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Complete Stability and Stabilization Analysis of Rolling Mill Main Drive Systems with Time Delay

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Abstract

This paper addresses the stability and stabilization problems of rolling mill main drive systems with time delay. The complete stability problem with respect to the delay parameter is investigated for linearized systems with and without control. The systems are first formulated in the characteristic function form, and the corresponding complete stability problem is introduced. By employing the frequency-sweeping approach, the complete stability properties are systematically analyzed. As a result, several analytical properties related to the local asymptotic behavior of critical imaginary roots are derived. Finally, numerical examples demonstrate that the whole stability set with respect to delay can be accurately determined.

Keywords: complete stability problem; stabilization; rolling mill main drive systems; frequency-sweeping approach; time-delay

MSC: 93C95; 93D20

1. Introduction

Rolling mill vibration models have evolved from simple linear representations to complex nonlinear dynamical systems [1–3]. By incorporating key features such as time delay [4], nonlinear stiffness [5,6], and nonlinear damping, based on the structure and loading characteristics of the actual equipment, the nonlinear torsional vibration model can be established, revealing more vibration mechanism and phenomena. In practical operation, frictional vibrations are commonly observed and have a significant impact on system stability. For rolling mill main drive systems with time delay, the characteristic function plays a central role in stability analysis.

In view of system instability, various control strategies have been proposed to stabilize the system and achieve desired dynamic performance. For example, reference [7] employed state feedback control to stabilize unstable equilibria and periodic orbits in a congestion control model. In [8], a polynomial-based state feedback controller was designed for internet congestion systems. In addition, a nonlinear controller was developed in [9] for rolling mill main drive systems with time delay.

Existing studies on such systems typically focus on a stability interval of the form

$$\tau \in [0, \bar{\tau}), \tag{1}$$

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where $\bar{\tau} > 0$ is referred to as the delay margin [10].

The delay margin is a key concept in stability analysis (see, e.g., [11–13]) and has also been widely used in stabilization problems (see, e.g., [14–16]). If $\bar{\tau}$ can be accurately computed, then the equilibrium is locally asymptotically stable for all $\tau \in [0, \bar{\tau})$.

However, it has been shown in [17,18] that time-delay systems may admit multiple stability intervals, which may or may not include zero. In such cases, the delay margin becomes inherently conservative, and a more general framework is required.

In general, a time-delay system is asymptotically stable if

$$\tau \in [\underline{\tau}_1, \bar{\tau}_1) \cup \cdots \cup (\underline{\tau}_s, \bar{\tau}_s), 0 = \underline{\tau}_1 < \bar{\tau}_1 < \cdots < \underline{\tau}_s < \bar{\tau}_s, \tag{2}$$

or

$$\tau \in (\underline{\tau}_1, \bar{\tau}_1) \cup \cdots \cup (\underline{\tau}_s, \bar{\tau}_s), 0 \leq \underline{\tau}_1 < \bar{\tau}_1 < \cdots < \underline{\tau}_s < \bar{\tau}_s, \tag{3}$$

where $\bar{\tau}_s$ is referred to as the generalized delay margin [19].

For rolling mill main drive systems, multiple stability τ intervals may arise. In such situations, the concept of delay margin is no longer sufficient, indicating that further investigation is necessary.

This paper studies the complete stability and stabilization problems of rolling mill main drive systems with time delay. By applying the frequency-sweeping approach proposed in [20,21], the stability of the system can be analyzed along the whole semi-infinite τ axis. No specific constraints are imposed on the delayed systems. Numerical examples are provided to illustrate the effectiveness of the proposed approach. The results show that the whole stability set can be accurately detected. In particular, it is revealed that multiple stability τ intervals may exist for such systems.

Beyond stability analysis, the proposed framework also provides a systematic tool for controller parameter design. In particular, it enables the explicit characterization of admissible ranges of controller gains that guarantee stability. This allows one to evaluate how controller parameters influence the stability structure, including the enlargement of stability regions and the possible emergence of multiple stability intervals or delay-independent stability. Such results offer useful a priori guidance for controller tuning in practical implementations.

The main contributions of this paper are summarized as follows:

(1) The frequency-sweeping approach is firstly introduced into the stability analysis of rolling mill main drive systems, providing a systematic framework to address complete stability and stabilization problems over the whole delay domain.
(2) It is shown that the stability with respect to τ may exhibit a non-monotonic structure, where multiple stability intervals can arise and stability may be regained after certain critical delays.
(3) The controller parameter can be characterized. In particular, the controller parameter is shown to significantly influence the stability structure, enabling the enlargement of stability regions.

The remainder of this paper is organized as follows. Section 2 introduces the necessary preliminaries and prerequisites. Section 3 develops the main results on complete stability and stabilization problems. Section 4 presents illustrative examples for validation. Finally, Section 5 concludes the paper.

The following notation is used throughout this paper. Let $\mathbb{R}$ and $\mathbb{R}_+$ denote the sets of real numbers and positive real numbers, respectively, and let $\mathbb{C}$ denote the set of complex numbers. The open left and right half-planes of $\mathbb{C}$ are denoted by $\mathbb{C}_-$ and $\mathbb{C}_+$, respectively, while $\mathbb{C}_0$ represents the imaginary axis. The unit circle is denoted by $\partial\mathbb{D}$. The set of

nonnegative integers is denoted by $\mathbb{N}$. For any $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\lambda)$ denote its real and imaginary parts, respectively. Finally, ε represents an arbitrarily small positive scalar.

2. Preliminaries and Prerequisites

This section introduces some preliminaries and prerequisites for the stability analysis of the rolling mill main drive system and presents the corresponding mathematical models with time delay in both uncontrolled and controlled cases.

2.1. Mathematical Model of the Rolling Mill Main Drive Delay System

Consider the mathematical model of a rolling mill main drive time-delay system [4], which is described by the following differential equation:

$$\ddot{\theta} + \zeta\theta + \xi\dot{\theta} + \eta\dot{\theta}^2 + \beta'\dot{\theta}^3 = p'\theta(t-\tau), \tag{4}$$

where θ denotes the angular displacement and $\tau > 0$ is the time delay. The parameters ζ, ξ, η, β', and p' are real constants, defined as $\zeta = \frac{b_1+p}{J}$, $\xi = \frac{c-\alpha Pr_a^2+3\beta Pr_a^4\dot{\theta}_0^2}{J}$, $\eta = -3\beta'\dot{\theta}_0$, $\beta' = \frac{\beta Pr_a^4}{J}$, $p' = \frac{p}{J}$ with $p = \frac{(2-R)Er_a^2y_2B}{L}$, $R = \frac{y_1-y_2}{y_1}$, where R is the reduction ratio, and y_1, y_2 denote the inlet and outlet strip thicknesses, respectively. Here, E is the Young's modulus, B is the strip width, r_a is the roll radius, c is the damping coefficient, b_1 is the stiffness coefficient, P is the rolling force, and J is the rotational inertia of the roll.

In addition, the parameters α and β originate from the stick–slip friction model. Specifically, α represents the linear velocity-dependent friction coefficient, while β denotes the cubic nonlinear friction coefficient. These terms characterize the velocity-dependent friction behavior between the roll and the strip and play an essential role in the nonlinear vibration characteristics of the system.

The model captures the combined effects of structural stiffness, damping, nonlinear friction, and delayed tension feedback in the rolling mill main drive system. The detailed derivation of this model can be found in [4].

For analytical convenience, we introduce the state variables $x_1 = \theta$, $x_2 = \dot{\theta}$, and define the parameters $\alpha_1 = \xi$, $\alpha_2 = \eta$, $\alpha_3 = \beta'$, $\alpha_4 = \zeta$, $\alpha_5 = p'$.

Then, the system (4) can be transformed into the following state-space form:

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -\alpha_1 x_2(t) - \alpha_2 x_2^2(t) - \alpha_3 x_2^3(t) - \alpha_4 x_1(t) + \alpha_5 x_1(t-\tau). \end{cases} \tag{5}$$

Linearizing system (5) at the zero equilibrium point $E_0 = (0,0)^{\mathrm{T}}$, the quadratic and cubic terms vanish, yielding

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -\alpha_1 x_2(t) - \alpha_4 x_1(t) + \alpha_5 x_1(t-\tau). \end{cases} \tag{6}$$

Eliminating $x_2(t)$ leads to the equivalent second-order delay differential equation:

$$\ddot{x}_1 + \alpha_1\dot{x}_1 + \alpha_4 x_1 - \alpha_5 x_1(t-\tau) = 0. \tag{7}$$

Substituting $x_1(t) = e^{\lambda t}$ into (7) yields the characteristic function:

$$f(\lambda,\tau) = q_0(\lambda) + q_1(\lambda)e^{-\tau\lambda}, \tag{8}$$

where $q_0(\lambda) = \lambda^2 + \alpha_1\lambda + \alpha_4$ and $q_1(\lambda) = -\alpha_5$.

2.2. Design of Nonlinear State Feedback Controller

To stabilize the system (5), a nonlinear state feedback controller [9] is designed as follows:

$$u(t) = -k_1 x_1(t) - k_2 x_1^2(t) - k_3 x_1^3(t), \tag{9}$$

where k_1, k_2, k_3 are positive feedback gain parameters.

Remark 1. *The controller (9) is designed such that the control input vanishes at the equilibrium of system (4), thereby preserving the equilibrium point. This property allows the system dynamics to be modified without shifting the equilibrium, and has been widely adopted in the literature (see, e.g., [7–9]). This choice also leads to a simple controller structure, which facilitates analytical tractability in the subsequent stability analysis.*

With the introduction of the state feedback controller, the rolling mill main drive system (5) can be rewritten in state-space form as:

$$\begin{cases} \dot{x}_1(t) = x_2(t) - k_1 x_1(t) - k_2 x_1^2(t) - k_3 x_1^3(t) \\ \dot{x}_2(t) = -\alpha_1 x_2(t) - \alpha_2 x_2^2(t) - \alpha_3 x_2^3(t) - \alpha_4 x_1(t) + \alpha_5 x_1(t-\tau) \end{cases} \tag{10}$$

Linearizing the controlled system (10) at the zero equilibrium point E_0 leads to the following characteristic function:

$$f(\lambda, \tau) = q_0(\lambda) + q_1(\lambda) e^{-\tau\lambda}, \begin{cases} q_0(\lambda) = \lambda^2 + (\alpha_1 + k_1)\lambda + \alpha_4 + \alpha_1 k_1 \\ q_1(\lambda) = -\alpha_5 \end{cases}. \tag{11}$$

2.3. Complete Stability Analysis

For time-delay systems, the equilibrium point E_0 is locally asymptotically stable if and only if all characteristic roots of the characteristic equation $f(\lambda, \tau) = 0$ lie in the open left half-plane $\mathbb{C}_-$ (see [22]). Therefore, the stability problem can be equivalently transformed into the analysis of the distribution of characteristic roots.

For the rolling mill main drive time-delay systems (5) and (10) mentioned above, the objective of complete stability analysis is to analytically determine all the stability τ intervals by investigating the distribution of characteristic root for $f(\lambda, \tau) = 0$.

By convention, let $NU(\tau) \in \mathbb{N}$ denote the number of characteristic roots located in the right half-plane $\mathbb{C}_+$ as a function of τ. Clearly, the system is stable if and only if $NU(\tau) = 0$. Hence, the complete stability problem reduces to calculating all the whole stability τ intervals satisfying $NU(\tau) = 0$.

We now recall some related notations. As the delay parameter τ increases, $NU(\tau)$ can change only when the characteristic equation has purely imaginary roots $\lambda = j\omega$ ($\omega \in \mathbb{R}$). Such delay values are called critical delays. A pair (λ, τ) satisfying $f(\lambda, \tau) = 0$, consisting of a critical imaginary root $\lambda \in \mathbb{C}_0$ and its corresponding critical delay $\tau \in \mathbb{R}_+ \cup 0$, is called a critical pair.

Furthermore, according to the definition in [20], let $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ denote the variation in the number of unstable roots induced by the asymptotic behavior of a critical imaginary root λ_α as τ increases from $\tau_{\alpha,k} - \varepsilon$ to $\tau_{\alpha,k} + \varepsilon$. The asymptotic behavior of a critical imaginary root refers to the variation of the critical imaginary root as τ increases near the corresponding critical delay. This notation $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) \in \mathbb{N}$ can quantify the influence of the asymptotic behavior for the critical pair $(\lambda_\alpha, \tau_{\alpha,k} > 0)$ on $NU(\tau)$.

2.4. Limitations of Existing Methods

Most existing studies on the stability of rolling mill main drive systems assume that the critical imaginary roots are simple. Under this assumption, the implicit function theorem can be applied to compute

$$\frac{d\lambda}{d\tau} = -\frac{f_\tau}{f_\lambda}. \tag{12}$$

When τ increases near a critical delay, the sign of $\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)$ determines the crossing direction of the root across the imaginary axis. Specifically, if $\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right) > 0$ (or $\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right) < 0$), the root crosses the imaginary axis from the left half-plane to the right half-plane (or from the right half-plane to the left half-plane), indicating a change in stability.

However, it has been shown that critical imaginary roots may exhibit more complicated asymptotic behaviors, such as multiple roots or degenerate cases (see, e.g., [17,23–25]). In particular, when

$$\mathrm{Re}(\frac{d\lambda}{d\tau}) = 0, \tag{13}$$

the first-order derivative is insufficient to determine the crossing direction, and higher-order derivatives (e.g., $\frac{d^2\lambda}{d\tau^2}$, $\frac{d^3\lambda}{d\tau^3}$, ...) are required. In such cases, the implicit function theorem is no longer applicable, which significantly limits the effectiveness of conventional approaches.

Moreover, existing studies mainly focus on stability analysis in the form of a single interval (1), while the possibility of multiple disjoint stability intervals (2) or stability intervals not starting from zero (3) has not been systematically investigated. Nevertheless, such phenomena are common in general time-delay systems, and are therefore expected to also arise in rolling mill main drive systems.

To address these issues, the frequency-sweeping approach proposed in [20,21] will be introduced in the next section. This approach can directly obtain the explicit expression of $NU(\tau)$ and systematically determine all stability τ intervals of the rolling mill main drive time-delay system by plotting the "$NU(\tau)$ vs. τ" curve.

3. Main Results

This section reviews the fundamental principle of the frequency-sweeping approach, which serves as the basis for solving the complete stability problem of the rolling mill main drive time-delay system. Based on this framework, several analytical properties are derived for both the uncontrolled and controlled systems.

3.1. Complete Stability Analysis via the Frequency-Sweeping Approach

Consider the characteristic function $f(\lambda, \tau) = q_0(\lambda) + q_1(\lambda)e^{-\tau\lambda}$ defined in (8) and (11). To analyze its time-delay stability, introduce the variable transformation $z = e^{-\tau\lambda}$, which converts the characteristic function $f(\lambda, \tau)$ into

$$p(\lambda, z) = q_0(\lambda) + q_1(\lambda)z. \tag{14}$$

Without loss of generality, assume that there exist u critical pairs (λ, z) with $\lambda \in \mathbb{C}_0$ and $z \in \partial\mathbb{D}$ such that $p(\lambda, z) = 0$. These pairs are denoted by

$$(\lambda_0 = j\omega_0, z_0), \ldots, (\lambda_{u-1} = j\omega_{u-1}, z_{u-1}) \text{ and } \omega_0 \geq \cdots \geq \omega_{u-1} > 0.$$

Once all critical pairs $(\lambda_\alpha, z_\alpha)$, $\alpha = 0, \ldots, u-1$, are obtained, the corresponding values of critical delay for $f(\lambda, \tau) = 0$ can be determined. For each critical imaginary root λ_α, the associated critical delays are given by

$$\tau_{\alpha,k} = \tau_{\alpha,0} + \frac{2k\pi}{\omega_\alpha}, \quad k \in \mathbb{N}, \tag{15}$$

where $\tau_{\alpha,0}$ is the smallest critical delay, which can be calculated from $z_\alpha = e^{-j\omega_\alpha \tau_{\alpha,0}}$.

Since the coefficients of $q_0(\lambda)$ and $q_1(\lambda)$ are real, the characteristic roots are symmetric with respect to the real axis. Therefore, it suffices to consider only critical imaginary roots with nonnegative imaginary parts, i.e., $\lambda_\alpha = j\omega_\alpha$ with $\omega_\alpha > 0$.

The frequency-sweeping curve for $f(\lambda, \tau) = 0$ is generated as follows: For $\omega \geq 0$, let $\lambda = j\omega$ and solve $p(j\omega, z) = 0$ to obtain $z(j\omega)$. Then, the curve $\Gamma(\omega) : |z(j\omega)|$ is plotted as a function of ω. Define the horizontal line with ordinate equal to 1 as $\Im_1$. A critical pair $(\lambda_\alpha \neq 0, \tau_{\alpha,k})$ exists if and only if the curve $\Gamma(\omega)$ intersects $\Im_1$ at $\omega = \omega_\alpha$. Hence, all critical imaginary roots and the corresponding critical delays can be detected and calculated from the frequency-sweeping curve.

Based on the above construction, the following property can be directly obtained.

Property 1. *For the characteristic function $f(\lambda, \tau)$, if the frequency-sweeping curve does not intersect $\Im_1$ for any $\omega \geq 0$, then:*

(1) The systems (5) and (10) have no critical imaginary roots.
(2) The systems are asymptotically stable for all $\tau \in [0, \infty)$ if and only if $NU(0) = 0$.
(3) The systems are unstable for all $\tau \in [0, \infty)$ if and only if $NU(0) > 0$.

Define $\Delta NF_{z_\alpha}(\omega_\alpha)$ as the number change in the frequency-sweeping curve above $\Im_1$ as ω increases from $\omega_\alpha - \varepsilon$ to $\omega_\alpha + \varepsilon$. According to Theorems 8.5 and 9.4 in [20], the following theorems can be obtained.

Theorem 1. *For each critical imaginary root λ_α of the characteristic equation $f(\lambda, \tau) = 0$, when all $\tau_{\alpha,k} > 0$, then $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is a constant equal to $\Delta NF_{z_\alpha}(\omega_\alpha)$, denoted by U_{λ_α}.*

Theorem 2. *Consider the characteristic equation $f(\lambda, \tau) = 0$. For any noncritical delay $\tau > 0$, the number of unstable roots can be explicitly expressed as*

$$NU(\tau) = NU(+\varepsilon) + \sum_{\alpha=0}^{u-1} NU_\alpha(\tau), \tag{16}$$

where

$$NU_\alpha(\tau) = \begin{cases} 0, \tau < \tau_{\alpha,0}, \\ 2U_{\lambda_\alpha} \left\lceil \frac{\tau - \tau_{\alpha,0}}{2\pi/\omega_\alpha} \right\rceil, \tau > \tau_{\alpha,0}, \end{cases} \quad \text{if } \tau_{\alpha,0} \neq 0,$$

and

$$NU_\alpha(\tau) = \begin{cases} 0, \tau < \tau_{\alpha,1}, \\ 2U_{\lambda_\alpha} \left\lceil \frac{\tau - \tau_{\alpha,1}}{2\pi/\omega_\alpha} \right\rceil, \tau > \tau_{\alpha,1}, \end{cases} \quad \text{if } \tau_{\alpha,0} = 0.$$

Proof. First, determine the number of unstable roots at $\tau = +\varepsilon$, denoted by $NU(+\varepsilon)$. According to Theorem 5.1 in [20] or Theorem 1 in [21], if no critical imagiary root exists at $\tau = 0$, then $NU(+\varepsilon) = NU(0)$. Otherwise, if nonzero critical imaginary roots exist at $\tau = 0$, then $NU(+\varepsilon) - NU(0)$ equals the number of the values in $\mathbb{C}_+$ of Puiseux series associated with all the corresponding critical imaginary roots for $\Delta\tau = +\varepsilon$.

For each critical imaginary root λ_α, Theorem 1 guarantees that the variation $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ remains constant for all $\tau_{\alpha,k} > 0$, denoted by U_{λ_α}. Since each critical imaginary root λ_α appears together with its complex conjugate, $NU(\tau)$ is affected by $2U_{\lambda_\alpha}$.

Therefore, for any given $\tau > 0$, the value of $NU(\tau)$ can be obtained by summing the initial value $NU(+\varepsilon)$ and all variations associated with critical delays satisfying $0 < \tau_{\alpha,k} < \tau$. This leads to the explicit expression (16). □

Based on Theorem 2, since $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is constant for all $\tau_{\alpha,k} > 0$, the complete stability problem for the rolling mill main drive time-delay systems (5) and (10) can be solved. Specifically, if τ is not a critical delay and satisfies $NU(\tau) = 0$, then the equilibrium point E_0 of the linearized system associated with the characteristic equation $f(\lambda, \tau) = 0$ is locally asymptotically stable.

Moreover, the explicit expression of $NU(\tau)$ in (16) enables an accurate plot of $NU(\tau)$ with respect to τ, from which all stability τ intervals can be precisely determined. Owing to the periodicity of the delay (15), the critical pair $(\lambda_\alpha, z_\alpha)$ exhibits the same periodic property. Hence, in light of Theorems 1 and 2, there may exist multiple intervals of the delay that ensure stability.

It is worth noting that Theorems 1 and 2 can be extended to more general characteristic functions of the form

$$f(\lambda, \tau) = q_0(\lambda) + q_1(\lambda)e^{-\tau\lambda} + \cdots + q_n(\lambda)e^{-n\tau\lambda}. \tag{17}$$

Remark 2. *The frequency-sweeping approach is based on the analytic properties of the characteristic function (17) and is applicable to retarded-, neutral-, and advanced-type systems. The method imposes no restriction on the system order and can also handle multiple and degenerate roots. In this paper, we focus on the case of a single delay. The extension to multiple delays is more involved; for further details, we refer the reader to [26].*

In the following, we propose a lemma from Section VI-D of [21], which is important to characterizes the crossing behavior of the frequency-sweeping curve associated with the characteristic function considered in this paper.

Lemma 1. *Consider the characteristic function $f(\lambda, \tau)$ of the rolling mill main drive time-delay system. Let $\lambda_i = j\omega_i$ be a critical imaginary root with $\omega_i > 0$. Then the following statements hold:*

(1) If the multiplicity of λ_i plus the number of degenerate terms is even, then the frequency-sweeping curve is tangent to the line $\Im_1$ at ω_i, and no stability switching occurs. In this case, $\Delta NF_{z_i}(\omega_i) = 0$.

(2) If the multiplicity of λ_i plus the number of degenerate terms is odd, then a transversal crossing occurs at ω_i, leading to a change in stability. Moreover, by ordering all such critical frequencies associated with transversal crossings as $\omega_0 > \omega_1 > \cdots > \omega_k > 0$, then

$$\Delta NF_{z_k}(\omega_k) = \begin{cases} +1, & \text{if } k \text{ is even}, \\ -1, & \text{if } k \text{ is odd}. \end{cases}$$

From the definitions of $f(\lambda, \tau)$ and $F(W)$ in (8) and (the later-defined Equation (22)), and similarly for (11) and (26), $\deg(F(W)) = \deg(q_0(\lambda)) = 2$ always hold. Then, the number of degenerate terms is 0, according to Theorem 3 of [18].

Clearly, for the characteristic functions (8) and (11) in this paper, the multiplicity of the critical frequency is either 1 or 2. Hence, we have the following property.

Property 2. *For the rolling mill main drive time-delay system considered in this paper, the set of possible values of U_{λ_α} is $\{-1, 0, +1\}$.*

(1) *If the multiplicity of a critical imaginary root $\lambda_0 = j\omega_0$ is even, the frequency-sweeping curve Γ_1 only touches without crossing $\Im_1$ at ω_0. In this case, $\Delta NF_{z_0}(\omega_0) = 0$.*
(2) *If the multiplicity is odd, the curve Γ_1 crosses $\Im_1$ at ω_i. Let the critical frequencies be ordered as $\omega_0 > \omega_1 > 0$. Then, $\Delta NF_{z_0}(\omega_0) = +1$ and $\Delta NF_{z_1}(\omega_1) = -1$.*

Proof. Since the characteristic functions considered in this paper are of the form $f(\lambda, \tau) = q_0(\lambda) + q_1(\lambda)e^{-\tau\lambda}$, there exists only one frequency-sweeping curve Γ_1. Hence, $\Delta NF_{z_\alpha}(\omega_\alpha) \in \{-1, 0, +1\}$.

By Theorem 1, for all $\tau_{\alpha,k} > 0$, we have

$$U_{\lambda_\alpha} = \Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = \Delta NF_{z_\alpha}(\omega_\alpha). \tag{18}$$

Furthermore, according to the Lemma 1, if the multiplicity of λ_α is even, the frequency-sweeping curve Γ_1 touches, but does not cross, $\Im_1$, and hence $\Delta NFz_\alpha(\omega_\alpha) = 0$.

If the multiplicity is odd, the curve Γ_1 crosses $\Im_1$. Ordering the critical frequencies in descending order as $\omega_0 > \omega_1 > \cdots > \omega_k > 0$, continuity of the root locus implies alternating crossing directions: when k is even, $\Delta NF_{z_k}(\omega_k) = +1$ (Γ_1 crosses $\Im_1$ upward); when k is odd, $\Delta NF_{z_k}(\omega_k) = -1$ (Γ_1 crosses $\Im_1$ downward).

For (8) and (11), the multiplicity of any critical imaginary root is at most 2. If the multiplicity is 2 (even), Case (1) holds; if the multiplicity is 1 (odd), Case (2) holds. □

3.2. Specific Properties Without Control

In this subsection, we apply the frequency-sweeping approach to perform a complete stability analysis for the rolling mill main drive time-delay system (5) without control.

When $\tau = 0$, the characteristic function (8) reduces to

$$f(\lambda, 0) = \lambda^2 + \alpha_1\lambda + \alpha_4 - \alpha_5. \tag{19}$$

According to Theorem 1 in [21], the following property can be obtained.

Property 3. *For the rolling mill main drive time-delay system (5), $NU(0) = 0$ if $\alpha_1 > 0$ and $\alpha_4 > \alpha_5$; otherwise, $NU(0) = +1$ or $+2$.*

Proof. When $\tau = 0$, the system reduces to a second-order linear system (19), whose characteristic roots are determined by the quadratic equation

$$\lambda^2 + \alpha_1\lambda + (\alpha_4 - \alpha_5) = 0. \tag{20}$$

According to the Routh–Hurwitz stability criterion, all roots lie in the left half-plane if and only if $\alpha_1 > 0$ and $\alpha_4 - \alpha_5 > 0$. Otherwise, at least one root has a nonnegative real part. According to Theorem 1 in [21], $NU(0)$ can only take the values 0, $+1$, or $+2$. □

Following [27], if $\lambda = j\omega$ ($\omega > 0$) is a critical imaginary root, the characteristic function (8) can be reformulated into an auxiliary characteristic function [28]:

$$F(\omega) = \left(\mathrm{Re}((j\omega)^2 + \alpha_1 j\omega + \alpha_4)\right)^2 + \left(\mathrm{Im}((j\omega)^2 + \alpha_1 j\omega + \alpha_4)\right)^2 - (-\alpha_5)^2. \tag{21}$$

Letting $W = \omega^2$, the above auxiliary characteristic function can be further simplified as

$$F(W) = W^2 + (\alpha_1^2 - 2\alpha_4)W + (\alpha_4^2 - \alpha_5^2). \tag{22}$$

Thus, all critical frequencies $\omega = \sqrt{W}$ can be obtained by solving $F(W) = 0$ for positive real roots W.

Define the discriminant

$$\Delta = \alpha_1^4 - 4\alpha_1^2\alpha_4 + 4\alpha_5^2. \tag{23}$$

Combining Vieta's theorem with Property 2, we obtain the following property.

Property 4. *The distribution of positive real roots of the auxiliary characteristic function* (22) *exhibits the following three cases:*

(1) There are two distinct positive real roots if and only if $\Delta > 0$, $\alpha_1^2 < 2\alpha_4$, *and* $|\alpha_4| > |\alpha_5|$.
(2) There is a double positive real root if and only if $\Delta = 0$ *and* $\alpha_1^2 < 2\alpha_4$.
(3) There is only one positive real root if and only if $\Delta > 0$, $\alpha_1^2 < 2\alpha_4$, *and* $|\alpha_4| < |\alpha_5|$.

Furthermore, by combining Properties 2 and 4, the local asymptotic behavior near critical imaginary roots can be characterized.

Property 5. *For the rolling mill main drive time-delay system* (5), *the local asymptotic behavior near critical imaginary roots falls into the following three cases:*

(1) If and only if $\Delta > 0$, $\alpha_1^2 < 2\alpha_4$, *and* $|\alpha_4| > |\alpha_5|$, *the characteristic function* (8) *has two simple critical frequencies* $\omega_0 > \omega_1 > 0$, *with* $\Delta NF_{z_1}(\omega_1) = -1$ *and* $\Delta NF_{z_0}(\omega_0) = +1$.
(2) If and only if $\Delta = 0$ *and* $\alpha_1^2 < 2\alpha_4$, *the characteristic function* (8) *has one double critical frequency* $\omega_0 > 0$, *with* $\Delta NF_{z_0}(\omega_0) = 0$.
(3) If and only if $\Delta > 0$, $\alpha_1^2 < 2\alpha_4$, *and* $|\alpha_4| < |\alpha_5|$, *the characteristic function* (8) *has only one simple critical frequency* $\omega_0 > 0$, *with* $\Delta NF_{z_0}(\omega_0) = +1$.

Proof. Property 4 determines the existence and multiplicity of critical frequencies for (8). Combining this with Property 2, which establishes the relationship between the multiplicity of critical roots and $\Delta NF_{z_\alpha}(\omega_\alpha)$, the local asymptotic behavior can be determined.

For case (1), the two simple roots are ordered as $\omega_0 > \omega_1 > 0$. According to Property 2, the frequency-sweeping curve Γ_1 crosses $\Im_1$ upward at ω_0 and downward at ω_1, thus $\Delta NF_{z_0}(\omega_0) = +1$ and $\Delta NF_{z_1}(\omega_1) = -1$.

For case (2), the double critical imaginary root corresponds to the frequency-sweeping curve Γ_1 touches $\Im_1$ without crossing, hence $\Delta NF_{z_0}(\omega_0) = 0$.

For case (3), there is only one simple root. According to Property 2, the frequency-sweeping curve Γ_1 crosses $\Im_1$ upward at ω_0, leading to $\Delta NF_{z_0}(\omega_0) = +1$. □

3.3. Specific Properties with Control

This subsection investigates the rolling mill main drive time-delay system with the proposed controller (10). The frequency-sweeping approach is applied to analyze the corresponding complete stability properties.

When $\tau = 0$, the characteristic function (11) reduces to

$$f(\lambda, 0) = \lambda^2 + (\alpha_1 + k_1)\lambda + \alpha_4 + \alpha_1 k_1 - \alpha_5. \tag{24}$$

According to Theorem 1 in [21], the following property can be obtained.

Property 6. *For the rolling mill main drive time-delay system* (10), *if*

$$\alpha_1 + k_1 > 0 \quad \textit{and} \quad \alpha_4 - \alpha_5 + \alpha_1 k_1 > 0,$$

then $NU(0) = 0$. *Otherwise,* $NU(0) = +1$ *or* $+2$.

Proof. Similar to the proof of Property 3, when $\tau = 0$, the system reduces to a second-order linear system (24). By applying the Routh–Hurwitz stability criterion and Theorem 1 in [21], the property follows. □

Let $\lambda = j\omega$ ($\omega > 0$) be a critical imaginary root. Substituting it into the characteristic function (11) and separating the real and imaginary parts yields the following auxiliary function [28]:

$$\begin{aligned} F(\omega) = &(\mathrm{Re}((j\omega)^2 + (\alpha_1 + k_1)j\omega + \alpha_4 + \alpha_1 k_1))^2 + \\ &(\mathrm{Im}((j\omega)^2 + (\alpha_1 + k_1)j\omega + \alpha_4 + \alpha_1 k_1))^2 - (-\alpha_5)^2. \end{aligned} \tag{25}$$

Letting $W = \omega^2$, the above auxiliary characteristic function can be rewritten as

$$F(W) = W^2 + (\alpha_1^2 + k_1^2 - 2\alpha_4)W + (\alpha_4 + \alpha_1 k_1)^2 - \alpha_5^2. \tag{26}$$

All critical frequencies $\omega = \sqrt{W}$ can be obtained by solving $F(W) = 0$ for positive real roots W.

Define the discriminant

$$\Delta_K = (\alpha_1^2 - k_1^2)^2 - 4\alpha_4(\alpha_1 + k_1)^2 + 4\alpha_5^2. \tag{27}$$

Based on Vieta's theorem and Property 2, we directly obtain the following property.

Property 7. *The distribution of positive real roots of the auxiliary characteristic function* (26) *exhibits the following three cases:*

(1) There are two distinct positive real roots if and only if $\Delta_K > 0$, $\alpha_1^2 + k_1^2 < 2\alpha_4$, *and* $|\alpha_4 + \alpha_1 k_1| > |\alpha_5|$.

(2) There is a double positive real root if and only if $\Delta_K = 0$, $\alpha_1^2 + k_1^2 < 2\alpha_4$, *and* $(\alpha_4 + \alpha_1 k_1)^2 > \alpha_5^2$.

(3) There is only one positive real root exists if and only if $|\alpha_4 + \alpha_1 k_1| < |\alpha_5|$.

Furthermore, by combining Properties 2 and 7, we can characterize the local asymptotic behavior near critical imaginary roots.

Property 8. *For the rolling mill main drive time-delay system* (10), *the local asymptotic behavior near critical imaginary roots falls into the following three cases:*

(1) If and only if $\Delta_K > 0$, $\alpha_1^2 + k_1^2 < 2\alpha_4$, *and* $|\alpha_4 + \alpha_1 k_1| > |\alpha_5|$, *the characteristic function* (11) *has two simple critical frequencies* $\omega_0 > \omega_1 > 0$, *with* $\Delta N F_{z_1}(\omega_1) = -1$ *and* $\Delta N F_{z_0}(\omega_0) = +1$.

(2) If and only if $\Delta_K = 0$, $\alpha_1^2 + k_1^2 < 2\alpha_4$, *and* $(\alpha_4 + \alpha_1 k_1)^2 > \alpha_5^2$, *the characteristic function* (11) *has one double critical frequency* $\omega_0 > 0$, *with* $\Delta N F_{z_0}(\omega_0) = 0$.

(3) If and only if $|\alpha_4 + \alpha_1 k_1| < |\alpha_5|$, *the characteristic function* (11) *has only one simple critical frequency* $\omega_0 > 0$, *with* $\Delta N F_{z_0}(\omega_0) = +1$.

Proof. The proof is similar to that of Property 5. □

4. Case Study

In this section, we will present two numerical examples to analyze the complete stability problem of rolling mill main drive time-delay system and investigate the effect of the controller parameter k_1.

Example 1. *Consider the system (4) with parameters $\alpha_1 = \alpha_2 = \alpha_3 = 1$, $\alpha_4 = 3$, and $\alpha_5 = -4.98$. In the following, both the uncontrolled system and the controlled system are analyzed.*

For the system without control, the characteristic function (8) is given by

$$f(\lambda, \tau) = \lambda^2 + \lambda + 3 + 4.98e^{-\tau\lambda}. \tag{28}$$

When $\tau = 0$, the roots of the characteristic function (28) are $\lambda = -0.5000 \pm 2.7803j$, which lie in the open left half-plane $\mathbb{C}_-$. According to Theorem 2, the number of unstable roots is $NU(0) = 0$.

Then, the frequency-sweeping approach is employed to analytically determine the stable set of τ. The auxiliary characteristic function (22) is

$$F(W) = W^2 - 5W - 15.8004, \tag{29}$$

where $W = \omega^2$.

By calculating $F(W) = 0$, we obtain the critical frequency $\omega = \sqrt{W} = 2.6825$. That is, the critical imaginary root is $2.6825j$ and the corresponding critical delays are $\tau = 0.2121 + 2.3423k$, $k \in \mathbb{N}$. According to Property 5, $\Delta NU_{2.6825j}(0.2121 + 2.3423\mathrm{k}) = +1$ for all $k \in \mathbb{N}$.

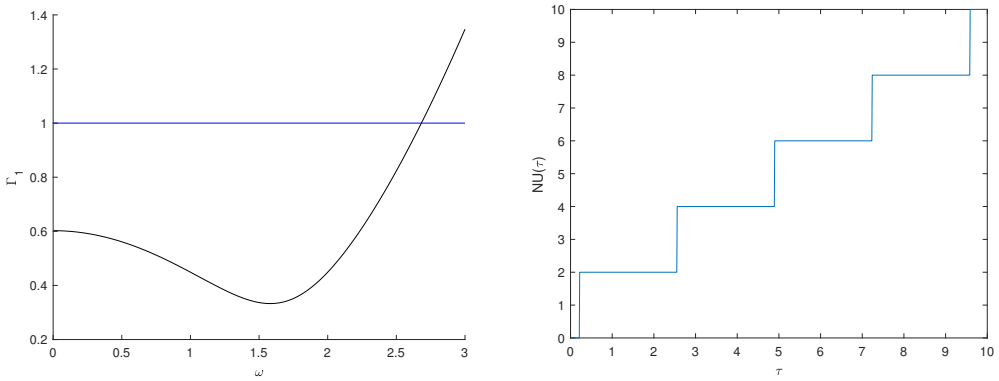
This result is illustrated by the frequency-sweeping curve shown in Figure 1a, where the curve Γ_1 crosses $\Im_1$ in the upward direction at $\omega = 2.6825$. According to Property 2, this upward crossing yields $U_{2.6825j} = +1$.

Since $NU(0) = 0$, the variation of $NU(\tau)$ with respect to τ can be explicitly determined by applying Theorem 2, as depicted in Figure 1b. It can be observed that $NU(\tau) = 0$ for $0 \leq \tau < 0.2121$. Consequently, all characteristic roots of the characteristic function (28) lie in the open left half-plane $\mathbb{C}_-$. For $\tau > 0.2121$, we have $NU(\tau) > 0$, indicating that some characteristic roots of (28) move into the right half-plane $\mathbb{C}_+$, thereby rendering the system unstable.

The above analysis leads that all roots of the characteristic function (28) lie in $\mathbb{C}_-$ if and only if

$$\tau \in [0,\ 0.2121).$$

In other words, there exists only one stability interval: $\tau \in [0, 0.2121)$.



(**a**) (**b**)

Figure 1. Frequency-sweeping curve and $NU(\tau)$ without control for Example 1. (**a**) Frequency-sweeping result. (**b**) $NU(\tau)$.

With the introduction of the controller gain k_1, the characteristic function (11) becomes

$$f(\lambda, \tau) = \lambda^2 + (1 + k_1)\lambda + 3 + k_1 + 4.98e^{-\tau\lambda}. \tag{30}$$

Since k_1 is a positive real parameter, Property 6 implies that when $\tau = 0$ the characteristic roots of (30) remain in $\mathbb{C}_-$. Hence, $NU(0) = 0$ still holds.

According to (26), the auxiliary characteristic function is given by

$$F(W) = W^2 + (k_1^2 - 5)W + k_1^2 + 2k_1 - 15.8004, \tag{31}$$

where $W = \omega^2$.

From Property 8, it follows that for $0 < k_1 < 1.98$ *the characteristic function (30) has a single critical frequency. By Theorem 2, one concludes that the system has exactly one stable interval originating from* $\tau = 0$. *In the case* $1.98 < k_1 < 2.0042$, *the characteristic function (30) has two distinct critical frequencies, yielding at least one stable interval. When* $k_1 = 2.0042$, *the characteristic function (30) exhibits a double critical frequency, and the system is delay-independent stable.*

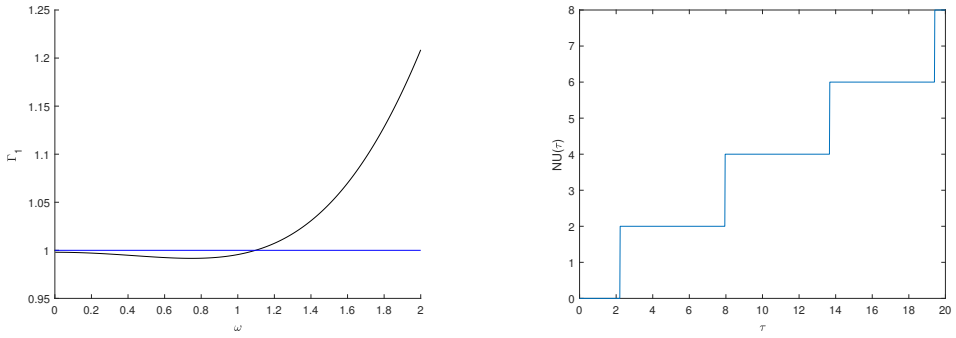
In the following, three representative cases are considered.

When $k_1 = 1.97$ *(*$k_1 = 2$*)[*$k_1 = 2.0042$*], the characteristic roots for* $\tau = 0$ *are* $\lambda = -1.5000 \pm 2.7803j$ *(*$\lambda = -1.4850 \pm 2.7829j$*) [*$\lambda = -1.5021 \pm 2.7799j$*], which are still located in* $\mathbb{C}_-$. *Hence,* $NU(0) = 0$.

For $k_1 = 1.97$, *there is a positive real characteristic root* $W_0 = 1.2019$ *from* $F(W) = 0$, *corresponding to* $\omega_0 = 1.0963$. *The associated critical delays are* $\tau = 2.2156 + 5.7312k$, $k \in \mathbb{N}$. *According to Property 8,* $\Delta NU_{1.0963j}(2.2156 + 5.7312k) = +1$ *for all* $k \in \mathbb{N}$, *which is consistent with the frequency-sweeping curve as shown in Figure 2a. Applying Theorem 2, we can obtain the variation of* $NU(\tau)$ *with respect to* τ *as shown in Figure 2b. Clearly, the system (4) is stable for*

$$\tau \in [0, 2.2156).$$

For $k_1 = 2$, *there are two positive real characteristic roots* $W_0 = 0.7245$ *and* $W_1 = 0.2755$ *from* $F(W) = 0$, *corresponding to* $\omega_0 = 0.8512$ *and* $\omega_1 = 0.5249$, *respectively.*



(**a**) (**b**)

Figure 2. Frequency-sweeping curve and $NU(\tau)$ with $k_1 = 1.97$ for Example 1. (**a**) Frequency-sweeping result. (**b**) $NU(\tau)$.

Based on Property 8, the critical delays and the corresponding local asymptotic behavior associated with each critical frequency are given as follows:

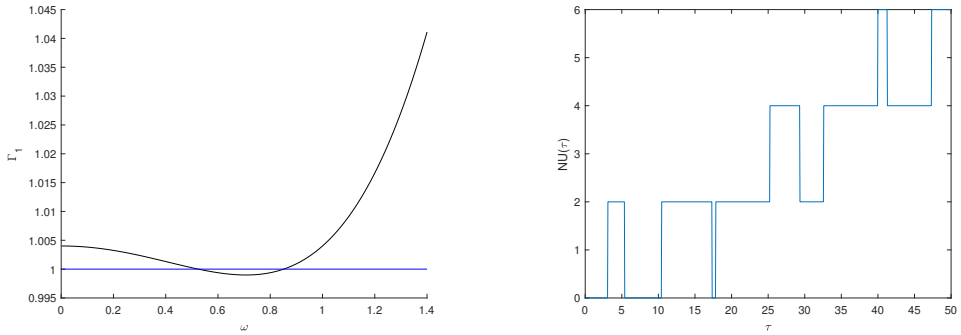
- *For* $\omega_0 = 0.8512$, *the corresponding delays are* $\tau_{0,k} = 3.0584 + 7.3818k$ *(*$k \in \mathbb{N}$*), and*

$$\Delta NU_{0.8512j}(3.0584 + 7.3818k) = +1. \tag{32}$$

- *For* $\omega_1 = 0.5249$, *the corresponding delays are* $\tau_{1,k} = 5.3724 + 11.9707k$ *(*$k \in \mathbb{N}$*), and*

$$\Delta NU_{0.5249j}(5.3724 + 11.9707k) = -1. \tag{33}$$

The frequency-sweeping curve in Figure 3a shows that Γ_1 *crosses* $\Im_1$ *downwards at* $\omega_1 = 0.5249$ *and upwards at* $\omega_0 = 0.8512$. *According to Property 8, the downward crossing yields* $U_{0.5249j} = -1$ *and the upward crossing yields* $U_{0.8512j} = +1$.



(**a**) (**b**)

Figure 3. Frequency-sweeping curve and $NU(\tau)$ with $k_1 = 2$ for Example 1. (**a**) Frequency-sweeping result. (**b**) $NU(\tau)$.

Combining Theorem 1 with (32), we obtain

$$\begin{aligned} \Delta NU_{0.8512j}(3.0584) = \Delta NU_{0.8512j}(10.4401) = \Delta NU_{0.8512j}(17.8219) \\ = \cdots = U_{0.8512j} = +1 \end{aligned} . \tag{34}$$

Similarly, combining Theorem 1 with (33), we have

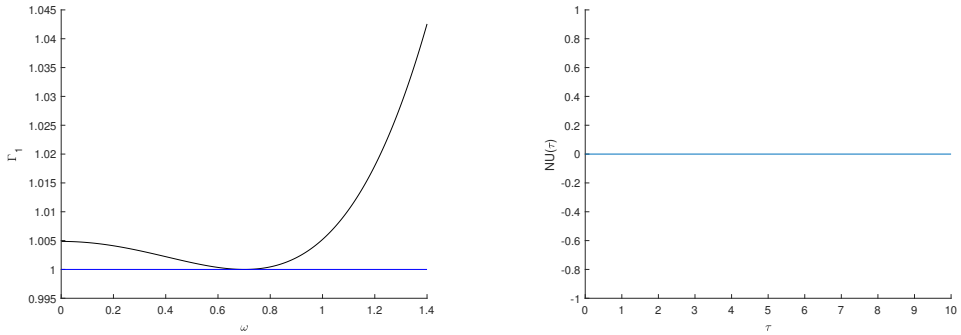
$$\Delta NU_{0.5249j}(5.3724) = \Delta NU_{0.5249j}(17.3431) = \cdots = U_{0.5249j} = -1 \tag{35}$$

According to Theorem 2, and noting that $\mathrm{NU}(0) = 0$*, the variation of* $NU(\tau)$ *with respect to* τ *can be explicitly determined from (34) and (35), as illustrated in Figure 3b.*

Consequently, all characteristic roots of the characteristic function (30) lie in the open left half-plane $\mathbb{C}_-$ *if and only if*

$$\tau \in [0, 3.0584) \cup (5.3724, 10.4401) \cup (17.3431, 17.8219).$$

For $k_1 = 2.0042$*, there is a double positive real characteristic root* $W_0 = 0.4916$*, corresponding to* $\omega_0 = 0.7011$*. Then, the critical imaginary root is* $0.7011j$ *and* $U_{0.7011j} = 0$*, which is consistent with the Frequency-sweeping curve shown in Figure 4a. According to Theorem 2, we obtain the variation of* $NU(\tau)$ *with respect to* τ *as shown in Figure 4b. Obviously,* $NU(\tau) = 0$ *for all* $\tau \geq 0$*, and the system (10) is delay-independent stable.*



(**a**) (**b**)

Figure 4. Frequency-sweeping curve and $NU(\tau)$ with $k_1 = 2.0042$ for Example 1. (**a**) Frequency-sweeping result. (**b**) $NU(\tau)$.

From the above analysis, it is clear that the controller gain k_1 *significantly affects the stability properties:*

- *In the uncontrolled case, only a single and small stability interval exists.*
- *For* $k_1 = 1.97$*, the stability interval is enlarged.*
- *For* $k_1 = 2$*, the presence of two critical frequencies leads to multiple stability intervals.*
- *For* $k_1 = 2.0042$*, the system becomes delay-independent stable.*

These results demonstrate that the controller can effectively enlarge the stability region of the system (4) and improve its delay tolerance.

Example 2. *Now consider the system (4) with parameters reset to* $\alpha_1 = -2$, $\alpha_2 = \alpha_3 = 1$, $\alpha_4 = 4$, *and* $\alpha_5 = \sqrt{3}$.

For the uncontrolled system, the characteristic function (8) is

$$f(\lambda, \tau) = \lambda^2 - 2\lambda + 4 - \sqrt{3}e^{-\tau\lambda}. \tag{36}$$

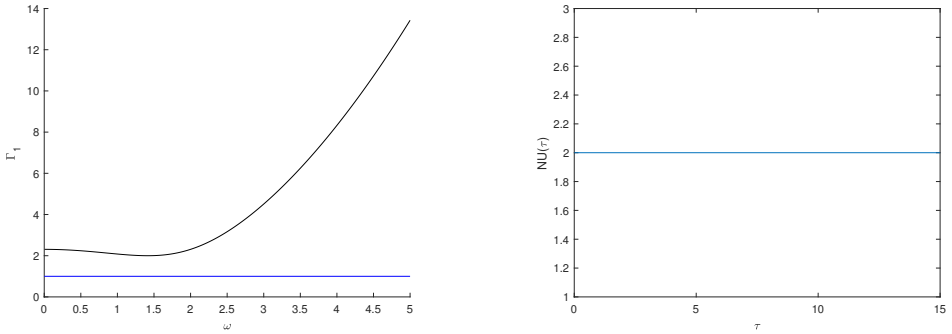
When $\tau = 0$*, the characteristic roots of (36) are* $1.0000 \pm 1.1260j$*, which lie in the open right half-plane* $\mathbb{C}_+$*. According to Theorem 2, the number of unstable roots is* $NU(0) = +2$.

Applying the frequency-sweeping approach, the auxiliary characteristic function (22) is given by

$$F(W) = W^2 - 4W + 13, \tag{37}$$

where $W = \omega^2$.

Since it has no positive real roots for $F(W) = 0$*, there are no critical imaginary root. This indicates that the frequency-sweeping curve* Γ_1 *has no intersection with* $\Im_1$*, as shown in Figure 5a. By Theorem 2, the number of unstable roots remains* $NU(\tau) = 2$ *for all* $\tau \geq 0$*, as depicted in Figure 5b. Therefore, the system is delay-independent unstable.*



(**a**) (**b**)

Figure 5. Frequency-sweeping curve and $NU(\tau)$ without control for Example 2. (**a**) Frequency-sweeping result. (**b**) $NU(\tau)$.

Now, a controller with gain k_1 *is introduced. The characteristic function (11) becomes*

$$f(\lambda, \tau) = \lambda^2 + (k_1 - 2)\lambda + 4 - 2k_1 - \sqrt{3}e^{-\tau\lambda}. \tag{38}$$

By Property 6, $NU(0) > 0$ *holds for all* $k_1 > 0$.

According to (26), the auxiliary characteristic function is:

$$F(W) = W^2 + (k_1^2 - 4)W + 4k_1^2 - 16k_1 + 13, \tag{39}$$

where $W = \omega^2$.

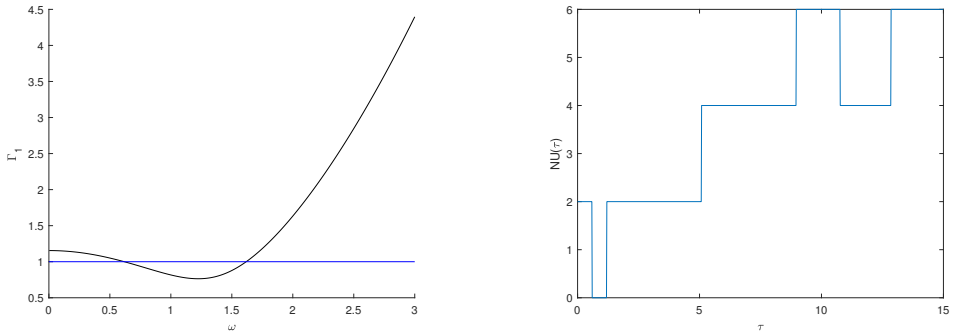
According to Property 8 and Theorem 2, the characteristic function (38) has at least one stable interval only when $0.792 < k_1 < 1.134$.

Let $k_1 = 1$. The characteristic roots for $\tau = 0$ are $\lambda = 0.5000 \pm 0.1340j$, still located in $\mathbb{C}_+$, and hence $NU(0) = +2$. There are two positive real roots $W_1 = 0.3820$ and $W_0 = 2.6180$, corresponding to critical frequencies $\omega_1 = 0.6180$ and $\omega_0 = 1.6180$, respectively.

Based on Property 8, the critical delays and their associated local asymptotic behavior are:

- *For $\omega_0 = 1.6180$: $\tau_0 = 1.1963 + 3.8832k$, $k \in \mathbb{N}$, $\Delta NU_{1.6180j}(1.1963 + 3.8832k) = +1$.*
- *For $\omega_1 = 0.6180$: $\tau_1 = 0.5904 + 10.1664k$, $k \in \mathbb{N}$, $\Delta NU_{0.6180j}(0.5904 + 10.1664k) = -1$.*

The frequency-sweeping curve in Figure 6a shows that Γ_1 crosses $\Im_1$ downwards at $\omega_1 = 0.6180$ and upwards at $\omega_0 = 1.6180$. Applying Theorem 2, the explicit expression of $NU(\tau)$ with respect to τ is illustrated in Figure 6b.



(**a**) (**b**)

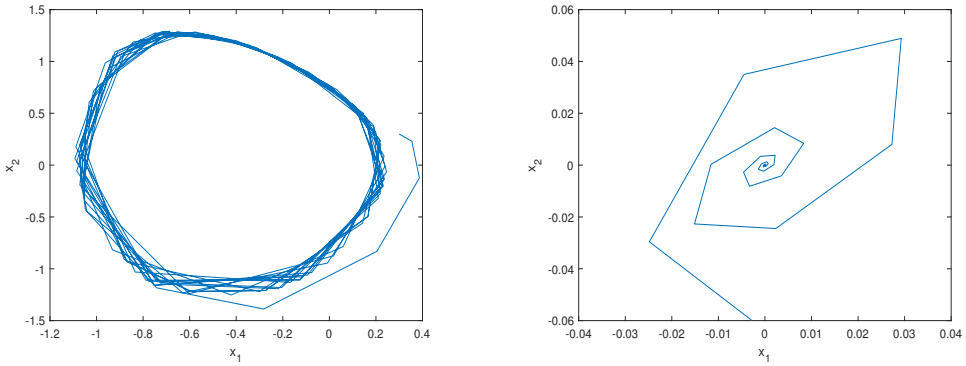
Figure 6. Frequency-sweeping curve and $NU(\tau)$ with $k_1 = 1$ for Example 2. (**a**) Frequency-sweeping result. (**b**) $NU(\tau)$.

Consequently, all characteristic roots of the characteristic function (38) lie in $\mathbb{C}_-$ if and only if

$$\tau \in (0.5904, 1.1963).$$

Compared with the uncontrolled system, the introduction of the controller ($k_1 = 1$ and $k_2 = k_3 = 0.01$) renders the system asymptotically stable for $\tau \in (0.5904, 1.1963)$.

To verify the above result, we simulate system (4) with $\tau = 0.8933$ under both uncontrolled and controlled cases. Figure 7a shows the phase portrait without control, where the trajectory does not converge to the equilibrium $(0, 0)$. In contrast, Figure 7b shows the phase portrait with controller $k_1 = 1$ and $k_2 = k_3 = 0.01$, where the trajectory converges to the equilibrium $(0, 0)$. The initial condition is $x_1(t) = 0.3$ for $t \in [-\tau, 0]$ and $x_2(0) = 0.3$. These results validate the effectiveness of the controller in stabilizing the unstable system (4).



(**a**) (**b**)

Figure 7. Phase portraits when $\tau = 0.8933$ without and with control for Example 2. (**a**) Without control. (**b**) With control.

5. Conclusions

In this paper, we have investigated the complete stability and stabilization problems of a rolling mill main drive system with time delay. By employing the frequency-sweeping approach, the stability intervals with respect to τ have been explicitly characterized for both the uncontrolled and controlled systems. The main contributions and findings are summarized as follows.

Existence and structure of stability intervals. This paper revealed the existence of multiple stability intervals, which may or may not include $\tau = 0$. It can be seen that increasing the delay does not necessarily destabilize the system, and stability may be regained after certain critical delays.

Controller parameter design. Instead of analyzing a fixed gain, this paper determines the admissible range of the controller gain k_1 that ensures the existence of stability intervals. It is shown that the controller can significantly enlarge the stability region and improve delay tolerance. Such a priori characterization provides practical guidance for controller tuning and reduces the need for trial-and-error procedures in implementation.

Theoretical contribution and applicability. The frequency-sweeping approach adopted in this paper offers an effective analytical perspective for stability analysis of rolling mill main drive time-delay systems. It offers explicit insight into the relationship between controller gain and time delay. The method can be extended to more general characteristic functions with multiple delays and to higher-dimensional systems.

Limitations and future work. The present study is primarily theoretical and does not include experimental validation on physical systems. In addition, the analytical procedure may become increasingly complex for higher-dimensional systems or for systems with multiple delays. Future work will focus on extending the proposed approach to more complex delay structures and developing computational tools to facilitate practical implementation.

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