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Weighted Approximation by Szász–Mirakyan-Type Operators Preserving Two Exponential Functions

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Abstract

In this paper, we study the weighted approximation properties of a family of Szász–Mirakyan-type operators preserving two exponential functions on the unbounded interval $[0, \infty)$. The operators act on exponential weighted spaces and are analyzed within the framework of positive linear operator theory. We first establish their well-definedness and boundedness between suitable weighted spaces. By applying a weighted Korovkin-type theorem, we prove convergence in the corresponding weighted norm. Furthermore, we obtain quantitative estimates in terms of a weighted modulus of continuity and derive an order of convergence result. A Voronovskaya-type asymptotic formula is also established, describing the precise asymptotic behavior of the operators. Numerical examples are included to support the theoretical results.

Keywords: Szász–Mirakyan operators; exponential preservation; positive linear operators; weighted approximation; Korovkin theorem

MSC: 41A36; 41A25; 41A30; 47B38

1. Introduction

Approximation theory is one of the fundamental areas of mathematics that deals with representing continuous functions by simpler and more manageable structures. In this context, positive linear operators constitute one of the most effective tools since they preserve the basic structural properties of functions. Therefore, investigating the constructive properties of positive linear operators has become an important research topic in approximation theory.

Bernstein [1] introduced Bernstein operators in order to provide a constructive proof of the Weierstrass approximation theorem. Later, Szász [2] extended this structure to infinite intervals and obtained the well-known Szász–Mirakyan operators. These operators preserve constant and linear functions, and their convergence is usually studied by means of Korovkin-type theorems; see [3]. Various modifications of Szász type operators have been widely studied in the literature. In particular, Kantorovich-type modifications, which represent integral versions of these operators, play an important role in approximation theory [4]. In addition, modified Szász operators associated with Hermite polynomials have also been studied and different aspects of their convergence behavior have been revealed [5].



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Various generalizations and variants of Szász type operators have also been proposed. For example, operators based on q -integers [6] and generalized Szász Gamma type operators [7] have provided a broader framework for the analysis of approximation processes and enriched the structure of these operators.

All these developments reveal the limitations of classical approximation methods and indicate the need for new operator structures suitable for more general classes of functions.

In particular, the approximation of functions defined on unbounded intervals involves significant difficulties both theoretically and practically. In such cases, since functions often exhibit exponential growth or decay, the effectiveness of classical approximation methods based on preserving polynomials remains limited. For this reason, the development of operators preserving exponential functions has become an important research direction in order to approximate such functions more accurately and effectively. Nowadays, many researchers study positive linear operators that preserve exponential functions instead of classical polynomial ones since this approach provides better approximation results especially for functions exhibiting exponential behavior.

In parallel with these developments, considerable attention has been devoted to positive linear operators preserving exponential-type functions. In this direction, Acar, Aral and Gonska [8] constructed Szász–Mirakyan-type operators preserving exponential functions and investigated their moment structure and approximation properties. Later, Gupta and Acu [9] developed Baskakov Szász type operators, preserving exponential functions, and obtained results on convergence and error estimates. Furthermore, in [10], more general operator constructions reproducing exponential functions were proposed and their structural and approximation properties were analyzed in detail. A comprehensive and systematic study on the structure, approximation behavior, and various generalizations of such operators was presented by Goyal [11], and this work serves as an important and guiding reference in the related literature.

More recently, different classes of operators preserving exponential functions have also been studied. For instance, Bodur [12] considered bivariate operators, while Herdem [13] investigated other classes of operators possessing the exponential preserving property. In this context, weighted approximation theory is of great importance since it allows controlling the growth behavior of functions defined on unbounded intervals. The main contributions in this area can be found in the works of Holhoş [14] and Ispir [15], as well as in the monograph of Gupta and Agarwal [16].

On the other hand, although various operators preserving exponential functions have been proposed in the literature, most of these constructions focus only on preserving a single exponential function and do not provide sufficient flexibility in adapting to different types of functions. In addition, the approximation properties of such operators in weighted spaces have not been sufficiently addressed.

The operators considered in this study were first introduced in [17], and their basic properties were investigated in the classical setting. These operators preserve two exponential functions, namely $e^{\alpha_1 x}$ and $e^{2\alpha_1 x}$, and provide an appropriate framework for the approximation of functions involving exponential components. Moreover, due to the presence of the parameter sequence (θ_n) , they ensure a continuous transition between different Szász type structures and provide additional flexibility in controlling the approximation behavior.

The main aim of this paper is to investigate the approximation properties of these operators in exponential weighted spaces. In this direction, weighted Korovkin-type theorems are obtained, quantitative estimates in terms of weighted moduli of continuity are derived, and Voronovskaya-type asymptotic results are established. Thus, this study

extends the theory of operators preserving exponential functions to weighted settings and fills an important gap in the literature.

The operators considered in this paper provide a more flexible framework compared to existing constructions. In particular, the presence of the parameter sequence (θ_n) enables a continuous transition between different operator structures and allows the approximation behavior to be adjusted according to the function under consideration. This flexibility makes it possible to obtain better approximation results for suitable choices of (θ_n) , especially for functions involving exponential components.

2. Auxiliary Results

Throughout the paper, we consider real-valued functions defined on $[0, \infty)$.

Let $n \in \mathbb{N}$, $y \in [0, \infty)$ and $k \in \mathbb{N}_0$, define the Poisson-type basis functions

$$p_{n,k}(y) = e^{-ny} \frac{(ny)^k}{k!}.$$

It is well known that

$$\sum_{k=0}^{\infty} p_{n,k}(y) = 1, \quad y \geq 0.$$

Let $c > 0$ be a fixed real parameter and define

$$\alpha_{n,c}(x) = \frac{cx}{n(1 - e^{-c/n})}, \quad x \geq 0.$$

Following [17], for a function $f : [0, \infty) \rightarrow \mathbb{R}$ we define

$$(S_n^{(c)}f)(x) = \sum_{k=0}^{\infty} p_{n,k}(\alpha_{n,c}(x)) f\left(\frac{k}{n}\right), \quad x \geq 0. \quad (1)$$

Let (θ_n) be a sequence of real numbers satisfying

$$0 < \theta_n < 1, \quad n \in \mathbb{N},$$

and, when needed, assume that $\theta_n \rightarrow \theta \in (0, 1)$ as $n \rightarrow \infty$:

$$(L_n f)(x) = \theta_n (S_n^{(1)} f)(x) + (1 - \theta_n) (S_n^{(2)} f)(x), \quad x \geq 0. \quad (2)$$

These operators were introduced in [17] and shown to preserve two exponential functions.

Lemma 1. Let $a \in \mathbb{R}$ and $c > 0$. Then, for every $x \geq 0$,

$$(S_n^{(c)}(e^{at}))(x) = \exp\left(\frac{cx}{1 - e^{-c/n}}(e^{a/n} - 1)\right).$$

Proof. By (1) and the definition of $p_{n,k}$, we write

$$(S_n^{(c)}(e^{at}))(x) = \sum_{k=0}^{\infty} e^{-n\alpha_{n,c}(x)} \frac{(n\alpha_{n,c}(x))^k}{k!} e^{ak/n}.$$

Thus

$$(S_n^{(c)}(e^{at}))(x) = e^{-n\alpha_{n,c}(x)} \sum_{k=0}^{\infty} \frac{(n\alpha_{n,c}(x)e^{a/n})^k}{k!}.$$

We obtain

$$(S_n^{(c)}(e^{at}))(x) = e^{-n\alpha_{n,c}(x)} \exp(n\alpha_{n,c}(x)e^{a/n}) = \exp(n\alpha_{n,c}(x)(e^{a/n} - 1)).$$

Finally, substituting

$$\alpha_{n,c}(x) = \frac{cx}{n(1 - e^{-c/n})}$$

yields the desired formula. $\square$

Remark 1. For $\theta_n = 0$, the operators reduce to $L_n = S_n^{(2)}$. Then, for $f(t) = e^{-2t}$, we have

$$L_n(e^{-2t}; x) = e^{-n\alpha_{n,2}(x)} \sum_{k=0}^{\infty} \frac{(n\alpha_{n,2}(x))^k}{k!} e^{-2k/n} = e^{-n\alpha_{n,2}(x)(1 - e^{-2/n})}.$$

Using

$$\alpha_{n,2}(x) = \frac{2x}{n(1 - e^{-2/n})} = \frac{2xe^{2/n}}{n(e^{2/n} - 1)},$$

we obtain

$$L_n(e^{-2t}; x) = e^{-2x}.$$

Thus, in this case, the operators coincide with the modification introduced in [18].

Similarly, for $\theta_n = 1$, the operators reduce to $S_n^{(1)}$ and reproduce the function e^{-x} , that is,

$$L_n(e^{-t}; x) = e^{-x}.$$

Lemma 2. Let $x \geq 0$ be fixed. For the operators

$$L_n = \theta_n S_n^{(1)} + (1 - \theta_n) S_n^{(2)}, \quad 0 < \theta_n < 1,$$

where (θ_n) is a sequence of real numbers such that $\theta_n \rightarrow \theta \in (0, 1)$ as $n \rightarrow \infty$, define the central moments

$$\mu_{n,r}(x) := L_n((t - x)^r; x), \quad r = 1, 2.$$

Then the following statements hold:

(i)

$$\lim_{n \rightarrow \infty} n\mu_{n,1}(x) = x \left(1 - \frac{\theta}{2}\right).$$

(ii)

$$\lim_{n \rightarrow \infty} n\mu_{n,2}(x) = x.$$

(iii)

$$L_n((t - x)^4; x) = O(n^{-2}) \quad (n \rightarrow \infty).$$

Proof. We first compute the first moments of the operators. Clearly,

$$L_n(1; x) = 1.$$

Moreover, by linearity and the definition of $S_n^{(c)}$, we have

$$L_n(t; x) = \theta_n \alpha_{n,1}(x) + (1 - \theta_n) \alpha_{n,2}(x),$$

and

$$L_n(t^2; x) = \theta_n \left(\alpha_{n,1}(x)^2 + \frac{\alpha_{n,1}(x)}{n} \right) + (1 - \theta_n) \left(\alpha_{n,2}(x)^2 + \frac{\alpha_{n,2}(x)}{n} \right).$$

From these expressions, the central moments can be written as

$$\mu_{n,1}(x) = L_n(t - x; x) = L_n(t; x) - x,$$

and

$$\mu_{n,2}(x) = L_n((t - x)^2; x) = L_n(t^2; x) - 2xL_n(t; x) + x^2.$$

Since

$$S_n^{(c)}(t; x) = \alpha_{n,c}(x),$$

we have

$$S_n^{(c)}(t - x; x) = \alpha_{n,c}(x) - x.$$

Using the expansion

$$\alpha_{n,c}(x) = \frac{cx}{n(1 - e^{-c/n})} = x + \frac{c}{2n}x + O(n^{-2}),$$

it follows that

$$\alpha_{n,c}(x) - x = \frac{c}{2n}x + O(n^{-2}).$$

Hence

$$\mu_{n,1}(x) = \theta_n(\alpha_{n,1}(x) - x) + (1 - \theta_n)(\alpha_{n,2}(x) - x) = \frac{x}{2n}(\theta_n + 2(1 - \theta_n)) + O(n^{-2}),$$

which yields

$$n\mu_{n,1}(x) = x \left(1 - \frac{\theta_n}{2} \right) + O(n^{-1}).$$

Letting $n \rightarrow \infty$ proves (i).

For the second central moment, the Poisson-type formula gives

$$S_n^{(c)}((t - x)^2; x) = (\alpha_{n,c}(x) - x)^2 + \frac{\alpha_{n,c}(x)}{n}.$$

Since $(\alpha_{n,c}(x) - x)^2 = O(n^{-2})$ and

$$\frac{\alpha_{n,c}(x)}{n} = \frac{x}{n} + O(n^{-2}),$$

we obtain

$$\mu_{n,2}(x) = \frac{x}{n} + O(n^{-2}),$$

and thus $n\mu_{n,2}(x) \rightarrow x$, proving (ii).

Finally, standard central moment computations for Poisson-type kernels yield (see [19])

$$S_n^{(c)}((t - x)^4; x) = O(n^{-2}),$$

uniformly for fixed $x \geq 0$. Since L_n is a convex combination of such operators, (iii) follows. $\square$

We briefly recall the weighted function spaces used in the sequel; see [3]

3. Weighted Approximation

Let $I = [0, \infty)$. For a continuous weight function $w : I \rightarrow [1, \infty)$ define

$$B_w(I) := \left\{ f : I \rightarrow \mathbb{R} : |f(x)| \leq M_f w(x) \text{ for all } x \in I \right\},$$

$$C_w(I) := \{f \in B_w(I) : f \text{ is continuous on } I\}.$$

endowed with the norm

$$\|f\|_w := \sup_{x \in I} \frac{|f(x)|}{w(x)}.$$

Let α_1 and α_2 be fixed positive real numbers such that $0 < \alpha_1 < \alpha_2$,

$$w_{\alpha_1}(x) = e^{2\alpha_1 x}, \quad w_{\alpha_2}(x) = e^{2\alpha_2 x}, \quad x \geq 0.$$

Then

$$\lim_{x \rightarrow \infty} \frac{w_{\alpha_1}(x)}{w_{\alpha_2}(x)} = \lim_{x \rightarrow \infty} e^{2(\alpha_1 - \alpha_2)x} = 0,$$

and the test functions $1, e^{\alpha_1 x}, e^{2\alpha_1 x}$ belong to $C_{w_{\alpha_1}}(I)$. In addition to $C_{w_{\alpha_1}}(I)$, we consider the subclass

$$C_{w_{\alpha_1}}^k(I) = \left\{ f \in C_{w_{\alpha_1}}(I) : \lim_{x \rightarrow \infty} \frac{f(x)}{w_{\alpha_1}(x)} \text{ exists as a finite constant} \right\}.$$

This class plays an essential role in the study of quantitative approximation on unbounded intervals; see [14].

Functions in $C_{w_{\alpha_1}}^k(I)$ are characterized by the existence of a finite weighted limit at infinity, that is,

$$\lim_{x \rightarrow \infty} \frac{f(x)}{w_{\alpha_1}(x)}$$

exists. This condition provides a control on the growth behavior of functions at infinity in the weighted sense.

Theorem 1 ([3]). Assume

$$\lim_{x \rightarrow \infty} \frac{w_{\alpha_1}(x)}{w_{\alpha_2}(x)} = 0.$$

Let (A_n) be a sequence of positive linear operators

$$A_n : C_{w_{\alpha_1}}(I) \rightarrow B_{w_{\alpha_2}}(I).$$

If

$$\lim_{n \rightarrow \infty} \|A_n(1) - 1\|_{w_{\alpha_2}} = 0, \quad \lim_{n \rightarrow \infty} \|A_n(e^{\alpha_1 t}) - e^{\alpha_1 x}\|_{w_{\alpha_2}} = 0,$$

and

$$\lim_{n \rightarrow \infty} \|A_n(e^{2\alpha_1 t}) - e^{2\alpha_1 x}\|_{w_{\alpha_2}} = 0,$$

then

$$\lim_{n \rightarrow \infty} \|A_n f - f\|_{w_{\alpha_2}} = 0 \quad \text{for every } f \in C_{w_{\alpha_1}}(I).$$

Recall from Lemma 1 that for $a \in \mathbb{R}$ and $c > 0$,

$$(S_n^{(c)}(e^{at}))(x) = \exp(x q_{n,c}(a)), \quad q_{n,c}(a) := \frac{c}{1 - e^{-c/n}} (e^{a/n} - 1). \quad (3)$$

Moreover $L_n = \theta_n S_n^{(1)} + (1 - \theta_n) S_n^{(2)}$ with $0 < \theta_n < 1$. Hence each L_n is a positive linear operator on $[0, \infty)$.

Lemma 3. For every fixed $a > 0$ and $c > 0$ one has

$$\lim_{n \rightarrow \infty} q_{n,c}(a) = a.$$

Consequently, if $0 < \alpha_1 < \alpha_2$, then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$q_{n,1}(2\alpha_1) < 2\alpha_2, \quad q_{n,2}(2\alpha_1) < 2\alpha_2,$$

and

$$q_{n,1}(\alpha_1) < 2\alpha_2, \quad q_{n,2}(\alpha_1) < 2\alpha_2.$$

Proof. Using the standard limits

$$\lim_{n \rightarrow \infty} \frac{e^{a/n} - 1}{a/n} = 1, \quad \lim_{n \rightarrow \infty} \frac{1 - e^{-c/n}}{c/n} = 1,$$

we write

$$q_{n,c}(a) = \frac{e^{a/n} - 1}{a/n} \cdot \frac{c/n}{1 - e^{-c/n}} \cdot a.$$

Since each of the first two factors converges to 1 as $n \rightarrow \infty$, we obtain

$$q_{n,c}(a) \rightarrow a,$$

which proves the first statement.

Now let $0 < \alpha_1 < \alpha_2$ be fixed. Since $q_{n,c}(\alpha_1) \rightarrow \alpha_1$, by the definition of limit, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|q_{n,c}(\alpha_1) - \alpha_1| < \varepsilon \quad \text{for all } n \geq n_0.$$

Since $0 < \alpha_1 < \alpha_2$, we have $2\alpha_2 - \alpha_1 > 0$. Define $\varepsilon := 2\alpha_2 - \alpha_1$. Then $\varepsilon > 0$, and for $n \geq n_0$,

$$|q_{n,c}(\alpha_1) - \alpha_1| < 2\alpha_2 - \alpha_1,$$

which implies

$$q_{n,c}(\alpha_1) < \alpha_1 + (2\alpha_2 - \alpha_1) = 2\alpha_2.$$

Thus $q_{n,c}(\alpha_1) < 2\alpha_2$ for all $n \geq n_0$.

The same argument applies to $q_{n,c}(2\alpha_1)$. Since $q_{n,c}(2\alpha_1) \rightarrow 2\alpha_1$ and $2\alpha_1 < 2\alpha_2$, choosing $\varepsilon := 2\alpha_2 - 2\alpha_1 > 0$ yields, for sufficiently large n ,

$$q_{n,c}(2\alpha_1) < 2\alpha_2.$$

Finally, taking $c = 1$ and $c = 2$ separately gives the stated inequalities. $\square$

Lemma 4. Let $0 < \alpha_1 < \alpha_2$. There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the operator

$$L_n : C_{w_{\alpha_1}}(I) \rightarrow B_{w_{\alpha_2}}(I)$$

is well-defined and bounded. More precisely, for every $f \in C_{w_{\alpha_1}}(I)$ and $x \geq 0$,

$$|L_n f(x)| \leq \|f\|_{w_{\alpha_1}} \left(\theta_n e^{x q_{n,1}(2\alpha_1)} + (1 - \theta_n) e^{x q_{n,2}(2\alpha_1)} \right), \quad (4)$$

and therefore

$$\|L_n f\|_{w_{\alpha_2}} \leq \|f\|_{w_{\alpha_1}}, \quad n \geq n_0. \quad (5)$$

Proof. Fix $f \in C_{w_{\alpha_1}}(I)$. By definition of the norm, for every $t \geq 0$,

$$|f(t)| \leq \|f\|_{w_{\alpha_1}} e^{2\alpha_1 t}.$$

Using positivity of $S_n^{(c)}$,

$$|S_n^{(c)} f(x)| \leq \|f\|_{w_{\alpha_1}} S_n^{(c)}(e^{2\alpha_1 t})(x).$$

By (3) with $a = 2\alpha_1$, we obtain

$$S_n^{(c)}(e^{2\alpha_1 t})(x) = e^{xq_{n,c}(2\alpha_1)},$$

where

$$q_{n,c}(a) := \frac{c}{1 - e^{-c/n}} (e^{a/n} - 1).$$

Hence,

$$|S_n^{(c)} f(x)| \leq \|f\|_{w_{\alpha_1}} e^{xq_{n,c}(2\alpha_1)}.$$

Combining $L_n = \theta_n S_n^{(1)} + (1 - \theta_n) S_n^{(2)}$ yields (4).

Now divide by $e^{2\alpha_2 x}$ and take the supremum over $x \geq 0$. By Lemma 3, we have $q_{n,c}(2\alpha_1) \rightarrow 2\alpha_1 < 2\alpha_2$ as $n \rightarrow \infty$. Hence, there exists n_0 such that for all $n \geq n_0$,

$$q_{n,c}(2\alpha_1) < 2\alpha_2.$$

Thus,

$$\sup_{x \geq 0} \frac{e^{xq_{n,c}(2\alpha_1)}}{e^{2\alpha_2 x}} = \sup_{x \geq 0} e^{-(2\alpha_2 - q_{n,c}(2\alpha_1))x} = 1.$$

This proves (5). $\square$

Lemma 5. Let $\lambda > 0$ and $\Delta \geq 0$ with $\Delta < \lambda$. Then

$$\sup_{x \geq 0} e^{-\lambda x} (e^{\Delta x} - 1) \leq \frac{\Delta}{e(\lambda - \Delta)}.$$

Proof. For $x \geq 0$ and $\Delta x \geq 0$, one has

$$e^{\Delta x} - 1 \leq \Delta x e^{\Delta x},$$

hence

$$e^{-\lambda x} (e^{\Delta x} - 1) \leq \Delta x e^{-(\lambda - \Delta)x}.$$

Consider the function

$$\phi(x) = x e^{-(\lambda - \Delta)x}, \quad x \geq 0.$$

Since $\lambda - \Delta > 0$, differentiating gives

$$\phi'(x) = e^{-(\lambda - \Delta)x} (1 - (\lambda - \Delta)x),$$

so the maximum is attained at $x = \frac{1}{\lambda - \Delta}$. Therefore

$$\sup_{x \geq 0} \phi(x) = \frac{1}{e(\lambda - \Delta)}.$$

Multiplying by Δ , we obtain the desired estimate. $\square$

Lemma 6. Let $0 < \alpha_1 < \alpha_2$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\|L_n(1) - 1\|_{w_{\alpha_2}} = 0,$$

and

$$\lim_{n \rightarrow \infty} \|L_n(e^{\alpha_1 t}) - e^{\alpha_1 x}\|_{w_{\alpha_2}} = 0, \quad \lim_{n \rightarrow \infty} \|L_n(e^{2\alpha_1 t}) - e^{2\alpha_1 x}\|_{w_{\alpha_2}} = 0.$$

Proof. First, $S_n^{(c)}(1) = 1$ implies $L_n(1) = 1$, hence $\|L_n(1) - 1\|_{w_{\alpha_2}} = 0$.

Next, by (3) and $L_n = \theta_n S_n^{(1)} + (1 - \theta_n) S_n^{(2)}$,

$$L_n(e^{at})(x) = \theta_n e^{xq_{n,1}(a)} + (1 - \theta_n) e^{xq_{n,2}(a)}.$$

Fix $a \in \{\alpha_1, 2\alpha_1\}$ and write $\Delta_{n,c}(a) := q_{n,c}(a) - a$. By Lemma 3, $\Delta_{n,c}(a) \rightarrow 0$ as $n \rightarrow \infty$.

Moreover, since $q_{n,c}(a) \rightarrow a < 2\alpha_2$, there exists n_0 such that for $n \geq n_0$, $q_{n,c}(a) < 2\alpha_2$, so $\Delta_{n,c}(a) < 2\alpha_2 - a$.

We also note that $\Delta_{n,c}(a) \geq 0$ for $a > 0$: indeed, with $u = \frac{a}{n}$ and $v = \frac{c}{n}$, using $e^u - 1 \geq u$ and $1 - e^{-v} \leq v$ for $u \geq 0$ and $v \geq 0$, which follow from the monotonicity and convexity properties of the exponential function.

$$q_{n,c}(a) = \frac{c}{1 - e^{-v}}(e^u - 1) \geq \frac{c}{v} u = a.$$

For each $c \in \{1, 2\}$,

$$\frac{|e^{xq_{n,c}(a)} - e^{ax}|}{e^{2\alpha_2 x}} = e^{-(2\alpha_2 - a)x} (e^{\Delta_{n,c}(a)x} - 1).$$

Applying Lemma 5 with $\lambda = 2\alpha_2 - a$ and $\Delta = \Delta_{n,c}(a)$ yields

$$\sup_{x \geq 0} \frac{|e^{xq_{n,c}(a)} - e^{ax}|}{e^{2\alpha_2 x}} \leq \frac{\Delta_{n,c}(a)}{e(2\alpha_2 - a - \Delta_{n,c}(a))} \xrightarrow{n \rightarrow \infty} 0.$$

Finally, since $0 < \theta_n < 1$,

$$\|L_n(e^{at}) - e^{ax}\|_{w_{\alpha_2}} \leq \theta_n \|e^{xq_{n,1}(a)} - e^{ax}\|_{w_{\alpha_2}} + (1 - \theta_n) \|e^{xq_{n,2}(a)} - e^{ax}\|_{w_{\alpha_2}} \rightarrow 0,$$

for $a = \alpha_1$ and $a = 2\alpha_1$. $\square$

Theorem 2. Let $0 < \alpha_1 < \alpha_2$. Then, for every $f \in C_{w_{\alpha_1}}(I)$,

$$\lim_{n \rightarrow \infty} \|L_n f - f\|_{w_{\alpha_2}} = 0.$$

Proof. By Lemma 4, there exists n_0 such that for all $n \geq n_0$,

$$L_n : C_{w_{\alpha_1}}(I) \rightarrow B_{w_{\alpha_2}}(I)$$

is a bounded positive linear operator.

By Lemma 6, the Korovkin test conditions in Theorem 1 hold for the operator $(L_n)_{n \geq n_0}$. Hence Theorem 1 yields

$$\lim_{n \rightarrow \infty} \|L_n f - f\|_{w_{\alpha_2}} = 0, \quad f \in C_{w_{\alpha_1}}(I).$$

Since the above inequality holds for all sufficiently large n , the proof is complete. $\square$

Remark 2. If we choose $\theta_n = 1$ for all n , then $L_n = S_n^{(1)}$ and the operators reduce to the Szász–Mirakyan-type operator corresponding to the parameter $c = 1$. If we choose $\theta_n = 0$ for all n , then $L_n = S_n^{(2)}$, which corresponds to the operator with parameter $c = 2$. Thus, the sequence (L_n) provides a convex interpolation between these two Szász type constructions, and Theorem 2 includes both extreme cases as special instances.

4. Order of Convergence

In quantitative weighted approximation on unbounded intervals, the additional condition $f \in C_{w_{\alpha_1}}^k(I)$ is often imposed in order to control the behaviour of f at infinity. This condition ensures that the weighted growth of f is asymptotically comparable to the weight function. For $f \in C_{w_{\alpha_1}}(I)$ and $\delta > 0$, define the weighted modulus

$$\Omega_{w_{\alpha_1}}(f; \delta) = \sup_{\substack{x, t \geq 0 \\ |t-x| \leq \delta}} \frac{|f(t) - f(x)|}{w_{\alpha_1}(x) + w_{\alpha_1}(t)}.$$

(See [20].)

Lemma 7. If $f \in C_{w_{\alpha_1}}^k(I)$, then

$$\lim_{\delta \rightarrow 0} \Omega_{w_{\alpha_1}}(f; \delta) = 0.$$

Lemma 8. For every $x \geq 0$ and $a > 0$, one has

$$L_n(1; x) = 1,$$

$$L_n(e^{at}; x) = \theta_n e^{xq_{n,1}(a)} + (1 - \theta_n) e^{xq_{n,2}(a)},$$

and

$$L_n(e^{2at}; x) = \theta_n e^{xq_{n,1}(2a)} + (1 - \theta_n) e^{xq_{n,2}(2a)}.$$

Proof. By definition, $L_n = \theta_n S_n^{(1)} + (1 - \theta_n) S_n^{(2)}$ and each $S_n^{(c)}$ is a positive linear operator with $S_n^{(c)}(1; x) = 1$. Hence $L_n(1; x) = 1$.

Moreover, the exponential formula

$$S_n^{(c)}(e^{at})(x) = e^{xq_{n,c}(a)}$$

yields

$$L_n(e^{at}; x) = \theta_n e^{xq_{n,1}(a)} + (1 - \theta_n) e^{xq_{n,2}(a)}.$$

The case $2a$ can be treated similarly. $\square$

Lemma 9. Let $a > 0$, $c > 0$ and $n \in \mathbb{N}$. Define

$$q_{n,c}(a) = \frac{c}{1 - e^{-c/n}} (e^{a/n} - 1), \quad \Delta_{n,c}(a) := q_{n,c}(a) - a.$$

$\Delta_{n,c}(a) \geq 0$ and

$$0 \leq \Delta_{n,c}(a) \leq \frac{a(a+c)e^{a+c}}{n}.$$

Proof. Put $u = a/n$ and $v = c/n$. Since $e^u - 1 \geq u$ and $1 - e^{-v} \leq v$,

$$q_{n,c}(a) = \frac{c}{1 - e^{-v}}(e^u - 1) \geq \frac{c}{v} u = a,$$

hence $\Delta_{n,c}(a) \geq 0$.

For the upper bound, use $e^u - 1 \leq ue^u$ and $1 - e^{-v} \geq ve^{-v}$:

$$q_{n,c}(a) = \frac{c}{1 - e^{-v}}(e^u - 1) \leq \frac{c}{ve^{-v}}(ue^u) = ae^{u+v}.$$

$$0 \leq q_{n,c}(a) - a \leq a(e^{u+v} - 1) \leq a(u+v)e^{u+v} = \frac{a(a+c)}{n}e^{u+v} \leq \frac{a(a+c)e^{a+c}}{n}.$$

□

Lemma 10. Assume that $0 < \alpha_1 < \alpha_2$. Then there exist $n_0 \in \mathbb{N}$ and a constant $C_{\alpha_1, \alpha_2} > 0$ such that for all $n \geq n_0$,

$$\sup_{x \geq 0} \frac{L_n((e^{\alpha_1 t} - e^{\alpha_1 x})^2; x)}{e^{2\alpha_2 x}} \leq \frac{C_{\alpha_1, \alpha_2}}{n}.$$

Proof. Fix $c \in \{1, 2\}$. Using $S_n^{(c)}(e^{at}; x) = e^{xq_{n,c}(a)}$, we obtain

$$S_n^{(c)}((e^{\alpha_1 t} - e^{\alpha_1 x})^2; x) = e^{xq_{n,c}(2\alpha_1)} - 2e^{\alpha_1 x} e^{xq_{n,c}(\alpha_1)} + e^{2\alpha_1 x}.$$

Write $q_{n,c}(a) = a + \Delta_{n,c}(a)$ with $\Delta_{n,c}(a) \geq 0$. Then

$$S_n^{(c)}((e^{\alpha_1 t} - e^{\alpha_1 x})^2; x) = e^{2\alpha_1 x} (e^{x\Delta_{n,c}(2\alpha_1)} - 2e^{x\Delta_{n,c}(\alpha_1)} + 1),$$

hence

$$\frac{S_n^{(c)}((e^{\alpha_1 t} - e^{\alpha_1 x})^2; x)}{e^{2\alpha_2 x}} = e^{-2(\alpha_2 - \alpha_1)x} (e^{x\Delta_{n,c}(2\alpha_1)} - 2e^{x\Delta_{n,c}(\alpha_1)} + 1).$$

Using $e^y - 1 \leq ye^y$ for $y \geq 0$, we get

$$\frac{S_n^{(c)}((e^{\alpha_1 t} - e^{\alpha_1 x})^2; x)}{e^{2\alpha_2 x}} \leq e^{-2(\alpha_2 - \alpha_1)x} (x\Delta_{n,c}(2\alpha_1)e^{x\Delta_{n,c}(2\alpha_1)} + 2x\Delta_{n,c}(\alpha_1)e^{x\Delta_{n,c}(\alpha_1)}).$$

By Lemma 9 and since $c \leq 2$,

$$\Delta_{n,c}(2\alpha_1) \leq \frac{K_{\alpha_1}}{n}, \quad \Delta_{n,c}(\alpha_1) \leq \frac{H_{\alpha_1}}{n},$$

where one may take

$$K_{\alpha_1} := 2\alpha_1(2\alpha_1 + 2)e^{2\alpha_1 + 2}, \quad H_{\alpha_1} := \alpha_1(\alpha_1 + 2)e^{\alpha_1 + 2}.$$

We choose n_0 such that for all $n \geq n_0$,

$$\max\left\{\frac{K_{\alpha_1}}{n}, \frac{H_{\alpha_1}}{n}\right\} \leq (\alpha_2 - \alpha_1).$$

Then, for $n \geq n_0$,

$$e^{-2(\alpha_2 - \alpha_1)x} e^{x\Delta_{n,c}(2\alpha_1)} \leq e^{-(\alpha_2 - \alpha_1)x}, \quad e^{-2(\alpha_2 - \alpha_1)x} e^{x\Delta_{n,c}(\alpha_1)} \leq e^{-(\alpha_2 - \alpha_1)x}.$$

Hence, for $n \geq n_0$,

$$\frac{S_n^{(c)}((e^{\alpha_1 t} - e^{\alpha_1 x})^2; x)}{e^{2\alpha_2 x}} \leq (\Delta_{n,c}(2\alpha_1) + 2\Delta_{n,c}(\alpha_1)) x e^{-(\alpha_2 - \alpha_1)x}.$$

Taking the supremum and using $\sup_{x \geq 0} x e^{-kx} = \frac{1}{ek}$, we obtain

$$\sup_{x \geq 0} \frac{S_n^{(c)}((e^{\alpha_1 t} - e^{\alpha_1 x})^2; x)}{e^{2\alpha_2 x}} \leq \frac{1}{e(\alpha_2 - \alpha_1)} (\Delta_{n,c}(2\alpha_1) + 2\Delta_{n,c}(\alpha_1)) \leq \frac{C_{\alpha_1, \alpha_2}}{n},$$

for

$$C_{\alpha_1, \alpha_2} := \frac{1}{e(\alpha_2 - \alpha_1)} (K_{\alpha_1} + 2H_{\alpha_1}).$$

Here, K_{α_1} and H_{α_1} depend only on α_1 , while C_{α_1, α_2} depends on both α_1 and α_2 ; all these constants are independent of n .

Finally, since $L_n = \theta_n S_n^{(1)} + (1 - \theta_n) S_n^{(2)}$ with $0 < \theta_n < 1$, the same bound holds for L_n . $\square$

Theorem 3. Assume that $0 < \alpha_1 < \alpha_2$. Then there exist $n_0 \in \mathbb{N}$ and a constant $C_{\alpha_1, \alpha_2} > 0$ such that for every $f \in C_{w_{\alpha_1}}^k(I)$ and all $n \geq n_0$,

$$\|L_n f - f\|_{w_{\alpha_2}} \leq 2\Omega_{w_{\alpha_1}}\left(f; \frac{1}{\sqrt{n}}\right) \left(2 + \frac{\sqrt{C_{\alpha_1, \alpha_2}}}{\alpha_1}\right).$$

Proof. Let $f \in C_{w_{\alpha_1}}(I)$ and $\delta > 0$. Using the standard inequality for the weighted modulus of continuity (see [20]), we have

$$|f(t) - f(x)| \leq 2(e^{\alpha_1 t} + e^{\alpha_1 x}) \left(1 + \frac{|e^{\alpha_1 t} - e^{\alpha_1 x}|}{\alpha_1 \delta}\right) \Omega_{w_{\alpha_1}}(f; \delta), \quad x, t \geq 0.$$

Applying $L_n(\cdot; x)$ and using positivity, we obtain

$$\begin{aligned} |L_n f(x) - f(x)| &\leq 2\Omega_{w_{\alpha_1}}(f; \delta) \left[L_n(e^{\alpha_1 t}; x) + e^{\alpha_1 x} \right. \\ &\quad \left. + \frac{1}{\alpha_1 \delta} L_n((e^{\alpha_1 t} + e^{\alpha_1 x})|e^{\alpha_1 t} - e^{\alpha_1 x}|; x) \right]. \end{aligned}$$

Using the Cauchy–Schwarz inequality,

$$L_n((e^{\alpha_1 t} + e^{\alpha_1 x})|e^{\alpha_1 t} - e^{\alpha_1 x}|; x) \leq \sqrt{L_n((e^{\alpha_1 t} + e^{\alpha_1 x})^2; x)} \sqrt{L_n((e^{\alpha_1 t} - e^{\alpha_1 x})^2; x)}.$$

Moreover,

$$(e^{\alpha_1 t} + e^{\alpha_1 x})^2 \leq 2(e^{2\alpha_1 t} + e^{2\alpha_1 x}),$$

hence

$$L_n((e^{\alpha_1 t} + e^{\alpha_1 x})^2; x) \leq 2(L_n(e^{2\alpha_1 t}; x) + e^{2\alpha_1 x}).$$

For $n \geq n_0$, we have $q_{n,c}(\alpha_1) < 2\alpha_2$ and $q_{n,c}(2\alpha_1) < 2\alpha_2$ for $c \in \{1, 2\}$. Hence, after dividing by $e^{2\alpha_2 x}$ and taking the supremum over $x \geq 0$,

$$\sup_{x \geq 0} \frac{L_n(e^{\alpha_1 t}; x)}{e^{2\alpha_2 x}} \leq 1, \quad \sup_{x \geq 0} \frac{L_n(e^{2\alpha_1 t}; x)}{e^{2\alpha_2 x}} \leq 1.$$

Finally, Lemma 10 yields

$$\sup_{x \geq 0} \frac{L_n((e^{\alpha_1 t} - e^{\alpha_1 x})^2; x)}{e^{2\alpha_2 x}} \leq \frac{C_{\alpha_1, \alpha_2}}{n} \quad (n \geq n_0).$$

Collecting these estimates, we obtain

$$\|L_n f - f\|_{w_{\alpha_2}} \leq 2 \Omega_{w_{\alpha_1}}(f; \delta) \left(2 + \frac{1}{\alpha_1 \delta} \sqrt{\frac{C_{\alpha_1, \alpha_2}}{n}} \right),$$

and choosing $\delta = 1/\sqrt{n}$ completes the proof. $\square$

5. Voronovskaya Type Theorem

Theorem 4. Let $0 < \alpha_1 < \alpha_2$ and assume that $f, f', f'' \in C_{w_{\alpha_1}}(I)$. Fix $x \geq 0$ and assume that f'' is continuous at x . Assume additionally that $f'' \in C_{w_{\alpha_1}}^k(I)$. If $\theta_n \rightarrow \theta \in (0, 1)$, then

$$\lim_{n \rightarrow \infty} n \left(L_n(f; x) - f(x) + e^x f'(x) (L_n(e^{-t}; x) - e^{-x}) - \frac{e^{2x}}{2} (f''(x) + f'(x)) (L_n(e^{-2t}; x) - e^{-2x}) \right) = 0.$$

The first limit identity represents a refined asymptotic expansion involving exponential test functions. By using the asymptotic behavior of the central moments given in Lemma 2, this expression reduces to the classical Voronovskaya-type formula stated below. In particular,

$$\lim_{n \rightarrow \infty} n (L_n(f; x) - f(x)) = x \left(1 - \frac{\theta}{2} \right) f'(x) + \frac{x}{2} f''(x).$$

Proof. Fix $x \geq 0$ and put $u_0 = e^{-x}$. Define $G : (0, 1] \rightarrow \mathbb{R}$ by $G(u) = f(-\ln u)$. Note that the transformation $u = e^{-t}$ maps $[0, \infty)$ onto $(0, 1]$, and thus G is well-defined since f is defined on $[0, \infty)$. For $t \geq 0$ set $u = e^{-t}$. Applying Taylor's formula to G at u_0 gives

$$G(u) - G(u_0) = G'(u_0)(u - u_0) + \frac{1}{2} G''(u_0)(u - u_0)^2 + R_x(t),$$

where

$$R_x(t) = \frac{1}{2} (G''(\eta_{x,t}) - G''(u_0))(u - u_0)^2$$

for some $\eta_{x,t}$ between u and u_0 . Since $u = e^{-t}$ and $u_0 = e^{-x}$, we obtain

$$f(t) - f(x) = G'(u_0) (e^{-t} - e^{-x}) + \frac{1}{2} G''(u_0) (e^{-t} - e^{-x})^2 + R_x(t). \quad (6)$$

A direct computation yields

$$G'(u_0) = -e^x f'(x), \quad G''(u_0) = e^{2x} (f''(x) + f'(x)).$$

Applying $L_n(\cdot; x)$ to (6) and using $L_n(1; x) = 1$, we obtain the estimate

$$L_n(f; x) - f(x) = -e^x f'(x) (L_n(e^{-t}; x) - e^{-x}) + \frac{e^{2x}}{2} (f''(x) + f'(x)) (L_n(e^{-2t}; x) - e^{-2x}) + L_n(R_x(t); x). \quad (7)$$

Multiplying by n and using the known moment limits, it remains to show that

$$\lim_{n \rightarrow \infty} n L_n(R_x(t); x) = 0.$$

To estimate $R_x(t)$, note that $\eta_{x,t} = e^{-\xi}$ for some ξ between x and t . Hence

$$|R_x(t)| = \frac{1}{2} |G''(e^{-\xi}) - G''(e^{-x})| (e^{-t} - e^{-x})^2.$$

Since

$$G''(e^{-s}) = e^{2s} (f''(s) + f'(s)), \quad s \geq 0,$$

we can write

$$|G''(e^{-\xi}) - G''(e^{-x})| \leq e^{2\xi} |f''(\xi) - f''(x)| + |e^{2\xi} - e^{2x}| \cdot |f''(x) + f'(x)| + e^{2\xi} |f'(\xi) - f'(x)|.$$

Since x is fixed and f', f'' are continuous at x , the last two terms are absorbed into a constant multiple of $(e^{-t} - e^{-x})^2$ in the final estimate, so it suffices to control the principal term involving $|f''(\xi) - f''(x)|$.

Let $\Omega_{w_{\alpha_1}}$ be the weighted modulus from Section 4. Then for every $\delta > 0$,

$$|f''(\xi) - f''(x)| \leq (w_{\alpha_1}(\xi) + w_{\alpha_1}(x)) \Omega_{w_{\alpha_1}}(f''; \delta) \left(1 + \frac{(t-x)^2}{\delta^2}\right),$$

because $|\xi - x| \leq |t - x|$ and w_{α_1} is increasing. Moreover, $w_{\alpha_1}(\xi) \leq w_{\alpha_1}(t) + w_{\alpha_1}(x)$. Consequently, there exists a constant $C_x > 0$ such that

$$|R_x(t)| \leq C_x \Omega_{w_{\alpha_1}}(f''; \delta) (w_{\alpha_1}(t) + w_{\alpha_1}(x)) \left(1 + \frac{(t-x)^2}{\delta^2}\right) (t-x)^2, \quad t \geq 0,$$

where we also use $|e^{-t} - e^{-x}| \leq |t - x|$.

Applying $L_n(\cdot; x)$ and using positivity yields

$$|L_n(R_x(t); x)| \leq C_x \Omega_{w_{\alpha_1}}(f''; \delta) \left[w_{\alpha_1}(x) \left(\mu_{n,2}(x) + \frac{1}{\delta^2} L_n((t-x)^4; x) \right) + L_n(w_{\alpha_1}(t)(t-x)^2; x) + \frac{1}{\delta^2} L_n(w_{\alpha_1}(t)(t-x)^4; x) \right],$$

where $\mu_{n,2}(x) = L_n((t-x)^2; x)$.

We now estimate the weighted terms. By Cauchy–Schwarz,

$$L_n(w_{\alpha_1}(t)(t-x)^2; x) = L_n(e^{2\alpha_1 t}(t-x)^2; x) \leq \sqrt{L_n(e^{4\alpha_1 t}; x)} \sqrt{L_n((t-x)^4; x)}.$$

For fixed x , the exponential moment $L_n(e^{4\alpha_1 t}; x)$ is bounded in n by Lemma 8, and $L_n((t-x)^4; x) = O(n^{-2})$ by Lemma 2 (iii), hence

$$L_n(w_{\alpha_1}(t)(t-x)^2; x) = O(n^{-1}).$$

Similarly, we have

$$L_n(w_{\alpha_1}(t)(t-x)^4; x) = L_n(e^{2\alpha_1 t}(t-x)^4; x) \leq \sqrt{L_n(e^{4\alpha_1 t}; x)} \sqrt{L_n((t-x)^8; x)} = O(n^{-2}),$$

using again boundedness of $L_n(e^{4\alpha_1 t}; x)$ and the standard Poisson-type estimate $L_n((t-x)^8; x) = O(n^{-4})$ for fixed x .

Now choose $\delta = n^{-1/2}$. Then $\delta \rightarrow 0$ and, since $f'' \in C_{w_{\alpha_1}}^k(I)$, we have $\Omega_{w_{\alpha_1}}(f''; \delta) \rightarrow 0$ as $\delta \rightarrow 0$. Multiplying the above bound by n and using

$$\lim_{n \rightarrow \infty} n\mu_{n,2}(x) = x, \quad L_n((t-x)^4; x) = O(n^{-2}),$$

we conclude that $n L_n(R_x(t); x) \rightarrow 0$. Substituting into (7) completes the proof. $\square$

6. Numerical Results

In this section, we present numerical examples illustrating the approximation behavior of the operators L_n . In order to provide a more comprehensive analysis, we consider two different test functions. The first one includes both polynomial and exponential components, while the second one is chosen as a purely exponential function in order to examine the behavior of the operators for functions of the exponential type.

6.1. Example 1

For numerical computations, we fix $\theta_n \equiv \theta = 0.5$ and consider the test function

$$f(x) = x^2 + 2e^{-x} - e^{-2x}, \quad x \geq 0.$$

The computations are carried out for

$$n \in \{5, 10, 20, 50\}$$

on the interval $[0, 10]$.

For brevity, we define

$$E_n(f) := \sup_{x \in [0, 10]} \frac{|L_n(f; x) - f(x)|}{e^{2\alpha_2 x}}, \quad M_n(f) := \sup_{\substack{x, t \in [0, 10] \\ |t-x| \leq \frac{1}{\sqrt{n}}}} \frac{|f(t) - f(x)|}{e^{2\alpha_1 x} + e^{2\alpha_1 t}}.$$

In this example, the approximation behavior of the operators L_n is examined by means of Figures 1 and 2 and Table 1. From Figure 1, it is observed that $L_n(f; x)$ approaches $f(x)$ as n increases. The absolute error graphs presented in Figure 2 support this observation, showing that the error decreases as n increases.

Table 1. Numerical illustration of Theorem 3 for $\theta = 0.5$. For the computations, we take $\alpha_1 = 1$ and $\alpha_2 = 2$.

n	$E_n(f)$	$M_n(f)$
5	0.0212921	0.0415771
10	0.00958081	0.0321685
20	0.00453538	0.0219382
50	0.00175439	0.0111218

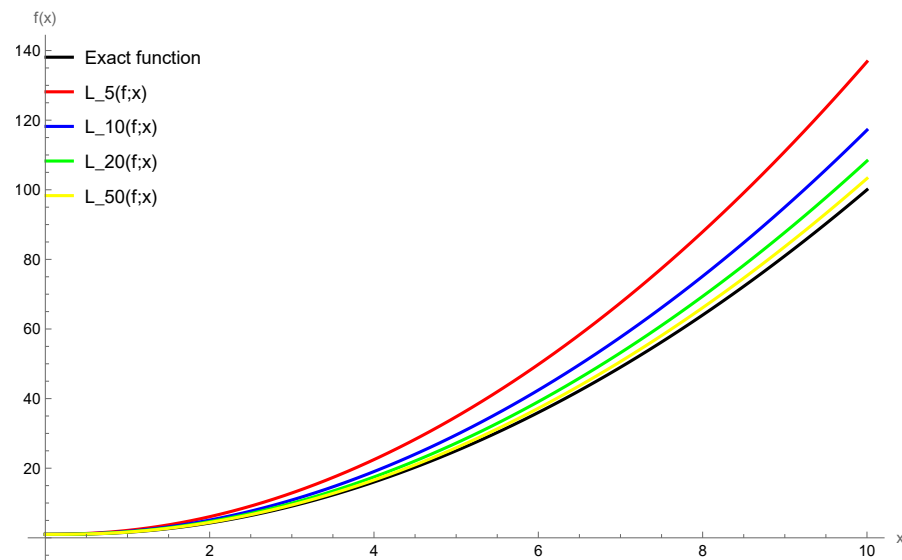


Figure 1. Graphs of $f(x) = x^2 + 2e^{-x} - e^{-2x}$ and its approximations $L_n(f; x)$ for $\theta = 0.5$ and $n = 5, 10, 20, 50$.

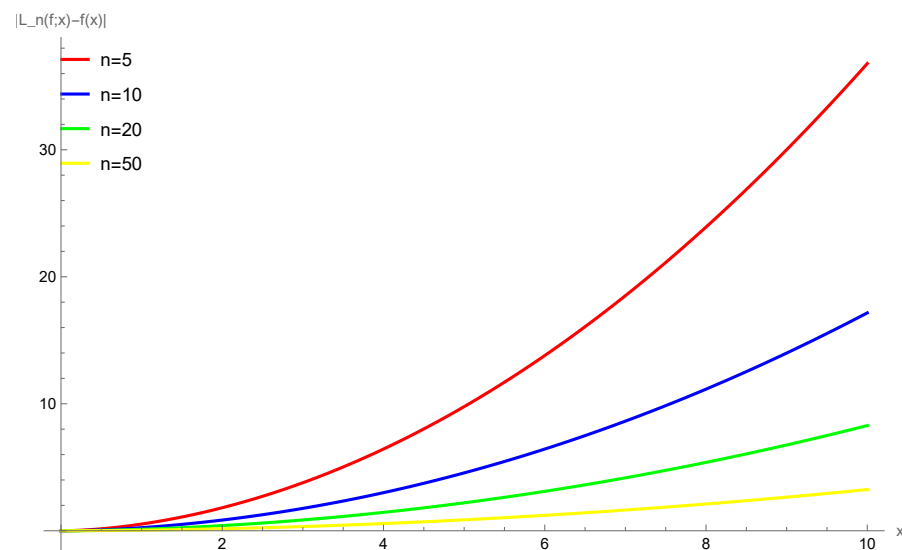


Figure 2. Absolute error functions $|L_n(f; x) - f(x)|$ for $\theta = 0.5$ and $n = 5, 10, 20, 50$.

Table 1 provides a numerical illustration of Theorem 3 by presenting the left-hand side and the weighted modulus term appearing on the right-hand side of the estimate. It is observed that the weighted approximation error remains below the corresponding weighted modulus values for all considered values of n . Moreover, both quantities decrease as n increases, which is consistent with the convergence behavior predicted by the theorem.

This example illustrates the general convergence behavior of the operators L_n , and shows that the approximation improves as n increases.

In addition, we compare the proposed operators with the classical Szász operator.

In this part, the proposed operators are compared with the classical Szász operator. Figure 3 presents the approximation curves obtained for different values of θ together with the classical Szász operator. This figure visually illustrates how the parameter θ affects the approximation behavior of the operators.

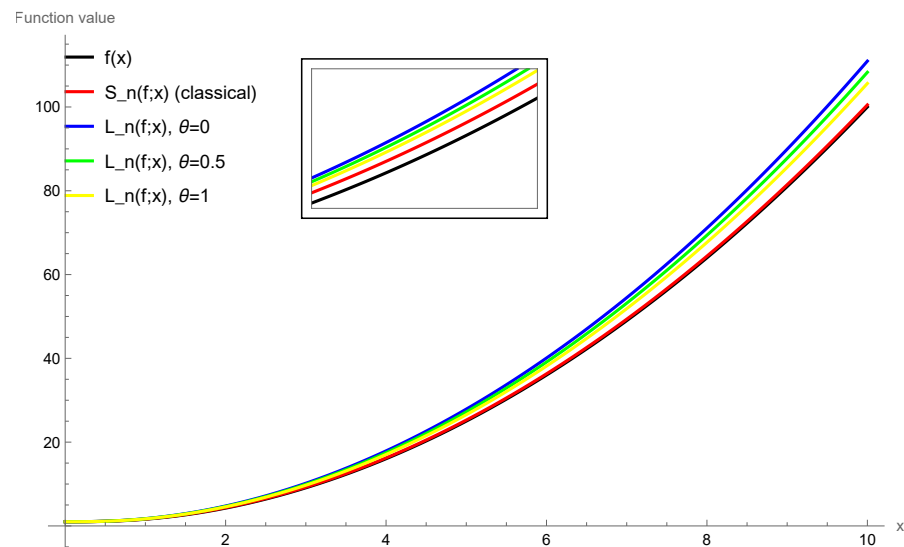


Figure 3. Comparison of the classical Szász operator and L_n for different values of θ with $n = 20$.

In Figure 4, the absolute error functions are presented comparatively. It is clearly seen that the approximation quality varies depending on the parameter θ . In addition, for this test function, the classical Szász operator yields smaller errors.

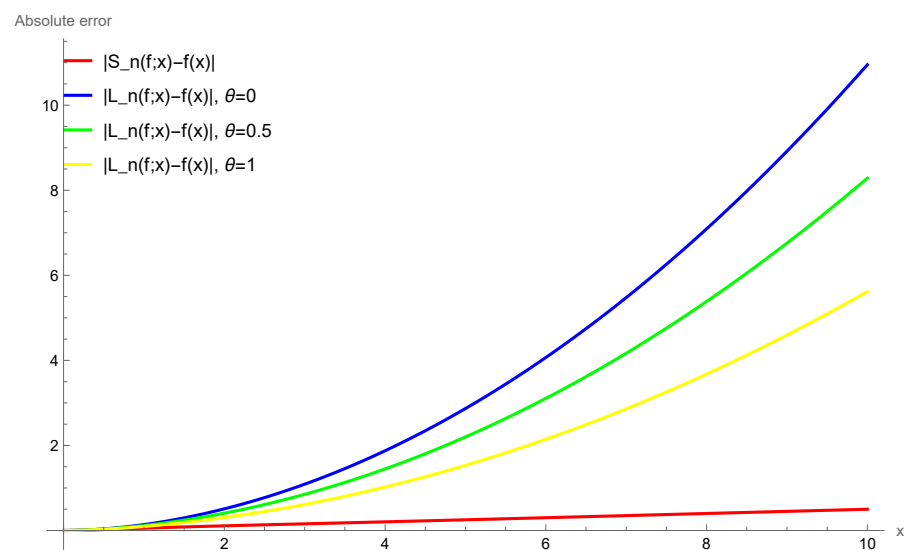


Figure 4. Comparison of the absolute errors of the classical Szász operator and the operators L_n for different values of θ with $n = 20$.

6.2. Example 2

At this stage, we consider the exponential-type test function

$$f_2(x) = e^{-x} + \frac{1}{2}e^{-2x}, \quad x \geq 0,$$

and examine the approximation behavior of the operators on functions of the exponential type.

In this example, the error behavior of the operators is examined by means of Figures 5 and 6 and Table 2. Figures 5 and 6 present the approximation curves and the corresponding absolute error graphs for different values of θ . From these figures, it is observed that the operators L_n generally yield smaller errors than the classical Szász operator.

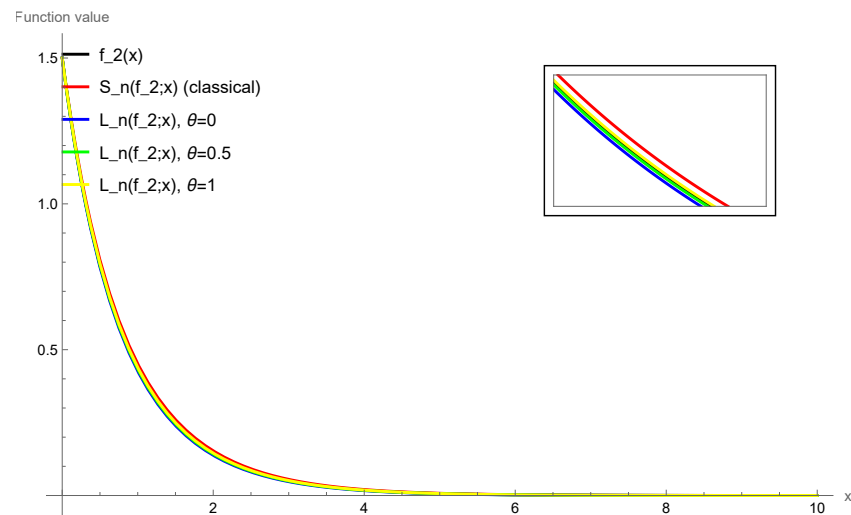


Figure 5. Comparison between the classical Szász operator and the operators L_n for different values of θ for the function f_2 with $n = 20$.

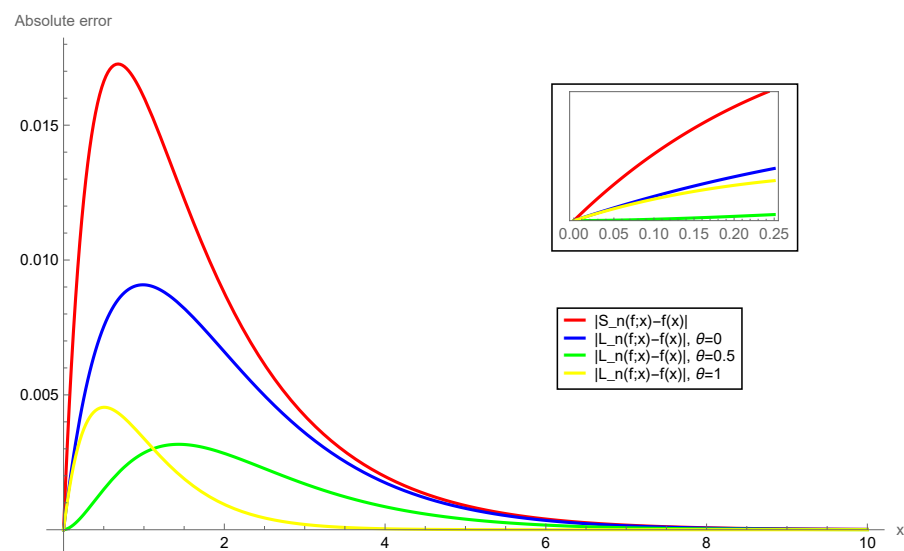


Figure 6. Absolute error comparison for the exponential test function f_2 for different values of θ and the classical Szász operator with $n = 20$.

Table 2. Maximum absolute errors for the classical Szász operator and the operators L_n for different values of θ and n in Example 2.

n	Classical Szász	$\theta = 0$	$\theta = 0.5$	$\theta = 1$
5	0.06805	0.03489	0.01166	0.01745
10	0.03437	0.01792	0.00616	0.00896
20	0.01726	0.00908	0.00316	0.00454
50	0.00691	0.00366	0.00128	0.00183

The maximum error values reported in Table 2 numerically support this observation. In addition, it is seen that the choice $\theta = 0.5$ provides smaller errors compared to the other values.

When the numerical results obtained in this section are evaluated together, it is observed that the approximation performance of the operators depends on the structure of the function. In the first example, for the test function containing both polynomial and exponential components, the classical Szász operator provides better results, whereas

in the second example, for the exponential-type function, the operators L_n yield smaller error values.

Examining the structure of the operator family, it is seen that in the cases $\theta = 0$ and $\theta = 1$, the functions e^{-2x} and e^{-x} are reproduced exactly, respectively. In particular, the case $\theta = 0$ corresponds to the operator introduced in [18]. On the other hand, for $0 < \theta < 1$, there is no direct preservation of these functions. However, since the function considered in the second example consists of a combination of these two exponential terms, the operators L_n yield very small errors for this function.

From the magnified (zoomed) regions presented in the figures, it is observed that the differences between the operators are more pronounced, especially for small values of x . Moreover, this behavior is not limited to a very narrow neighborhood of zero, but persists over the interval $0 < x < 1$, where the distinctions between the operators can still be clearly observed.

In these regions, a clearer ordering can be observed, while as x increases, the flexibility provided by the parameter θ becomes more evident, and the relative behavior of the curves corresponding to different values of θ may change. In particular, it is seen that the choice $\theta = 0.5$ provides better results in certain intervals, although this behavior is not uniform over the entire domain.

As a result, the parameter θ introduces an important flexibility into the operator family and allows the approximation behavior to be adjusted according to different types of functions.

6.3. Application to Exponential-Type Models

Functions involving exponential terms arise naturally in many applied models. In such settings, preserving the exponential structure of the approximated function is important for maintaining its qualitative behavior.

Since the operators considered in the numerical section involve infinite series, an appropriate truncation strategy has been employed in the computations. In particular, the summation interval is selected around the Poisson parameter λ with a width proportional to its standard deviation, and a minimum half-width is imposed to ensure numerical stability.

In addition, the effect of the truncation parameters has been examined. Numerical tests with different truncation settings yield identical error values, indicating that enlarging the truncation interval does not affect the results. This confirms that the chosen truncation strategy is sufficient and reliable for the computations.

7. Conclusions

In this study, we investigated the approximation properties of a family of Szász–Mirakyan-type operators associated with two exponential functions. The well-definedness of the operators and their behavior between appropriate weighted spaces were established, and convergence results were obtained by means of a weighted Korovkin-type theorem.

Furthermore, estimates for the rate of convergence were derived using a weighted modulus of continuity, and a Voronovskaya-type result was presented to describe the asymptotic behavior of the operators. It was observed that the parameter sequence (θ_n) plays a significant role in determining the behavior of the operators and provides a transition between different constructions.

The numerical examples illustrate the approximation behavior of the operators for different types of functions and show the influence of the parameter θ on this behavior. The results indicate that the operators provide a suitable approximation, particularly for functions of the exponential type.

The approach considered in this study can be extended to other families of positive linear operators with similar properties, as well as to more general weighted settings.

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