mathematics

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Extended Two-Parameter *F*-Controlled Asymptotically Contractive Self-Mappings in Metric Spaces

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Abstract

Certain extensions of *F*-controlled self-mappings in metric spaces to the, as called in this manuscript, $F_{\tau\prime\tau}$ and modified $*F_{\tau\prime\tau}$ controlled self-mappings, which are parameterized by two parameters, are addressed. Those parameters govern the properties of local expansivity, asymptotic nonexpansivity, and contractivity properties of the generated sequences. Also, further generalizations to parameterizations by two real sequences of parameters, which are referred to as $F_{\{\tau_j'\}_{j=0}^{\infty}\{\tau_j\}_{j=0}^{\infty}}$-controlled self-mappings, are studied. The main formulated results rely on the asymptotic contractivity and the asymptotic nonexpansivity in metric spaces and some of their relevant properties. In particular, the properties of boundedness of the sequences of distances, as well as those of boundedness of the elements of the sequences themselves, are investigated under asymptotic contractivity or nonexpansivity related to the various types of the above-mentioned $F_{(.)}$-controlled self-mappings. Also, existence and uniqueness results of fixed points are proved if the metric space is complete, and the resulting Cauchyness properties of sequences and properties of the convergence of such sequences to fixed points are also proved. Finally, two illustrative examples are described if the $F_{(.)}$-controlled self-mappings are of a cyclic nature when defined using the union of two nonempty closed subsets of the metric space, in the case that those sets intersect, and also in the case when they are disjointed.

Keywords: *F*-controlled maps; cyclic mappings; *F*-contractions; fixed points; best proximity points

MSC: 54E35; 47H10; 37C25

1. Introduction

The study of fixed points provides essential tools for establishing the existence and uniqueness of solutions to complex problems. The study is relevant in geometry and topology, such as in the analysis of isometric maps of convex sets, as well as in analysis in the applied sciences, like economics, engineering, and computer science. In particular, this is an important tool in the derivation of iterative computational schemes and in the study of the location and analysis of equilibrium points of continuous-time and discrete-time dynamic systems, as well as in the parallel investigation of the convergence of the solutions to fixed points, provided that they are asymptotically stable equilibrium points. This is also relevant in the study of existence, uniqueness, and boundedness of the solutions, in particular, in the case of complex nonlinear differential systems with solutions that are

Check for updates

Academic Editor: Mihai Postolache

Received: 23 March 2026
Revised: 12 April 2026
Accepted: 17 April 2026
Published: 21 April 2026

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difficult or impossible to find explicitly. A class of mappings of very relevant interest is that of contractive mappings.

There are different types of contractions in the context of fixed-point theory. One of them, of important interest in recent years, is the so-called F-contraction, which was proposed by Wardowski in the now frequently used reference [1]. Such a contraction type generalizes the Banach contraction principle. Generalizations, modifications, and applications of F-contractions in several formal frameworks have been described [2–11], in particular, in the fields of differential and difference equations and in fractional calculus . It is demonstrated in [2] that a wider class of F-contractions might be introduced, which has been referred to as sF-contraction. Such a contraction is proved to have a unique fixed point in a complete metric space. A modified concept of F-Suzuki contraction is proved to be endowed with very close results. A survey and discussion of results regarding F-contractions is given in [3], where it is clearly visualized that F-contractions are generalizations of strict contractions by defining the F-function as $F(t) = logt$, while its associated one-step iteration-decrease in the F-function of distances is taken as $\tau = log(K^{-1}) > 0$, where $K \in (0, 1)$ is the contractive constant of the strict contraction under comparison. Several alternative F-functions are proposed as well. On the other hand, the relation between F-contraction and some of its modifications are discussed, as well as that of the Boyd–Wong contraction and Matkowski contraction. Conditions for those classes of contractions to possess a unique fixed point on a complete metric space are given. Also, some assumptions are given which ensure that an F-contraction is a Boyd–Wong contraction or a Matkowski contraction. The relations between Meir–Keeler contractions and F-contractions are discussed in [5]. It is shown that an F-contraction is a Meir–Keeler contraction, but a given example in that article proves that the converse is not true. In [6], the focus is on the links between F-contractions and Meir–Keeler contractions. It is proved that, for a nondecreasing function F with a point of discontinuity on the right, there exists an F-contraction that is not a Meir–Keeler contraction. In [7], the fixed points in ordered F-contractions are investigated. In [8], the concept of modified F-contractions through an α-admissible pair of mappings is proposed. Also, some common fixed-point results are proved in the framework of metric spaces. On the other hand, in [9,10], further fixed point theorems of F-contractions in complete metric spaces are proved. In [11], the iterated function systems composed of F-contractions are investigated by extending some fixed-point results from the classical Hutchinson–Barnsley theory of iterated function systems consisting of Banach contractions.

In this paper, eventual local deviations are allowed at certain iterations of the iterated sequences of F-functions in metric spaces from their step-by-step contractive or nonexpansive characteristics. Some conditions at infinity are introduced that might guarantee the F-asymptotic contractive or asymptotic nonexpansive properties. For that purpose, two separate parameters are introduced, one under the claim of governing the step-by-step decrease in the F-function of distances and another one, of opposed sign to the above one, claiming to govern its eventual local increase. As a result, the generated sequences can be locally expansive while keeping F-asymptotic contractivity or nonexpansivity properties under certain conditions. Further extensions are addressed for the concourse of iteration-dependent sequences of such parameters to govern the F-asymptotic either contractivity or nonexpansivity properties under modified conditions that modulate the relative influences of those sequences. Existence and uniqueness of fixed-point results are also proved for those various F-controlled maps if they are F-asymptotically contractive and the metric space is complete, in which case the generated sequences of the controlled self-mappings are proved to converge to such fixed points. Two illustrative examples are also discussed in detail if the F-controlled self-mappings are cyclic and defined on the union of two nonempty closed subsets of the metric space.

The paper is organized as follows. Section 2 recalls the celebrated Wardowski F-function. By convenience, its associated F-contraction is denoted and referred to as an F_τ contraction, where $\tau > 0$ is the parameter that governs the one-step decrease in the F-function of distances of sequences generated by a self-mapping $T : X \to X$ in a metric space (X, d). As extensions of the above F-controlled maps, the so-called two-parameter $F_{\tau\prime\tau}$ and modified $*F_{\tau\prime\tau}$ controlled self-mappings $T : X \to X$ are introduced, as well as their extensions to parameterizations by sequences instead of by parameters, that is, $F_{\{\tau_j'\}_{j=0}^\infty \{\tau_j\}_{j=0}^\infty}$-controlled self-mappings. Some properties of boundedness of the associated sequences of distances and properties of boundedness of the elements of the sequences themselves are investigated under asymptotic contractivity/nonexpansivity of the various above-mentioned $F_{(.)}$-controlled self-mappings. Section 3 establishes some existence and uniqueness results for fixed-point results if the metric space is complete, as well as Cauchyness properties and results on the convergence of such sequences to the fixed points. Also, two illustrative examples are given if the self-mappings are cyclic and defined on the union of two nonempty closed subsets of the metric space in the cases that those sets intersect and in the case when they are disjointed. Finally, our conclusions end the paper.

2. Boundedness and Convergence Properties of Self-Mappings That Have *F*-Contractive Subsequences

The following notation is used:

$\boldsymbol{Z}_+ = \{z \in \boldsymbol{Z} : z > 0\}$; $\boldsymbol{Z}_{0+} = \boldsymbol{Z}_+ \cup \{0\}$; $\boldsymbol{Z}_{0-} = \boldsymbol{Z}_- \cup \{0\}$; $\boldsymbol{Z}_- = \{z \in \boldsymbol{Z} : z < 0\}$;
$\boldsymbol{R}_+ = \{z \in \boldsymbol{R} : z > 0\}$; $\boldsymbol{R}_{0+} = \boldsymbol{R}_+ \cup \{0\}$; $\boldsymbol{R}_{0-} = \boldsymbol{R}_- \cup \{0\}$; $\boldsymbol{R}_- = \{z \in \boldsymbol{R} : z < 0\}$;
$\boldsymbol{Z}$, $\boldsymbol{R}$ and $\mathbf{C}$ are, respectively, the sets of integer, real and complex numbers;
$\bar{n} = \{1, 2, \cdots, n\}$;
I_d is the identity matrix of order d;

The symbol $\vee$ stands for the logic disjunction "or"; the symbol $\wedge$ stands for the logic conjunction "and".

$Fix(T)$ denotes the set of fixed points of a self-mapping T on X, where (X, d) is a metric space.

We first borrow from [1] the following essential definition for its later use.

Definition 1. *Let $F : \boldsymbol{R}_+ \to \boldsymbol{R}$ be a function which satisfies the subsequent conditions:*

(C1) It is strictly increasing.
(C2) $\exists \lim_{x \to 0+} F(x) = -\infty$.
(C3) For any sequence $\{x_n\}_{n=0}^\infty \subset \boldsymbol{R}_+$, $\lim_{n\to\infty} x_n = 0$ iff $\lim_{n\to\infty} F(x_n) = -\infty$.
(C4) $\lim_{x\to 0+} x^k F(x) = 0$ for some real constant $k \in (0\ ,\ 1)$.
Then, $F : \boldsymbol{R}_+ \to \boldsymbol{R}$ is said to be an F-function.

Definition 2. *Let $F : \boldsymbol{R}_+ \to \boldsymbol{R}$ be an F-function, (X, d) be a metric space and $T : X \to X$ be a self-mapping on X which satisfies:*

(C5) $[d(Tx, Ty) > 0] \Rightarrow [\tau + F(d(Tx, Ty)) \le F(d(x, y))]$ for some $\tau \in \boldsymbol{R}_+$ and all $x, y \in X$.
Then, $T : X \to X$ is said to be an F_τ-contraction.

The following extension of Definition 2 is concerned with the eventual local deviations at certain iterations of the iterated sequences of F-functions from their step-by-step contractive or nonexpansive characteristics. For that purpose, an extra parameter $\tau\prime$, which can govern an eventual local growing of the F-function of distances at certain iterations,

is introduced that coexists with the usual one τ, which governs its standard step-by-step decrease in the F-function in the contraction case.

Definition 3. *Let* $F : \boldsymbol{R}_+ \to \boldsymbol{R}$ *be an F-function,* (X, d) *be a metric space and* $T : X \to X$ *be a self-mapping on* X *that satisfies the following recursive relation:*

(C6) $[\mathrm{d}(\mathrm{T^n x}, \mathrm{T^n y}) > 0] \Rightarrow \left[\mathrm{F}(\mathrm{d}(\mathrm{T^n x}, \mathrm{T^n y})) \leq \mathrm{F}\left(\mathrm{d}\left(\mathrm{T^{n-1} x}, \mathrm{T^{n-1} y}\right)\right) + \delta(\mathrm{n})\tau\prime - \tau\right]$, *where* $\{\delta(\mathrm{n})\}_{\mathrm{n}=0}^{\infty} \in \{0, 1\}$ *for some* $\tau\prime, \tau \in \mathbf{R}_{0+}$; $\forall \mathrm{x}, \mathrm{y} \in \mathrm{X}$; $\forall \mathrm{n} \in \mathbf{Z}_+$.

Then, $T : X \to X$ *is said to be an* $F_{\tau\prime\tau}$*-controlled self-mapping.*

Note that if $d(T^n x, T^n(Tx)) = d(T^n x, T(T^n x)) = 0$ for some finite $n \in \boldsymbol{Z}_+$, then the implication of the condition C6 does not apply, and $x = Tx = T^n x$ is the unique fixed point of T.

Definition 3 applies, for instance, to the following concrete examples:

(a) If $F(a) = lna$ is $F_{\tau\prime\tau}$-controlled with $\tau, \tau\prime \in \boldsymbol{R}_+$, then lone has:

$$d(T^n x\ ,\ T^n y) \leq e^{\delta(n)\tau\prime - \tau} d\left(T^{n-1} x\ ,\ T^{n-1} y\right);\ \forall x, y \in X\ ,\ \forall n \in \mathbf{Z}_+.$$

(b) If $F(a) = lna + a$ is $F_{\tau\prime\tau}$-controlled with $\tau, \tau\prime \in \boldsymbol{R}_+$, then one has:

$$d(T^n x\ ,\ T^n y) \leq e^{d(T^{n-1}x, T^{n-1}y) - d(T^n x, T^n y) + \delta(n)\tau\prime - \tau} d\left(T^{n-1} x\ ,\ T^{n-1} y\right);\ \forall x, y \in X\ ,\ \forall n \in \mathbf{Z}_+$$

The following result follows directly via recursion from the condition C6:

Proposition 1. *If* $T : X \to X$ *is an* $F_{\tau\prime\tau}$*-controlled self-mapping, then:*

$$[d(T^n x, T^n y) > 0] \Rightarrow$$
$$\left[F(d(T^n x, T^n y)) \leq F(d(x, y)) + \sum_{j=1}^{n} (\delta(j)\tau\prime - \tau) = F(d(x, y)) + \left(\sum_{j=1}^{n} \delta(j)\right)\tau\prime - n\tau\right] \quad (1)$$
$$;\forall x, y \in X, \forall n \in \mathbf{Z}_+$$

Remark 1. *Note in Definition 2 that if* $\tau\prime = \tau = 0$, *then the* F_{00}*-controlled self-mapping* $T : X \to X$ *is* F_{00}*-nonexpansive, which satisfies* $F(d(Tx\ ,\ Ty)) \leq F(d(x, y))$ *if* $d(Tx, Ty) > 0$, *which* via *the constraint C1, implies that* $d(Tx, Ty) \leq d(x, y)$; $\forall x, y \in X$. *If* $\tau\prime = 0$ *and* $\tau > 0$ *then the* $F_{0\tau}$*-controlled self-mapping* $T : X \to X$ *is an* $F_{0\tau}$*-contraction since* $F(d(Tx\ ,\ Ty)) \leq F(d(x, y)) - \tau$; $\forall x, y \in X$, *if* $d(Tx, Ty) > 0$, *which is also obviously nonexpansive.*

On the other hand, note from (1) that if $\delta(n) = 1$, *then* $\delta(n)\tau\prime - \tau = \tau\prime - \tau$, *and if* $\delta(n) = 0$, *then* $\delta(n)\tau\prime - \tau = -\tau$. *Thus, the major interest of the condition C5 arises when* $\tau\prime > \tau$ *since, otherwise, the condition reduces to an F-contractive condition parameterized for each iteration by* $min(\tau, \tau\prime)$ *with initial step* $min(\tau, \tau\prime) + F(d(Tx,\ Ty)) \leq F(d(x, y))$.

The concepts of an $F_{\tau\prime\tau}$-asymptotically nonexpansive and $F_{\tau\prime\tau}$-asymptotically contractive self-mapping $T : X \to X$ are established as follows.

Definition 4. *Let* $F : \boldsymbol{R}_+ \to \boldsymbol{R}$ *be an F-function,* (X, d) *be a metric space and* $T : X \to X$ *be a self-mapping on* X. *Then,* $T : X \to X$ *is said to be* $F_{\tau\prime\tau}$*-asymptotically nonexpansive if:*

$$\limsup_{n\to\infty} (F(d(T^n x, T^n y)) - F(d(x, y))) \leq 0; \forall x, y \in X.$$

$F : \boldsymbol{R}_+ \to \boldsymbol{R}$ *is said to be nonexpansive if* $F(d(T^n x, T^n y)) \leq F(d(x, y))$; $\forall x, y \in X$, $\forall n \in \boldsymbol{Z}_+$.

Note that if $F : \boldsymbol{R}_+ \to \boldsymbol{R}$ is $F_{\tau\prime\tau}$-nonexpansive, then it is $F_{\tau\prime\tau}$-asymptotically nonexpansive as well, but, in general, the converse is not true.

Definition 5. *Let $F : \boldsymbol{R}_+ \to \boldsymbol{R}$ be an F-function, (X, d) be a metric space and $T : X \to X$ be a self-mapping on X which satisfies the condition C6 for all $x, y \in X$ subject to $\lim_{n\to\infty}\left(n\tau - \left(\sum_{j=1}^{n}\delta(j)\right)\tau\prime\right) = +\infty$ for some pair $\tau\prime \in \boldsymbol{R}_{0+}$, $\tau \in \boldsymbol{R}_+$ with $\{\delta(n)\}_{n=0}^{\infty} \in \{0, 1\}$. Then, $T : X \to X$ is said to be $F_{\tau\prime\tau}$-asymptotically contractive.*

Note that if for $\tau, \tau\prime \in \boldsymbol{R}_+$, it holds according to Definition 5 that:

$$\lim_{n\to\infty}\left(n\tau - \left(\sum_{j=1}^{n}\delta(j)\right)\tau\prime\right) = \lim_{n\to\infty}\left(n - \left(\sum_{j=1}^{n}\delta(j)\right)(\tau\prime/\tau)\right) = +\infty$$

And then $\lim_{n\to\infty}\left(\sum_{j=1}^{n}\delta(j)/n\right) < \tau/\tau\prime$, which implies that local expansions are dominated by the global contraction. Valid conditions for the above constraint to hold, for instance, are:
$\sum_{j=1}^{n}\delta(j) \leq lnn$; $\sum_{j=1}^{n}\delta(j) \leq \sqrt{n}$; $\sum_{j=1}^{n}\delta(j) \leq Cn^k$ *with* $C, k(<1) \in \boldsymbol{R}_+$; $\sum_{j=1}^{n}\delta(j) \leq C$ *with* $C \in \boldsymbol{R}_+$; $\sum_{j=1}^{n}\delta(j) \leq (k\tau/\tau\prime)n + o(n)$ *with* $k(<1) \in \boldsymbol{R}_+$.

Note that if $T : X \to X$ is $F_{\tau\prime\tau}$-asymptotically contractive, then $\lim_{n\to\infty} F(d(T^n x, T^n y)) = -\infty$; $\forall x, y \in X$.

Proposition 2. *If (X, d) is a metric space and $T : X \to X$ is $F_{\tau\prime\tau}$-asymptotically nonexpansive (respectively, $F_{\tau\prime\tau}$ asymptotically contractive), then it is asymptotically nonexpansive (respectively, asymptotically regular).*

Proof. It follows directly from the conditions C1 and C3 that $\lim_{n\to\infty} sup(F(d(T^n x, T^n y)) - F(d(x, y))) \leq 0$; $\forall x, y \in X$ (Definition 4) implies that $\lim_{n\to\infty} sup\, d(T^n x, T^n y) \leq d(x, y)$ and that $\exists \lim_{n\to\infty} F(d(T^n x, T^n y)) = -\infty$; $\forall x, y \in X$ (Definition 5) implies that $\lim_{n\to\infty} d(T^n x, T^n y) = 0$; $\forall x, y \in X$ from C3. □

An $F_{\tau\prime\tau}$-asymptotically nonexpansive or contractive self-mapping $T : X \to X$ verifies the subsequent properties.

Lemma 1. *Let $F : \boldsymbol{R}_+ \to \boldsymbol{R}$ be an F-function, (X, d) be a metric space and $T : X \to X$ be a self-mapping on X that satisfies the condition C5. Then, the following propositions hold:*

(i) *If $F_{\tau\prime\tau}$-asymptotically nonexpansive then:*

$$\lim_{n\to\infty} sup\,(d(T^n x, T^n y) - d(x, y)) \leq 0; \forall x, y \in X.$$

(ii) *A sufficient condition for $T : X \to X$ to be $F_{\tau\prime\tau}$-asymptotically nonexpansive is:*

$$\lim_{n\to\infty} sup\left(n\tau - \left(\sum_{j=1}^{n}\delta(j)\right)\tau\prime\right) \geq 0; \forall x, y \in X.$$

(iii) *If $T : X \to X$ is $F_{\tau\prime\tau}$-asymptotically contractive, then there exists $\lim_{n\to\infty} d(T^n x,\ T^n y) = 0$; $\forall x, y \in X$ so that the self-mapping $T : X \to X$ is asymptotically regular.*

(iv) *A sufficient condition for $T : X \to X$ to be $F_{\tau\prime\tau}$-asymptotically contractive is that $\lim_{n\to\infty}\left(n\tau - \left(\sum_{j=1}^{n}\delta(j)\right)\tau\prime\right) = +\infty;\ \forall x, y \in X$.*

Proof. Property (i) follows from:

$$\left[F\left(\limsup_{n\to\infty} d(T^n x, T^n y)\right) = \limsup_{n\to\infty} F(d(T^n x, T^n y)) \le F(d(x,y))\right] \Rightarrow \left[\limsup_{n\to\infty} d(T^n x, T^n y) \le d(x,y)\right] \quad (2)$$

; $\forall x, y \in X$, since the function $F : \boldsymbol{R}_+ \to \boldsymbol{R}$ is uniformly continuous, and the use of the condition C1 is:
Property (ii) follows from (1) since $\limsup_{n\to\infty} \left(n\tau - \left(\sum_{j=1}^{n} \delta(j)\right)\tau\prime\right) \ge 0$ leads to:

$$\limsup_{n\to\infty} \left(F(d(T^n x,\, T^n y)) - F(d(x,\, y))\right) \le \limsup_{n\to\infty} \left(\left(\sum_{j=1}^{n}\delta(j)\right)\tau\prime - n\tau\right) \le 0; \forall x, y \in X.$$

Property (iii) follows from:

$$\left[\lim_{n\to\infty} F(d(T^n x, T^n y)) = F\left(\lim_{n\to\infty} d(T^n x, T^n y)\right) = -\infty\right] \Rightarrow \left[\exists \lim_{n\to\infty} d(T^n x, T^n y) = 0\right] \quad (3)$$

; $\forall x, y \in X$ from the continuity of $F : \boldsymbol{R}_+ \to \boldsymbol{R}$ and the conditions C2–C3.
Property (iv) follows from (1) and Property (iii) since:

$$\lim_{n\to\infty} F(d(T^n x,\, T^n y)) \le F(d(x,\, y)) + \limsup_{n\to\infty} \left(\left(\sum_{j=1}^{n}\delta(j)\right)\tau\prime - n\tau\right)$$
$$\le F(d(x,\, y)) + \lim_{n\to\infty} \left(\left(\sum_{j=1}^{n}\delta(j)\right)\tau\prime - n\tau\right) = F(d(x,\, y)) - \infty = -\infty; \forall x, y \in X.$$

□

The cases of interest to address the concepts of $F_{\tau\prime\tau}$ asymptotically nonexpansive and contractive self-mappings are $T : X \to X$ if $\limsup_{n\to\infty}\left(\sum_{j=1}^{n}\delta(j)\right) = +\infty$ under the condition $\tau\prime > \tau > 0$ since if $\tau\prime < \tau$, then the formalism can be reduced to replace $\tau \to \tau - \tau\prime$ and not to use $\tau\prime$ as a separate second parameter. If $\sum_{j=1}^{\infty}\delta_j < +\infty$, then $T : X \to X$ is asymptotically nonexpansive if $\tau \ge 0$. It is asymptotically contractive if $\tau > 0$ since $\left(\sum_{j=1}^{\infty}\delta_j\right)\tau\prime - n\tau \to -\infty$ and $F(d(T^n x, T^n y)) \to -\infty$ as $n \to \infty$, which leads to $d(T^n x, T^n y) \to 0$ as $n \to \infty$. In summary, the following consequence follows from Proposition 1 (Equation (1), Lemma 1 (ii), and Lemma 1 (iv).

Theorem 1. *Let $F : \boldsymbol{R}_+ \to \boldsymbol{R}$ be an F-function, (X, d) be a metric space and $T : X \to X$ be a self-mapping which satisfies the condition C5. Then, the following properties hold:*

(i) *If $\lim_{n\to\infty}\left(\sum_{j=1}^{n}\delta(j)\right) = +\infty$, then a sufficient condition for $T : X \to X$ to be $F_{\tau\prime\tau}$-asymptotically nonexpansive is that:*

$$\tau\prime/\tau \le \liminf_{n\to\infty}\left(n/\left(\sum_{j=1}^{n}\delta(j)\right)\right)$$

with $\tau > 0$.

(ii) *If $\lim_{n\to\infty}\left(\sum_{j=1}^{n}\delta(j)\right) = +\infty$, then a sufficient condition for $T : X \to X$ to be $F_{\tau\prime\tau}$-asymptotically contractive is that there exists:*

$$\lim_{n\to\infty}\left(n/\left(\sum_{j=1}^{n}\delta(j)\right)\right) = +\infty$$

and

$$\tau\prime/\tau < \lim_{n\to\infty}\left(n/\left(\sum_{j=1}^{n}\delta(j)\right)\right) < +\infty$$

with $\tau > 0$.

(iii) *If* $\sum_{j=1}^{\infty}\delta_j < +\infty$, *then* $T : X \to X$ *is* $F_{\tau\prime\tau}$*-asymptotically nonexpansive irrespective of* $\tau\prime$ *if* $\tau \geq 0$ *and asymptotically contractive irrespective of* $\tau\prime$ *if* $\tau > 0$.

Also, the following further result holds about boundedness.

Theorem 2. *Let* (X, d) *be a metric space and let* $T : X \to X$ *be an* $F_{\tau\prime\tau}$*-controlled self-mapping for some* $\tau, \tau\prime \in \boldsymbol{R}_{0+}$ *such that the following condition holds:*

(C7) $\sup_{n\in\boldsymbol{Z}_{0+}} \left(\left(\sum_{j=1}^{n}\delta(j)\right)\tau\prime - n\tau\right) \leq M$

Then, $\{d(T^n x, T^n y)\}_{n=0}^{\infty}$ *is bounded for any given* $x, y \in X$.

Proof. The result is direct if $\tau = \tau\prime = 0$, since then $F(d(T^n x\ ,\ Ty)) \leq F(d(x\ ,\ y))$ from (1), which implies that $d(T^n x\ ,\ Ty) \leq d(x, y)$ from C1; $\forall n \in \boldsymbol{Z}_{0+}, \forall x, y \in X$ so that $T : X \to X$ is F_{00}-nonexpansive.

Note from (1) that:

$$F(d(T^n x, T^n y)) \leq F(d(x, y)) + \mu(n)\left(\left(\sum_{j=1}^{n}\delta(j)\right)\tau\prime - n\tau\right) \tag{4}$$

with $\{\mu(n)\}_{n=0}^{\infty} \subset [0\ ,\ 1]$ being such that $\mu(n) = 1$ if and only if $d(T^n x, T^n y) > 0$ and $\mu(n) = 0$ if and only if $d(T^n x, T^n y) = 0$. Assume that $\{d(T^n x, T^n y)\}_{n=0}^{\infty}$ is unbounded for some $x, y \in X$. Then, we can have:

Case a: there is a strictly increasing subsequence of distances $\{d(T^{n_k} x, T^{n_k} y)\}_{k=0}^{\infty} \subset \{d(T^n x, T^n y)\}_{n=0}^{\infty}$, for some strictly increasing sequence $\{n_k\}_{k=1}^{\infty} \subset \boldsymbol{Z}_{0+}$, such that $\{d(T^{n_k} x, T^{n_k} y)\}_{k=0}^{\infty} \to \infty$. Thus, from (1) and the condition C7, one gets:

$$F(d(T^{n_k} x, T^{n_k} y)) \leq F(d(x, y)) + \mu(n_k)\left(\left(\sum_{j=1}^{n_k}\delta(j)\right)\tau\prime - n_k\tau\right) \leq F(d(x, y)) + M \tag{5}$$

for some finite $M \in \boldsymbol{R}_+$ and, from the condition C1, that is, since $\{d(T^{n_k} x, T^{n_k} y)\}_{k=0}^{\infty} \to \infty$ and since F is strictly increasing on its definition domain $\boldsymbol{R}_+$, the following contradiction holds:

$$+\infty = \lim_{k\to\infty} F(d(T^{n_k} x,\ T^{n_k} y)) \leq F(d(x,\ y)) + \limsup_{k\to\infty}\left(\mu(n_k)\left(\left(\sum_{j=1}^{n_k}\delta(j)\right)\tau\prime - n_k\tau\right)\right) \leq F(d(x,\ y)) + M < +\infty \tag{6}$$

As a result, all subsequences $\{d(T^{n_k} x, T^{n_k} y)\}_{k=0}^{\infty}$ of any sequence of distances $\{d(T^n x, T^n y)\}_{n=0}^{\infty}$ for any given $x, y \in X$ are bounded.

Case b: There is a finite $m \in \boldsymbol{Z}_{0+}$ iteration value of escape to a $+\infty$ value of the distance $d(T^m x, T^m y)$ for some $x, y \in X$, then $F(d(T^m x, T^m y)) = +\infty$ so that $F\left(d\left(T^{m+j} x, T^{m+j} y\right)\right) = +\infty$; $\forall j \in \boldsymbol{Z}_{0+}$ and then, since F is strictly increasing from the condition C1 of Definition 1, $\lim_{n\to\infty} F(d(T^n x, T^n y)) = +\infty$. As a result, the subsequence $\left\{d\left(T^j x, T^j y\right)\right\}_{j=m}^{\infty}$ of $\{d(T^n x, T^n y)\}_{n=0}^{\infty}$ consists of infinitely many infinity values, since $d\left(T^j x, T^j y\right) = +\infty$ for all $j \geq m$, and a contradiction follows since then from Condition C7 we have:

$$+\infty = \lim_{n\to\infty} F(d(T^n x, T^n y)) \leq F(d(x, y)) + \limsup_{n\to\infty}\left(\mu(n)\left(\left(\sum_{j=1}^{n}\delta(j)\right)\tau\prime - n\tau\right)\right) \leq F(d(x, y)) + M < +\infty \tag{7}$$

As a result, any sequence of distances $\{d(T^n x, T^n y)\}_{n=0}^{\infty}$ cannot have an infinite escape value at a finite iteration point for any given $x, y \in X$ that are bounded. □

The following result is a direct consequence of Theorem 2.

Corollary 1. *In Theorem 2, assume that* $T : X \to X$ *is an* $F_{\tau\prime\tau}$*-controlled self-mapping for* $\tau = 0$ *and some* $\tau\prime \geq 0$. *Then,* $\{d(T^n x, T^n y)\}_{n=0}^{\infty} < +\infty$; $\forall x, y \in X$.

Remark 2. *Note that* $\tau = 0$ *and* $\tau\prime \geq 0$ *(Corollary 1), and then the subsequent* $\{d(T^{n_k}x, T^{n_k}y)\}_{k=0}^{\infty}$ *of* $\{d(T^{n}x, T^{n}y)\}_{n=0}^{\infty}$ *of Theorem 3 (ii) and its subsequent* $\left\{d\left(T^{j_{n_k}}x,\ T^{j_{n_k}}y\right)\right\}_{k=0}^{\infty} \subset \{d(T^{n_k}x,\ T^{n_k}y)\}_{k=0}^{\infty}$ *fulfils:*

$$F(d(T^{n_{k+1}}x,\ T^{n_{k+1}}y)) \leq F(d(x,\ y)) + n_1\tau\prime < +\infty; \forall x, y \in X$$

$$F\left(d\left(T^{n_k+j_k}x,\ T^{n_k+j_k}y\right)\right) \leq F(d(T^{n_k}x,\ T^{n_k}y)) + \hat{n}\tau' \leq F(d(x,\ y)) + (\hat{n} + n_1)\tau\prime; \forall x, y \in X$$

for $j_k \in [n_k,\ n_{k+1}) \cap \mathbf{Z}_{0+}$, *and one concludes that they are bounded, and then the whole sequence* $\{d(T^{n}x, T^{n}y)\}_{n=0}^{\infty} < +\infty$ *is bounded.*

Now Definition 3 is modified to introduce the now so-called $*F_{\tau\prime\tau}$-controlled $T : X \to X$ mappings where each iteration is governed either by a non-positive parameter τ or by a non-negative parameter $\tau\prime$.

Definition 6. *Let* $F : \mathbf{R}_+ \to \mathbf{R}$ *be an F-function,* (X, d) *be a metric space and* $T : X \to X$ *be a self-mapping on* X *that satisfies the following recursive relation:*
(C8)

$$[d(T^{n}x, T^{n}y) > 0] \Rightarrow \left[F(d(T^{n}x, T^{n}y)) \leq F\left(d\left(T^{n-1}x, T^{n-1}y\right)\right) + \delta(n)\tau\prime - (1 - \delta(n))\tau\right] \tag{8}$$

where $\{\delta(n)\}_{n=0}^{\infty} \in \{0\ ,\ 1\}$ *for some* $\tau\prime, \tau \in \mathbf{R}_{0+};\ \forall x, y \in X.$
Then, $T : X \to X$ *is said to be an* $*F_{\tau\prime\tau}$*-controlled self-mapping.*

Definitions 4 and 5 extend directly for $T : X \to X$ *being an* $*F_{\tau\prime\tau}$-controlled self-mapping which are $*F_{\tau\prime\tau}$-asymptotically nonexpansive and $*F_{\tau\prime\tau}$-asymptotically contractive.

The following result follows directly via recursion from the condition C8:

Proposition 3. *If* $T : X \to X$ *is an* $*F_{\tau\prime\tau}$*-controlled self-mapping, then:*

$$[d(T^{n}x, T^{n}y) > 0] \Rightarrow \left[F(d(T^{n}x, T^{n}y)) \leq F(d(x,y)) + \textstyle\sum_{j=1}^{n}(\delta(j)\tau\prime - (1 - \delta(j))\tau)\right] ; \forall x, y \in X \tag{9}$$

It turns out that (9) is identical to (10) below if $\hat{\tau}\prime = \tau\prime + \tau$:

$$[d(T^{n}x, T^{n}y) > 0] \Rightarrow \left[F(d(T^{n}x, T^{n}y)) \leq F(d(x,y)) + \textstyle\sum_{j=1}^{n}(\delta(j)\hat{\tau}\prime - \tau)\right]; \forall x, y \in X \tag{10}$$

so that, directly, we have the following:

Assertion 1. $T : X \to X$ *is an* $*F_{\tau\prime\tau}$*-controlled self-mapping, and then, conversly, it is also an* $F_{\tau+\tau\prime,\tau}$*-controlled self-mapping and c.*

Theorem 3. *Let* (X, d) *be a metric space and let* $T : X \to X$ *be an* $*F_{\tau\prime\tau}$*-controlled self-mapping for some* $\tau, \tau\prime \in \mathbf{R}_{0+}$ *such that the condition C8 holds.*

Then, the following properties hold:

- ***(i)*** *For any* $n \in \mathbf{Z}_{0+}$ *there exist unique integers* $n_1 = n_1(n) = \sum_{j=1}^{n}\delta(j) \in \mathbf{Z}_{0+}$, $n_2 = n_2(n) = n - n_1 = n - \sum_{j=1}^{n}\delta(j) \in \mathbf{Z}_{0+}$.
- ***(ii)*** *The sequence of distances* $\{d(T^{n}x\ ,\ T^{n}y)\}_{n=0}^{\infty}$ *is bounded, and* $\forall x, y \in X$ *if the condition* $n_1(n)/n \leq \tau/(\tau + \tau\prime)$ *holds for any* $n(\geq n_0) \in \mathbf{Z}_0$ *and some finite* $n_0 \in \mathbf{Z}_{0+}$, *provided that* $(\tau + \tau\prime) \in \mathbf{R}_+$ *and* $(n_1(n)/n) = \left(\left(\sum_{j=1}^{n}\delta(j)\right)/n\right) \leq \tau/(\tau + \tau\prime)$; $\forall n \in \mathbf{Z}_{0+}$, *or if* $\tau = \tau\prime = 0$.

(iii) *Assume that there is a bounded subsequence of distances* $\{d(T^{n_k}x, T^{n_k}y)\}_{k=0}^{\infty} \subset \{d(T^nx, T^ny)\}_{n=0}^{\infty}$, *with* $\hat{N} = \{n_k\}_{k=1}^{\infty}(\subset \mathbf{Z}_{0+})$ *being strictly increasing with a finite* n_1 *and such that* $\hat{n}(\in \mathbf{Z}_{0+}) = \sup_{k\in\mathbf{Z}_+}(n_{k+1} - n_k) < +\infty$. *Then,* $\{d(T^nx, T^ny)\}_{n=0}^{\infty}$ *is bounded.*

(iv) *If for* $\tau > 0$, $\lim_{n\to\infty}\sum_{j=1}^{n}((1-\delta(j))\tau - \delta(j)\tau\prime) = +\infty$ $\lim_{n\to\infty}\left(n - \sum_{j=1}^{n}\delta(j)\right)(\tau + \tau\prime) = +\infty$, *and then* $T : X \to X$ *is asymptotically regular and* $*F_{\tau\prime\tau}$*-asymptotically contractive.*

Proof. Property (i) follows directly from the condition C8.

Now, one also has from the condition C8 that:

$$F(d(T^nx, T^ny)) \le F(d(T^{n_0}x, T^{n_0}y)) + (n_1(n) - n_0)\hat{\tau}\prime - (n_2(n) - n_0)\tau$$

$$\le F(d(T^{n_0}x, T^{n_0}y)) + n_0(\tau - \hat{\tau}\prime) + n_1(n)\hat{\tau}\prime - n_2(n)\tau$$

$$\le F(d(x,y)) + n_1(n_0)\hat{\tau}\prime - n_2(n_0)\tau + n_0(\tau - \hat{\tau}\prime) + n_1(n)\hat{\tau}\prime - n_2(n)\tau$$

$$\le F(d(x,y)) + |n_1(n_0)\hat{\tau}\prime - n_2(n_0)\tau + n_0(\tau - \hat{\tau}\prime)| + n_1(n)\hat{\tau}\prime - n_2(n)\tau$$

$$\le M(d(x,y), n_0) + n_1(n)\hat{\tau}\prime - n_2(n)\tau$$

$$\le M(d(x,y), n_0); \forall x, y \in X$$

where $M(d(x,y), n_0) = F(d(x,y)) + |n_1(n_0)\hat{\tau}\prime - n_2(n_0)\tau + n_0(\tau - \hat{\tau}\prime)| < +\infty$, since n_0 is finite, provided that $n_1(n) \le n_2\tau/\tau\prime = n\tau/\tau\prime - n_1(n)\tau/\tau\prime$, and, equivalently, if $n_1(n)/n \le \tau/(\tau + \tau\prime)$. Since $F : \mathbf{R}_0^+ \to \mathbf{R}$ is strictly increasing, it cannot be unbounded for any finite real value $n \in \mathbf{Z}_{0+}$ in view of (9). Since $d(x,y)$ is finite, then $F(d(x,y))$ is finite, then $F(d(T^nx, T^ny))$ is finite, and then $d(T^nx, T^ny)$ is finite; $\forall n \in \mathbf{Z}_{0+}$. Property (ii) has been proved. To prove Property (iii), note that:

$$F(d(T^nx,\ T^ny)) - F(d(T^{n_k}x,\ T^{n_k}y)) \le \left|\left(\sum_{j=n_k+1}^{n}\delta(j)\right)\hat{\tau}\prime - (n - n_{k+1})\tau\right| \le \hat{N}(\tau + \tau\prime) < +\infty; \forall x, y \in X,$$
$$\forall n \in (n_k, n_{k+1}] \cap \mathbf{Z}_{0+}, \forall k \in \mathbf{Z}_+$$

since $\hat{N}$ is finite. Then, $F(d(T^nx,\ T^ny))$ is bounded if $F(d(T^{n_k}x,\ T^{n_k}y))$ is bounded and $d(T^nx,\ T^ny)$ is bounded; $\forall x, y \in X$, $\forall n \in (n_k, n_{k+1}] \cap \mathbf{Z}_{0+}$. Otherwise, if $d(T^nx,\ T^ny)$ is infinity for some $n \in (n_k, n_{k+1}) \cap \mathbf{Z}_{0+}$, some $k \in \mathbf{Z}_+$ and some $x, y \in X$, then $F(d(T^mx,\ T^my))$ is also infinity from the condition C2 for all $m(\ge n) \in \mathbf{Z}_{0+}$, which contradicts the hypothesis that $F(d(T^{n_{k+1}}x,\ T^{n_{k+1}}y))$ is bounded. If $d(T^nx,\ T^ny) \to \infty$ as $n \to \infty$, a similar conclusion is attained. Since this property holds for all $k \in \mathbf{Z}_+$, then $\{d(T^nx, T^ny)\}_{n=0}^{\infty}$ is bounded. Property (iii) has been proved.

Property (iv) follows by first noting from Equation (8) of the condition C8 that:

$$\delta(n)\tau\prime - (1 - \delta(n))\tau = -\tau + \delta(n)(\tau + \tau\prime); \forall n \in \mathbf{Z}_{0+}.$$

Then, one has that:

$$\lim_{n\to\infty}\left(n\tau - \sum_{j=1}^{n}\delta(j)(\tau + \tau\prime)\right) = \lim_{n\to\infty}\sum_{j=1}^{n}((1-\delta(j))\tau - \delta(j)\tau\prime) = +\infty$$

And, from the conditions C7, C1 and C3, one has:

$$-\infty = \lim_{n\to\infty}F(d(T^nx,\ T^ny)) = \lim_{n\to\infty}F(d(T^nx,\ T^ny)) \le F(d(x,y)) + \lim_{n\to\infty}\left(\left(\sum_{j=1}^{n}\delta(j)\right)(\tau + \tau\prime) - n\tau\right) = -\infty$$

; $\forall x, y \in X$ implies that $d(T^nx,\ T^ny) \to 0$ as $n \to \infty$; $\forall x, y \in X$, then $T : X \to X$ is asymptotically regular and $*F_{\tau\prime\tau}$-asymptotically contractive. □

Theorem 3 may be extended directly for F-controlled mappings that involve two bounded real sequences $\{\tau_j\}_{j=1}^{\infty}$ and $\left\{\tau'_j\right\}_{j=1}^{\infty}$, rather than two constant real values, and which are referred to as $F_{\{\tau'_j\}_{j=0}^{\infty}\{\tau_j\}_{j=0}^{\infty}}$-controlled self-mappings.

Corollary 2. *Assume that in Theorem 3, τ and $\tau\prime$ are generalized to real sequences $\{\tau_j\}_{j=1}^{\infty} \subset [\tau_{min}\,,\,\tau_{max}]$ and $\left\{\tau'_j\right\}_{j=1}^{\infty} \subset [\tau'_{min}\,,\,\tau'_{max}]$ with $\tau_{min} > 0$, $\tau_{max} \geq \tau_{min}$, $\tau'_{min} \geq 0$ and $\tau'_{max} \geq \tau'_{min}$ such that $a_j = \tau'_j/\tau_j \in \left[a_{min} = \tau_{min}/\tau'_{min}\,,\; a_{max} = \tau_{max}/\tau'_{max}\right]$; $\forall j \in \mathbf{Z}_+$. The self-mapping $T : X \to X$ is $F_{\{\tau'_j\}_{j=0}^{\infty}\{\tau_j\}_{j=0}^{\infty}}$-asymptotically-contractive if:*

$$\left\{\sum_{j=1}^{n}\left(\delta_j - \delta'_j a_j\right)\tau_j\right\}_{n=1}^{\infty}(\subset [0,+\infty)) \to +\infty \tag{11}$$

Two sufficient conditions for the above condition to hold are as follows:

$$\left\{\sum_{j=1}^{n}\left(\delta_j - \delta'_j a_j\right)\right\}_{n=1}^{\infty}(\subset [0,\,+\infty)) \to +\infty \tag{12}$$

$$\left\{\sum_{j=1}^{n}\left(\delta_j \tau_{min} - \delta'_j \tau'_{max}\right)\right\}_{n=1}^{\infty}(\subset [0,\,+\infty)) \to +\infty \tag{13}$$

Remark 3. *Note that $\left\{\sum_{j=1}^{n}\delta_j\right\}_{n=1}^{\infty} \to +\infty$ is a necessary condition for the self-mapping $T : X \to X$ to be $F_{\{\tau'_j\}_{j=0}^{\infty}\{\tau_j\}_{j=0}^{\infty}}$-asymptotically contractive.*

Corollary 3. *The self-mapping $T : X \to X$ is $F_{\{\tau'_j\}_{j=0}^{\infty}\{\tau_j\}_{j=0}^{\infty}}$-asymptotically-contractive if $\left\{\sum_{j=1}^{n}\left(\delta_j - \delta'_j a_j\right)\right\}_{n=1}^{\infty}(\subset [0,\,+\infty))$ and $\left(\left(\sum_{j=1}^{n}\delta'_j\tau'_j/\tau_j\right)/\left(\sum_{j=1}^{n}\delta_j\right)\right) \to K$ for some $K \in (0\,,\,1)$.*

Proof. From Corollary 2, the sufficiency-type condition $0 \leq \left\{\sum_{j=1}^{n}\left(\delta_j - \delta'_j a_j\right)\right\}_{n=1}^{\infty} \to +\infty$ implies that:

$$+\infty \leftarrow \sum_{j=1}^{n}\delta_j \geq \sum_{j=1}^{n}\left(\tau'_j/\tau_j\right)\delta'_j \text{ as } n \to \infty,$$

$$\limsup_{n\to\infty}\sum_{j=1}^{n}\left(\tau'_j/\tau_j\right)\delta'_j \leq +\infty,$$

$$\lim_{n\to\infty}\left(\sum_{j=1}^{n}\delta_j\right)/\limsup_{n\to\infty}\left(\sum_{j=1}^{n}\left(\tau'_j/\tau_j\right)\delta'_j\right) = +\infty$$

so that

$$1 - \left[\lim_{n\to\infty}\left(\sum_{j=1}^{n}\left(\tau'_j/\tau_j\right)\delta'_j\right)/\left(\sum_{j=1}^{n}\delta_j\right)\right] = K(<1)$$

□

Corollary 4. *The self-mapping $T : X \to X$ is $F_{\{\tau'_j\}_{j=0}^{\infty}\{\tau_j\}_{j=0}^{\infty}}$-asymptotically contractive irrespective of the sequences $\{\tau_j\}_{j=0}^{\infty} \subset [\tau_{min}\,,\,\tau_{max}] \subset (0\,,\,+\infty)$, $\left\{\tau'_j\right\}_{j=1}^{\infty} \subset [\tau'_{min}\,,\,\tau'_{max}]$ if $\left\{\sum_{j=1}^{n}\left(\delta_j\tau_{min} - \delta'_j\tau'_{max}\right)\right\}_{n=1}^{\infty}(\subset [0,\,+\infty))$, $\tau_{min} < \tau'_{max}$ and $\left(\left(\sum_{j=1}^{n}\delta'_j\right)/\left(\sum_{j=1}^{n}\delta_j\right)\right) \to W(\in (0\,,\,\tau_{min}/\tau'_{max}))$.*

Proof. From Corollary 2, the sufficiency-type condition $\left\{\sum_{j=1}^{n}\left(\delta_j\tau_{min} - \delta'_j\tau'_{max}\right)\right\}_{n=1}^{\infty}$ $(\subset [0,\,+\infty)) \to +\infty$ implies that $+\infty \leftarrow \tau_{min}\left(\sum_{j=1}^{n}\delta_j\right) \geq \tau'_{max}\left(\sum_{j=1}^{n}\delta'_j\right)$ as $n \to \infty$,

$\limsup_{n\to\infty}\delta'_j \leq +\infty$ and $1-\lim_{n\to\infty}\left[\left(\tau'_{max}\sum_{j=1}^{n}\delta'_j\right)/\left(\tau_{min}\sum_{j=1}^{n}\delta_j\right)\right] = W_1(<1)$, then $\lim_{n\to\infty}\left[\left(\sum_{j=1}^{n}\delta'_j\right)/\left(\sum_{j=1}^{n}\delta_j\right)\right] = W < \tau_{min}/\tau'_{max} < 1$. □

A weaker result than Corollary 2 that simply guarantees the $F_{\{\tau'_j\}_{j=0}^{\infty}\{\tau_j\}_{j=0}^{\infty}}$-nonexpansivity follows directly below.

Corollary 5. *Assume that in Corollary 3,* $\tau_{min} \geq 0$, $\tau_{max} \geq \tau_{min}$, $\tau'_{min} \geq 0$ *and* $\tau'_{max} \geq \tau'_{min}$ *such that* $a_j = \tau'_j/\tau_j$; $\forall j \in \mathbf{Z}_+$. *The self-mapping* T *is* $F_{\{\tau'_j\}_{j=0}^{\infty}\{\tau_j\}_{j=0}^{\infty}}$*-asymptotically nonexpansive if, for some finite* $M \in \mathbf{R}_{0+}$, *the following condition holds:*

$$\left\{\sum_{j=1}^{n}\left(\delta_j-\delta'_j a_j\right)\tau_j\right\}_{j=1}^{n} \leq M \text{ and } \limsup_{n\to\infty}\left(\sum_{j=1}^{n}\left(\delta_j-\delta'_j a_j\right)\tau_j - M\right) \leq 0.$$

Two sufficient conditions for the above condition to hold are the following ones:

(1) $\left\{\sum_{j=1}^{n}\left(\delta_j-\delta'_j a_j\right)\right\}_{n=1}^{\infty}(\subset[0,+\infty)) \leq \Omega$; $\limsup_{n\to\infty}\left(\sum_{j=1}^{n}\left(\delta_j-\delta'_j a_j\right)-M\right) \leq 0$ for some finite $\Omega \in \mathbf{R}_{0+}$;

(2) $\left\{\sum_{j=1}^{n}\left(\delta_j\tau_{min}-\delta'_j\tau'_{max}\right)\right\}_{n=1}^{\infty}(\subset[0,+\infty)) \leq \Psi$; $\limsup_{n\to\infty}\left(\sum_{j=1}^{n}\left(\delta_j\tau_{min}-\delta'_j\tau'_{msx}\right)-\Psi\right) \leq 0$ for some finite $\Psi \in \mathbf{R}_{0+}$.

The following result is concerned with the boundedness of the sequences generated by $*F_{\tau\prime\tau}$, $F_{\tau\prime\tau}$ and $F_{\{\tau'_j\}_{j=0}^{\infty}\{\tau_j\}_{j=0}^{\infty}}$-controlled self-mappings $T : X \to X$, provided that they are asymptotically contractive holds.

Theorem 4. *Let* (X,d) *be a metric space. The following properties hold:*

(i) *Let* $T : X \to X$ *be an* $*F_{\tau\prime\tau}$*-controlled self-mapping for some* $\tau,\tau\prime \in \mathbf{R}_{0+}$ *such that the condition C8 holds. Thus, a sufficient condition for the sequence* $\{T^n x\}_{n=0}^{\infty}$ *to be bounded for any given* $x \in X$ *is that* $\lim_{n\to\infty}(n\tau - 2(\sum_{i=1}^{n}\delta(i))(\tau+\tau\prime)) = +\infty$, *that is, that* $T : X \to X$ *be* $*F_{\tau\prime\tau}$*-asymptotically contractive.*

(ii) *If* $T : X \to X$ *is* $F_{\tau\prime\tau}$*-asymptotically contractive, then the sequence* $\{T^n x\}_{n=0}^{\infty}$ *is bounded for any* $x \in X$.

(iii) $T : X \to X$ *is* $F_{\{\tau'_j\}_{j=0}^{\infty}\{\tau_j\}_{j=0}^{\infty}}$*-asymptotically contractive, then the sequence* $\{T^n x\}_{n=0}^{\infty}$ *is bounded for any* $x \in X$.

Proof. Assume that $\{d(x\,,\,T^n x)\}\overleftarrow{\longrightarrow}_{n=0}^{\infty}$ is an unbounded sequence for some $x \in X$. Then, there is a strictly increasing subsequence $\{n_k\}_{k=0}^{\infty} \subset \mathbf{Z}_{0+}$, thus $n_k \to \infty$ and $\left(n_{k+j}/n_k\right) \to \infty$ as $k,j \to \infty$ such that $d(x\,,\,T^{n_{k+1}}x) > d(x\,,\,T^{n_k}x)$ with $\lim_{k\to\infty} d(x\,,\,T^{n_k}x) = +\infty$, $\lim_{j\to\infty,k\to\infty}(d(x\,,\,T^{n_{k+j}}x)/d(x\,,\,T^{n_k}x)) = +\infty$ and then $\lim_{j\to\infty,k\to\infty}(d(x,T^{n_{k+j}}x) - d(x,T^{n_k}x)) = +\infty$. Since F is strictly increasing, then:

$$\lim_{k\to\infty} F(d(x\,,\,T^{n_k}x)) = +\infty;\ \lim_{j\to\infty,k\to\infty}(F(d(x\,,\,T^{n_{k+j}}x))/F(d(x\,,\,T^{n_k}x))) = +\infty;$$
$$\lim_{j\to\infty,k\to\infty} F(d(x\,,\,T^{n_{k+j}}x) - d(x\,,\,T^{n_k}x)) = +\infty$$

By the triangle inequality of distances, one has that:

$$d(T^{n_{k+j}}x\,,\,x) \leq d(T^{n_k}x\,,\,x) + d(T^{n_k}x\,,\,T^{n_{k+j}}x) = d(T^{n_k}x\,,\,x) + d\left(T^{n_k}x\,,\,T^{n_k}\left(T^{n_{k+j}-n_k}x\right)\right)$$

$$\leq d(T^{n_k}x\,,\,x) + \sum_{i=1}^{n_k-1} d\left(T^i x\,,\,T^{i-1}\left(T^{n_{k+j}-n_k+1}\right)x\right)$$

Thus, since $F : \boldsymbol{R}_+ \to \boldsymbol{R}$ is strictly increasing, and since $d(T^{n_k}x\ ,\ x) \to \infty$ as $k \to \infty$, then one has:

$$F(d(T^{n_{k+j}}x\ ,\ x)) \to +\infty,\ \ F(d(T^{n_k}x\ ,\ x)) \to +\infty,\ \ d(T^{n_{k+j}}x\ ,\ x)/d(T^{n_k}x,\ x) \to \infty,$$

$$d(T^{n_{k+j}}x\ ,\ x)/d\left(T^{n_{k+j}-n_k}x\ ,\ x\right) \to +\infty,\ \ F(d(T^{n_{k+j}}x\ ,\ x))/F(d(T^{n_k}x\ ,\ x)) \to +\infty,$$

$$F(d(T^{n_{k+j}}x\ ,\ x))/F\left(d\left(T^{n_{k+j}-n_k}x\ ,\ x\right)\right) \to +\infty \text{ as } k,\ j \to \infty$$

so that:

$$F(d(T^{n_{k+j}}x\ ,\ x)) \le F\left(d(T^{n_k}x\ ,\ x) + d\left(T^{n_k}x\ ,\ T^{n_k}\left(T^{n_{k+j}-n_k}x\right)\right)\right)$$

$$\le F\left(2\ max\left(d(T^{n_k}x\ ,\ x), d\left(T^{n_k}x\ ,\ T^{n_k}\left(T^{n_{k+j}-n_k}x\right)\right)\right)\right)$$

The above conditions lead to the following cases at infinity:

Case a: $+\infty = \lim\limits_{k,j\to\infty} F(d(T^{n_{k+j}}x\ ,\ x)) \le \lim\limits_{k\to\infty} F(2\ d(T^{n_k}x\ ,\ x)) = +\infty$. This implies that $+\infty \leftarrow d(T^{n_{k+j}}x\ ,\ x)/\ d(T^{n_k}x\ ,\ x) \le 2$ as $k, j \to \infty$, a contradiction.

Case b: $+\infty = \lim\limits_{k,j\to\infty\to\infty} F(d(T^{n_{k+j}}x\ ,\ x)) \le \lim\limits_{k,j\to\infty} F\left(2\ d\left(T^{n_k}x\ ,\ T^{n_k}\left(T^{n_{k+j}-n_k}x\right)\right)\right) = +\infty$.

This implies that:

$$+\infty = \lim_{k,j\to\infty\to\infty} F(d(T^{n_{k+j}}x\ ,\ x)) \le \lim_{k,j\to\infty} F\left(2\ d\left(x\ ,\ T^{n_{k+j}-n_k}x\right)\right) - \lim_{k\to\infty}\left(2n_k\tau - 2\left(\sum_{i=1}^{n_k}\delta(i)\right)\ (\tau+\tau\prime)\right)$$

so that

$$1 \le \lim_{k,j\to\infty} F\left(2\ d\left(x\ ,\ T^{n_{k+j}-n_k}x\right)\right) / \lim_{k,j\to\infty} F(\ d(x\ ,\ T^{n_{k+j}}x))$$

$$-2\lim_{k\leftarrow\infty}\left(n_k\tau - \left(\sum_{i=1}^{n_k}\delta(i)\right)\ (\tau+\tau\prime)\right) / \lim_{k,j\to\infty} F(\ d(x\ ,\ T^{n_{k+j}}x))$$

$$= 0 - 2\lim_{k\leftarrow\infty}\left(n_k\tau - \left(\sum_{i=1}^{n_k}\delta(i)\right)\ (\tau+\tau\prime)\right) / \lim_{k,j\to\infty} F(\ d(x\ ,\ T^{n_{k+j}}x))$$

$$\le 0.$$

Hence, we get a contradiction again. As a result, there is no subsequence $\{d(x, T^{n_k}x)\}_{k=0}^{\infty}(\to +\infty) \subset \{d(x, T^n x)\}_{k=0}^{\infty}$ for some strictly increasing sequence $\{n_k\}_{k=0}^{\infty} \subset \boldsymbol{Z}_{0+}$ $x \in X$ so that no subsequence $\{T^{n_k}x\}_{k=0}^{\infty} \subset \{T^n x\}_{n=0}^{\infty}$ is unbounded. As a result, all the sequences $\{T^n x\}_{n=0}^{\infty}$ are bounded for any given $x \in X$ if:

$$\lim_{n\to\infty}(n\tau - 2(\textstyle\sum_{i=1}^{n}\delta(i))\ (\tau+\tau\prime)) = +\infty$$

Property (i) has been proved. Properties [(ii)-(iii)] are proved by using similar arguments as the above ones in view of Lemma 1 (iv) and, in view of Corollary 2, Equations (11)–(13). □

In the event that the F-controlled mapping is nonexpansive and it has a fixed point, the boundedness of $\{T^n x\}_{n=0}^{\infty}$ is bounded for any $x \in X$ and is also guaranteed. Note the following.

Corollary 6. *Assume that in Theorem 4 [(i)–(iii)] any of the three respective mentioned F-controlled mappings $T : X \to X$ is nonexpansive and has a fixed point $z(= Tz) \in X$. Then, the sequence $\{T^n x\}_{n=0}^{\infty}$ is bounded for any given $x \in X$.*

Proof. For any $x \in X$, one has:

$$F(d(x\ T^n x)) \le F(d(x\ ,z) + d(z,\ T^n x)) = F(d(x\ ,z) + d(T^n z,\ T^n x)) \le F(2max(d(x,z)\ ,\ d(z,\ T^n x)))$$

$$F\left(\sup_{n\in\boldsymbol{Z}_+} d(x\ T^n x)\right) \le F\left(2max\left(d(x,z)\ ,\ \sup_{n\in\boldsymbol{Z}_+} d(z,\ T^n x)\right)\right)$$

Case a: $F\left(\sup_{n\in\mathbf{Z}_+} d(x\ T^n x)\right) \le F(2d(x,z))$. Then, $\sup_{n\in\mathbf{Z}_+} F(d(x\ T^n x)) \le F(2d(x\ ,z))$ then $\sup_{n\in\mathbf{Z}_+} d(x\ T^n x) \le 2d(x\ ,z) < +\infty$. Then, $\{T^n x\}_{n=0}^{\infty}$ is bounded for any $x \in X$.

Case b: $F\left(\sup_{n\in\mathbf{Z}_+} d(x\ T^n x)\right) \le F(2d(z,\ T^n x))$. Then, $\sup_{n\in\mathbf{Z}_+} F(d(x\ T^n x)) \le F(2d(x\ ,z))$, and $\sup_{n\in\mathbf{Z}_+} d(x\ T^n x) \le 2d(z,\ T^n x) = 2d(T^n z,\ T^n x) \le 2(d(z,x) - M_n) < +\infty$:

$$\limsup_{n\to\infty} d(x\ T^n x) \le 2\limsup_{n\to\infty} d(z\ ,\ T^n x) = 2\limsup_{n\to\infty} d(T^n z\ ,\ T^n x) \le 2\left(d(z,x) - \lim_{n\to\infty} \inf M_n\right) \le 2d(z,x) < +\infty$$

for some bounded real sequence $\{M_n\}_{n=0}^{\infty} \subset \mathbf{R}_{0+}$ is bounded for any $x \in X$. □

The subsequent result is concerned with a modified F_τ-asymptotic contraction where there is an iteration-dependent weight of τ, which is not necessarily positive for all the iterations. This arrangement allows the replacement of the two-parameter $F_{\tau\prime\tau}$-asymptotic contraction by a single weighted parameterized F_τ-asymptotic contraction.

Theorem 5. *Let* $F : \mathbf{R}_+ \to \mathbf{R}$ *be an F-function,* (X,d) *be a metric space and* $T : X \to X$ *be a self-mapping on* X *which satisfies:*

(C5) $[d(T^n x, T^n y) > 0] \Rightarrow \left[F(d(T^n x, T^n y)) \le F(d(x,y)) + \sum_{j=1}^{n} (a(j)\delta(j) - \delta\prime(j))\tau\right]$ *for some* $\tau \in \mathbf{R}_+$, $\forall n \in \mathbf{Z}_+$, *where* $\delta(j)\ ,\ \delta\prime(j) \in \{0\ ,\ 1\}$; $\forall j \in \mathbf{Z}_+$, *and* $\{a(j)\}_{j=1}^{\infty} \subset [0,\ a]$ *for some* $a \in \mathbf{R}_+$. *The following properties hold:*

(i) *Assume that the limit exists* $\lim_{k\to\infty}\left(\sum_{j=1}^{n}(a(j)\delta(j) - \delta\prime(j))\right) = -\infty$.
Then, $T : X \to X$ *is a modified asymptotic* F_τ*-contraction for any* $\tau > 0$, *and* $T : X \to X$ *is asymptotically regular.*

(ii) *(Assume that, for some strictly increasing sequence* $\{n_k\}_{k=1}^{\infty} \subset \mathbf{Z}_+$ *such that* $n_{k+1} - n_k \le \bar{n}$; *for* $\forall k \in \mathbf{Z}_+$ *and some* $\bar{n} \in \mathbf{Z}_+$, *there is a real sequence* $\{\alpha_{n_k}\}_{k=0}^{\infty} \subset [\alpha_1\ ,\ \alpha]$ *for some real constants* α_1 *and* α *with* $0 < \alpha_1 \le \alpha$, *and the following condition holds:*

$$\sum_{k=1}^{n_k}(\delta\prime(k) - a(k)\delta(k)) \ge \alpha_{n_k} n_k; \forall k \in \mathbf{Z}_+$$

Then, $T : X \to X$ *is a modified asymptotic* F_τ*-contraction for any* $\tau > 0$, *and* $T : X \to X$ *is asymptotically regular.*

Proof. Property (i) follows from C3 since $\lim_{k\to\infty}\left(\sum_{j=1}^{n}(a(j)\delta(j) - \delta\prime(j))\right) = -\infty$ implies that $F(d(T^n x, T^n y)) \to 0$ as $n \to \infty$ then $d(T^n x, T^n y) \to 0$ as $n \to \infty$ for any $\tau > 0$. Also, the condition of the property (ii) implies that $\sum_{k=1}^{n_k}(a(k)\delta(k) - \delta\prime(k)) \le -\alpha_{n_k} n_k \le -\alpha_1 n_k$, so that:

$$\limsup_{k\to\infty} \sum_{k=1}^{n_k}(a(k)\delta(k) - \delta\prime(k)) \le -\alpha_1 \lim_{k\to\infty} n_k = -\infty$$

And so that the following limit exists:

$$\lim_{k\to\infty} \sum_{k=1}^{n_k}(a(k)\delta(k) - \delta\prime(k)) = \limsup_{k\to\infty} \sum_{k=1}^{n_k}(a(k)\delta(k) - \delta\prime(k)) = -\infty$$

Thus, as in the property (i), one concludes that $F(d(T^{n_k} x, T^{n_k} y)) \to 0$ as $k \to \infty$; so $n_k \to \infty$ and then $d(T^{n_k} x, T^{n_k} y) \to 0$ as $k \to \infty$. Now, take any integer $n \in (n_k,\ n_{k+1})$ so that if $d(T^n x, T^n y) > 0$, then one has:

$$F(d(T^n x, T^n y)) \le F(d(x,y)) + \sum_{j=1}^{n_k}(a(j)\delta(j) - \delta\prime(j))\tau + \sum_{j=n_k+1}^{n}(a(j)\delta(j) - \delta\prime(j))\tau$$

then,

$$F(d(T^n x, T^n y)) - a\overline{n}\tau \leq F(d(T^n x, T^n y)) - a(n-n_k)\tau \leq F(d(x,y)) + \sum_{j=1}^{n_k} (a(j)\delta(j) - \delta\prime(j))\tau$$

so that

$$\limsup_{k\to\infty}\left(F\left(d\left(T^{n_k+\overline{n}}x,\, T^{n_k+\overline{n}}y\right)\right) - a\overline{n}\tau\right) \leq F(d(x,\,y)) + \lim_{k\to\infty}\left(\sum_{j=1}^{n_k}(a(j)\delta(j) - \delta\prime(j))\,\tau\right) = -\infty$$
$$;\forall n \in (n_k\,,\, n_{k+1})$$

so that there exists the limit $\lim_{n\to\infty} F(d(T^n x,\, T^n y)) = -\infty$, and the property (ii) is proved under the same arguments as those used in the proof of the property (i). □

3. Fixed-Point Results and Examples

This section discusses the boundedness and Cauchyness of the generated sequences of any initial point $x \in X$ for the various F-controlled self-mappings $T : X \to X$ that are F-asymptotically contractive, and the existence and uniqueness of fixed points as well as the convergence of the Cauchy sequences to them.

Theorem 6. *Let (X,d) be a metric space and let $T : X \to X$ be an $F_{\tau\prime\tau}$-controlled self-mapping, such that the following condition holds:*

(C9)

$$\begin{array}{l}\exists \lim_{n\to\infty}\left(n/\left(\sum_{j=1}^{n}\delta(j)\right)\right) = +\infty \\ ,\tau\prime/\tau < +\infty, \tau > 0, \tau\prime \geq 0\end{array} \tag{14}$$

Then, $T : X \to X$ is $F_{\tau\prime\tau}$-asymptotically contractive, it has a unique fixed point $z(= Tz) \in X$, and $\{T^n x\}_{n=0}^{\infty}$ is a Cauchy sequence; $\forall x \in X$. Furthermore, if (X,d) is complete then $\{T^n x\}_{n=0}^{\infty} \to z$ for any $x \in X$.

Proof. First, note that for C9, Equation (14) implies that $T : X \to X$ is $F_{\tau\prime\tau}$-asymptotically contractive from Theorem 1 (ii). It is now proven that $T : X \to X$ has, at most, one fixed point. Assume that $Fix(T) = \{z_1, z_2\}$. Then,

$$n\tau - \left(\sum_{j=1}^{n}\delta(j)\right)\tau\prime \leq F(d(z_1,z_2)) - F(d(T^n z_1, T^n z_2)) = F(d(z_1,z_2)) - F(d(z_1,z_2)) = 0$$

and, from C8, the subsequent contradiction holds:

$$0 \leq \lim_{n\to\infty}\left(n\tau - \left(\sum_{j=1}^{n}\delta(j)\right)\tau\prime\right) \leq F(d(z_1\,,\,z_2)) - F\left(d\left(T^k z_1\,,\,T^k z_2\right)\right) = F(d(z_1\,,\,z_2)) - F(d(z_1\,,\,z_2)) = 0$$

$;\forall k \in \mathbf{Z}_{0+}$, so that $z_1 = z_2 = z$. Now, note from C3 and C8 that:

$$\lim_{n\leftarrow\infty} F\left(d\left(T^{n+1}x,\, T^n x\right)\right) \leq F(d(x,Tx)) + \lim_{n\to\infty}\left(\tau\prime\left(\sum_{j=1}^{n}\delta(j)\right) - n\tau\right) = -\infty \Rightarrow \lim_{n\to\infty} d\left(T^{n+1}x,\, T^n x\right) = 0$$

since C8 reads equivalently to $\lim_{n\to\infty}\left(n\tau - \left(\sum_{j=1}^{n}\delta_j\right)\tau\prime\right) = +\infty$ and from Theorem 1 (ii), $\lim_{n\to\infty}\left(n/\left(\sum_{j=1}^{n}\delta_j\right)\tau\prime/\tau\right) = +\infty$ if $\tau > 0$ for any finite $\tau\prime \geq 0$. Now,

$$d\left(T^{n+1}x,\, T^n x\right) \leq d\left(T^{n+1}x,\, T^{n+1}z\right) + d\left(T^{n+1}z,\, T^n x\right) = d\left(T^{n+1}x,\, T^{n+1}z\right) + d(T^n z,\, T^n x)$$

$$\begin{aligned} d\left(z\,,\,T^{n+1}x\right) &= d\left(T^{n+1}z\,,\,T^{n+1}x\right) = d\left(T^{n+1}x, T^{n}x\right) + d\left(T^{n}x,\,T^{n+1}z\right) \\ &= d\left(T^{n+1}x, T^{n}x\right) + d(T^{n}x,\,z) \\ &= d(T(T^{n}x),\,Tz) + d(z,\,T^{n}x) \end{aligned}$$

$$F\left(d\left(z\,,\,T^{n+1}x\right)\right) = F(d(T(T^{n}x),\,Tz) + d(z,\,T^{n}x)) \leq F\left(2max\left(d\left(z,\,T^{n+1}x\right), d(z,\,T^{n}x)\right)\right).$$

For each $n \in Z_+$ and any $x \in X$:

Case a: $d\left(z\,,\,T^{n}x\right) \geq max\left(d\left(z\,,\,T^{n+1}x\right)/2, d\left(z\,,\,T^{n+1}x\right)\right) = d\left(z\,,\,T^{n+1}x\right)$. Then, $F(d(z\,,\,T^{n}x)) \geq F\left(d\left(z\,,\,T^{n+1}x\right)\right) = F\left(d\left(T^{n+1}z\,,\,T^{n+1}x\right)\right) \leq F(d(z\,,\,T^{n}x))$ from the contractive condition of the F-controlled map. Then, $F\left(d\left(z\,,\,T^{n+1}x\right)\right) \leq F(d(z\,,\,T^{n}x))$ so that $d\left(z\,,\,T^{n+1}x\right) \leq d\left(z\,,\,T^{n}x\right)$ for sufficiently large n, and since the mapping is $F_{\tau/\tau}$-asymptotically contractive, one has that:

$$\limsup_{n\to\infty}\left(d\left(z\,,\,T^{n+1}x\right) - d\left(z\,,\,T^{n}x\right)\right) < 0$$

This contradicts the initial assumption for this case for sufficiently large n so that Case a is unfeasible for sufficiently large n.

Case b: $d\left(z\,,\,T^{n+1}x\right) \geq max\left(d\left(z\,,\,T^{n+1}x\right)/2, d\left(z\,,\,T^{n}x\right)\right) = d\left(z\,,\,T^{n}x\right)$. Then, $F\left(d\left(z\,,\,T^{n+1}x\right)\right) - u_n \geq F\left(d\left(T^{n+1}z\,,\,T^{n+1}x\right)\right) = F\left(d\left(z\,,\,T^{n+1}x\right)\right) \geq F(d(z\,,\,T^{n}x))$ with $\{u_n\}_{n=0}^{\infty}(\subset [0\,,\,+\infty)) \to +\infty$. This implies that $d(z\,,\,T^{n}x) \leq d\left(z\,,\,T^{n+1}x\right); \forall n(\geq n_0) \in Z_{0+}$ and some $n_0 \in Z_{0+}$, and, for sufficiently large n, and since the mapping is $F_{\tau/\tau}$-asymptotically contractive so that $\{u_n\}_{n=0}^{\infty}(\subset [0\,,\,+\infty)) \to +\infty$, the above inequality is strict, that is, $d(z\,,\,T^{n}x) < d\left(z\,,\,T^{n+1}x\right)$ and

$$\limsup_{n\to\infty}\left(d(z\,,\,T^{n}x) - d\left(z\,,\,T^{n+1}x\right)\right) < 0$$

As a result, $\{T^{n}x\}_{n=0}^{\infty}$ is bounded for any $x \in X$. Also, $F(d(T^{n}z\,,\,T^{n}x)) = F(d(z\,,\,T^{n}x)) \to -\infty$ as $n \to \infty$ and from C3, $d(z\,,\,T^{n}x) \to 0$ as $n \to \infty$, and since the metric is a continuous function, then $\{T^{n}x\}_{n=0}^{\infty}(\subset X) \to z$ for any $x \in X$. It has been proved that if $T : X \to X$ has a fixed point then it is unique (equivalently, $T : X \to X$ has at most a fixed point z) and, furthermore, the sequence $\{T^{n}x\}_{n=0}^{\infty}$ is bounded for any $x \in X$. Furthermore, it is now proved by contradiction that $\{T^{n}x\}_{n=0}^{\infty} \to z; \forall x \in X$. Assume that there exists some $x \in X$ such that the limit $\lim_{n\to\infty} d(x\,,\,T^{n}x)$ either does not exist or it is nonzero. Both above cases are covered by the assumption that:

$$\limsup_{n\to\infty} d(x\,,\,T^{n}x) > 0$$

and, by using C3, one gets:

$$-\infty < \limsup_{n\to\infty} F(d(T^{n}x\,,\,z)) = \limsup_{n\to\infty}\ F(d(T^{n}x\,,\,T^{n}z))$$

$$\leq F\left(d(x\,,\,z)\right) - \lim_{n\to\infty}\left(n\tau - \left(\sum_{j=1}^{n}(\delta(j))\,\tau\prime\right)\right) = -\infty$$

Hence, one reaches a contradiction. Then, for any $x \in X$,

$$\lim_{n\to\infty}\ F(d(T^{n}x\,,\,z)) = \limsup_{n\to\infty}\ \ F(d(T^{n}x\,,\,z)) = -\infty$$

$$d(T^{n}x\,,\,z) \to 0 \text{ as } n \to \infty$$

from C3, and since the distance is a continuous function, then $\lim_{n\to\infty} d(T^{n}x\,,\,z) = d\left(\lim_{n\to\infty} T^{n}x\,,\,z\right) = 0$ so that $\{T^{n}x\}_{n=0}^{\infty} \to z$.

It remains to be proved that such a unique fixed point exists. One has from C3 that:

$$\lim_{n\to\infty} F\left(d\left(T^{n+1}x\ ,\ T^{n}x\right)\right) - F\left(d(x\ ,\ Tx)\right) = -\lim_{n\to\infty}\left(n\tau - \left(\sum_{j=1}^{n}(\delta(j))\ \tau\prime\right)\right) = -\infty \quad (15)$$

which implies that $d\left(T^{n+1}x\ ,\ T^{n}x\right)$ as $n\to\infty$. Also, one has:

$$d\left(T^{n+1}x\ ,\ T^{n}x\right)^{k}\left(F\left(d\left(T^{n+1}x\ ,\ T^{n}x\right)\right) - F(d(x\ ,\ Tx))\right) = -d\left(T^{n+1}x\ ,\ T^{n}x\right)^{k}\left(n\tau - \left(\sum_{j=1}^{n}(\delta(j))\ \tau\prime\right)\right) = -\infty. \quad (16)$$

and, for some real constant $k\in(0\ ,\ 1)$ from C4, one has that:

$$\lim_{n\to\infty} d\left(T^{n+1}x\ ,\ T^{n}x\right)^{k}\lim_{n\to\infty} F\left(d\left(T^{n+1}x\ ,\ T^{n}x\right)\right) - \lim_{n\to\infty} d\left(T^{n+1}x\ ,\ T^{n}x\right)^{k} F(d(x\ ,\ Tx))$$

$$0+0 = -\lim_{n\to\infty} d\left(T^{n+1}x\ ,\ T^{n}x\right)^{k}\left(n\tau - \left(\sum_{j=1}^{n}(\delta(j))\ \tau\prime\right)\right) = 0. \quad (17)$$

so that $\lim_{n\to\infty} d\left(T^{n+1}x\ ,\ T^{n}x\right)^{k}\left(n - \left(\sum_{j=1}^{n}(\delta(j))\ \tau\prime/\tau\right)\right) = 0$. Thus, there exists some $n_0 = n_0(x)\in\mathbf{Z}_+$ such that for any $n(\geq n_0)\in\mathbf{Z}_+$, $d\left(T^{n+1}x\ ,\ T^{n}x\right)^{k}\left(n - \left(\sum_{j=1}^{n}(\delta(j))\ \tau\prime/\tau\right)\right) < 1$, equivalently, $d\left(T^{n+1}x\ ,\ T^{n}x\right) \leq 1/\left(n - \left(\sum_{j=1}^{n}(\delta(j))\ \tau\prime/\tau\right)\right)^{1/k}$ for any $n(\geq n_0)\in\mathbf{Z}_+$. Then, for any $n(\geq n_0)\in\mathbf{Z}_+$ and any $m\in\mathbf{Z}_+$, one gets:

$$d\left(T^{n+m+1}x\ ,\ T^{n}x\right) \leq \sum_{j=n+1}^{n+m} d\left(T^{j}x\ ,\ T^{j-1}x\right) \leq \sum_{j=n+1}^{n+m}\left(\lambda_j/\left(j - \left(\sum_{i=1}^{j}(\delta(i))\ \tau\prime/\tau\right)\right)\right)^{1/k}\tau \quad (18)$$

where $\lambda_j \geq \sqrt[k]{\frac{j}{\left(j-\left(\sum_{i=1}^{j}(\delta(i))\ \tau\prime/\tau\right)\right)}}$. The real sequence $\left\{\lambda_j\right\}_{j=0}^{\infty}$ is bounded so that $\underset{j\geq n_0}{sup}\lambda_j \leq \lambda < +\infty$ and $\underset{n\to\infty}{lim\ sup}\lambda_n \leq 1$ since $\lim_{n\to\infty}\left(n/\left(\sum_{j=1}^{n}\delta(j)\right)\right) = +\infty$. Then, (18) implies that $\{T^n x\}_{n=0}^{\infty}$ is a Cauchy sequence and, since (X,d) is complete, there exists a fixed point $z\in X$ such that $\{T^n x\}_{n=0}^{\infty}\to z$. □

Remark 4. *Note that one of the results of Theorem 8 is that if $T: X\to X$ is $F_{\tau\prime\tau}$-asymptotically contractive, and $\{T^n x\}_{n=0}^{\infty}$ is bounded; $\forall x\in X$, if by using that, it has a fixed point. In Theorem 4 (ii), the existence of a fixed point was not invoked to achieve such a boundedness result through an alternative proof.*

Note that $F_{\left\{\tau_j'\right\}_{j=0}^{\infty}\left\{\tau_j\right\}_{j=0}^{\infty}}$-controlled self-mappings $T: X\to X$ are $F_{\tau\prime\ \tau}$-controlled self-mappings if $\left\{\tau_j'\right\}_{j=0}^{\infty}, \left\{\tau_j\right\}_{j=0}^{\infty}$ are constant sequences of respective values $\tau\prime$ and τ. Therefore, the class of $F_{\left\{\tau_j'\right\}_{j=0}^{\infty}\left\{\tau_j\right\}_{j=0}^{\infty}}$-controlled self-mappings $T: X\to X$ contains that of $F_{\tau\prime\ \tau}$-controlled self-mappings. Note also that $*F_{\tau\prime\ \tau}$-controlled self-mappings are $F_{\tau\prime+\tau,\ \tau}$-controlled self-mappings (Assertion 1), while obviously $F_{0\tau}$-controlled self-mappings with $\tau>0$ are standard F-mappings controlled by the parameter τ.

The following results are immediate from Theorem 8.

Corollary 7. *Let (X,d) be a metric space and let $T: X\to X$ be an $*F_{\tau\prime\tau}$-controlled self-mapping for such that (11) holds. Then, $T: X\to X$ is $*F_{\tau\prime\tau}$-asymptotically contractive with a unique fixed point $z\in X$ and $\{T^n x\}_{n=0}^{\infty}$ is a Cauchy sequence; $\forall x\in X$. Furthermore, if (X,d) is complete, then $\{T^n x\}_{n=0}^{\infty}\to z$ for any $x\in X$.*

Corollary 8. *Let (X,d) be a metric space and let $T: X\to X$ be an $F_{\left\{\tau_j'\right\}_{j=0}^{\infty}\left\{\tau_j\right\}_{j=0}^{\infty}}$-controlled self-mapping for such that the conditions of any of the Corollaries 2, 3 or 4*

holds. Then, $T : X \to X$ *is* $F_{\{\tau_j'\}_{j=0}^{\infty}\{\tau_j\}_{j=0}^{\infty}}$*-asymptotically contractive with a unique fixed point* $z \in X$*, and* $\{T^n x\}_{n=0}^{\infty}$ *is a Cauchy sequence;* $\forall x \in X$*. Furthermore, if* (X,d) *is complete, then* $\{T^n x\}_{n=0}^{\infty} \to z$ *for any* $x \in X$*.*

Example 1. *Let* (X,d) *be a complete metric space, let* A_1 *and* A_2 *be two nonempty closed subsets of* X *wthat intersect and let* $T : A_1 \cup A_2 \to A_1 \cup A_2$ *be a cyclic self-mapping, that is, satisfying* $T(A_1) \subset A_2$ *and* $T(A_2) \subset A_1$*, which is* $F_{\tau\prime\tau}$*-asymptotically contractive, that is, it satisfies the condition C6 for all* $(x,y) \in A_1 \times A_2 \cup A_2 \times A_1$ *subject to the condition* $\lim_{n\to\infty}\left(n\tau - \left(\sum_{j=1}^{n}\delta(j)\right)\tau\prime\right) = +\infty$ *for some* $\tau\prime \in \boldsymbol{R}_{0+}$ *,* $\tau \in \boldsymbol{R}_{0+}$ *(Definition 5). Then, one has for any* $n \in \boldsymbol{Z}_+$ *that:*

$$F(d(T^n x\ ,\ T^n y)) \le F(d(x,y)) + \left(\sum_{j=1}^{n}\delta(j)\right)\tau\prime - n\tau; \forall (x,y) \in A_1 \times A_2 \cup A_2 \times A_1.$$

$$F\left(d\left(T^n x\ ,\ T^{n+1}x\right)\right) \le F(d(x,Tx)) + \left(\sum_{j=1}^{n}\delta(j)\right)\tau\prime - n\tau; \forall x \in A_1 \cup A_2.$$

From the above relations, it follows by taking limits as $n \to \infty$ *that* $F(d(T^n x\ ,\ T^n y)) \to -\infty$ *implies that* $d(T^n x\ ,\ T^n y) \to 0; \forall (x,y) \in A_1 \times A_2 \cup A_2 \times A_1$ *and* $F\left(d\left(T^n x\ ,\ T^{n+1}x\right)\right) \to -\infty$ *imply that* $d\left(T^n x\ ,\ T^{n+1}x\right) \to 0; \forall x \in A_1 \cup A_2$*. Then, from Theorem 6,* $T : X \to X$ *is* $F_{\tau\prime\tau}$*-asymptotically contractive, and it has a unique fixed point* $z(= Tz) \in A_1 \cap A_2$*, and* $\{T^n x\}_{n=0}^{\infty} \subset A_1 \cup A_2$ *is a Cauchy sequence;* $\forall x \in A_1 \cup A_2$*. Again from Theorem 6, since* $T : A_1 \cup A_2 \to A_1 \cup A_2$ *is cyclic, then* $\left\{T^{2n}x\right\}_{n=0}^{\infty} \subset A_1$ *and* $\left\{T^{2n+1}x\right\}_{n=0}^{\infty} \subset A_2$ *if* $x \in A_1$ *are Cauchy subsequences of* $\{T^n x\}_{n=0}^{\infty}$*. Similarly,* $\left\{T^{2n}x\right\}_{n=0}^{\infty} \subset A_2$ *and* $\left\{T^{2n+1}x\right\}_{n=0}^{\infty} \subset A_1$ *if* $x \in A_2$ *are Cauchy subsequences of* $\{T^n x\}_{n=0}^{\infty}$*. Furthermore, if* (X,d) *is complete, then* $(A_1 \cup A_2\ ,\ d)$*, and since* $A_1 \cup A_2$ *is closed, it is a complete metric space too, which is a complete submetric space of* (X,d)*. From Theorem 6,* $T : A_1 \cup A_2 \to A_1 \cup A_2$ *has a unique fixed point* $z \in A_1 \cup A_2$ *to which* $\{T^n x\}_{n=0}^{\infty}$ *converges,* $\forall x \in A_1 \cup A_2$*. Since the subsequences* $\left\{T^{2n}x\right\}_{n=0}^{\infty}$ *,* $\left\{T^{2n+1}x\right\}_{n=0}^{\infty} \subset \left\{T^2 x\right\}_{n=0}^{\infty}$ *are also Cauchy and convergent;* $\forall x \in A_1 \cup A_2$*, and each one of them in one of the subsets has the same fixed point* z *as the one from the whole sequence, and it follows that* $z \in A_1 \cap A_2$*.*

A simple example for the two relevant sets of this example is $A_1 = \boldsymbol{R}_{0+} = \boldsymbol{R}_+ \cup \{0\}$*,* $A_2 = \boldsymbol{R}_{0-} = \boldsymbol{R}_- \cup \{0\}$*, whose intersection is 0. A direct extension is when the sets are the non-negative and non-positive n-dimensional real orthants whose intersection is the n-vector of null components.*

Example 2. *Now, assume in Example 1 that* A_1 *and* A_2 *are disjointed so that the distance between them is* $D = dist(A_1\ ,\ A_2) = \inf_{(x\ ,\ y)\in A_1\times A_2} d(x,y) > 0$*, and assume also that at least one of the adjacent subsets* A_1 *or* A_2 *in the cyclic disposal is boundedly compact with a unique best proximity point to its adjacent subset, and that the* $F_{\tau\prime\tau}$*-asymptotically contraction condition of the cyclic self-mapping* $T : A_1 \cup A_2 \to A_1 \cup A_2$ *is modified as follows:*

$$F(d(T^n x, T^n y) - D) \le F\left(d\left(T^{n-1}x, T^{n-1}y\right) - D\right) + \delta(n)\tau\prime - \tau; \forall (x,y) \in A_1 \times A_2 \cup A_2 \times A_1$$

which results in

$$F(d(T^n x\ ,\ T^n y) - D) \le F(d(x,y) - D) + \left(\sum_{j=1}^{n}\delta(j)\right)\tau\prime - n\tau; \forall (x,y) \in A_1 \times A_2 \cup A_2 \times A_1,$$

$$\boldsymbol{F}\left(d\left(T^n x\ ,\ T^{n+1}x\right) - D\right) \le F(d(x,Tx) - D) + \left(\sum_{j=1}^{n}\delta(j)\right)\tau\prime - n\tau; \forall x \in A_1 \cup A_2,$$

This is after using iteration to the initial distances. Since $\lim_{n\to\infty}\left(n\tau - \left(\sum_{j=1}^{n}\delta(j)\right)\tau\prime\right) = +\infty$*, one gets, by taking limits such as* $n \to \infty$*, that* $F(d(T^n x\ ,\ T^n y) - D) \to -\infty$ *implies that* $d(T^n x\ ,\ T^n y) \to D; \forall (x,y) \in A_1 \times A_2 \cup A_2 \times A_1$ *and* $F\left(d\left(T^n x\ ,\ T^{n+1}x\right) - D\right) \to -\infty$ *implies that* $d\left(T^n x\ ,\ T^{n+1}x\right) \to D; \forall x \in A_1 \cup A_2$*. By taking any* $m \in \boldsymbol{Z}_{0+}$*, it is also seen that:*

$$\boldsymbol{F}\big(d\big(T^n x\,,\,T^{n+m}y\big)-D\big)\le F(d(x,T^m y)-D)+\Big(\textstyle\sum_{j=1}^{n}\delta(j)\Big)\tau\prime-n\tau;$$

$$F\Big(d\Big(T^n x\,,\,T^{n+m+1}x\Big)-D\Big)\le F\Big(d\Big(x,T^{m+1}x\Big)-D\Big)+\Big(\textstyle\sum_{j=1}^{n}\delta(j)\Big)\tau\prime-n\tau;\forall x\in A_1\cup A_2,$$

which leads to the following extended results:

$$d\big(T^n x\,,\,T^{n+m}y\big)\to D;\ \forall(x,y)\in A_1\times A_2\cup A_2\times A_1;$$

$$d\Big(T^n x\,,\,T^{n+m+1}x\Big)\to D;\forall x\in A_1\cup A_2\ \text{as}n\to\infty\ \textit{for any}\ m\in\mathbf{Z}_{0+}\ \textit{and also as}\ n,m\to\infty.$$

Assume, with no loss in generality, that A_1 *is boundedly compact with a unique best proximity point* z_1 *to* A_2 *[12–14], and then there is a subsequence* $\big\{T^{2n_k}x\big\}_{k=0}^{\infty}\subset\big\{T^{2n}x\big\}_{n=0}^{\infty}\subset\big\{T^{n}x\big\}_{n=0}^{\infty}$ *for any* $x\in A_1$ *and* $\{n_k\}_{k=0}^{\infty}\subset\mathbf{Z}_{0+}$ *that strictly increases with* $\sup\limits_{k\ge0}(n_{k+1}-n_k)\le\hat{n}<+\infty$ *such that* $\big\{T^{2n_k}x\big\}_{k=0}^{\infty}\to z_1$ *and, furthermore,* $D=d(z_1\,,A_2)$. *And, since* T *is single-valued,* $Tz_1\in A_{20}\subset A_2$, *with* A_{20} *being the best proximity set in* A_2 *to* A_1, *so that* $D=d(z_1\,,Tz_1)$; *otherwise, if* $Tz_1\notin A_{20}$, *then* $d(z_1\,,Tz_1)>D$, *which would contradict* $d\big(T^{2n_k}x\,,\,T^{2n_k+1}x\big)\to D;\ \forall x\in A_1\cup A_2$. *Note that, although* $A_{10}=\{z_1\}$, A_1 *has a unique best proximity point, and it is not needed or assumed that* $A_{20}=\{Tz_1\}$, *that is, that* A_2 *has a unique best proximity point.*

Now, take $n\in(n_k\,,\,n_{k+1})\cap\mathbf{Z}_{0+}$ *and assume that for some* $x\in A_1$, $\limsup\limits_{k\to\infty}d\big(T^{2n_k}x\,,\,T^{2n}x\big)>0$, *where clearly* $\big\{T^{2n_k}x\big\}_{k=0}^{\infty}\subset\big\{T^{2n}x\big\}_{n=0}^{\infty}\subset A_1$. *Then, one has to take* $n=n_k+j$ *for some integer* $1\le j<n_{k+1}-n_k$ *and since* $\big\{T^{2n_k}x\big\}_{k=0}^{\infty}\to z_1$ *and* $z_1\in A_{10}$ *is the unique point in* A_1 *such that* $Tz_1\in A_2$ *and* $d(z_1,Tz_1)=D$, *one has that* $\limsup\limits_{k\to\infty}d\big(T^{2n_k}x\,,\,T^{2n+1}x\big)>D$ *so that:*

$$\begin{aligned}
&D<\limsup_{k\to\infty}d\big(T^{2n_k}x\,,\,T^{2n+1}x\big)=\limsup_{k\to\infty}d\Big(\lim_{k\to\infty}T^{2n_k}x\,,\,T^{2(n_k+j)+1}x\Big)\\
&=\limsup_{k\to\infty}d\Big(T^{2n_k}x\,,\,T^{2(n_k+j)+1}x\Big)=\limsup_{k\to\infty}d\Big(\lim_{k\to\infty}T^{2n_k}x\,,\,T^{2(n_k+j)+1}x\Big)\\
&=\limsup_{k\to\infty}d\Big(z_1\,,\,T^{2(n_k+j)+1}x\Big)\le\limsup_{k\to\infty}d\big(z_1\,,\,T^{2n_{k+i}}x\big)+\limsup_{k\to\infty}d\Big(T^{2n_{k+i}}x\,,\,T^{2(n_k+j)+1}x\Big)\\
&=d\Big(z_1\,,\,\lim_{k\to\infty}T^{2n_{k+i}}x\Big)+\limsup_{k\to\infty}d\Big(T^{2n_{k+i}}x\,,\,T^{2(n_k+j)+1}x\Big);\forall i\in\mathbf{Z}_{0+}.
\end{aligned}\tag{19}$$

Note that the last equality in the above set of Equation (19) is justified from its former one as follows. Since $T:A_1\cup A_2\to A_1\cup A_2$ *is an* $F_{\tau\prime\tau}$*-asymptotic contraction and the* F*-function is strictly increasing,* $\lim\limits_{n\to\infty}F\big(d\big(T^{n+1}x\,,\,T^{n}x\big)\big)=-\infty$, *and then there is some* $N\in\mathbf{Z}_+$ *such that for* $n(\ge N)\in\mathbf{Z}_+$, $d\big(T^{n+1}x\,,\,T^{n}x\big)<d\big(T^{n}x\,,\,T^{n-1}x\big)$ *if* $T^nx\ne T^{n-1}x$. *Then, the* $F_{\tau\prime\tau}$*-asymptotic contraction* $T:A_1\cup A_2\to A_1\cup A_2$ *is also an asymptotic weak contraction on* $A_1\cup A_2$, *andthen it is asymptotically continuous everywhere in its definition domain. Thus, if* $2(n_k+j)+1\ge N$, *and since* $\lim\limits_{k\to\infty}T^{2n_{k+i}}x=z_1$, *we can interchange the limit and the distance operators as follows:*

$$\limsup_{k\to\infty}d\Big(z_1\,,\,T^{2n_{k+i}}x\Big)=\limsup_{k\to\infty}d\Big(T^{2n_{k+i}}z_1\,,T^{2n_{k+i}}x\Big)=\lim_{k\to\infty}d\Big(T^{2n_{k+i}}z_1\,,T^{2n_{k+i}}x\Big)=d\Big(T^{2n_{k+i}}z_1\,,\,\lim_{k\to\infty}T^{2n_{k+i}}x\Big).$$

Since $\{n_k\}_{k=0}^{\infty} \subset \mathbf{Z}_{0+}$ *is strictly increasing, we can take* $i \to \infty$ *in the above expression to yield:*

$$\begin{aligned}
D &< \lim_{k\to\infty\,,\,i\to\infty} d\left(z_1\,,\,T^{2n_{k+i}}x\right) + \lim_{k\to\infty\,,\,i\to\infty} d\left(T^{2n_{k+i}}x\,,\,T^{2(n_k+j)+1}x\right) \\
&< d\left(z_1\,,\,\lim_{k\to\infty}\lim_{i\to\infty} T^{2n_{k+i}}x\right) + \lim_{k\to\infty\,,\,i\to\infty} d\left(T^{2n_{k+i}}x\,,\,T^{2(n_k+j)+1}x\right) \\
&= d\left(z_1\,,\,\lim_{k\to\infty}\lim_{i\to\infty} T^{2n_{k+i}}x\right) + \lim_{k\to\infty\,,\,i\to\infty} d\left(T^{2n_{k+i}}x\,,\,T^{2(n_k+j)+1}x\right) \\
&= d(z_1\,,\,z_1) + \lim_{k\to\infty\,,\,i\to\infty} d\left(T^{2n_{k+i}}x\,,\,T^{2(n_k+j)+1}x\right) \\
&= d(z_1\,,\,z_1) + \lim_{k\to\infty\,,\,i\to\infty} d\left(\lim_{k\to\infty\,,\,i\to\infty} T^{2n_{k+i}}x\,,\,T^{2(n_k+j)+1}x\right) \\
&= d(z_1\,,\,z_1) + d\left(z_1\,,\,T\left(\lim_{k\to\infty\,,\,i\to\infty} T^{2(n_k+j)}x\right)\right) \\
&= d(z_1\,,\,z_1) + d(z_1\,,\,Tz_1) = 0 + D = D
\end{aligned}$$

since $\left\{T^{2n_{k+i}}x\right\}_{k=0}^{\infty} \subset A_1$ *and* $\left\{d\left(T^{2n_{k+i}}x\,,\,T^{2(n_k+j)+1}x\right)\right\}_{k=0}^{\infty} \subset A_2$ *if* $x \in A_1$*, and hence, a contradiction of:*

$$D < \limsup_{k\to\infty} d\left(T^{2n_k}x\,,\,T^{2n+1}x\right)$$

if

$$\limsup_{k\to\infty} d\left(T^{2n_k}x\,,\,T^{2n}x\right) > 0$$

and then

$$\limsup_{k\to\infty} d\left(T^{2n_k}x\,,\,T^{2n}x\right) = 0$$

for $n \in (n_k\,,\,n_{k+1})$*, which leads to:*

$$\lim_{n\to\infty} d\left(T^{2n}x\,,\,T^{2(n+m)}x\right) = 0,\ \lim_{n\to\infty} d\left(T^{2n}x\,,\,T^{2(n+m)+1}x\right) = D$$

; $\left\{T^{2n}x\right\}_{n=0}^{\infty} \to z_1$*,* $\left\{T^{2n+1}x\right\}_{n=0}^{\infty} \to Tz_1$*;* $\forall x \in A_1$*;* $m \in \mathbf{Z}_{0+}$*, and it turns out as well that* $\left\{T^{2n}x\right\}_{n=0}^{\infty} \to Tz_1$*,* $\left\{T^{2n+1}x\right\}_{n=0}^{\infty} \to z_1$*;* $\forall x \in A_2$*. Note that* $Tz_1 \in A_{20}$*, and it is not required that* $A_{20} = \{Tz_1\}$*. These results also make evident that the composed mapping* $T^2 : A_1 \to A_1$ *and* $T^2 : A_2 \to A_2$ *have, respectively, unique fixed points* $z_1 \in A_1$ *and* Tz_1*.*

A simple example for the two relevant sets of this example is when A_1 *and* A_2 *are the subsets of the non-negative and non-positive real n-dimensional sets of all absolute components being larger than or equal to* $d > 0$*. Then, the sets are disjointed and the Euclidean distance between them is* $D = d > 0$*.*

It can be pointed out that discrete models of dynamic systems, as well as the resulting ones from the discretization of their continuous-time counterparts, are relevant in several fields of science and engineering, as, for instance, in switched parameterizations of certain processes [15] or in some biological models [16,17]. Such models often include the presence of delays and nonlinearities while they need to keep basic properties, like well-posedness, stability, existence, and characterization of their solutions. In this context, the formalism of F-contractions might be a useful and promising tool as an alternative to other techniques like the Lyapunov theory or classical analysis methodologies of the solutions, as has already been discussed in some of the existing background literature. Note that such a formalism may allow the direct investigation of the existence and uniqueness of solutions, as well as the monitored evaluation of the rate of convergence of the solution trajectories to the equilibrium points. See, for instance [18,19]. On the other hand, it can also be pointed out that a survey of results on F-contractions is developed in [20], while an adaptation of

such contractions to probabilistic Menger spaces with new ad hoc proved results has been reported in [21].

4. Conclusions and Some Future Work

Some extensions of Wardowski F-controlled self-mappings $T : X \to X$, which are referred to as $F_{\tau/\tau}$ and modified $*F_{\tau/\tau}$ controlled self-mappings, are proposed and investigated in metric spaces, as well as some extensions to F-controlled self-mappings parameterized by two sequences of parameters. The main formulated results rely on asymptotic contractivity and asymptotic nonexpansivity in metric spaces and some of their relevant properties. In particular, the properties of boundedness of the sequences of distances, as well as those of boundedness of the elements of the sequences themselves, are investigated under asymptotic contractivity or nonexpansivity properties related to the various above-mentioned $F_{(.)}$-controlled self-mappings. Also, existence and uniqueness fixed-point results are obtained if the metric space is complete, as well as results of Cauchyness properties and properties of convergence of such sequences to the fixed points. Finally, two illustrative examples are described if the F-controlled self-mappings are cyclic on the union of two nonempty closed subsets of the metric space in the case that those sets intersect, and in the case when they are disjointed.

Future work, under the basis of the above formal analysis, could focus on the study of the stability and stabilization via F-contractions of discrete, eventually delayed, and nonlinear dynamic systems in the presence of switching actions of their parameterizations through time . Also, and alternatively, switching actions could take place on the working operation states by combining the introduction of F-contractions of their state norms with Perov-type techniques of analysis of contractions for non-scalar discrete dynamic systems.

Funding: This research received no funding.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The author declares no conflicts of interest.

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