systems that admit a finite-order input–output differential representation and satisfy the property of differential flatness, such as electrical circuits, electromechanical systems, and linear mechanical systems. In the context of control engineering, these systems are typically operated under closed-loop conditions, where parameter estimation plays a key role in ensuring performance and stability. In general, the method requires measurable signals, sufficiently smooth behavior, and adequate excitation to guarantee parameter identifiability. Despite these favorable properties, several limitations must be acknowledged. First, the method assumes disturbances that admit local polynomial representations through Taylor series expansions over sufficiently small time intervals. Second, the selection of β is currently performed empirically. Third, the computational cost may become relevant for higher-order systems or implementations on resource-constrained platforms.

These observations suggest several directions for future research. In particular, the development of systematic strategies for selecting β , the extension to nonlinear systems and more complex disturbance structures, and the design of efficient numerical implementations constitute relevant topics for further investigation. In addition, the applicability of the proposed framework to microscale systems such as MEMS, where parameter estimation under small-scale disturbances and uncertainties is particularly challenging, represents a promising direction for future work.

6. Conclusions

This paper presented a fractional-order algebraic parameter estimation methodology for linear differentially flat systems subject to external disturbances. By combining algebraic estimation with fractional-order differentiation in the Laplace domain, the proposed approach reduces the influence of unknown initial conditions and disturbances that admit local polynomial representations through Taylor series expansions over sufficiently small time intervals, while preserving a closed-form algebraic structure.

The effectiveness of the estimator was experimentally validated on two electrical circuits of increasing complexity under multiple disturbance scenarios. The results indicate that introducing a fractional differentiation order $\beta > 0$ leads to improved estimation accuracy in the considered cases when compared with the conventional integer-order formulation. In particular, the inclusion of β provides an additional degree of freedom that allows balancing smoothing effects and responsiveness to signal variations, leading to enhanced estimation performance under the tested conditions.

The analysis further revealed that the optimal value of β is not universal, but depends on both the system dynamics and the characteristics of the disturbance. In the two-mesh circuit, higher values of β yielded better performance, whereas in the three-mesh circuit, intermediate values provided the best trade-off. This behavior highlights the role of β as a key tuning parameter that governs the estimator performance. Moreover, the proposed methodology does not require explicit numerical differentiation in the time domain nor prior knowledge of the disturbance signal. Even in the presence of more demanding

disturbance profiles, such as multi-frequency signals, the estimator remained stable and provided satisfactory accuracy under the considered conditions.

From a control engineering perspective, the proposed framework provides a structured approach for parameter estimation in controlled linear engineering systems, where disturbances and uncertainties are inherently present. Future research will focus on the systematic selection of the fractional differentiation order, potentially through optimization-based or adaptive strategies. A more comprehensive comparison with classical estimation techniques, such as least squares or Kalman filtering, constitutes a relevant direction for future work. Additionally, the extension of the proposed framework to nonlinear differentially flat systems, real-time implementations, and systems inherently described by fractional-order models will be investigated.

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