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Advancing the Application of the Rayleigh-Schrödinger Method for Identifying Key Parameters in Fractional-Order Begley–Torvik Type Models

Andrey Sedelnikov * and Hedi Aleroeva

Department of Metal Forming, Institute of Aviation and Rocket and Space Technology, Samara National Research University, 34 Moskovskoye Shosse, Samara 443086, Russia; binsabanur@gmail.com
* Correspondence: axe_backdraft@inbox.ru

Abstract

This study develops an extension of the classical Rayleigh–Schrödinger method for solving fractional-order differential equations. The primary objective is to derive the eigenvalues of a Begley–Torvik-type equation. The proposed analytical expression for the eigenvalues, obtained through this methodological advancement, shows excellent agreement with their exact values. This result is obtained by developing the Rayleigh–Schrödinger method and can be used for a wide range of applied problems. As an illustrative example, the Begley–Torvik type equation is used to describe the deformation-strength characteristics of polymer concrete and other modern granular road materials. It should be noted, however, that this represents just one of many potential applications for such fractional-order models.

Keywords: Rayleigh–Schrödinger method; Begley–Torvik equation; fractional derivative; perturbation theory

MSC: 34L10; 35G05; 35P15; 35R11; 41A30

1. Introduction

The Rayleigh–Schrödinger method [1,2] in perturbation theory is well-established and widely used for the analysis of perturbed systems [3–5]. Researchers are consistently drawn to this method due to its straightforward and logical concept of describing a complex system in terms of a simpler one. This approach of utilizing a simpler reference system is not unique to the Rayleigh–Schrödinger method and is, in fact, extensively applied to solve a broad range of problems [6–9]. The core idea is to start with a simple system for which the solution is known and which is related to the complex system under investigation. A small additional (perturbing) Hamiltonian is then introduced as a weak perturbation to this simple system. These perturbations must be sufficiently small. The solution for the complex system can then be expressed through corrections to the solution of the simple system. These corrections, which are small compared to the magnitudes of the principal quantities, can be calculated using approximate methods.

However, a significant amount of time has passed since the development of the Rayleigh–Schrödinger method. Many new fields have emerged in mathematics. One of them is fractional calculus [10–12]. This mathematical framework has proven highly effective for solving a wide range of problems, from the study of oscillations [13–15] to heat conduction [16–18]. Notably, when modeling the deformation and strength characteristics

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of polymer concrete and other modern granular road materials, many authors [19–22] suggest using the Begley–Torvik equation [23]:

$$mu''(x) + \nu\, D^{\alpha}_{0x} u + ku(x) = \xi(x), \tag{1}$$

where $u(x)$ is the granule displacement under load ($x \in [0,\ 1]$), m is the granule mass, ν is the viscosity modulus of the polymer concrete filler, α is the viscoelasticity parameter of the medium, and $\xi(x)$ is the external load acting on the polymer concrete during the road surface operation.

When α = 0.7 Equation (1) takes the following form:

$$mu''(x) + \nu\, D^{0.7}_{0x} u + ku(x) = \xi(x). \tag{2}$$

In reference [24], Equation (2) is used to describe the oscillations of elastomeric bearings. These are modeled as an oscillator with viscoelastic damping, commonly referred to as a fractional oscillator [25,26].

The aim of the paper is to extend the Rayleigh-Schrödinger method to obtain the eigenvalues of equations with fractional derivatives. This study develops an extension of the Rayleigh–Schrödinger method and derives the eigenvalue dependencies for the Begley–Torvik type Equation (2).

2. Problem Statement and Identification of Perturbation Theory Limitations

Reference [27] presents a well-known method for estimating the key parameters of Begley–Torvik-type models. It is performed for a particular problem with a small ε parameter, which is described by the following equation:

$$\begin{cases} u'' - \varepsilon D^{\alpha}_{0x} u = -\lambda u,\ \alpha \in (0,\ 1), \\ u(0) = u(1) = 0. \end{cases} \tag{3}$$

Two theorems are proven [27]:

Theorem 1. *If* $\varepsilon < 0.24$, *then all eigenvalues of problem (1) are simple and real.*

Theorem 2. *All eigenvalues of problem (1) are negative when* $\varepsilon > 0$.

The corollary to the Theorem 1 has been proven:

Investigation. *Problem (1) does not generate associated functions.*

However, the given method does not allow for the study models of the following type:

$$\begin{cases} \frac{d^2u(x)}{dx^2} - \frac{\mu}{E} D^{\alpha} u(x) = -\frac{f(x)}{E}\, u(x), \\ u(0) = u(1) = 0. \end{cases} \tag{4}$$

The refinement of mathematical models for processes across various scientific and engineering fields, accounting for the fractal structure of the studied processes, leads to a new formulation of the parametric identification problem. The use of fractional orders in integro-differential operators transforms identification problems into structural-parametric ones. This is because an additional degree of freedom is introduced into the system—the fractional order of the integrals and derivatives that constitute the equations of the mathematical models. One of the most pressing and currently unsolved challenges in fractional calculus is the development of methods for the structural and parametric identification of dynamical systems whose mathematical models contain fractional-order integro-differential operators. Therefore, this work aims to develop an approximate method for solving such problems. This method constitutes an extension of the well-known Rayleigh-Schrödinger method [28] and is distinguished by its incorporation of fractional-order derivatives.

3. Development of the Rayleigh-Schrödinger Method

To derive the necessary formulas in perturbation theory, we use the method of successive approximations [28]. Let us consider the problem in its general formulation that was represented by Equations (2) and (3):

$$\begin{cases} \frac{d^2\psi(x)}{dx^2} + \varepsilon D_{0x}^{\alpha}\psi(x) + \rho^2\psi(x) + \lambda\nu(x)\psi(x) = 0, \\ \psi(0) = \psi(1) = 0, \end{cases} \tag{5}$$

where ε; ρ^2 and λ are some numerical parameters.

Let us take the following problem as the unperturbed problem:

$$\begin{cases} \frac{d^2\psi(x)}{dx^2} + \varepsilon D_{0x}^{\alpha}\psi(x) + \rho^2\psi(x) = 0, \\ \psi(0) = \psi(1) = 0. \end{cases} \tag{6}$$

We obtain the integral formulation of problem (5) using the corresponding Green's function Gρ (x, t):

$$\psi(x) = -\lambda \int_0^1 G_\rho(x,\ t)\,\nu(t)\psi(t)dt. \tag{7}$$

Since the Skolem standard form of problem (6) forms a basis in L2(0, 1), then

$$G_\rho(x,\ t) = \sum_p \frac{\varphi_p(t)\varphi_p^*(t)}{\rho^2 - \mu_n} \tag{8}$$

where$\{\varphi_p{}^*(t)\}$ is the system biorthogonal to $\{\varphi_p(t)\}$.

Substituting (8) into (7), we obtain:

$$\psi_n(x) = \varphi_n(x) + \lambda \sum_p \frac{\int_0^1 \varphi_n(x)\nu(x)\psi_n(x)\,dx}{\rho^2 - \mu_n}\varphi_p(x). \tag{9}$$

Equation (9) is solved by the method of successive approximations. For the zero-order approximation of $\psi_n(x)$, we take $\varphi_n(x)$. Consequently, the first-order approximation for $\psi_n(x)$ takes the form [28]:

$$\psi_n^{(1)}(x) = \varphi_n(x) + \lambda \sum_{p\neq n} \frac{\nu_{pn}}{\rho^2 - \mu_n}\varphi_p(x). \tag{10}$$

The first two approximations for ρ^2 have the form [28]:

$$\begin{cases} \left(\rho^2\right)^{(1)} = \rho_n^2 + \lambda\nu_{nn}, \\ \left(\rho^2\right)^{(2)} = \rho_n^2 + \lambda\nu_{nn} + \lambda^2 \sum\limits_{p\neq n} \frac{\nu_{pn}\,\nu_{np}}{\rho^2 - \mu_n}, \end{cases} \tag{11}$$

where $\nu_{pn} = \int_0^1 \varphi_p(x)\nu(x)\psi_n(x)\,dx$.

In practical applications of perturbation theory, the first-order approximation for eigenfunctions and the second-order approximation for eigenvalues are typically used. It should be noted, however, that the method presented here is only valid when the series of successive approximations converges. The applicability conditions of perturbation theory can be expressed as follows [28]:

$$|\nu_{lm}| = |\mu_l - \mu_m|. \tag{12}$$

The exact test of condition (12) is related to eigenfunctions and corresponding biorthogonal functions that depend on the magnitude of the eigenvalues. However, the asymptotics of the eigenfunctions are known. It depends on the asymptotics of the eigenvalues. That evaluation of the integral $\nu_{pn} = \int\limits_0^1 \varphi_p(x)\nu(x)\psi_n(x)\,dx$ depends on the $\nu(x)$ function. In each specific model, this function is different. It should be taken small enough to satisfy condition (12).

4. Main Results of the Work

Let us consider the application of the Rayleigh-Schrödinger method to a more specific formulation of problem (5). To this end, we prove the following theorem.

Theorem 3. *The eigenvalues of the model of the form*

$$\begin{cases} -u''(x) + cD^{\alpha}u(x) + \lambda u(x) = 0, \ x \in [0;\ \aleph], \\ u(0) = u(\aleph) = 0. \end{cases} \tag{13}$$

for $0 < \alpha < 1$ have the form:

$$\lambda_n = \left(\frac{\pi n}{\aleph}\right)^2 - c\frac{2}{\pi}\left(\frac{\pi n}{\aleph}\right)^{1-\alpha} \sin\frac{\alpha\pi}{2}, \tag{14}$$

Proof of Theorem 1. We will conduct the proof for the case $\aleph = \pi$ without loss of generality. The transition to the general case can be made by a change of variable. Thus, we rewrite (13) in the form:

$$\begin{cases} u''(x) + cD^{1-\alpha}u(x) + \lambda u(x) = 0, \ x \in [0;\ \pi], \\ u(0) = u(\pi) = 0, \end{cases} \tag{15}$$

where $D^{1-\alpha}$ is the fractional differentiation Riemann-Liouville operator of order $1-\alpha$; c is some constant.

Imagine the operator $D^{1-\alpha}R(\zeta, T)$ as an amount [29]:

$$D^{1-\alpha}R(\zeta,\ T) = xT'R(\zeta,\ T) + \varepsilon T''R(\zeta,\ T). \tag{16}$$

Since the operator:

$$Tu = \begin{cases} -u'', \\ u(0) = u(\pi) = 0, \end{cases} \tag{17}$$

a complete self-adjoint operator, then

$$R(\zeta, T)\vartheta = \sum_{k=1}^{\infty} \frac{(\vartheta, \sin kx)}{k^2 - \zeta} \sin kx \tag{18}$$

Therefore

$$D^{1-\alpha}R(\zeta,\ T)\vartheta = \frac{2}{\pi}D^{1-\alpha}\left(\sum_{k=1}^{\infty}\frac{(\vartheta, \sin kx)}{k^2-\zeta}\right)\sin kx = \frac{2}{\pi}\frac{(\vartheta, \sin kx)}{k^2-\zeta}D^{1-\alpha}\sin kx. \tag{19}$$

Since

$$\begin{aligned} D^{1-\alpha}\sin kx &= \tfrac{d}{dx}\left[D^{1-\alpha}\sin kx\right] = \tfrac{d}{dx}\left[n^{1-\alpha}\sin\left(nx-\tfrac{\alpha\pi}{2}\right)\right] = \\ &= n^{1-\alpha}\left[\cos nx\cos\tfrac{\alpha\pi}{2}+\sin\tfrac{\alpha\pi}{2}\sin nx\right] \end{aligned}. \quad (20)$$

Then we have

$$\begin{aligned} D^{1-\alpha}R(\zeta,\,T)\vartheta &= \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(\vartheta,\sin kx)}{k^2-\zeta}\left[\cos kx\cos\frac{\alpha\pi}{2}+\sin kx\sin\frac{\alpha\pi}{2}\right] = \\ &= \frac{2}{\pi}\cos\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}\frac{(\vartheta,\sin kx)}{k^2-\zeta}\cos kx+\frac{2}{\pi}\sin\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}\frac{(\vartheta,\sin kx)}{k^2-\zeta}\sin kx \end{aligned}. \quad (21)$$

Thus

$$D^{1-\alpha}R(\zeta,\,T)\vartheta = A\,R(\zeta,\,T)\vartheta + B\,R(\zeta,\,T)\vartheta. \quad (22)$$

where

$$\begin{aligned} A\vartheta &= \frac{2}{\pi}\sin\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}n^{1-\alpha}(\vartheta,\sin kx)\sin kx; \\ B\vartheta &= \frac{2}{\pi}\cos\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}n^{1-\alpha}(\vartheta,\sin kx)\cos kx. \end{aligned}. \quad (23)$$

Really

$$\begin{aligned} R(\zeta,T)\vartheta &= \frac{2}{\pi}\sin\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}k^{1-\alpha}\left[\left(\sum_{j=1}^{\infty}\frac{(\vartheta,\sin kx)}{j^2-\zeta}\sin jx\right)\sin kx\right]\sin kx = \\ &= \frac{2}{\pi}\sin\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}k^{1-\alpha}\frac{(\vartheta,\sin kx)}{k^2-\zeta}\sin kx \end{aligned}, \quad (24)$$

$$B\,R(\zeta,T)\vartheta = \frac{2}{\pi}\cos\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}k^{1-\alpha}\left[\left(\sum_{j=1}^{\infty}\frac{(\vartheta,\sin kx)}{j^2-\zeta}\sin jx\right)\sin kx\right]\cos kx = \frac{2}{\pi}\cos\frac{\alpha\pi}{2} \quad (25)$$

$$[A\,R+B\,R]^{2n+1}\vartheta = \left[(A\,R)^{2n+1}+B\,A^{2n}R^{2n+1}\right]\vartheta. \quad (26)$$

That the operators A; $R(\zeta, T)$ are permutable is checked directly:

$$\begin{aligned} A\,R(\zeta,T)\vartheta &= \frac{2}{\pi}\sin\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}k^{1-\alpha}\left[\left(\sum_{j=1}^{\infty}\frac{(\vartheta,\sin kx)}{j^2-\zeta}\sin jx\right)\sin kx\right]\sin kx = \\ &= \frac{2}{\pi}\sin\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}k^{1-\alpha}\frac{(\vartheta,\sin kx)}{k^2-\zeta}\sin kx \end{aligned}. \quad (27)$$

$$\begin{aligned} R(\zeta,T)A\vartheta &= \frac{2}{\pi}\sin\frac{\alpha\pi}{2}\sum_{n=1}^{\infty}\frac{(A\vartheta,\sin nx)}{n^2-\zeta}\sin nx = \\ &= \frac{2}{\pi}\sin\frac{\alpha\pi}{2}\sum_{n=1}^{\infty}\frac{\left\{\left[\sum_{k=1}^{\infty}n^{1-\alpha}(\vartheta,\sin kx)\right]\sin nx\right\}}{n^2-\zeta}\sin nx = \\ &= \frac{2}{\pi}\sin\frac{\alpha\pi}{2}\sum_{n=1}^{\infty}n^{1-\alpha}\frac{(\vartheta,\sin nx)}{n^2-\zeta}\sin nx = A\,R(\zeta,T)\vartheta \end{aligned}. \quad (28)$$

That is

$$A\,R(\zeta,T)\vartheta = R(\zeta,T)A\vartheta. \quad (29)$$

Let us now show the nilpotency of the operator $B\,R(\zeta, T)$

$$\begin{aligned} &[B\,R(\zeta,\,T)]^2\vartheta = B\,R(\zeta,\,T)[B\,R(\zeta,\,T)]\vartheta = \frac{2}{\pi}\cos\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}\frac{k^{1-\alpha}[B\,R(\zeta,\,T)\vartheta,\,\sin kx]}{k^2-\zeta}\cos kx = \\ &= \frac{2}{\pi}\cos\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}\frac{k^{1-\alpha}\left\{\left[\frac{2}{\pi}\cos\frac{\alpha\pi}{2}\sum_{j=1}^{\infty}\frac{(\vartheta,\,\sin jx)}{j^2-\zeta}\cos jx\right],\sin kx\right\}}{k^2-\zeta}\cos kx = \\ &= \left(\frac{2}{\pi}\cos\frac{\alpha\pi}{2}\right)^2\sum_{k=1}^{\infty}\frac{k^{1-\alpha}\sum_{j=1}^{\infty}\left[\frac{(\vartheta,\,\sin jx)}{j^2-\zeta}\cos jx,\sin kx\right]}{k^2-\zeta}\cos kx = 0 \end{aligned}. \quad (30)$$

The eigenvalues for the problem under consideration (15) have the form [30]:

$$\lambda_k = n^2 + \hat{\lambda}_{0m} + \hat{\lambda}_{1m}. \tag{31}$$

Let us calculate

$$\hat{\lambda}_{0m} = \frac{1}{2\pi i} \int\limits_{\Gamma_n} R(\zeta, T)\, d\zeta. \tag{32}$$

Then P_n is an integral operator with a kernel:

$$P_n(x, y) = \frac{2}{\pi} \sin nx \sin ny. \tag{33}$$

So

$$P_n\vartheta = \frac{2}{\pi} \int\limits_0^{\pi} \sin nx \sin ny\, \vartheta(y) dy = \frac{2}{\pi} (\vartheta, \sin nx) \sin nx. \tag{34}$$

Now we calculate:

$$\begin{aligned} A\, P_n\vartheta &= \frac{2}{\pi} \sin \frac{\alpha\pi}{2} \sum_{k=1}^{\infty} k^{1-\alpha} (\vartheta, \sin kx)(\sin nx, \sin kx) \sin kx = \\ &= \frac{2}{\pi} \sin \frac{\alpha\pi}{2} (\vartheta, \sin nx) n^{1-\alpha} \sin nx \end{aligned}. \tag{35}$$

Let us find the eigenvalues of the operator AP_n:

$$A\, P_n\vartheta = \lambda\vartheta \tag{36}$$

Or

$$\frac{2}{\pi} \sin \frac{\alpha\pi}{2} n^{1-\alpha} (\vartheta, \sin nx) = \lambda\vartheta. \tag{37}$$

From here

$$\lambda_1 = \frac{2}{\pi} \sin \frac{\alpha\pi}{2} n^{1-\alpha}. \tag{38}$$

Means

$$spAP_n = \lambda_1 = \frac{2}{\pi} \sin \frac{\alpha\pi}{2} n^{1-\alpha}. \tag{39}$$

Further

$$\begin{aligned} BP_n\vartheta &= \frac{2}{\pi} \sin \frac{\alpha\pi}{2} \sum_{n=1}^{\infty} (P_n\vartheta, \sin nx) \sin nx = \\ &= \frac{2}{\pi} \sin \frac{\alpha\pi}{2} \sum_{n=1}^{\infty} n^{1-\alpha} \left[\frac{2}{\pi} \sum_{k=1}^{\infty} (\vartheta, \sin kx) \sin kx,\ \sin kx \right] \cos nx = \\ &= \frac{2}{\pi} \sin \frac{\alpha\pi}{2} (\vartheta, \sin nx) \cos nx = \frac{2}{\pi} \sin \frac{\alpha\pi}{2} \cos nx \int\limits_0^{\pi} \vartheta(t) \sin nt\, dt \end{aligned}. \tag{40}$$

Calculate the trace of the operator BP_n:

$$spBP_n = \sum_{n=1}^{\infty} \lambda_n\vartheta. \tag{41}$$

Or, what's the same thing

$$\frac{2}{\pi} \sin \frac{\alpha\pi}{2} \cos nx \int\limits_0^{\pi} \vartheta(t) \sin nt\, dt = \lambda\vartheta(t). \tag{42}$$

It is obvious that

$$spBP_n = 0. \tag{43}$$

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Therefore

$$\lambda_n = n^2 - n^{1-\alpha}\frac{2}{\pi}\sin\frac{\alpha\pi}{2}. \tag{44}$$

For the general case, where $\aleph \neq \pi$ (44) takes the form (14).
The theorem is proved. □

Thus, the proven theorem extends the application of the Rayleigh-Schrödinger method to perturbed systems containing fractional-order derivatives. The following section presents numerical simulations that demonstrate the effectiveness of the proposed approach.

5. Numerical Simulation

Let us consider an important special case for applied problems [31–33] for $\aleph = \pi$, α = 0.5. Then we obtain the formula for the eigenvalues from (14):

$$\lambda_n = n^2 - c\frac{\sqrt{2}}{\pi}\sqrt{n}. \tag{45}$$

The papers [34,35] prove that eigenvalues of the problem (13) are zeros of the following function:

$$y(\lambda) = \pi - \sum_{n=1}^{\infty}(-1)^{n+1}\sum_{m=0}^{n}\frac{\binom{n}{m}c^m\lambda^{n-m}}{\Gamma(2n+2-m/2)}\pi^{2n+1-m/2}. \tag{46}$$

However, since infinite summation is impossible to perform, these values will also be approximate. We compare the eigenvalues of the form (14), obtained by developing the Rayleigh–Schrödinger method, with the approximate solutions of Equation (46). To compute the latter, we use the MATLAB 2015b software package. We will consider the first 80 terms to calculate the sum in Formula (46). The error of such replacement does not exceed 10^{-5}%. Table 1 presents the first 5 eigenvalues for problem (13) for various values of the parameter c.

Table 1. Presents the first 5 eigenvalues of problem (13).

Parameter c	Eigenvalue	Formula (46) with 80 Terms	Formula (45)	Relative Error
−0.7	λ_1	1.275	1.315	0.0314
	λ_2	4.655	4.446	0.0449
	λ_3	9.814	9.546	0.0273
	λ_4	16.951	16.630	0.0189
	λ_5	26.072	25.705	0.0141
0.7	λ_1	0.651	0.685	0.0522
	λ_2	3.350	3.554	0.0609
	λ_3	8.195	8.454	0.0316
	λ_4	15.049	15.370	0.0213
	λ_5	23.931	24.295	0.0152

Calculations using Formulas (45) and (46) revealed repeatability of results within 10% in the range of parameter values: $-0.8 < c < 0.8$, $0 < \alpha < 1$. No restrictions were imposed on the ε parameter during calculations. In tasks of type (1), the parameter c is associated with the viscosity of the material related to the mass of the granule. This parameter for real materials lies in the positive part of the range c.

This advancement in the application of the Rayleigh–Schrödinger method for solving problems of type (5) and (13) allows for the determination of eigenvalues at significantly lower computational cost. This is because estimating these eigenvalues using (14) is much simpler than solving the implicit Equation (46).

This is a new approach, because the unperturbed operator used in the work is non-self-adjoint. Therefore, the eigenfunction system, although it forms a basis, is not orthogonal. In this context, the Relay-Schrödinger method has not been previously considered.

6. Conclusions

Thus, this work presents a methodology for applying the enhanced Rayleigh-Schrödinger method to solve problems of types (5) and (13), which contain fractional-order derivatives. This methodology is significantly more efficient than classical approaches. It allows for the straightforward determination of eigenvalues for problems of type (5) and (13) using Equation (14).

Numerical simulations revealed a limitation in applying the proposed methodology for estimating the first eigenvalue λ1. For positive values of parameter c, the eigenvalue estimates from (14) are overestimated compared to the more accurate estimate based on the first 80 terms of expansion (46). For negative values of parameter c, the opposite trend is observed: the estimates from (14) are lower than those from (46).

These results have practical applications. For example, in describing deformation processes of road surfaces using models of type (5) and (13) [36–38], as well as in other fields where problems of types (5) and (13) may be applicable.

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