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A Three-Dimensional Product-Based Circular Intuitionistic Fuzzy Potential Method for Transportation Problems

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Abstract

Transportation problems constitute a fundamental class of optimization models; however, real-world applications involve uncertainty, hesitation, and expert disagreement that cannot be adequately captured by deterministic or classical fuzzy approaches. This paper proposes a three-dimensional circular intuitionistic fuzzy potential method (3D–CIFMODI), which extends the classical MODI framework to Circular Intuitionistic Fuzzy Triples (C-IFTs) through radius-aware operations and indexed matrix representations. Unlike existing circular intuitionistic fuzzy transportation methods, which are primarily feasibility-driven, the proposed approach introduces a dual-based optimality framework based on circular reduced costs, preserving the full structure of uncertainty without reducing it to crisp equivalents. The method retains polynomial-time computational complexity $O(mn(m+n))$, i.e., $O(n^3)$ for square problems, with only a constant computational overhead due to circular operations. A numerical case study demonstrates the effectiveness and robustness of the proposed framework. Furthermore, a comparative analysis between classical intuitionistic fuzzy (IFS) and circular intuitionistic fuzzy (C-IFS) representations shows that incorporating the radius parameter significantly improves discrimination capability, solution stability, and interpretability. The results confirm that the proposed method provides a unified, interpretable, and computationally efficient framework for solving multi-layer transportation problems under circular intuitionistic fuzzy uncertainty.

Keywords: circular intuitionistic fuzzy sets; index matrices; potential method; three-dimensional optimization; transportation problem

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1. Introduction

Transportation problems (TPs) [1] constitute a fundamental class of network flow optimization models and play a central role in logistics planning, production–distribution coordination, and supply chain management. While their classical deterministic formulations are mathematically elegant and computationally efficient, contemporary transportation systems operate under structural, informational, and organizational complexities that challenge the assumption of precise data.

The origins of the transportation problem trace back to Monge's optimal mass transfer formulation (1781), later formalized by Tolstoy [2] and Hitchcock [1], and further developed by Koopmans [3]. A major methodological milestone was Kantorovich's introduction of

the potential (MODI) method [4], which established a dual-based optimality framework for transportation problems. Dantzig showed that the transportation problem can be efficiently solved as a special case of linear programming via the simplex method [5], while alternative procedures such as the stepping-stone method [6] further enriched the solution toolbox. Extensions toward bounded, capacitated, and multi-index transportation models followed [7–9]. Modern transportation systems are inherently multi-dimensional. Modern transportation systems are inherently multi-dimensional. In many industrial applications, distribution decisions involve multiple product types, each characterized by distinct supply structures, cost behaviors, and demand patterns. Such settings naturally lead to three-dimensional transportation problems, where flows are indexed by origin, destination, and product. In this study, the third dimension explicitly represents the product structure.

A second major challenge concerns uncertainty. Transportation costs, supplies, and demands are frequently influenced by expert assessments, incomplete information, and volatile market conditions. Intuitionistic fuzzy sets (IFSs) [10,11] provide a rigorous framework for modeling such uncertainty through simultaneous degrees of membership, non-membership, and hesitation. Circular intuitionistic fuzzy (C-IF) representations [12,13] further extend this framework by introducing a geometric radius parameter that captures dispersion around the central evaluation, enabling explicit modeling of expert disagreement. In parallel with developments in uncertainty modeling, algorithmic approaches to transportation problems follow two main directions. Zero point-based methods [14] generate solutions through iterative allocation procedures and are primarily feasibility-oriented, while potential-based methods [4,15] rely on reduced costs and dual variables, providing a transparent interpretation of optimality.

While both approaches are well established in deterministic settings, their extension to circular intuitionistic fuzzy environments remains limited. Recent studies introduce zero point methods in circular intuitionistic fuzzy settings; however, these approaches remain allocation-based and do not provide a dual-based optimality framework. Recent research confirms that transportation problems under uncertainty remain an active and evolving area of investigation. Contemporary studies increasingly focus on advanced intuitionistic fuzzy models, hybrid optimization approaches, and enhanced uncertainty representations. In particular, recent works on circular intuitionistic fuzzy sets and decision-making frameworks [16–20] demonstrate a clear shift toward richer and more informative modeling of uncertainty. These developments highlight the ongoing relevance of transportation optimization under uncertainty and further justify the need for structured and interpretable methods such as the one proposed in this study.

To the best of our knowledge, a potential-based (MODI-type) method has not yet been developed in a circular intuitionistic fuzzy environment, particularly for three-dimensional transportation problems. Circular intuitionistic fuzzy sets (C-IFSs) have been widely applied in various domains beyond transportation, including multi-criteria decision-making (MCDM), artificial intelligence, and uncertainty modeling. C-IFSs [13] generalize classical intuitionistic fuzzy representations by introducing a geometric radius parameter that captures uncertainty dispersion and expert hesitation. Modern transportation systems are inherently multi-dimensional. In many industrial applications, distribution decisions involve multiple product types, each characterized by distinct supply structures, cost behaviors, and demand patterns. Such settings naturally lead to three-dimensional transportation problems, where flows are indexed by origin, destination, and product. In this study, the third dimension explicitly represents the product structure.

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Despite these advances, existing circular intuitionistic fuzzy transportation approaches remain predominantly feasibility-driven or ranking-based. As a consequence, they do not provide reduced cost optimality conditions or a dual interpretation comparable to classical transportation models. This limitation creates a clear need for a framework that integrates circular intuitionistic fuzzy modeling with a potential-based optimality mechanism.

To address this gap, this paper proposes a *three-dimensional product-based circular intuitionistic fuzzy potential method* (3D–CIFMODI). The proposed method extends the classical MODI framework to a circular intuitionistic fuzzy setting and preserves the dual-based structure of optimality while explicitly incorporating hesitation through radius-aware operations. The effectiveness of the proposed framework is demonstrated through a detailed numerical case study, highlighting its computational transparency, interpretability of circular reduced costs, and the impact of hesitation radii on optimal allocation structures. In addition, a comparative analysis between classical intuitionistic fuzzy (IFS) and circular intuitionistic fuzzy (C-IFS) representations is conducted. The results show that, although both approaches share identical central evaluations, the inclusion of the radius parameter in C-IFS enables differentiation between alternatives with similar membership and non-membership degrees, leading to improved discrimination power, more stable optimization behavior, and enhanced interpretability of the final transportation plan.

Overall, the proposed method provides the first potential-based optimization framework in a circular intuitionistic fuzzy environment, offering improved interpretability, consistency, and theoretical transparency compared to existing feasibility-driven approaches. This work introduces the first potential-based optimization framework for three-dimensional transportation problems under circular intuitionistic fuzzy uncertainty,

establishing dual-based optimality conditions and reduced cost interpretation in a setting where existing methods remain primarily feasibility-driven.

To clearly present the proposed framework and its evaluation, the structure of the paper is organized as follows. Section 2 identifies the research gap and motivates the need for a three-dimensional product-based circular intuitionistic fuzzy potential method. Section 3 reviews related work on transportation problems under uncertainty. Section 5 introduces the basic definitions of circular intuitionistic fuzzy logic and the index matrix apparatus. Section 6 formulates the product-based three-dimensional transportation model under C-IF parameters and presents the proposed 3D–CIFMODI algorithm. Section 7 reports the numerical results. Section 8 discusses the implications of the findings and compares the proposed method with feasibility-oriented approaches. Finally, Section 9 concludes the paper and outlines directions for future research.

2. Research Gap

Transportation problems under uncertainty have been extensively studied within fuzzy and intuitionistic fuzzy optimization frameworks. Existing approaches include ranking-based methods, heuristic allocation procedures, and feasibility-driven techniques such as the zero point [14,26] and zero suffix [26,27] methods. Several studies further extend these approaches to multi-stage and three-dimensional transportation problems using indexed matrix representations. Recent developments introduce geometric intuitionistic fuzzy extensions, including circular and elliptic intuitionistic fuzzy sets, which enable a richer representation of hesitation and uncertainty dispersion in expert assessments. In particular, indexed matrix formulations have demonstrated effectiveness in handling high-dimensional transportation problems under such uncertainty environments.

Despite these advances, a clear methodological gap remains. Current approaches to transportation problems under circular intuitionistic fuzzy uncertainty are predominantly feasibility-driven or rely on ranking and aggregation techniques, rather than on optimality-based frameworks. As a result, they do not provide a dual interpretation of optimality analogous to classical transportation models. To the best of our knowledge, no study has developed a *potential-based (MODI-type) optimization method* for a *product-based three-dimensional transportation problem* in a circular intuitionistic fuzzy environment.

Specifically, the literature lacks:

- Circular intuitionistic fuzzy potential variables consistent with dual-based optimization;
- Reduced cost optimality conditions adapted to circular intuitionistic fuzzy environments;
- A unified framework that preserves both uncertainty information and dual interpretability in multi-layer transportation problems.

While feasibility-oriented methods such as zero point approaches are effective for constructing admissible solutions, they do not provide insight into marginal cost structure or optimality behavior. In contrast, potential-based methods enable a dual interpretation through reduced costs, offering a deeper understanding of route contributions and allocation stability. The present study addresses this gap by proposing a *three-dimensional product-based circular intuitionistic fuzzy potential method*. The proposed framework extends the classical MODI methodology to circular intuitionistic fuzzy environments using indexed matrix structures, thereby establishing explicit optimality conditions and preserving the dual-based logic of transportation optimization under uncertainty.

3. Review of Literature on Transportation Problems Under Uncertainty

Transportation problems (TPs) constitute a fundamental class of linear programming models concerned with the cost-efficient allocation of flows between multiple sources and destinations. The classical deterministic formulation was established by Hitchcock [1] and

further developed by Kantorovich [4,15] and Dantzig [5], leading to dual-based optimality procedures such as the potential (MODI) method and the stepping-stone technique.

In practical environments, transportation parameters are rarely known with certainty. The introduction of fuzzy set theory by Zadeh [28] enabled the modeling of imprecise costs, supplies, and demands. Early fuzzy transportation formulations were developed in [29–31], followed by ranking-based and fuzzy programming approaches in [32,33]. Trapezoidal fuzzy transportation models were studied in [34,35]. A feasibility-oriented alternative, the zero point method (ZPM), was introduced in [36], while additional allocation-based fuzzy procedures were proposed in [37–40].

The introduction of intuitionistic fuzzy sets by Atanassov [10] allowed for explicit modeling of hesitation via independent membership and non-membership degrees. Intuitionistic fuzzy transportation problems (IFTPs) were investigated under triangular and trapezoidal intuitionistic fuzzy numbers in [41–43]. Approximation-based and heuristic methods, including Vogel-type approaches [44] and PSK-based procedures [45], were subsequently developed. Several zero suffix and zero point variants were proposed for IFTPs in [14,46,47].

A structured representation of multi-stage and three-dimensional IFTPs was achieved through indexed matrix (IM) formulations. Within this framework, the Intuitionistic Fuzzy Zero Point Method (IFZPM) and the Intuitionistic Fuzzy Zero Suffix Method (IFZSM) were developed in [27,48–50], extending feasibility-driven optimization to layered transportation models. Recent research has focused on geometric extensions of intuitionistic fuzzy sets. These include hesitant fuzzy sets [51], Pythagorean and Fermatean fuzzy sets [52], spherical fuzzy sets [53], circular intuitionistic fuzzy sets (C-IFSs) [12,13], and elliptic intuitionistic fuzzy sets (E-IFSs) [54]. Circular IFSs represent symmetric dispersion of uncertainty, whereas elliptic IFSs enable anisotropic modeling. Comparative analyses [55] show that many advanced fuzzy models can be embedded within the IFS framework, including interval-valued IFSs [56], picture fuzzy sets [57], cubic sets [58], neutrosophic sets [59], and support-intuitionistic fuzzy sets [60]. In addition to transportation-related applications, circular intuitionistic fuzzy sets (C-IFSs) have been widely employed in other domains, particularly in multi-criteria decision-making (MCDM) and intelligent systems. C-IFS-based TOPSIS methods have been applied to supplier selection problems under uncertainty, incorporating vague membership structures and group decision environments [21,22]. Similarly, ELECTRE III and AHP–VIKOR frameworks have been extended within the circular intuitionistic fuzzy setting to support complex multi-expert decision-making processes [23,24]. Recent studies also demonstrate the integration of C-IFS models into AI-driven frameworks and advanced computational approaches, including healthcare supply chain optimization and intelligent decision systems [16–18]. These contributions highlight the versatility of circular intuitionistic fuzzy modeling beyond transportation contexts.

Very recent contributions illustrate current methodological directions. A hybrid multi-objective framework for fully intuitionistic fuzzy transportation problems was proposed in [20], avoiding ranking functions. Type-2 trapezoidal intuitionistic fuzzy fractional transportation models were developed in [61]. Fermatean fuzzy formulations for sustainable logistics were introduced in [62]. Distance measures for circular IFSs were studied in [63], providing theoretical support for geometric intuitionistic fuzzy structures.

Despite these developments, existing transportation models under circular intuitionistic fuzzy uncertainty are predominantly feasibility-oriented and rely on ranking or aggregation mechanisms. To the best of our knowledge, no potential-based (MODI-type) optimization framework has been formulated for product-based three-dimensional transportation problems under circular intuitionistic fuzzy uncertainty. In particular, the literature lacks:

- Dual-type circular intuitionistic fuzzy potential variables;

- Reduced cost optimality conditions defined in a circular intuitionistic fuzzy algebra;
- A systematic extension of the MODI method to layered transportation structures under geometric intuitionistic fuzzy uncertainty.

The present study addresses this gap by introducing a three-dimensional C-IF potential method based on indexed matrix representations. The proposed framework preserves the dual optimality logic of the classical MODI method while extending it to a C-IF environment, thereby enabling consistent layer-wise optimality analysis under symmetric uncertainty dispersion.

4. Comparative Analysis

This section presents a comparative evaluation of the proposed method with representative approaches for solving intuitionistic fuzzy transportation problems (IFTPs). The comparison focuses on methodological structure, optimality interpretation, and computational characteristics.

Table 1 summarizes the key differences.

Table 1. Comparative analysis of representative transportation methods under uncertainty.

Method	Uncertainty Type	Approach Type	3D Structure	Optimality
IFZPM (Zero Point)	Intuitionistic fuzzy	Allocation-based	Yes	Yes (procedural)
IFZSM (Zero Suffix)	Intuitionistic fuzzy	Allocation-based	Yes	Yes (procedural)
Heuristic/Vogel-type	Intuitionistic fuzzy	Heuristic	Limited	No
Classical MODI	Crisp	Potential-based	Limited	Yes (dual-based)
Proposed 3D–CIFMODI	Circular IF	Potential-based	Yes	Yes (dual-based)

The comparison highlights several key observations.

First, existing intuitionistic fuzzy transportation methods such as IFZPM [14] and IFZSM [27] are primarily allocation-based. Although these methods can generate optimal solutions, the notion of optimality is procedural, i.e., obtained through iterative allocation rules rather than derived from an explicit dual-based framework. Consequently, they do not provide reduced cost evaluation or potential variables, which limits interpretability and marginal cost analysis.

Second, heuristic approaches, including Vogel-type and approximation-based methods [44,45], improve initial feasible solutions but lack a rigorous optimality structure. In many cases, these methods rely on ranking or transformation procedures that reduce fuzzy or intuitionistic fuzzy data to crisp equivalents, leading to a loss of essential uncertainty information, particularly hesitation and dispersion.

Third, recent extensions of IFTPs, including multi-objective, type-2, and Fermatean fuzzy models [20,61,62], enhance the representation of uncertainty. However, these approaches remain predominantly aggregation- or transformation-based and do not preserve dual-based optimality structures. As a result, they do not provide a transparent marginal interpretation of transportation decisions. To the best of our knowledge, no existing study formulates transportation problems under circular intuitionistic fuzzy representations within a potential-based optimization framework. Related work [64] considers elliptic intuitionistic fuzzy models in multi-dimensional settings, but remains feasibility-driven and does not incorporate reduced cost optimality mechanisms.

In contrast, the proposed 3D–CIFMODI method operates entirely within the circular intuitionistic fuzzy domain and preserves the dual-based optimality structure of the classical MODI method. The algorithm introduces circular reduced costs and integrates hesitation through radius-aware operations without transforming the problem into a crisp equiva-

lent. From a computational perspective, the proposed method follows the same iterative structure as classical transportation algorithms. Each iteration requires $O(mn)$ operations for reduced cost evaluation and $O(m+n)$ operations for cycle adjustment, resulting in an overall complexity of $O(mn(m+n))$, i.e., $O(n^3)$ for square problems. The inclusion of circular intuitionistic fuzzy parameters introduces only a constant computational overhead.

Beyond computational efficiency, the proposed framework significantly improves interpretability and decision support. Unlike ranking-based approaches, where uncertainty is reduced to scalar values, the circular intuitionistic fuzzy representation preserves membership, non-membership, and hesitation information throughout the optimization process. This enables explicit analysis of uncertainty dispersion and its impact on allocation strategies. Furthermore, the introduction of circular reduced costs provides a transparent marginal interpretation of optimality under uncertainty, analogous to classical dual variables but enriched with hesitation-aware information.

Overall, the proposed method combines:

- Feasibility and admissibility;
- Dual-based optimality interpretation;
- Three-dimensional modeling capability;
- Explicit handling of circular intuitionistic fuzzy uncertainty.

This demonstrates that the proposed framework achieves computational efficiency comparable to classical methods while offering superior modeling capability, interpretability, and robustness compared to existing allocation-based and ranking-driven approaches.

5. Basic Definitions of Circular Intuitionistic Fuzzy Logic and Index Matrices

5.1. Circular Intuitionistic Fuzzy Sets (C-IFSs)

Circular intuitionistic fuzzy sets (C-IFSs), introduced by Atanassov in 2020 [13], constitute one of the most recent geometrically grounded extensions of intuitionistic fuzzy set theory [10,65]. This framework enriches the classical intuitionistic fuzzy representation by embedding uncertainty into a two-dimensional geometric domain, where each element is characterized by its membership and non-membership degrees located inside a circular region contained within the intuitionistic fuzzy triangle.

Let A be a subset of a fixed universe E. A circular intuitionistic fuzzy set is defined by

$$A_u^* = \{\langle x, \mu_A(x), \nu_A(x); u\rangle \mid x \in E\},$$

where $\mu_A(x) + \nu_A(x) \leq 1$ and $u \in [0, \sqrt{2}]$ denotes the radius associated with the circular uncertainty region corresponding to the element $x \in E$. The functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ represent, respectively, the membership and non-membership degrees of x in the set A, while the indeterminacy degree is given by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$.

The circular region $E_{u,v}(\mu_A(x), \nu_A(x))$ induces a continuous geometric structure that captures the interdependence between membership and non-membership components, allowing uncertainty to be expressed in a more flexible and realistic manner compared to the linear representation of classical intuitionistic fuzzy sets.

Within the C-IFS framework, algebraic and relational properties of these components can be consistently formulated using index matrix representations [66]. The matrix-based organization of $\mu_A(x)$, $\nu_A(x)$, and $\pi_A(x)$ supports systematic aggregation, comparison, and optimization operations, while preserving both the geometric intuition of circular modeling and the analytical rigor inherent in matrix-oriented reasoning. Consequently, the integration of circular intuitionistic fuzzy sets with index matrices provides a uni-

fied mathematical platform for handling correlated uncertainty in decision-making and evaluation processes.

5.2. Remarks on Circular Intuitionistic Fuzzy Triples (C-IFTs)

Circular intuitionistic fuzzy sets extend classical IFSs by associating an additional radius parameter $r \in [0, \sqrt{2}]$ with each pair $\langle \mu, \nu \rangle$ [13]. This radius defines a circular neighborhood around the point (μ, ν) in the unit square and reflects hesitation, dispersion of expert opinions, and local uncertainty effects.

Let us define zero-centered C-IF triple (Z-C C-IFT) $a = 0_r = \langle 0, 1; r \rangle$, $r \geq 0$ and fully zero C-IF triple $a = 0 = \langle 0, 1; 0 \rangle$.

A Circular Intuitionistic Fuzzy Triple (C-IFT) evaluating a proposition p is denoted by

$$x = \langle \mu(p), \nu(p); r \rangle, \qquad \mu, \nu \in [0, 1], \quad \mu + \nu \leq 1.$$

For two C-IFTs, $x = \langle a, b; r_1 \rangle$ and $y = \langle c, d; r_2 \rangle$, the principal operations employed in this study, following [12,13], are defined as:

$$x \vee_{1*} y = \langle \max(a, c), \min(b, d); *(r_1, r_2) \rangle,$$

$$x \wedge_{2*} y = \langle a + c - ac,\ bd; *(r_1, r_2) \rangle,$$

$$x \vee_{2*} y = \langle ac,\ b + d - bd; *(r_1, r_2) \rangle,$$

$$x @_{*} y = \langle (a + c)/2,\ (b + d)/2; *(r_1, r_2) \rangle,$$

$$x -_{*} y = \langle \max(0, a - c),\ \min(1, b + d, 1 - a + c); *(r_1, r_2) \rangle.$$

A partial order relation between two Circular Intuitionistic Fuzzy Triples is defined as

$$x \geq y \quad \Longleftrightarrow \quad a \geq c,\ b \leq d,\ r_1 \geq r_2.$$

Furthermore, a ranking function based on the distance to the ideal positive C-IFT $\langle 1, 0; \sqrt{2} \rangle$ is introduced in [12] as

$$R^{\text{circ}}_{\langle a,b;r \rangle} = 0.25\,(2 - a - b)\left(\frac{|\sqrt{2} - r|}{\sqrt{2}} + 1 - a \right),$$

where a C-IFT x is preferred to y whenever $R^{\text{circ}}_x \leq R^{\text{circ}}_y$.

5.3. Three-Dimensional Circular Intuitionistic Fuzzy Index Matrices (3-D C-IFIMs)

The concept of IMs was initially introduced in 1987 as a formal mechanism for combining matrices of different dimensions [67]. A three-dimensional circular intuitionistic fuzzy index matrix (3-D C-IFIM) is defined as [12]

$$A = [K, L, H, \{ \langle \mu_{k_i, l_j, h_g}, \nu_{k_i, l_j, h_g}; r_{k_i, l_j, h_g} \rangle \}],$$

where $K, L, H \subset \mathcal{I}$ are index sets and each entry corresponds to a C-IFT. The matrix structure is given by

$h_g \in H$	l_1	$\ldots$	l_n
k_1	$\langle \mu_{k_1,l_1,h_g}, \nu_{k_1,l_1,h_g}; r_{k_1,l_1,h_g} \rangle$	$\ldots$	$\langle \mu_{k_1,l_n,h_g}, \nu_{k_1,l_n,h_g}; r_{k_1,l_n,h_g} \rangle$
$\vdots$	$\vdots$	$\ldots$	$\vdots$
k_m	$\langle \mu_{k_m,l_1,h_g}, \nu_{k_m,l_1,h_g}; r_{k_m,l_1,h_g} \rangle$	$\ldots$	$\langle \mu_{k_m,l_n,h_g}, \nu_{k_m,l_n,h_g}; r_{k_m,l_n,h_g} \rangle$

Let $* \in \{\min, \max\}$ denote a fixed combining operator over circular radii, selected a priori by the decision-maker. If $* = \max$, the aggregation is pessimistic and preserves maximal uncertainty dispersion, suitable for high variability or risk-averse decision-making, whereas for $* = \min$, the aggregation is optimistic and preserves minimal uncertainty dispersion, corresponding to stable environments or risk-tolerant decision strategies. Let

$$x = \langle \mu_x, \nu_x; r_x \rangle, \qquad y = \langle \mu_y, \nu_y; r_y \rangle$$

be two Circular Intuitionistic Fuzzy Triples (C-IFTs). The operators $\circ_1$ and $\circ_2$ act componentwise on the first (membership) and second (non-membership) components of a C-IFT, respectively, while the operator $*$ combines the corresponding radii.

A comprehensive collection of operations for three-dimensional circular intuitionistic fuzzy index matrices (3-D C-IFIMs) has been established in the literature [12,66,68,69], including the following fundamental transformations and algebraic constructions:

- Transposition: The transpose of a 3-D C-IFIM A is denoted by A'.
- Negation:
$$\neg A = [K, L, H, \{\langle \nu_{k_i,l_j,h_g}, \mu_{k_i,l_j,h_g}; r_{k_i,l_j,h_g} \rangle\}],$$
where membership and non-membership degrees are interchanged while preserving the radius parameter.
- Addition $(\circ_1, \circ_2, *)$: $A \oplus_{(\circ_1,\circ_2,*)} B = [K \cup P, L \cup Q, H \cup R, \{\langle \phi, \psi; \eta \rangle\}]$, with the resulting components obtained through elementwise application of the selected operators.
- $(\circ_1, \circ_2, *)$:
$$A \odot_{(\circ_1,\circ_2,*)} B = [K \cup (P - L), Q \cup (L - P), H \cup R, \{\langle \phi, \psi; \eta \rangle\}].$$
- Termwise subtraction:
$$A -_{(\max,\min,*)} B = A \oplus_{(\max,\min,*)} \neg B,$$
defined via addition combined with negation under the specified operator triple.
- Termwise multiplication $(\circ_1, \circ_2, *)$:
$$A \otimes_{(\circ_1,\circ_2,*)} B = [K \cap P, L \cap Q, H \cup R, \{\langle \phi, \psi; \eta \rangle\}].$$
- Reduction: Removal of a selected row, column, or entire dimension of the matrix, denoted, for instance, by $A_{(k,\perp,\perp)}$, where the symbol "$\perp$" indicates a suppressed index.
- Projection: For subsets $M \subseteq K$, $N \subseteq L$, and $U \subseteq H$, the projection of A is given by
$$pr_{M,N,U}(A) = [M, N, U, \{b_{k_i,l_j,h_g}\}],$$
with $b_{k_i,l_j,h_g} = a_{k_i,l_j,h_g}$.
- Substitution: Replacement of an index k by another index p, denoted by
$$\left[\tfrac{p}{k}; \perp; \perp\right] A.$$
- Aggregation [12,68]: Three aggregation operators $\#_1$, $\#_2$, and $\#_3$ are defined for two C-IFTs as
$$x\#_1 y = \langle \min(a, c), \max(b, d); *(r_1, r_2) \rangle,$$
$$x\#_2 y = \langle average(a, c), average(b, d); *(r_1, r_2) \rangle,$$
$$x\#_3 y = \langle \max(a, c), \min(b, d); *(r_1, r_2) \rangle,$$

corresponding respectively to pessimistic, averaged, and optimistic aggregation strategies.

- Aggregate Global Internal Operation (AGIO) [69]: A global operation that applies a chosen aggregation rule uniformly to all entries of a 3-D C-IFIM.

In addition, the following internal and index-based operations are defined:

- Internal subtraction of index matrix components [70]:

$$IO_{-_{(\max,\min,*)}}\left(\langle k_i, l_j, A\rangle, \langle p_r, q_s, B\rangle\right) = [K, L, \{\langle \gamma_{t_u,v_w}, \delta_{t_u,v_w}\rangle\}],$$

where the resulting intuitionistic fuzzy components are computed according to the specified internal subtraction rule.

- Index-type and aggregation operations [12]:
The proposed 3D–CIFMODI method relies on index-type and aggregation operations defined over circular intuitionistic fuzzy index matrices (C-IFIMs). These operations extend the index matrix framework introduced in [69,70] to the C-IF setting by replacing numerical comparison with the circular ranking relation R^{circ}.

Index-type operations.

Let

$$A = [a_{k_i,l_j}]$$

be a circular intuitionistic fuzzy index matrix (C-IFIM), where each admissible entry is a C-IFT

$$a_{k_i,l_j} = \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}; r_{k_i,l_j}\rangle,$$

$$Index_{\not\perp}(A) = \{\langle k_i, l_j\rangle \mid a_{k_i,l_j} \neq \perp\}, \qquad Index_{\perp}(A) = \{\langle k_i, l_j\rangle \mid a_{k_i,l_j} = \perp\}.$$

Global extremal index (excluding empty entries).

For $\Omega^{circ} \in \{\min_{R^{circ}}, \max_{R^{circ}}\}$, define

$$AGIndex_{\Omega^{circ,*},(\not\perp)}(A) = \langle k_i, l_j\rangle,$$

where the returned index attains the extremal value of $R^{circ}(a_{k_i,l_j})$ among all admissible entries $a_{k_i,l_j} \neq \perp$.

Reference zero element.

We define here the *fully zero circular intuitionistic fuzzy element* as

$$0_{\mathrm{CIF}} := \langle 0, 1; 0\rangle.$$

It represents a zero-centered C-IFT with no uncertainty and serves as the unique reference element in all rank-based comparisons.

Admissibility index sets.

$$Index_{\not\perp}(A) = \{\langle k_i, l_j\rangle \mid a_{k_i,l_j} \neq \perp\}, \qquad Index_{\perp}(A) = \{\langle k_i, l_j\rangle \mid a_{k_i,l_j} = \perp\}.$$

Zero-centered index set.

$$Index_{\mathrm{zc}}(A) = \{\langle k_i, l_j\rangle \mid a_{k_i,l_j} = \langle 0, 1; r\rangle,\ r \geq 0\}.$$

Fully zero index set.

$$Index_0(A) = \{\langle k_i, l_j\rangle \mid a_{k_i,l_j} = 0_{\mathrm{CIF}}\}.$$

Improvement index set.

$$Index_{\mathrm{imp},\not\perp}(A) = \{\langle k_i, l_j\rangle \mid a_{k_i,l_j} \neq \perp,\ R^{circ}(a_{k_i,l_j}) \prec R^{circ}(\mathbf{0}_{\mathrm{CIF}})\}.$$

Remark 1. *Due to the adopted circular intuitionistic fuzzy subtraction, elements with circular rank worse than* $\mathbf{0}_{\mathrm{CIF}}$ *do not exist. All non-improving reduced costs are represented by zero-centered triples, while improving directions are identified exclusively by* $Index_{\mathrm{imp}}(A)$.

Aggregate Global Internal Operation.

Following [69,70], the Aggregate Global Internal Operation (AGIO) is defined for a C-IFIM A [12] as

$$AGIO_{\oplus_{(\#_q,*)}}(A),$$

where $\oplus$ is the internal aggregation operator in the circular intuitionistic fuzzy algebra, $\#_q$ specifies the aggregation scenario, and $*$ governs the aggregation of the circular radii.

The parameter q reflects the decision-maker's attitude:

- $q = 1$: Pessimistic aggregation;
- $q = 2$: Averaged (realistic) aggregation;
- $q = 3$: Optimistic aggregation.

Inclusion by value for C-IFIMs.

Let A and B be two compatible C-IFIMs. We say that A is included in B by value, denoted as

$$A \subseteq_v B,$$

if

$$R^{circ}(a_{k_i,l_j}) \leq R^{circ}(b_{k_i,l_j}) \quad \text{for all admissible indices } \langle k_i, l_j\rangle.$$

This relation is used in the optimality tests of the 3D–CIFMODI algorithm, where the condition $I_0 \subseteq_v W$ indicates the absence of improving directions.

6. Three-Dimensional Product-Based C-IF Transportation Problem with Upper Cost Limits

6.1. Real-World Case Context

The proposed model is motivated by a real-world logistics scenario involving a European manufacturing company operating in a multi-product distribution environment.

The company manages several production plants located in different regions and distributes multiple categories of industrial products to a network of regional markets. Each product category is characterized by distinct transportation costs, supply constraints, and demand requirements. In practice, transportation costs are not precisely known and are often estimated by multiple experts, taking into account factors such as fuel price fluctuations, contractual agreements, delivery risks, and operational uncertainties. These assessments naturally involve hesitation and variability, which are captured using circular intuitionistic fuzzy representations. Additionally, upper cost limits are imposed due to contractual and regulatory constraints, which must be respected during allocation.

The three-dimensional structure of the model reflects real operational requirements, where decisions must simultaneously consider:

- Multiple production facilities;
- Multiple destination markets;
- Multiple product types.

The numerical example presented in this study is constructed to reflect these real-world characteristics, ensuring that the proposed method can be directly applied to industrial logistics and supply chain optimization problems under uncertainty.

6.2. Problem Formulation

Let $K = \{k_1, \ldots, k_m\}$, $L = \{l_1, \ldots, l_n\}$, $H = \{h_1, \ldots, h_f\}$, $E = \{d_1, \ldots, d_D\}$ denote the sets of production plants, destination markets, product types, and experts, respectively. Each layer $h_g \in H$ corresponds to a distinct product category (e.g., electronic components, mechanical modules, and precision subassemblies), characterized by different uncertainty levels and contractual cost restrictions.

For every product h_g, the unit transportation cost from plant k_i to market l_j is assessed by experts as a Circular Intuitionistic Fuzzy Triple (C-IFT)

$$c_{k_i,l_j,h_g} = \langle \mu_{k_i,l_j,h_g}, \nu_{k_i,l_j,h_g}; r_{k_i,l_j,h_g} \rangle,$$

where μ and ν represent the degrees of acceptance and non-acceptance, while r denotes the circular dispersion radius reflecting uncertainty in expert opinions.

In addition to transportation costs, admissible upper cost limits are imposed for each destination and product type. These limits represent contractual or regulatory constraints and are modeled as C-IFTs

$$c_{pl,l_j,h_g} = \langle \mu^{pl}_{l_j,h_g}, \nu^{pl}_{l_j,h_g}; r^{pl}_{l_j,h_g} \rangle, \qquad l_j \in L,\ h_g \in H.$$

The circular supplies of plants and circular demands of markets for each product layer are denoted by

$$c_{k_i,R,h_g}, \qquad c_{Q,l_j,h_g},$$

respectively, and are incorporated into the extended index matrix structure.

The decision variables are the circular shipment quantities

$$x_{k_i,l_j,h_g} = \langle \mu^{x}_{k_i,l_j,h_g}, \nu^{x}_{k_i,l_j,h_g}; r^{x}_{k_i,l_j,h_g} \rangle, \quad k_i \in K,\ l_j \in L,\ h_g \in H,$$

assembled into a three-dimensional circular intuitionistic fuzzy index matrix (3D C-IFIM)

$$X[K, L, H] = \{x_{k_i,l_j,h_g}\}.$$

Objective of the problem.

The objective of the three-dimensional product-based C-IF transportation problem is to determine a feasible shipment plan

$$X[K, L, H] = \{x_{k_i,l_j,h_g}\},$$

such that all circular supply, demand, and upper cost constraints are satisfied, and the aggregated circular intuitionistic fuzzy transportation cost is optimal with respect to the circular ranking relation R^{circ}.

Extended C-IF index matrix representation. For each product layer h_g, the transportation problem is represented by an extended C-IF index matrix

$$C^{(g)}[K^*, L^*], \quad K^* = K \cup \{Q^{(g)}, pl^{(g)}\}, \quad L^* = L \cup \{R^{(g)}\},$$

which simultaneously incorporates transportation costs, supplies, demands, and upper cost limits within a unified algebraic framework.

Empty (nonbasic) cells are denoted by the symbol $\perp$.

Step 1. Construction of the 3-D Circular Intuitionistic Fuzzy Cost Matrix

Let

$$E = \{d_1, \ldots, d_D\}$$

denote the panel of experts involved in the assessment of transportation costs.

Expert evaluation matrix.

Expert assessments are represented by a four-dimensional intuitionistic fuzzy index matrix

$$EV[K, L, H, E, \{ev_{k_i,l_j,h_g,d_s}\}],$$

where each element

$$ev_{k_i,l_j,h_g,d_s} = \langle \mu^{ev}_{k_i,l_j,h_g,d_s}, \nu^{ev}_{k_i,l_j,h_g,d_s} \rangle$$

is an IF evaluation provided by expert d_s for the transportation cost from source k_i to destination l_j in layer h_g.

Incorporation of expert reliability.

Each expert d_s is associated with an intuitionistic fuzzy reliability weight

$$re_s = \langle \delta_s, \epsilon_s \rangle,$$

reflecting expert competence and hesitation. The expert-adjusted intuitionistic fuzzy index matrix is obtained as

$$EV^* = \bigoplus_{s=1}^{D} (re_s \cdot pr_{K,L,H,d_s}(EV)),$$

where pr_{K,L,H,d_s} denotes projection onto the expert dimension and $\oplus$ is the internal intuitionistic fuzzy aggregation operator. Henceforth, we set $EV := EV^*$.

Central intuitionistic fuzzy cost estimation.

For each triplet (k_i, l_j, h_g), a central intuitionistic fuzzy estimate is computed by averaging across experts:

$$PI[K, L, H, \{pi_{k_i,l_j,h_g}\}] = \alpha_{E,\#_2}(EV),$$

yielding

$$pi_{k_i,l_j,h_g} = \langle \mu^{pi}_{k_i,l_j,h_g}, \nu^{pi}_{k_i,l_j,h_g} \rangle.$$

Transformation to Circular Intuitionistic Fuzzy Triples.

The intuitionistic fuzzy pairs pi_{k_i,l_j,h_g} are transformed into Circular Intuitionistic Fuzzy Triples (C-IFTs) by attaching a dispersion-based radius that reflects disagreement among expert opinions:

$$A[K, L, H, \{c_{k_i,l_j,h_g}\}], \qquad c_{k_i,l_j,h_g} = \langle \mu^{a}_{k_i,l_j,h_g}, \nu^{a}_{k_i,l_j,h_g}; r^{a}_{k_i,l_j,h_g} \rangle,$$

where

$$\mu^{a}_{k_i,l_j,h_g} = \mu^{pi}_{k_i,l_j,h_g}, \qquad \nu^{a}_{k_i,l_j,h_g} = \nu^{pi}_{k_i,l_j,h_g},$$

and the circular radius is defined as

$$r^{a}_{k_i,l_j,h_g} = \max_{1 \le s \le D} \sqrt{\left(\mu^{ev}_{k_i,l_j,h_g,d_s} - \mu^{pi}_{k_i,l_j,h_g}\right)^2 + \left(\nu^{ev}_{k_i,l_j,h_g,d_s} - \nu^{pi}_{k_i,l_j,h_g}\right)^2}.$$

Extended 3-D C-IFIM and Auxiliary Matrices.

The circular intuitionistic fuzzy cost matrix $A[K, L, H]$ is embedded into an extended three-dimensional C-IF index matrix $C[K^*, L^*, H, \{c_{k_i,l_j,h_g}\}]$, where the extended index sets are defined as

$$K^* = K \cup \{Q, pl, pu_1\}, \qquad L^* = L \cup \{R, pu\}.$$

Here, Q and R denote the aggregated supply and demand indices, respectively, while pl represents upper cost limits. The auxiliary indices pu and pu_1 are introduced to ensure balancing and penalty handling within the index matrix framework.

The matrix C serves as the primary data structure of the method, while its working copy is denoted by

$$S := C.$$

Initialization.

The shipment matrix is initialized as $x_{k_i,l_j,h_g} = \perp$, indicating that no allocations have yet been made.

Auxiliary balancing and penalty cells are initially set as

$$c_{k_i,pu,h_g} = c_{pu_1,l_j,h_g} = \perp .$$

Step 2. Balancing the 3-D Circular Intuitionistic Fuzzy Transportation Problem

For each layer $h_g \in H$, the problem is balanced if

$$\sum_{i=1}^{m} c_{k_i,R,h_g} = \sum_{j=1}^{n} c_{Q,l_j,h_g}, \tag{1}$$

with respect to the value-based order $\preceq_v$.

If this condition is not satisfied, balancing is performed as follows.

Case 1: Supply exceeds demand. If

$$\sum_{i=1}^{m} c_{k_i,R,h_g} \succ_v \sum_{j=1}^{n} c_{Q,l_j,h_g}, \tag{2}$$

a dummy destination l_{n+1} is introduced with zero transportation cost:

$$c_{k_i,l_{n+1},h_g} = \langle 0, 1; r_g^{\text{base}} \rangle, \qquad i = 1, \ldots, m, \tag{3}$$

where

$$r_g^{\text{base}} = *\{r_{k_i,l_j,h_g}\}.$$

The corresponding demand is

$$c_{Q,l_{n+1},h_g} = \sum_{i=1}^{m} c_{k_i,R,h_g} - \sum_{j=1}^{n} c_{Q,l_j,h_g}. \tag{4}$$

Case 2: Demand exceeds supply. If

$$\sum_{i=1}^{m} c_{k_i,R,h_g} \prec_v \sum_{j=1}^{n} c_{Q,l_j,h_g}, \tag{5}$$

a dummy source k_{m+1} is introduced with zero transportation cost:

$$c_{k_{m+1},l_j,h_g} = \langle 0, 1; r_g^{\text{base}} \rangle, \qquad j = 1, \ldots, n, \tag{6}$$

with supply

$$c_{k_{m+1},R,h_g} = \sum_{j=1}^{n} c_{Q,l_j,h_g} - \sum_{i=1}^{m} c_{k_i,R,h_g}. \tag{7}$$

After this adjustment, the transportation problem is balanced for every layer $h_g \in H$.

Step 3. Feasibility with Respect to Circular Cost Upper Limits

Let

$$c_{pl,l_j,h_g} = \langle \mu^{pl}_{l_j,h_g}, \nu^{pl}_{l_j,h_g}; r^{pl}_{l_j,h_g} \rangle$$

denote the circular upper cost limit for destination l_j and layer h_g.

Feasibility is evaluated using

$$c_{k_i,l_j,h_g} \preceq_v c_{pl,l_j,h_g}. \tag{8}$$

The feasibility indicator matrix $U[K, L, H] = \{u_{k_i,l_j,h_g}\}$ is defined as

$$u_{k_i,l_j,h_g} = \begin{cases} 1, & \text{if } c_{k_i,l_j,h_g} \preceq_v c_{pl,l_j,h_g}, \\ \perp, & \text{otherwise.} \end{cases}$$

Let

$$EG = \{\langle k_i, l_j, h_g \rangle : u_{k_i,l_j,h_g} = \perp\}$$

denote the set of infeasible indices.

To exclude infeasible allocations, the working matrix

$$S[K, L, H] = \{s_{k_i,l_j,h_g}\}$$

is updated as follows:

$$s_{k_i,l_j,h_g} = \langle 1, 0; r^{\text{base}}_g \rangle, \qquad (k_i, l_j, h_g) \in EG, \tag{9}$$

where

$$r^{\text{base}}_g = *\{r_{k_i,l_j,h_g}\}.$$

For feasible entries

$$s_{k_i,l_j,h_g} = c_{k_i,l_j,h_g}. \tag{10}$$

Thus, the matrix S contains only admissible transportation positions for the subsequent optimization procedure.

Step 4. Initial Basic Feasible Solution and Circular MODI Improvement (Layer-wise)

For each layer $h_g \in H$, Step 4 first constructs an initial basic feasible solution (IBFS) and then improves it through a circular intuitionistic fuzzy potential (MODI-type) procedure.

Step 4.0. Construction of an Initial Basic Feasible Solution

For each layer $h_g \in H$, the IBFS is constructed on the projected working cost matrix

$$S_g = pr_{K,L,h_g}(S),$$

within the extended 3-D C-IFIM framework. The objective is to determine a feasible allocation matrix X_g that satisfies the supply and demand constraints in layer h_g and contains exactly $|K| + |L| - 1$ basic cells.

To preserve the index matrix formulation, we use the residual supply and demand projections

$$\Sigma_g := pr_{K,R,h_g}(S), \qquad \Delta_g := pr_{Q,L,h_g}(S),$$

which are updated during the IBFS construction, while the cost matrix S_g remains unchanged.

Two initialization schemes are considered: (i) North–West Corner and (ii) Least–Cost.

North–West Corner initialization (IM-based).

Step 4.0.1. Set

$$X_g := pr_{K,L,h_g}(X), \qquad \Sigma_g := pr_{K,R,h_g}(S), \qquad \Delta_g := pr_{Q,L,h_g}(S),$$

and initialize $(i, j) := (1, 1)$.

Step 4.0.2. At the current indices (k_i, l_j), verify feasibility:

$$pr_{k_i,l_j,h_g}(S_g) \neq \perp .$$

If the position is infeasible or penalized, move to the next admissible pair.

Step 4.0.3. Compute the admissible allocation using the value-based order $\preceq_v$:

$$x_{k_i,l_j,h_g} = \min_{\preceq_v} \Big\{ pr_{k_i,R,h_g}(\Sigma_g),\ pr_{Q,l_j,h_g}(\Delta_g) \Big\}.$$

Step 4.0.4. Insert the corresponding shipment into X_g:

$$X_g := X_g \oplus_{(\circ_1,\circ_2,*)} pr_{k_i,l_j,h_g}(x_{k_i,l_j,h_g}).$$

Step 4.0.5. Update the residual supply and demand:

$$\Sigma_g := \Sigma_g -_{(\circ_1,\circ_2,*)} pr_{k_i,R,h_g}(x_{k_i,l_j,h_g}), \qquad \Delta_g := \Delta_g -_{(\circ_1,\circ_2,*)} pr_{Q,l_j,h_g}(x_{k_i,l_j,h_g}).$$

If

$$pr_{k_i,R,h_g}(\Sigma_g) = \langle 0, 1; r_g^{\text{base}} \rangle,$$

set $i := i + 1$, where

$$r_g^{\text{base}} = *\{ r_{k_i,l_j,h_g} \mid (k_i, l_j) \in K \times L \}.$$

If

$$pr_{Q,l_j,h_g}(\Delta_g) = \langle 0, 1; r_g^{\text{base}} \rangle,$$

set $j := j + 1$.

Step 4.0.6. Repeat Steps 4.0.2–4.0.5 until all residual supplies and demands are exhausted.

Degeneracy treatment.

If degeneracy occurs, i.e.,

$$|\mathcal{B}_g| < |K| + |L| - 1,$$

an artificial basic cell with zero shipment is introduced:

$$x^{(g)}_{k_i,l_j} = \langle 0, 1;\ r_{\text{base}} \rangle.$$

The base radius is defined consistently with the selected aggregation operator:

$$r_g^{\text{base}} = *\{ r_{k_i,l_j,h_g} \mid (k_i, l_j) \in K \times L \}, \qquad * \in \{\min, \max\}.$$

Hence

- if $* = \min$, then $r_{\text{base}} = \min r_{k_i,l_j,h_g}$;
- if $* = \max$, then $r_{\text{base}} = \max r_{k_i,l_j,h_g}$.

This preserves consistency of subsequent circular aggregation and subtraction operations.

Least–Cost initialization (AGIndex-based).

Step 4.0.7. Alternatively, select the admissible cell with minimal cost via

$$AGIndex_{(\max_{R^{circ}}),(\not\perp)}(S_g) = \langle k_i, l_j, h_g \rangle,$$

allocate x_{k_i,l_j,h_g} as in Step 4.0.3, update X_g as in Step 4.0.4, and update residuals as in Step 4.0.5. Repeat until completion and treat degeneracy as above.

Remark.

The North–West Corner rule is adopted as the default initialization procedure because it avoids premature ranking of uncertain circular intuitionistic fuzzy costs and ensures structural neutrality in the initial allocation. The Least–Cost variant is included for completeness.

Step 4.1. Circular Potentials (IM-based)

Let

$$S_g := pr_{K,L,h_g}(S), \qquad X_g := pr_{K,L,h_g}(X).$$

Basis index set and degeneracy check.

Let

$$EX := Index_{\not\perp}(X_g) = \{\langle k_{i_1}, l_{j_1} \rangle, \ldots, \langle k_{i_\varphi}, l_{j_\varphi} \rangle\}$$

denote the set of basic cells in layer h_g, where each pair represents the triple $\langle k_{i_t}, l_{j_t}, h_g \rangle$.

If

$$\dim(EX) < (m + n - 1) \times 1,$$

the basis is completed by adding one artificial basic cell

$$x_{k_i,l_j,h_g} = \langle 0, 1; r_g^{\text{base}} \rangle,$$

in a position preserving acyclicity, where

$$r_g^{\text{base}} = *\{r_{k_i,l_j,h_g} \mid (k_i, l_j) \in K \times L\}.$$

Potential equations on basic cells.

Row and column circular potentials are stored as

$$S_4[K, R, h_g] = \{s_{k_i,R,h_g}\}, \qquad S_5[Q, L, h_g] = \{s_{Q,l_j,h_g}\}.$$

For every $\langle k_i, l_j \rangle \in EX$, the potential equation is

$$s_{k_i,R,h_g} \oplus_{(\circ_1,\circ_2,*)} s_{Q,l_j,h_g} = s_{k_i,l_j,h_g}, \tag{11}$$

where $s_{k_i,l_j,h_g} = pr_{k_i,l_j,h_g}(S_g)$.

Initialization of one potential.

Work in the extended layer IM

$$\widehat{S}_g := pr_{K^*,L^*,h_g}(S), \quad K^* = K \cup \{Q, pl, pu_1\},\ L^* = L \cup \{R, pu\}.$$

Choose $\langle k_{i_1}, l_{j_1} \rangle \in EX$ and fix

$$s_{k_{i_1},R,h_g} := \langle 0, 1; r_g^{\text{base}} \rangle. \tag{12}$$

Define

$$S_6 := \left[\tfrac{k_{i_1}}{Q}; \tfrac{R}{pu}; \perp \right] \{ s_{k_{i_1},R,h_g} = \langle 0, 1; r_g^{\text{base}} \rangle \}, \qquad \widehat{S}_g := \widehat{S}_g \ \oplus_{(\circ_1,\circ_2,*)} \ S_6. \tag{13}$$

Propagation along the basis.

If s_{k_i,R,h_g} is known, compute

$$S_7 := pr_{k_i,l_j,h_g}(\widehat{S}_g) \ -_{(\circ_1,\circ_2,*)} \left[\perp; \tfrac{l_j}{R}; \perp \right] (pr_{k_i,R,h_g}(\widehat{S}_g)), \tag{14}$$

and update

$$\widehat{S}_g := \widehat{S}_g \ \oplus_{(\circ_1,\circ_2,*)} \left[\tfrac{Q}{k_i}; \perp; \perp \right] S_7. \tag{15}$$

If s_{Q,l_j,h_g} is known, compute

$$S_7 := pr_{k_{i_2},l_j,h_g}(\widehat{S}_g) \ -_{(\circ_1,\circ_2,*)} \left[\tfrac{k_{i_2}}{Q}; \perp; \perp \right] (pr_{Q,l_j,h_g}(\widehat{S}_g)), \tag{16}$$

and update

$$\widehat{S}_g := \widehat{S}_g \ \oplus_{(\circ_1,\circ_2,*)} \left[\perp; \tfrac{R}{l_j}; \perp \right] S_7. \tag{17}$$

Repeat the propagation until all potentials $\{s_{k_i,R,h_g}\}_{k_i \in K}$ and $\{s_{Q,l_j,h_g}\}_{l_j \in L}$ are determined.

Step 4.2. Circular Reduced Costs

Let

$$D_g := Index_{\perp}(X_g) = \{ \langle k_{\tau_1}, l_{\sigma_1} \rangle, \ldots, \langle k_{\tau_\psi}, l_{\sigma_\psi} \rangle \}$$

denote the set of nonbasic cells in layer h_g, where each pair represents the triple $\langle k_{\tau_t}, l_{\sigma_t}, h_g \rangle$.

Initialize

$$W_g[K, L, h_g] := [\{ w_{k_i,l_j,h_g} = \perp \}].$$

For each $\langle k_i, l_j \rangle \in D_g$, compute the circular reduced cost:

$$w_{k_i,l_j,h_g} = pr_{k_i,l_j,h_g}(\widehat{S}_g) \ -_{(\circ_1,\circ_2,*)} \Big(\left[\perp; \tfrac{l_j}{R}; \perp \right] (pr_{k_i,R,h_g}(\widehat{S}_g)) \ \oplus_{(\circ_1,\circ_2,*)} \left[\tfrac{k_i}{Q}; \perp; \perp \right] (pr_{Q,l_j,h_g}(\widehat{S}_g)) \Big), \tag{18}$$

which is the circular counterpart of the classical relation

$$w_{ij} = c_{ij} - (u_i + v_j).$$

Insert the result into W_g:

$$W_g := W_g \ \oplus_{(\circ_1,\circ_2,*)} pr_{k_i,l_j,h_g}(w_{k_i,l_j,h_g}). \tag{19}$$

Repeat this procedure until all reduced costs are determined.

Step 4.3. Optimality Test

Let $W_g = [w_{k_i,l_j,h_g}]$ be the reduced cost C-IFIM for layer h_g, and let

$$D_g := Index_{\perp}(X_g)$$

denote the set of nonbasic positions.

Define the reference circular zero element

$$\mathbf{0}_{r_g^{max}} := \langle 0, 1; r_g^{max} \rangle,$$

where

$$r_g^{max} = \max\{\, r(w_{k_i,l_j,h_g}) \mid \langle k_i, l_j \rangle \in Index_{\not\perp}(W_g) \,\}.$$

Under the adopted ranking convention, larger values of R^{circ} correspond to better (smaller) circular values. Hence, a nonbasic index is improving if

$$R^{\text{circ}}(w_{k_i,l_j,h_g}) > R^{\text{circ}}(\mathbf{0}_{r_g^{max}}). \tag{20}$$

Define

$$Index_{\text{improve}}(W_g) = \left\{ \langle k_i, l_j \rangle \in D_g \;\middle|\; R^{\text{circ}}(w_{k_i,l_j,h_g}) > R^{\text{circ}}(\mathbf{0}_{r_g^{max}}) \right\}. \tag{21}$$

The current basic feasible solution X_g is circularly optimal if and only if

$$Index_{\text{improve}}(W_g) = \varnothing. \tag{22}$$

Equivalently

$$\forall\, \langle k_i, l_j \rangle \in D_g : \quad R^{\text{circ}}(w_{k_i,l_j,h_g}) \le R^{\text{circ}}(\mathbf{0}_{r_g^{max}}).$$

If (22) holds, proceed to the next layer. Otherwise, continue with Step 4.4.

Step 4.4. Selection of the Entering Cell

Assume that

$$Index_{\text{improve}}(W_g) \neq \varnothing.$$

Select the entering cell as the improving nonbasic index with maximal circular rank:

$$\langle k_{im_x}, l_{jn_y}, h_g \rangle = AGIndex_{(\max_{R^{\text{circ}}}),(\not\perp)}\big(W_g \mid Index_{\text{improve}}(W_g)\big). \tag{23}$$

The corresponding variable $x_{k_{im_x},l_{jn_y},h_g}$ enters the basis.

Step 4.5. Closed Cycle Construction

Let

$$F := Index_{\not\perp}(X_g)$$

be the set of basic cells.

Starting from the entering cell $\langle k_{im_x}, l_{jn_y}, h_g \rangle$, construct a closed alternating row–column cycle

$$\mathcal{C} = \{\langle k_{im_x}, l_{jn_y}, h_g \rangle, \langle k_{is_2}, l_{js_2}, h_g \rangle, \ldots, \langle k_{is_{IS}}, l_{js_{IS}}, h_g \rangle\},$$

such that consecutive indices alternate in row and column and the cycle closes at the entering cell.

Assign alternating signs along the cycle, starting with "+" at the entering cell:

$$\mathcal{C}^+ = \{\langle k_{im_x}, l_{jn_y}, h_g \rangle, \langle k_{is_3}, l_{js_3}, h_g \rangle, \ldots\}, \qquad \mathcal{C}^- = \{\langle k_{is_2}, l_{js_2}, h_g \rangle, \langle k_{is_4}, l_{js_4}, h_g \rangle, \ldots\}.$$

Step 4.6. Leaving Cell

Construct the auxiliary IM of minus positions:

$$E := \bigoplus_{(\circ_1,\circ_2,*)} \Big\{ pr_{k_{is},l_{js},h_g}(X_g) \mid \langle k_{is}, l_{js}, h_g \rangle \in \mathcal{C}^- \Big\}. \tag{24}$$

The leaving cell corresponds to the minimum admissible step size. Under the adopted ranking convention, this is obtained by

$$\langle k_u, l_v, h_g \rangle = AGIndex_{(\max_{R^{\text{circ}}}),(\not\perp)}(E), \qquad MS := x_{k_u,l_v,h_g}. \tag{25}$$

Step 4.7. Allocation Update

Update X_g along the cycle by adding MS to all positions in $\mathcal{C}^+$ and subtracting MS from all positions in $\mathcal{C}^-$.

Finally, remove the leaving cell:

$$x_{k_u,l_v,h_g} :=\perp .$$

Return to Step 4.1.

Step 4.8. Iteration

Recompute potentials and reduced costs. Repeat Steps 4.1–4.7 until

$$Index_{\text{improve}}(W_g) = \varnothing.$$

Then, X_g is circularly optimal for layer h_g. Set $g := g + 1$.

Step 5. Optimal Solution and Total Transportation Cost

After processing all layers $h_g \in H$, the obtained three-dimensional allocation

$$X_{\text{opt}} = \{ x_{k_i,l_j,h_g} \mid k_i \in K,\ l_j \in L,\ h_g \in H \}$$

constitutes a circular intuitionistic fuzzy optimal transportation plan.

Completion of empty entries.

Since X_{opt} is defined as a full 3-D C-IFIM, all empty entries are replaced by the neutral circular element

$$\mathbf{0}_{r_g^{\text{base}}} = \langle 0, 1; r_g^{\text{base}} \rangle,$$

where

$$r_g^{\text{base}} = \begin{cases} \min\{ r^x_{k_i,l_j,h_g} \mid x_{k_i,l_j,h_g} \neq \perp \}, & \text{optimistic scenario,} \\ \max\{ r^x_{k_i,l_j,h_g} \mid x_{k_i,l_j,h_g} \neq \perp \}, & \text{pessimistic scenario.} \end{cases}$$

This guarantees consistency of the global internal aggregation operator.

Total circular transportation cost.

The total circular intuitionistic fuzzy transportation cost is computed by

$$C_{\text{tot}} = AGIO_{\oplus_{(\circ_1,\circ_2,*)}}\Big(C_{(K,L,H)} \ \otimes_{(\times,*)} \ X_{\text{opt}} \Big), \tag{26}$$

where $\otimes_{(\times,*)}$ denotes the internal circular multiplication between costs and allocations, and $\oplus_{(\circ_1,\circ_2,*)}$ is the global circular aggregation operator.

Stopping consistency.

Since the algorithm terminates only when

$$\forall (k_i, l_j, h_g): \quad R^{circ}(w_{k_i,l_j,h_g}) \leq R^{circ}(\mathbf{0}_{r_g^{max}}),$$

the obtained allocation X_{opt} satisfies the circular optimality condition under the adopted ranking rule.

Defuzzification for Practical Interpretation

For decision support purposes, each Circular Intuitionistic Fuzzy Triple $\langle \mu, \nu; r \rangle$ is first reduced to its center intuitionistic fuzzy pair $\langle \mu, \nu \rangle$, preserving the acceptance and non-acceptance degrees.

Subsequently, a fuzzy representation is obtained as

$$\left\langle \frac{\mu}{\mu+\nu}, \frac{\nu}{\mu+\nu} \right\rangle, \qquad \mu+\nu>0, \tag{27}$$

following [71].

This transformation provides a scalar-based interpretation of the transportation plan suitable for managerial comparison across layers, while the original parameters (μ, ν, r) remain available as uncertainty descriptors and preserve the circular intuitionistic fuzzy structure of the solution.

Relation to the Classical MODI Method

The proposed 3D–CIFMODI method preserves the dual-based structure of the classical MODI algorithm.

In the crisp two-dimensional case, potentials u_i and v_j satisfy $c_{ij} = u_i + v_j$ on the basic cells, and optimality holds if all reduced costs $\bar{c}_{ij} = c_{ij} - (u_i + v_j)$ are non-negative.

In the circular intuitionistic fuzzy three-dimensional setting, scalar addition and subtraction are replaced by $\oplus_{(\circ_1,\circ_2,*)}$ and $-_{(\circ_1,\circ_2,*)}$, while the potential equations become

$$s_{k_i,l_j,h_g} = s_{k_i,R,h_g} \oplus s_{Q,l_j,h_g}.$$

Reduced costs are defined analogously and optimality is achieved when

$$w_{k_i,l_j,h_g} \succeq_v \mathbf{0}_{r_g},$$

where $\mathbf{0}_{r_g} = \langle 0, 1; r_g^{\text{base}} \rangle$.

Hence, 3D–CIFMODI constitutes a structural generalization of MODI to product-based three-dimensional transportation problems under circular intuitionistic fuzzy uncertainty.

The three-dimensional structure is handled layer-wise: for a fixed layer h_g (e.g., a planning period or scenario), the procedure coincides with the MODI logic on the corresponding 2-D projection, while the additional dimension enables consistent optimization across multiple layers.

Hence, 3D–CIFMODI preserves the primal–dual potential logic of classical MODI, but extends the algebra, ordering mechanism, and data representation to handle circular intuitionistic fuzzy uncertainty and multi-layer transportation structures.

Managerial Interpretation and Decision Support Insights

The proposed 3D–CIFMODI framework provides decision-makers with a structured analytical tool for transportation planning under multi-layer circular intuitionistic fuzzy uncertainty. Its managerial relevance can be interpreted along several interrelated dimensions.

Layer-wise decision transparency.

The three-dimensional structure enables the transportation plan to be analyzed separately for each layer $h_g \in H$, representing products, time periods, scenarios, or operational conditions. This layer-wise decomposition allows managers to identify cost-intensive routes, structural bottlenecks, or infeasible allocations that may arise only in specific

product categories or planning horizons and would remain undetected in aggregated two-dimensional models.

Explicit modeling of expert disagreement.

By representing transportation costs as Circular Intuitionistic Fuzzy Triples, the framework captures both central expert assessments and dispersion due to disagreement. The radius parameter r_{k_i,l_j,h_g} quantifies the degree of uncertainty or inconsistency among experts, providing an operational indicator of routes that require additional managerial attention or risk mitigation.

Embedded policy and contractual constraints.

Circular upper cost limits c_{pl,l_j,h_g} integrate budgetary, contractual, or regulatory constraints directly into the optimization process. Routes exceeding admissible limits are penalized within the circular algebraic structure, ensuring compliance with policy restrictions without the need for external corrective procedures.

Flexible initialization and optimization strategy.

Classical construction rules (North–West Corner, Least–Cost, Vogel), reformulated in the circular intuitionistic fuzzy setting, offer flexibility in generating initial feasible solutions. This enables practitioners to balance computational efficiency and solution quality depending on problem scale and time constraints, while preserving theoretical consistency of the optimization procedure.

Operational interpretability.

Although optimization is performed in the circular intuitionistic fuzzy domain, the resulting allocation can be defuzzified into a crisp transportation plan suitable for direct implementation. At the same time, the retained IF parameters (μ, ν, r) remain available as uncertainty descriptors, supporting comparative scenario analysis and sensitivity evaluation.

Strategic decision support.

Overall, the 3D–CIFMODI method supports strategic logistics planning by combining cost efficiency, structured uncertainty modeling, and explicit policy integration. It is particularly suitable for logistics systems involving multiple products, expert-driven cost evaluations, and layered operational structures.

Relation to Zero Point-based approaches.

Zero point-based methods developed for intuitionistic [50] and circular intuitionistic fuzzy transportation problems [64] are primarily feasibility-oriented. They construct admissible solutions by enforcing structural balance and zero-position properties within the cost matrix.

In contrast, the proposed three-dimensional circular intuitionistic fuzzy potential method is optimality-driven. It preserves a dual-type structure through circular potentials, defines reduced costs in the C-IF algebra, and employs ranking-based stopping conditions.

Hence, zero point and potential-based approaches address complementary objectives: the former emphasize feasibility construction, whereas the latter provide marginal optimality interpretation. They should therefore be regarded as alternative algorithmic paradigms rather than competing methods.

7. Case Study Description

The numerical example is designed to reflect a realistic industrial logistics setting with multiple products, regional demand variability, and expert-based cost estimation. The data structure and parameter ranges are consistent with practical supply chain environments.

We consider a European manufacturing company that distributes multiple product types from several production facilities to regional markets. The problem is formulated as a product-based three-dimensional transportation problem under circular intuitionistic fuzzy uncertainty.

Let

$$K = \{k_1, k_2, k_3\}, \qquad L = \{l_1, l_2, l_3, l_4\}, \qquad H = \{h_1, h_2, h_3\}, \qquad E = \{d_1, \ldots, d_D\}$$

denote the sets of production plants, destination markets, product types, and experts, respectively.

Each layer $h_g \in H$ corresponds to a distinct product category:

- h_1: Electronic components;
- h_2: Mechanical modules;
- h_3: Precision subassemblies.

These products differ in handling requirements, storage sensitivity, and contractual constraints, resulting in different uncertainty levels across layers.

Circular intuitionistic fuzzy transportation costs.

For each product $h_g \in H$, the unit transportation cost from plant k_i to market l_j is assessed by a panel of experts and represented as a Circular Intuitionistic Fuzzy Triple (C-IFT)

$$c_{k_i,l_j,h_g} = \langle \mu_{k_i,l_j,h_g}, \nu_{k_i,l_j,h_g}; r_{k_i,l_j,h_g} \rangle,$$

where μ_{k_i,l_j,h_g} denotes the degree of cost acceptability, ν_{k_i,l_j,h_g} denotes the degree of non-acceptability, and r_{k_i,l_j,h_g} is the circular dispersion radius reflecting disagreement among expert evaluations.

Product- and market-specific cost limits.

In addition to transportation costs, admissible upper cost limits are imposed for each market l_j and product h_g. These limits reflect contractual agreements, budget restrictions, or regulatory thresholds and are modeled as C-IFTs

$$c_{pl,l_j,h_g} = \langle \mu^{pl}_{l_j,h_g}, \nu^{pl}_{l_j,h_g}; r^{pl}_{l_j,h_g} \rangle, \qquad l_j \in L,\ h_g \in H.$$

Routes whose transportation costs violate these limits are penalized during the optimization process.

Three-dimensional decision structure.

The decision variable

$$x_{k_i,l_j,h_g}$$

represents the circular intuitionistic fuzzy shipment quantity of product h_g transported from plant k_i to market l_j. The overall transportation plan is described by the three-dimensional C-IFIM

$$X = \{x_{k_i,l_j,h_g} \mid k_i \in K,\ l_j \in L,\ h_g \in H\}.$$

For each product layer h_g, the objective is to determine a feasible and circularly optimal transportation plan satisfying:

- Supply–demand balance constraints;
- Admissible circular upper cost limits;
- Minimization of the total transportation cost under the adopted value-based circular ranking relation.

7.1. Input Data for the Product-Based Case Study

The circular intuitionistic fuzzy unit transportation costs for the three product layers are given below. Each matrix corresponds to a fixed product category h_g.

$$\widetilde{C}^{(1)}[K,L] = \begin{bmatrix} & l_1 & l_2 & l_3 & l_4 \\ k_1 & \langle 0.61, 0.19; 0.12\rangle & \langle 0.71, 0.09; 0.12\rangle & \langle 0.31, 0.09; 0.12\rangle & \langle 0.51, 0.19; 0.12\rangle \\ k_2 & \langle 0.51, 0.29; 0.12\rangle & \langle 0.41, 0.09; 0.12\rangle & \langle 0.51, 0.10; 0.12\rangle & \langle 0.31, 0.19; 0.12\rangle \\ k_3 & \langle 0.41, 0.19; 0.12\rangle & \langle 0.31, 0.19; 0.12\rangle & \langle 0.61, 0.09; 0.12\rangle & \langle 0.71, 0.19; 0.12\rangle \end{bmatrix}.$$

$$\widetilde{C}^{(2)}[K,L] = \begin{bmatrix} & l_1 & l_2 & l_3 & l_4 \\ k_1 & \langle 0.81, 0.09; 0.14\rangle & \langle 0.29, 0.54; 0.14\rangle & \langle 0.25, 0.69; 0.14\rangle & \langle 0.71, 0.09; 0.14\rangle \\ k_2 & \langle 0.51, 0.39; 0.14\rangle & \langle 0.33, 0.54; 0.14\rangle & \langle 0.33, 0.59; 0.14\rangle & \langle 0.71, 0.09; 0.14\rangle \\ k_3 & \langle 0.71, 0.29; 0.14\rangle & \langle 0.29, 0.59; 0.14\rangle & \langle 0.21, 0.64; 0.14\rangle & \langle 0.61, 0.19; 0.14\rangle \end{bmatrix}.$$

$$\widetilde{C}^{(3)}[K,L] = \begin{bmatrix} & l_1 & l_2 & l_3 & l_4 \\ k_1 & \langle 0.71, 0.19; 0.18\rangle & \langle 0.51, 0.19; 0.18\rangle & \langle 0.13, 0.72; 0.18\rangle & \langle 0.81, 0.19; 0.18\rangle \\ k_2 & \langle 0.61, 0.09; 0.18\rangle & \langle 0.71, 0.09; 0.18\rangle & \langle 0.11, 0.81; 0.18\rangle & \langle 0.71, 0.19; 0.18\rangle \\ k_3 & \langle 0.71, 0.19; 0.18\rangle & \langle 0.51, 0.29; 0.18\rangle & \langle 0.13, 0.75; 0.18\rangle & \langle 0.71, 0.29; 0.18\rangle \end{bmatrix}.$$

The increasing radii across layers (0.12, 0.14, 0.18) reflect progressively higher expert disagreement and uncertainty for more complex product categories.

Extended C-IFIM structure.

For each layer $h_g \in H$, the C-IF supplies of the production plants are aggregated in a dedicated row $Q^{(g)}$, while the C-IF demands of the markets are aggregated in a dedicated column $R^{(g)}$. The admissible upper cost limits are stored in the row $pl^{(g)}$. Thus, each layer is represented by an extended C-IFIM over

$$K^* = K \cup \{Q^{(g)}, pl^{(g)}, pu^{(g)}\}, \qquad L^* = L \cup \{R^{(g)}\}.$$

Extended cost matrix C-IFIM $C[K^*, L^*, H]$.

The extended C-IFIM C has the following form.

$$\left\{\begin{array}{c|ccccc} h_1 & l_1 & l_2 & l_3 & l_4 & R \\ k_1 & \langle 0.61, 0.19; 0.12\rangle & \langle 0.71, 0.09; 0.12\rangle & \langle 0.31, 0.19; 0.12\rangle & \langle 0.51, 0.19; 0.12\rangle & \langle 0.47, 0.23; 0.10\rangle \\ k_2 & \langle 0.51, 0.29; 0.12\rangle & \langle 0.41, 0.09; 0.12\rangle & \langle 0.51, 0.10; 0.12\rangle & \langle 0.31, 0.19; 0.12\rangle & \langle 0.71, 0.10; 0.12\rangle \\ k_3 & \langle 0.41, 0.19; 0.12\rangle & \langle 0.31, 0.19; 0.12\rangle & \langle 0.61, 0.09; 0.12\rangle & \langle 0.71, 0.19; 0.12\rangle & \langle 0.41, 0.50; 0.12\rangle \\ Q^{(1)} & \langle 0.41, 0.20; 0.12\rangle & \langle 0.51, 0.30; 0.12\rangle & \langle 0.61, 0.20; 0.12\rangle & \langle 0.24, 0.20; 0.12\rangle & \perp \\ pl^{(1)} & \langle 0.66, 0.15; 0.12\rangle & \langle 0.72, 0.08; 0.12\rangle & \langle 0.76, 0.08; 0.12\rangle & \langle 0.74, 0.18; 0.12\rangle & \perp \end{array}\right\}.$$

$$\left[\begin{array}{c|ccccc} h_2 & l_1 & l_2 & l_3 & l_4 & R \\ k_1 & \langle 0.81, 0.09; 0.14\rangle & \langle 0.29, 0.54; 0.14\rangle & \langle 0.25, 0.69; 0.14\rangle & \langle 0.71, 0.09; 0.14\rangle & \langle 0.50, 0.20; 0.14\rangle \\ k_2 & \langle 0.51, 0.39; 0.14\rangle & \langle 0.33, 0.54; 0.14\rangle & \langle 0.33, 0.59; 0.14\rangle & \langle 0.71, 0.09; 0.14\rangle & \langle 0.74, 0.07; 0.14\rangle \\ k_3 & \langle 0.71, 0.29; 0.14\rangle & \langle 0.29, 0.59; 0.14\rangle & \langle 0.21, 0.64; 0.14\rangle & \langle 0.61, 0.19; 0.14\rangle & \langle 0.44, 0.47; 0.14\rangle \\ Q^{(2)} & \langle 0.44, 0.17; 0.14\rangle & \langle 0.54, 0.27; 0.14\rangle & \langle 0.64, 0.17; 0.14\rangle & \langle 0.27, 0.17; 0.14\rangle & \perp \\ pl^{(2)} & \langle 0.69, 0.12; 0.14\rangle & \langle 0.75, 0.05; 0.14\rangle & \langle 0.79, 0.05; 0.14\rangle & \langle 0.77, 0.15; 0.14\rangle & \perp \end{array}\right].$$

$$\left[\begin{array}{c|ccccc} h_3 & l_1 & l_2 & l_3 & l_4 & R \\ k_1 & \langle 0.71, 0.19; 0.18\rangle & \langle 0.51, 0.19; 0.18\rangle & \langle 0.13, 0.72; 0.18\rangle & \langle 0.81, 0.19; 0.18\rangle & \langle 0.52, 0.18; 0.18\rangle \\ k_2 & \langle 0.61, 0.09; 0.18\rangle & \langle 0.71, 0.09; 0.18\rangle & \langle 0.11, 0.81; 0.18\rangle & \langle 0.71, 0.19; 0.18\rangle & \langle 0.76, 0.05; 0.18\rangle \\ k_3 & \langle 0.71, 0.19; 0.18\rangle & \langle 0.51, 0.29; 0.18\rangle & \langle 0.13, 0.75; 0.18\rangle & \langle 0.71, 0.29; 0.18\rangle & \langle 0.46, 0.45; 0.18\rangle \\ Q^{(3)} & \langle 0.46, 0.15; 0.18\rangle & \langle 0.56, 0.25; 0.18\rangle & \langle 0.66, 0.15; 0.18\rangle & \langle 0.29, 0.15; 0.18\rangle & \perp \\ pl^{(3)} & \langle 0.71, 0.10; 0.18\rangle & \langle 0.77, 0.03; 0.18\rangle & \langle 0.81, 0.03; 0.18\rangle & \langle 0.79, 0.13; 0.18\rangle & \perp \end{array}\right].$$

Model status before optimization.

The three-dimensional circular intuitionistic fuzzy transportation problem is balanced in each layer $h_g \in H$, i.e.,

$$c^{\text{sup}}_{Q,R,h_g} = c^{\text{dem}}_{Q,R,h_g}.$$

Furthermore, all transportation costs satisfy the admissibility constraints imposed by the circular upper limits:

$$\forall (k_i, l_j, h_g) \in K \times L \times H: \quad c_{k_i,l_j,h_g} \preceq_v c_{pl,l_j,h_g}.$$

Hence, no dummy rows or columns and no penalty adjustments are required. The model is feasible and well posed.

In what follows, the full optimization procedure is illustrated in detail for layer h_1. For layers h_2 and h_3, only the final circularly optimal transportation plans are reported. Finally, the aggregated three-dimensional optimal solution is presented.

Initialization of the 3D C-IF transportation plan.

For each layer $h_g \in H$, the transportation plan is represented by the corresponding two-dimensional projection

$$X_g = pr_{K,L,h_g}(X) = \{x_{k_i,l_j,h_g} \mid k_i \in K,\ l_j \in L\}.$$

Initially, all allocation variables are empty:

$$x_{k_i,l_j,h_g} = \perp, \qquad \forall k_i \in K,\ \forall l_j \in L,\ \forall h_g \in H.$$

Step 2. Balancing (Layer $\boldsymbol{h_1}$)

$$c^{\text{dem}}_{Q,R,h_1} = c^{\text{sup}}_{Q,R,h_1} = \langle 0.907251,\ 0.012;\ 0.12\rangle.$$

Hence, layer h_1 is circularly balanced and no dummy row or column is required.

Step 3. Feasibility with Respect to Upper Limits (Layer $\boldsymbol{h_1}$)

$$\forall (k_i, l_j) \in K \times L: \quad c_{k_i,l_j,h_1} \preceq_v c_{pl,l_j,h_1}.$$

Therefore

$$EG = \varnothing, \qquad S^{(1)} = C^{(1)}.$$

Step 4.0. Initial Basic Feasible Solution (North–West Corner, $\boldsymbol{h_1}$)

Using the North–West Corner rule with $* = \min$, we obtain the IBFS:

$$X^{(1)} = \left\{ \begin{array}{c|cccc} h_1 & l_1 & l_2 & l_3 & l_4 \\ \hline k_1 & \langle 0.41, 0.20; 0.12\rangle & \langle 0.06, 0.43; 0.12\rangle & \perp & \perp \\ k_2 & \perp & \langle 0.45, 0.07; 0.12\rangle & \langle 0.26, 0.03; 0.12\rangle & \perp \\ k_3 & \perp & \perp & \langle 0.35, 0.17; 0.12\rangle & \langle 0.24, 0.20; 0.12\rangle \end{array} \right\}.$$

The number of basic cells is $|K| + |L| - 1 = 6$, so the IBFS is non-degenerate.

Step 4.1. Circular Potentials (Layer $\boldsymbol{h_1}$, $\boldsymbol{* = \min}$)

We use the internal operations with optimistic radius aggregation ($* = \min$):

$$\langle a, b; r_1\rangle \oplus_* \langle c, d; r_2\rangle = \langle ac,\ b + d - bd;\ \min(r_1, r_2)\rangle,$$

$$\langle a, b; r_1\rangle -_* \langle c, d; r_2\rangle = \langle \max(0, a - c),\ \min(1,\ b + d,\ 1 - a + c);\ \min(r_1, r_2)\rangle.$$

Fix

$$u_{k_1} = \mathbf{0}_{0.12} = \langle 0, 1; 0.12\rangle.$$

For the basic set

$$\mathcal{B}_1 = \{(k_1, l_1), (k_1, l_2), (k_2, l_2), (k_2, l_3), (k_3, l_3), (k_3, l_4)\},$$

the potentials are propagated via

$$v_{l_j} = c_{k_i,l_j,h_1} -_* u_{k_i}, \qquad u_{k_i} = c_{k_i,l_j,h_1} -_* v_{l_j}.$$

This yields

$$v_{l_1} = \langle 0.61, 0.19; 0.12\rangle, \qquad v_{l_2} = \langle 0.71, 0.09; 0.12\rangle,$$

$$u_{k_2} = \langle 0.41, 0.09; 0.12\rangle -_* \langle 0.71, 0.09; 0.12\rangle = \langle 0, 0.18; 0.12\rangle,$$

$$v_{l_3} = \langle 0.51, 0.10; 0.12\rangle -_* \langle 0, 0.18; 0.12\rangle = \langle 0.51, 0.28; 0.12\rangle,$$

$$u_{k_3} = \langle 0.61, 0.09; 0.12\rangle -_* \langle 0.51, 0.28; 0.12\rangle = \langle 0.10, 0.37; 0.12\rangle,$$

$$v_{l_4} = \langle 0.71, 0.19; 0.12\rangle -_* \langle 0.10, 0.37; 0.12\rangle = \langle 0.61, 0.39; 0.12\rangle.$$

Hence, the circular potentials are

$u_{k_1} = \langle 0, 1; 0.12\rangle$, $u_{k_2} = \langle 0, 0.18; 0.12\rangle$, $u_{k_3} = \langle 0.10, 0.37; 0.12\rangle$,
$v_{l_1} = \langle 0.61, 0.19; 0.12\rangle$, $v_{l_2} = \langle 0.71, 0.09; 0.12\rangle$, $v_{l_3} = \langle 0.51, 0.28; 0.12\rangle$, $v_{l_4} = \langle 0.61, 0.39; 0.12\rangle$.

Step 4.2. Reduced Costs (Layer h_1)

The nonbasic set is

$$D = \{(k_1, l_3), (k_1, l_4), (k_2, l_1), (k_2, l_4), (k_3, l_1), (k_3, l_2)\}.$$

For each $(k_i, l_j) \in D$, compute

$$w_{k_i,l_j} = c_{k_i,l_j,h_1} -_* \left(u_{k_i} \oplus_* v_{l_j}\right).$$

The reduced cost C-IFIM is therefore

$$\left\{\begin{array}{c|cccc} & l_1 & l_2 & l_3 & l_4 \\ \hline k_1 & \perp & \perp & \langle 0.31, 0.69; 0.12\rangle & \langle 0.51, 0.49; 0.12\rangle \\ k_2 & \langle 0.51, 0.49; 0.12\rangle & \perp & \perp & \langle 0.31, 0.6898; 0.12\rangle \\ k_3 & \langle 0.349, 0.651; 0.12\rangle & \langle 0.239, 0.6167; 0.12\rangle & \perp & \perp \end{array}\right\}.$$

Step 4.3. Optimality Test (Layer h_1)

We rank reduced costs using

$$R^{\text{circ}}_{\langle a,b;r\rangle} = \frac{1}{4}(2 - a - b)\left(\frac{|\sqrt{2} - r|}{\sqrt{2}} + 1 - a\right).$$

For the layer threshold, we take

$$\mathbf{0}_{r^1_{\max}} = \langle 0, 1; r^1_{\max}\rangle, \qquad r^1_{\max} = 0.12, \qquad R^{\text{circ}}(\mathbf{0}_{0.12}) \approx 0.4788.$$

The current plan is circularly optimal for layer h_1 if and only if

$$\forall (k_i, l_j) \in D: \quad R^{\text{circ}}(w_{k_i,l_j}) \le R^{\text{circ}}(\mathbf{0}_{0.12}).$$

If this condition holds, no entering cell exists and the algorithm stops for h_1; otherwise, proceed to Step 4.4.

Final Optimal Plan for Layer h_1

Since all reduced costs satisfy

$$R^{\text{circ}}(w_{k_i,l_j,h_1}) \leq R^{\text{circ}}(\mathbf{0}_{0.12}),$$

no improving cell exists and the IBFS is already circularly optimal for layer h_1. Hence

$$X_{\text{opt}}^{(1)} = \left\{ \begin{array}{c|cccc} h_1 & l_1 & l_2 & l_3 & l_4 \\ \hline k_1 & \langle 0.41, 0.20; 0.12\rangle & \langle 0.06, 0.43; 0.12\rangle & \perp & \perp \\ k_2 & \perp & \langle 0.45, 0.07; 0.12\rangle & \langle 0.26, 0.03; 0.12\rangle & \perp \\ k_3 & \perp & \perp & \langle 0.35, 0.17; 0.12\rangle & \langle 0.24, 0.20; 0.12\rangle \end{array} \right\}.$$

Final Optimal Transportation Plans and Layer-wise Summary

Layer $\boldsymbol{h_1}$

$$X_{\text{opt}}^{(1)} = \left\{ \begin{array}{c|cccc} & l_1 & l_2 & l_3 & l_4 \\ \hline k_1 & \langle 0.41, 0.20; 0.12\rangle & \langle 0.06, 0.43; 0.12\rangle & \perp & \perp \\ k_2 & \perp & \langle 0.45, 0.07; 0.12\rangle & \langle 0.26, 0.03; 0.12\rangle & \perp \\ k_3 & \perp & \perp & \langle 0.35, 0.17; 0.12\rangle & \langle 0.24, 0.20; 0.12\rangle \end{array} \right\}.$$

Layer $\boldsymbol{h_2}$

$$X_{\text{opt}}^{(2)} = \left\{ \begin{array}{c|cccc} & l_1 & l_2 & l_3 & l_4 \\ \hline k_1 & \langle 0.44, 0.17; 0.14\rangle & \perp & \perp & \langle 0.06, 0.40; 0.14\rangle \\ k_2 & \perp & \langle 0.54, 0.27; 0.14\rangle & \langle 0.20, 0.39; 0.14\rangle & \perp \\ k_3 & \perp & \perp & \langle 0.44, 0.25; 0.14\rangle & \langle 0.27, 0.17; 0.14\rangle \end{array} \right\}.$$

Layer $\boldsymbol{h_3}$

$$X_{\text{opt}}^{(3)} = \left\{ \begin{array}{c|cccc} & l_1 & l_2 & l_3 & l_4 \\ \hline k_1 & \langle 0.46, 0.15; 0.18\rangle & \perp & \perp & \langle 0.06, 0.37; 0.18\rangle \\ k_2 & \perp & \langle 0.56, 0.25; 0.18\rangle & \langle 0.20, 0.41; 0.18\rangle & \perp \\ k_3 & \perp & \perp & \langle 0.46, 0.26; 0.18\rangle & \langle 0.29, 0.15; 0.18\rangle \end{array} \right\}.$$

Global 3D optimal allocation

$$X_{\text{opt}} = \{X_{\text{opt}}^{(1)}, X_{\text{opt}}^{(2)}, X_{\text{opt}}^{(3)}\}.$$

Numerical Results under Three Aggregation Scenarios

Layer-wise total C-IF costs.

- Layer $\boldsymbol{h_1}$:

$$\langle 0.7403,\ 0.0171;\ 0.12\rangle, \quad \langle 0.583,\ 0.047;\ 0.12\rangle, \quad \langle 0.3997,\ 0.09;\ 0.12\rangle.$$

- Layer $\boldsymbol{h_2}$:

$$\langle 0.75,\ 0.02;\ 0.14\rangle, \quad \langle 0.56,\ 0.19;\ 0.14\rangle, \quad \langle 0.397,\ 0.378;\ 0.14\rangle.$$

- Layer $\boldsymbol{h_3}$:

$$\langle 0.766,\ 0.0135;\ 0.18\rangle, \quad \langle 0.61,\ 0.099;\ 0.18\rangle, \quad \langle 0.301,\ 0.23;\ 0.18\rangle.$$

Final aggregation over layers $H = \{h_1, h_2, h_3\}$.

- Pessimistic aggregation:

$$\mu = \max_g \mu_g, \quad \nu = \min_g \nu_g, \quad r = \min_g r_g.$$

- Realistic aggregation (average):

$$\mu = \frac{1}{3}\sum_{g=1}^{3} \mu_g, \quad \nu = \frac{1}{3}\sum_{g=1}^{3} \nu_g, \quad r = \frac{1}{3}\sum_{g=1}^{3} r_g.$$

- Optimistic aggregation:

$$\mu = \min_g \mu_g, \quad \nu = \max_g \nu_g, \quad r = \max_g r_g.$$

Hence, the aggregated total C-IF transportation cost is

$$C_{\text{tot}}^{\text{pess}} = \langle 0.77,\ 0.0135;\ 0.12\rangle, \qquad C_{\text{tot}}^{\text{real}} = \langle 0.58,\ 0.11;\ 0.15\rangle, \qquad C_{\text{tot}}^{\text{opt}} = \langle 0.31,\ 0.38;\ 0.18\rangle.$$

Crisp projections [11,65,71].

Each C-IF triple is first projected to a normalized fuzzy pair

$$\langle \widehat{\mu}, \widehat{\nu}\rangle = \left\langle \frac{\mu}{\mu+\nu}, \frac{\nu}{\mu+\nu} \right\rangle, \qquad \mu + \nu > 0. \tag{28}$$

Using the inverse of Atanassov's linear normalization [11,65] with $x_{\min} = 15{,}000$ and $x_{\max} = 30{,}000$

$$\widehat{\mu} = \frac{x - x_{\min}}{x_{\max} - x_{\min}}, \qquad \widehat{\nu} = \frac{x_{\max} - x}{x_{\max} - x_{\min}}, \tag{29}$$

we obtain the crisp projection

$$x = x_{\min} + \widehat{\mu}\,(x_{\max} - x_{\min}) = x_{\max} - \widehat{\nu}\,(x_{\max} - x_{\min}). \tag{30}$$

Therefore

$$C_{\text{pess}}^{\text{crisp}} = 15{,}000 + \frac{0.77}{0.77 + 0.0135} \cdot 15{,}000 \approx 29{,}742,$$

$$C_{\text{real}}^{\text{crisp}} = 15{,}000 + \frac{0.58}{0.58 + 0.11} \cdot 15{,}000 \approx 27{,}609,$$

$$C_{\text{opt}}^{\text{crisp}} = 15{,}000 + \frac{0.31}{0.31 + 0.38} \cdot 15{,}000 \approx 21{,}739.$$

Comparative IFS vs. Circular IFS Illustration

To further illustrate the advantages of the proposed circular intuitionistic fuzzy framework, we analyze the obtained optimal transportation plan in layer h_1 under both circular intuitionistic fuzzy (C-IFS) and classical intuitionistic fuzzy (IFS) representations.

Specifically, each Circular Intuitionistic Fuzzy Triple (C-IFT)

$$\langle \mu, \nu; r\rangle$$

is reduced to the corresponding intuitionistic fuzzy pair

$$\langle \mu, \nu\rangle,$$

by omitting the radius parameter [71]. This produces a standard IFS representation with identical central values but without explicit modeling of uncertainty dispersion.

Table 2 presents the nonzero allocation values from the optimal transportation plan for layer h_1 in both representations.

Table 2. Comparison between IFS and C-IFS representations of the nonzero optimal allocations in layer h_1.

Route	IFS Representation	C-IFS Representation
(k_1, l_1, h_1)	$\langle 0.41, 0.20 \rangle$	$\langle 0.41, 0.20; 0.12 \rangle$
(k_1, l_2, h_1)	$\langle 0.06, 0.43 \rangle$	$\langle 0.06, 0.43; 0.12 \rangle$
(k_2, l_2, h_1)	$\langle 0.45, 0.07 \rangle$	$\langle 0.45, 0.07; 0.12 \rangle$
(k_2, l_3, h_1)	$\langle 0.26, 0.03 \rangle$	$\langle 0.26, 0.03; 0.12 \rangle$
(k_3, l_3, h_1)	$\langle 0.35, 0.17 \rangle$	$\langle 0.35, 0.17; 0.12 \rangle$
(k_3, l_4, h_1)	$\langle 0.24, 0.20 \rangle$	$\langle 0.24, 0.20; 0.12 \rangle$

Although both representations share identical central values (μ, ν), the classical IFS model does not capture the dispersion of expert opinions.

This limitation becomes critical when comparing alternatives with similar or identical intuitionistic fuzzy values. In such cases, the IFS framework cannot distinguish between solutions with different levels of uncertainty.

In contrast, the circular intuitionistic fuzzy representation incorporates the radius parameter r, which quantifies the degree of disagreement among experts. This enables a finer discrimination between alternatives and leads to more stable and reliable decision-making.

For example, two alternatives with identical IFS values $\langle 0.61, 0.19 \rangle$ but different circular representations $\langle 0.61, 0.19; 0.12 \rangle$ and $\langle 0.61, 0.19; 0.18 \rangle$ are indistinguishable in the IFS framework, while the C-IFS model prefers the alternative with a smaller radius due to lower uncertainty. Therefore, incorporating the radius parameter significantly enhances the discriminatory power, interpretability, and robustness of the transportation solution.

Numerical Impact of IFS vs. C-IFS Representation

To further illustrate the practical impact of circular intuitionistic fuzzy modeling, we consider a simplified decision scenario within layer h_1.

Suppose two feasible transportation alternatives have identical intuitionistic fuzzy evaluations:

$$A_1 = \langle 0.61, 0.19 \rangle, \qquad A_2 = \langle 0.61, 0.19 \rangle.$$

Within the classical IFS framework, these alternatives are indistinguishable and therefore equivalent from a decision-making perspective.

However, under the circular intuitionistic fuzzy representation, the same alternatives are expressed as:

$$A_1 = \langle 0.61, 0.19; 0.12 \rangle, \qquad A_2 = \langle 0.61, 0.19; 0.18 \rangle.$$

Applying the circular ranking measure

$$R^{\text{circ}}_{\langle a,b;r \rangle} = 0.25(2 - a - b)\left(\frac{|\sqrt{2} - r|}{\sqrt{2}} + 1 - a\right),$$

yields:

$$R^{\text{circ}}(A_1) < R^{\text{circ}}(A_2).$$

Thus, A_1 is preferred over A_2 due to its lower uncertainty dispersion.

This distinction cannot be captured within the classical intuitionistic fuzzy framework, where both alternatives remain equivalent.

Implication.

The absence of a dispersion parameter in IFS models may lead to multiple indistinguishable solutions and ambiguous decision outcomes. In contrast, the circular intuitionistic fuzzy framework introduces an additional discriminative dimension through the radius parameter, enabling a clear ordering of alternatives.

This directly affects reduced cost evaluation, entering-variable selection, and ultimately the structure of the optimal transportation plan.

Consequently, incorporating circular hesitation improves discrimination capability, enhances stability of the optimization process, and leads to more robust and reliable solutions.

7.2. Practical Validation of the Results

Beyond formal optimality conditions, practical validation of the obtained transportation plan is essential for real-world applicability.

The quality and reliability of the proposed solution can be assessed through the following criteria:

(1) Feasibility verification.

All circular supply, demand, and upper cost constraints must be satisfied in each layer $h_g \in H$, ensuring that the solution is admissible under operational and contractual requirements.

(2) Optimality consistency.

For all nonbasic cells, the reduced costs must satisfy

$$R^{\mathrm{circ}}(w_{k_i,l_j,h_g}) \leq R^{\mathrm{circ}}(\mathbf{0}),$$

confirming that no further improvement is possible within the circular intuitionistic fuzzy framework.

(3) Scenario stability.

The solution is evaluated under pessimistic, realistic, and optimistic aggregation scenarios. Stability is confirmed when the allocation structure remains consistent across scenarios.

(4) Sensitivity to uncertainty dispersion.

The influence of the radius parameter r is analyzed by examining how variations in dispersion affect reduced costs and allocation decisions. Stable behavior under such variations indicates robustness of the model.

(5) Comparison with IFS-based solutions.

A comparative analysis with classical IFS models serves as an additional validation layer. If the IFS formulation produces multiple equivalent solutions while the circular model yields a unique or more stable solution, this confirms the added value of incorporating uncertainty dispersion.

(6) Managerial interpretability.

The resulting transportation plan must be interpretable in practical terms, enabling decision-makers to identify critical routes, cost-sensitive allocations, and structurally important connections.

These validation criteria provide a systematic framework for assessing the feasibility, optimality, robustness, and practical relevance of the proposed circular intuitionistic fuzzy transportation model. Overall, the results demonstrate that the proposed circular intuitionistic fuzzy framework not only preserves the theoretical consistency of dual-based

optimization, but also provides superior practical performance in terms of robustness, discrimination capability, and decision interpretability under uncertainty.

8. Discussion

The proposed three-dimensional product-based circular intuitionistic fuzzy potential method extends the classical MODI framework to a circular intuitionistic fuzzy environment while preserving its dual-based optimality structure.

Unlike feasibility-driven zero point approaches, the proposed method provides an explicit marginal interpretation of optimality through circular intuitionistic fuzzy potentials and reduced costs. This enables the evaluation of transportation routes not only in terms of feasibility, but also with respect to their contribution to the global cost structure.

The layer-wise implementation confirms that the algorithm consistently maintains feasibility, supply–demand balance, and upper cost constraints across all product dimensions. The circular ranking mechanism ensures a stable ordering of reduced costs under uncertainty, allowing for meaningful comparison of alternatives even in the presence of expert disagreement.

In contrast to allocation-based methods, the proposed approach preserves the full circular intuitionistic fuzzy structure throughout the optimization process. This avoids information loss caused by ranking-based reductions and enables a transparent analysis of solution stability and sensitivity.

A key advantage of the framework is its ability to capture uncertainty dispersion through the radius parameter. While classical intuitionistic fuzzy (IFS) models rely only on central values (μ, ν), the circular model differentiates alternatives with similar evaluations but different levels of uncertainty, leading to improved discrimination and more reliable decision-making.

From a computational perspective, the indexed matrix representation provides a structured and scalable approach for handling three-dimensional transportation problems. The method remains consistent with classical transportation logic, while extending it to uncertainty-aware environments without increasing asymptotic computational complexity.

From a practical standpoint, the proposed framework is directly applicable to real-world logistics and supply chain systems involving multiple products, uncertain cost evaluations, and expert-based inputs. It enables decision-makers to explicitly incorporate variability and disagreement, which are inherent in real operational environments.

Overall, the results demonstrate that the proposed circular intuitionistic fuzzy framework provides superior discrimination capability, enhanced stability of the optimization process, and improved interpretability compared to classical intuitionistic fuzzy approaches.

9. Conclusions

This study introduces a three-dimensional product-based circular intuitionistic fuzzy potential method for transportation problems under uncertainty.

The main contribution lies in extending the classical MODI framework to circular intuitionistic fuzzy environments using indexed matrix representations and radius-aware operations. The proposed method preserves the dual-based optimality structure while explicitly incorporating hesitation through circular intuitionistic fuzzy modeling.

The numerical study demonstrates that the proposed method:

- Ensures feasibility and admissibility across all product layers;
- Provides interpretable reduced cost optimality conditions;
- Preserves uncertainty information without ranking-based reduction;
- Operates with polynomial-time complexity $O(mn(m+n))$, i.e., $O(n^3)$ for square problems.

The comparative analysis confirms that, unlike allocation-based zero point and zero suffix methods, which achieve optimality procedurally, the proposed approach establishes a dual-based optimality framework with explicit marginal interpretation.

Furthermore, the integration of the radius parameter significantly enhances the discrimination capability of the model. While classical IFS-based approaches may lead to indistinguishable alternatives, the circular intuitionistic fuzzy framework differentiates solutions based on uncertainty dispersion, resulting in more stable optimal allocations and improved decision sensitivity.

Therefore, the proposed 3D–CIFMODI method provides a unified optimization framework that integrates feasibility, optimality, and advanced uncertainty modeling in a consistent and interpretable way. The framework is directly applicable to real-world logistics and supply chain systems characterized by multi-product structures, uncertain parameters, and expert-based evaluations.

In addition, a structured validation procedure is proposed to assess the feasibility, optimality consistency, robustness, and practical interpretability of the obtained solutions.

Future research may focus on hybrid metaheuristic extensions, sensitivity analysis [72], and applications to large-scale transportation and logistics systems.

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Abbreviations

The following abbreviations are used in this manuscript:

Abbreviation	Full Form	Description
TP	Transportation Problem	Classical optimization model for distribution
IF	Intuitionistic Fuzzy	Uncertainty model with membership and non-membership
IFS	Intuitionistic Fuzzy Set	Set with degrees of membership, non-membership, hesitation
C-IF	Circular Intuitionistic Fuzzy	IF model with geometric radius of hesitation
C-IFT	Circular Intuitionistic Fuzzy Triple	Representation $\langle \mu, \nu; r \rangle$
C-IFIM	Circular Intuitionistic Fuzzy Index Matrix	Matrix with C-IFT elements
3D	Three-Dimensional	Multi-layer structure (products/scenarios/time)
MODI	Modified Distribution Method	Potential-based optimization method
CIFMODI	Circular IF MODI	MODI extended to circular IF environment
3D–CIFMODI	Proposed Method	3D circular IF potential-based algorithm

ZPM	Zero Point Method	Allocation-based transportation method
IFZPM	Intuitionistic Fuzzy ZPM	ZPM in intuitionistic fuzzy setting
C-IFZPM	Circular IF ZPM	ZPM in circular IF setting
ZSM	Zero Suffix Method	Allocation-based transportation method
IFZSM	Intuitionistic Fuzzy ZSM	ZSM under IF environment
VAM	Vogel's Approximation Method	Heuristic initial solution method
MVAM	Modified VAM	Fuzzy/IF extension of VAM
IM	Index Matrix	Structured matrix representation of data
AGIO	Aggregate Global Internal Operation	Aggregation operator over matrices

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