

Article

Theta Function Identities of Level 16 and Their Applications to Colored Partitions

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Abstract

Theta function identities play an important role in the theory of modular forms and partitions, with Ramanujan's q -series providing a fundamental analytical framework. This work derives several new identities of theta functions at level 16 by employing classical transformation techniques from the theory of q -series initiated by Ramanujan. Four main theorems are proved, yielding explicit algebraic relations involving the functions $\phi(q)$, $\psi(q)$, and their associated transforms. These identities are applied to partition theory through the introduction of colored partition functions $\sigma_i(m)$ defined by arithmetic conditions modulo 16. The resulting formulas provide direct combinatorial interpretations of the analytic identities and highlight connections between theta functions, modular forms, and partition functions. The results offer a unified analytic combinatorial framework at level 16 and contribute to the study of modular congruences and related arithmetic structures.

Keywords: theta function identities; q series transformations; integer partitions; colored partitions; generating functions

MSC: 11P81; 11P82; 11F11; 11P83; 05A17

1. Introduction

We assume $|q| < 1$ throughout and follow the standard notation for the infinite q -Pochhammer symbol as follows:

$$(a; q)_{\infty} := \prod_{s=0}^{\infty} (1 - aq^s),$$

which is essential to the study of q -series and modular forms.

Assuming $|xy| < 1$, the definition of Ramanujan's theta function $f(x, y)$ is:

$$f(x, y) := \sum_{s=-\infty}^{\infty} x^{s(s+1)/2} y^{s(s-1)/2}. \quad (1)$$

This function includes numerous special cases that arise naturally in the study of q -series, modular forms, and partition-theoretic identities.



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Jacobi's triple-product identity [1,2], rediscovered by Ramanujan, can be expressed as follows:

$$f(x, y) = (-x; xy)_\infty (-y; xy)_\infty (xy; xy)_\infty, \quad (2)$$

or equivalently:

$$\begin{aligned} \sum_{s=-\infty}^{\infty} q^{s^2} z^s &= \prod_{s=1}^{\infty} (1 - q^{2s})(1 + zq^{2s-1}) \left(1 + \frac{1}{z} q^{2s-1}\right) \\ &= (q^2; q^2)_\infty (-zq; q^2)_\infty \left(-\frac{q}{z}; q^2\right)_\infty, \quad \text{for } |q| < 1, z \neq 0. \end{aligned}$$

To simplify expressions, we denote $f(-q^s)$ by f_s , for any positive integer s . Using this convention, the following relations hold and are frequently used throughout this paper:

$$\begin{aligned} \phi(q) &= \frac{f_2^5}{f_1^2 f_4^2}, & \phi(-q) &= \frac{f_1^2}{f_2}, & \psi(q) &= \frac{f_2^2}{f_1}, & \psi(-q) &= \frac{f_1 f_4}{f_2}, \\ f(q) &= \frac{f_2^3}{f_1 f_4}, & \chi(q) &= \frac{f_2^2}{f_1 f_4}, & \chi(-q) &= \frac{f_1}{f_2}, \end{aligned} \quad (3)$$

where:

$$f_u = (q^u; q^u)_\infty,$$

for positive integers u .

These identities form the algebraic backbone of our investigations and appear ubiquitously in the analysis of theta and eta products. Among these, the following identities, recorded by Ramanujan, are particularly noteworthy:

$$\phi(q) = f(q, q) = \sum_{s=-\infty}^{\infty} q^{s^2} = (-q; q^2)_\infty (q^2; q^2)_\infty, \quad (4)$$

$$\psi(q) = f(q, q^3) = \sum_{s=0}^{\infty} q^{s(s+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \quad (5)$$

$$f(-q) = f(-q, -q^2) = \sum_{s=-\infty}^{\infty} (-1)^s q^{s(3s-1)/2} = (q; q)_\infty. \quad (6)$$

Euler's pentagonal number theorem [3] is given by:

$$(q; q)_\infty = \sum_{s=-\infty}^{\infty} (-1)^s q^{\frac{s(3s-1)}{2}}, \quad |q| < 1.$$

Proof. In particular, setting $x = -q$ and $y = -q^2$, in (1), we obtain:

$$f(-q, -q^2) = \sum_{s=-\infty}^{\infty} (-q)^{\frac{s(s+1)}{2}} (-q^2)^{\frac{s(s-1)}{2}} = \sum_{s=-\infty}^{\infty} (-1)^s q^{\frac{s(3s-1)}{2}}$$

For brevity, we write:

$$f(-q) := f(-q, -q^2).$$

Now, putting $x = -q$ and $y = -q^2$ in Equation (2), we get:

$$\begin{aligned} f(-q, -q^2) &= (q; q^3)_\infty (q^2; q^3)_\infty (q^3; q^3)_\infty = \prod_{s=0}^{\infty} (1 - q^{3s+1})(1 - q^{3s+2})(1 - q^{3s+3}) \\ &= [(1 - q)(1 - q^4)(1 - q^7) \cdots][(1 - q^2)(1 - q^5) \cdots][(1 - q^3)(1 - q^6) \cdots] \\ &= (1 - q)(1 - q^2)(1 - q^3)(1 - q^4) \cdots = \prod_{s=1}^{\infty} (1 - q^s) = (q; q)_\infty \end{aligned}$$

□

Ramanujan defined the auxiliary function:

$$\chi(q) = (-q; q^2)_\infty.$$

If $q = e^{2\pi i\tau}$, then for $\text{Im}(\tau) > 0$, we have:

$$f(-q) = q^{-1/24} \eta(\tau),$$

where $\eta(\tau)$ denotes the Dedekind eta function, an essential object in number theory and related fields. Its explicit form is given by:

$$\eta(\tau) = q^{1/24} \prod_{s=1}^{\infty} (1 - q^s), \quad \text{for } \text{Im}(\tau) > 0. \quad (7)$$

Identities that express relations between $f(-q)$ and $f(-q^s)$, where s is a positive integer, are known as theta function identities (TFI) of level s . Ramanujan documented numerous such identities, particularly involving $f(-q)$, $f(-q^2)$, $f(-q^3)$, and $f(-q^{2s})$ in both his second notebook [4] and the ‘Lost Notebook’ [5]. These identities capture deep arithmetic and combinatorial structures, especially those related to partition theory.

Over the last few years, there has been significant computational advancement in this field. Notably, Somos [6] employed computer algebra systems and algorithms in PARI/GP to experimentally discover a wide range of eta-products and TFI at various levels. He compiled extensive lists of such identities, many of which have since been rigorously verified or generalized using classical analytical techniques.

For instance, Yuttanan [7] provided analytic proofs of several conjectural identities posed by Somos by applying modular equations from Ramanujan’s notebooks. These efforts also yielded a number of new partition identities. Similarly, Vasuki and Veerasha [8] studied TFI of level 14, while Srivatsa Kumar and Veerasha [9] investigated the corresponding partition-theoretic consequences. Mahadeva Naika [10] further explored related identities arising from Ramanujan’s continued fraction theory.

The present study adds to the existing literature by deriving new TFI specifically at level 16, relying solely on classical techniques such as q series transformations, modular identities and known relations among theta functions. Our approach is elementary, yet it yields rich structural insights and meaningful arithmetic consequences.

The transition from level 8 to level 16 in colored partition constructions does not entail a change in the underlying methodology, but may instead be viewed as a natural enhancement in the resolution of the residue-class decomposition [11]. At level 8, multiple residue classes are combined into a relatively small number of types, resulting in a comparatively coarse organization of the associated generating functions. In the level 16 framework, representations of the form $16n \pm r$ (where r ranges over the relevant residue classes modulo 16) allow for a more detailed separation of these classes, leading to a finer

factorization of the corresponding q -products and enabling the identification of additional algebraic relations among partition components.

Furthermore, while higher levels, such as 32, yield further subdivisions of residue classes, they do not appear to introduce essentially new structural features beyond those already visible at level 16, but rather lead to increased technical complexity. In this sense, level 16 may be regarded as the minimal level at which the residue-class decomposition achieves sufficient refinement to support the identities developed in this work.

Section 2 contains our principal analytical findings that were gained through transformations and manipulations of identities of theta functions. These include new relations among $\phi(q)$, $\phi(q^2)$, $\psi(q^4)$ and $\psi(q^8)$, such as:

$$\frac{\phi(q)\phi(q^4) - \phi^2(q^2)}{2} = q\psi^2(q^4) - 2q^2\psi^2(q^8),$$

which exemplify the types of identities that naturally emerge from the theta function framework.

In Section 3, we present combinatorial interpretations of our findings, demonstrating their significance in enumerative partition theory, particularly for partitions with congruence or coloring conditions. We also propose a further extension of these identities to the study of modular equations and mock modular forms.

Altogether, the outcomes provided in this paper contribute to the theory of q -series and partition identities and provide a framework that may be developed to the higher level of identities and generalized mock theta functions (MTFs).

We start the arguments of our main findings by urging the following basic lemma:

Lemma 1. *We have:*

$$\phi(q) + \phi(-q) = 2\phi(q^4), \quad (8)$$

$$\phi(q) - \phi(-q) = 4q\psi(q^8), \quad (9)$$

$$\phi^2(q) + \phi^2(-q) = 2\phi^2(q^2), \quad (10)$$

$$\phi^2(q) - \phi^2(-q) = 8q\psi^2(q^4), \quad (11)$$

$$\phi^4(q) - \phi^4(-q) = 16q\psi^4(q^2), \quad (12)$$

$$\phi(q)\phi(-q) = \phi^2(-q^2), \quad (13)$$

$$\psi(q)\psi(-q) = \psi(q^2)\phi(-q^2), \quad (14)$$

$$\phi(q)\psi(q^2) = \psi^2(q). \quad (15)$$

Proof. For proof, refer to Entry 25 in Chapter 16 of [1]. $\square$

We briefly describe the aim and scope of this section before presenting the main results. Their main goal is to obtain new level 16 identities of theta functions using classical q -series transformation, and known theta functions identities of Ramanujan. These identities are created by systematic manipulations of algebra such as addition, subtraction and factorization of basic identities. The findings in this section are the basis of analytical treatment of further combinatorial interpretations that were made later in the paper, especially in the case of colored partition functions.

2. A Set of Main Results

The first key result of this study is presented below, introducing a new and nontrivial identity for level 16 theta functions.

Theorem 1. *The following identities hold:*

$$\phi^3(q) \phi^2(-q) = \phi^4(-q^2) \left\{ \phi(q^4) + 2q \psi(q^8) \right\}, \quad (16)$$

$$\phi^2(q^2) = \phi^2(q^4) + 4q^2 \psi^2(q^8), \quad (17)$$

$$3\phi^2(-q^2) + \phi^2(q) = 4 \left\{ \phi^2(q^4) + q \psi(q^8) \phi(-q) \right\}, \quad (18)$$

$$\phi^2(q^2) + q \psi(q^8) \phi(-q) = \frac{1}{2} \left\{ \phi^2(q) + \phi(q^4) \phi(-q) \right\}. \quad (19)$$

Proof. Squaring both sides of Equation (13), we obtain:

$$\phi^2(q) = \frac{\phi^4(-q^2)}{\phi^2(-q)}. \quad (20)$$

Multiplying both sides of Equation (20) by $\phi(q)\phi^2(-q)$, we get:

$$\phi^3(q) \phi^2(-q) = \phi(q) \phi^4(-q^2). \quad (21)$$

Adding Equations (8) and (9), we obtain:

$$\phi(q) = \phi(q^4) + 2q \psi(q^8). \quad (22)$$

Substituting Equation (22) into the right-hand side of Equation (21) yields (16).

Next, subtracting Equation (8) from Equation (9), squaring both sides and applying Equation (15), then multiplying by 2, we get:

$$2\phi^2(-q) = 2\phi^2(q^4) + 8q^2 \psi^2(q^8) - 8q \psi^2(q^4). \quad (23)$$

Using Equation (11) on the right-hand side of Equation (23), we obtain:

$$\phi^2(-q) = 2\phi^2(q^4) + 8q^2 \psi^2(q^8) - \phi^2(q). \quad (24)$$

Now, using Equation (10), we arrive at (17).

Further, squaring both sides of Equation (8) and then multiplying by $2q$, we obtain:

$$2q \phi^2(-q) + 2q \phi^2(q) + 4q \phi(q) \phi(-q) = 8q \phi^2(q^4). \quad (25)$$

Multiplying Equation (9) by $\phi(-q)$, then substituting into equation (25) and applying Equation (13), we obtain the desired (18).

Finally, subtracting Equation (8) from Equation (9) and then multiplying both sides by $\phi(-q)$, we get:

$$-\phi^2(-q) = 2q \phi(-q) \psi(q^8) - \phi(-q) \phi(q^4). \quad (26)$$

Substituting Equation (26) into Equation (10) yields (19). $\square$

3. Application to the Theory of Partitions

The identities established in Section 2 find applications in partition theory, as illustrated through their combinatorial interpretations.

The colored partition constructions introduced in this section are not ad hoc, but arise naturally from the factorization of the generating functions obtained in Section 2. In particular, the decomposition of q -products into residue classes modulo 16 leads to a natural classification of parts into types a , b , c , and d .

The number of colors assigned to each type corresponds directly to the exponent of the associated q -product factor. Therefore, the algebraic structure of the TFI is seen in the combinatorial interpretations. This approach provides a systematic method for translating analytic identities into partition-theoretic statements and can be extended to other levels and related q -series identities.

Integer partitions: Let s be a non-negative integer. A partition of s is a representation of s as a sum of positive integers, where each summand is called a part and order is disregarded (e.g., $4 = 3 + 1 = 2 + 2$). Let $p(s)$ denote the number of such partitions.

As $s \rightarrow \infty$, the asymptotic behavior of $p(s)$ is described by the Hardy–Ramanujan formula [12]:

$$p(s) \sim \frac{1}{4\sqrt{3}s} e^{\pi\sqrt{2s/3}}. \quad (27)$$

The generating function for integer partitions is given by:

$$\sum_{s=0}^{\infty} p(s)q^s = \frac{1}{(q; q)_{\infty}}, \text{ where } (q; q)_{\infty} = \prod_{j=1}^{\infty} (1 - q^j). \quad (28)$$

The theory of integer partitions is fundamental in number theory with broad mathematical significance [11,13–18].

Colored partitions: The colored partitions theory is the extension of the classical concept of integer partitions, which permits each part to be identified by a given color. In this generalized framework, each positive integer part in a partition may appear in one of κ different colors, thereby increasing the total number of distinct configurations.

Formally, a colored partition of a non-negative integer m is an ordered pair $(\zeta, \varkappa)$, where:

$\zeta = (\zeta_1, \zeta_2, \dots, \zeta_j)$ is a partition of m , satisfying $\zeta_1 \geq \zeta_2 \geq \dots \geq \zeta_j > 0$ and $\sum_{i=1}^j \zeta_i = m$, and $\varkappa = (\varkappa_1, \varkappa_2, \dots, \varkappa_j)$ is a sequence with $\varkappa_i \in \{1, 2, \dots, \kappa\}$, indicating the color assigned to each part ζ_i .

Two colored partitions $(\zeta, \varkappa)$ and $(\zeta', \varkappa')$ are considered distinct if either the underlying partitions $\zeta \neq \zeta'$ or the color assignments $\varkappa \neq \varkappa'$.

This concept was formally studied by Huang [19], who developed the foundation for analyzing such partitions using generating functions and modular forms. Variations in the definition of colored partitions may appear across different studies, depending on specific constraints or applications.

The generating function for the number of colored partitions $p_{\kappa}(m)$ of an integer m using κ colors is given by:

$$\sum_{m=0}^{\infty} p_{\kappa}(m)q^m = \prod_{n=1}^{\infty} \left(\frac{1}{1 - q^n} \right)^{\kappa}, \quad \text{for } |q| < 1,$$

reflecting the fact that each part size n contributes κ distinct choices, one for each color.

For example, Merca [20] allowed the integer s to appear in s different color instances in his analysis of partition sums. Bandyopadhyay and Baruah [21], along with other researchers [22–25], introduced s as a color variable appearing alongside $m + t$ colors to derive results for the s -colored partition function.

A particularly significant class of colored partitions allows each partition component to be assigned any of the k available colors. For instance, when partitioning the integer 3 using two colors, red and green, the following 10 distinct partitions arise:

$$\begin{aligned} &1_r + 1_r + 1_r, \quad 1_r + 1_r + 1_g, \quad 1_r + 1_g + 1_g, \quad 1_g + 1_g + 1_g, \\ &1_r + 2_r, \quad 1_r + 2_g, \quad 1_g + 2_r, \quad 1_g + 2_g, \quad 3_r, \quad 3_g, \end{aligned}$$

where subscripts r and g denote red and green, respectively. In this framework, numbers such as 1_r and 1_g indicate the integer 1 colored red and green, respectively. Likewise, $2_r, 2_g$ and $3_r, 3_g$ denote the integers 2 and 3 with their assigned colors. Allowing each part to take any of k colors freely defines what is called a uniformly k -colored partition.

Several authors, including [26–29], have studied this concept in depth. In some modified versions of the colored partition method, certain parts are restricted to selection from a subset of k colors, while others follow alternative color rules. For example, if the number 1 is permitted to exist in only two colors, the partitions of 2 reduce to:

$$2, \quad 1_b + 1_b, \quad 1_w + 1_w, \quad 1_b + 1_w,$$

where b and w represent two distinct colors. The methodology has been extensively applied to determine the links between q -product identities and colored partitions, as demonstrated in the works of researchers [30–32].

Modern partitioning techniques offer deep insights into number-based combinatorial structures, and further results on Dedekind η -function identities of various levels and related analytical developments are available in [33–37].

The number of colored partitions of a non-negative integer s into k distinct colors is given by the following determinant formula (see [26]):

$$p_k(s) = \frac{1}{s!} \begin{vmatrix} k\sigma(1) & k\sigma(2) & k\sigma(3) & \cdots & k\sigma(s) \\ -1 & k\sigma(1) & k\sigma(2) & \cdots & k\sigma(s-1) \\ 0 & -2 & k\sigma(1) & \cdots & k\sigma(s-2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 \cdots & -(s-1) & k\sigma(1) \end{vmatrix},$$

where $\sigma(s) = \sum_{d|s} d$ denotes the sum of the positive divisors of s .

This classical form is also provided to be complete and to have a structural context for colored partition enumeration. Although it is not used directly in the proofs, it highlights the connection with established combinatorial frameworks.

Remark: Setting $k = 1$ in Equation (3) yields the total number of partitions of s .

For convenience, we use the notation:

$$(q^{s\pm}; q^p)_\infty = (q^s, q^{s-p}; q^p)_\infty \quad \text{for } s < p, \quad s, p \in \mathbb{N}. \quad (29)$$

For brevity, we adopt a residue-based notation modulo 16. Specifically, parts of the form $16n \pm r$ with $r \in \{1, 3, 5, 7\}$ are denoted by a ; those with $r \in \{2, 6\}$ by b ; those with $r = 4$ by c ; and those with $r = 8$ by d .

i.e.,

$$q^a \equiv q^{1\pm}, q^{3\pm}, q^{5\pm}, q^{7\pm}, \quad q^b \equiv q^{2\pm}, q^{6\pm}, \quad q^c \equiv q^{4\pm}, \quad q^d \equiv q^8, \quad (\text{mod } q^{16}), \quad (30)$$

where:

$$\begin{aligned}(q^{1\pm}; q^{16})_{\infty} &\equiv (q^1, q^{15}; q^{16})_{\infty}, \\(q^{2\pm}; q^{16})_{\infty} &\equiv (q^2, q^{14}; q^{16})_{\infty}, \\(q^{3\pm}; q^{16})_{\infty} &\equiv (q^3, q^{13}; q^{16})_{\infty}, \\(q^{4\pm}; q^{16})_{\infty} &\equiv (q^4, q^{12}; q^{16})_{\infty}, \\(q^{5\pm}; q^{16})_{\infty} &\equiv (q^5, q^{11}; q^{16})_{\infty}, \\(q^{6\pm}; q^{16})_{\infty} &\equiv (q^6, q^{10}; q^{16})_{\infty}, \\(q^{7\pm}; q^{16})_{\infty} &\equiv (q^7, q^9; q^{16})_{\infty}.\end{aligned}$$

Before stating the main theorem, we establish the notation for the colored partition functions that will be used throughout this section. For each integer $m \in \mathbb{N}$, the functions $\sigma_i(m)$ enumerate partitions of m subject to prescribed congruence conditions modulo 16, together with fixed color multiplicities assigned to each residue class. The parameters a, b, c, d denote representatives of residue classes modulo 16, while the associated coefficients indicate the number of distinct colors available for parts in each class.

Throughout, expressions of the form $16n + r$, where $r = a, b, c$ and d , refer to integers lying in the residue class $r \pmod{16}$, as specified in (30). This notation provides the foundation for the combinatorial interpretation of the identities that follow.

Remark: (color-indexed q -notation). In this work, the effect of color in partition generating functions is incorporated through an indexed form of the variable q . The subscript attached to q specifies the number of color choices available for a given part. More precisely, writing q_i indicates that the corresponding part may appear in any one of i distinct colors. For example, the symbol q_6 represents parts that are allowed exactly six color variations. This indexing convention enables color information to be directly embedded into the q -product representations of generating functions for colored partitions. Also, the symbol q_0^d is used to indicate that parts of type d do not appear in the corresponding infinite product expansion.

Theorem 2. Let $m, n \in \mathbb{N}$.

$$\begin{aligned}\sigma_1(m) &: \text{parts } 16n + b, 16n + c, \text{ each in 3 colors;} \\ \sigma_2(m) &: \text{parts } 16n + b, 16n + c, 16n + d, \text{ with 3, 1, and 4 colors, respectively;} \\ \sigma_3(m) &: \text{parts } 16n + a, 16n + d, \text{ each in 2 colors.}\end{aligned}$$

For $m \geq 1$:

$$\sigma_1(m) + 2\sigma_2(m-1) - \sigma_3(m) = 0.$$

Proof. By substituting (3) into (16), we obtain:

$$f_1^2 f_8^6 + 2q f_1^2 f_4^2 f_{16}^4 - f_2^5 f_8 f_{16}^2 = 0. \quad (31)$$

Dividing Equation (31) by f_1^8 , we obtain:

$$\frac{f_1^2 f_8^6}{f_1^8} + 2q \frac{f_1^2 f_4^2 f_{16}^4}{f_1^8} - \frac{f_2^5 f_8 f_{16}^2}{f_1^8} = 0. \quad (32)$$

First, consider the expression $\frac{f_8^6}{f_1^6}$. Using Equations (29) and (30), we derive:

$$\frac{f_8^6}{f_1^6} = \frac{1}{(q_6^{1\pm, 3\pm, 5\pm, 7\pm}, q_6^{2\pm, 6\pm}, q_6^{4\pm}, q_0^8; q^{16})_\infty} = \frac{1}{(q_6^a, q_6^b, q_6^c, q_0^d; q^{16})_\infty}.$$

Similarly:

$$\begin{aligned}\frac{f_4^2 f_{16}^4}{f_1^6} &= \frac{1}{(q_6^a, q_6^b, q_4^c, q_4^d; q^{16})_\infty}, \\ \frac{f_2^5 f_8 f_{16}^2}{f_1^8} &= \frac{1}{(q_8^a, q_3^b, q_3^c, q_2^d; q^{16})_\infty}.\end{aligned}$$

Substituting these expressions into (32), we obtain:

$$\frac{1}{(q_6^a, q_6^b, q_6^c, q_0^d; q^{16})_\infty} + 2 \frac{q}{(q_6^a, q_6^b, q_4^c, q_4^d; q^{16})_\infty} - \frac{1}{(q_8^a, q_3^b, q_3^c, q_2^d; q^{16})_\infty} = 0.$$

Multiplying by $(q_6^a, q_3^b, q_3^c, q_0^d; q^{16})_\infty$ yields:

$$\frac{1}{(q_0^a, q_3^b, q_3^c, q_0^d; q^{16})_\infty} + 2 \frac{q}{(q_0^a, q_3^b, q_4^c, q_4^d; q^{16})_\infty} - \frac{1}{(q_2^a, q_0^b, q_0^c, q_2^d; q^{16})_\infty} = 0. \quad (33)$$

We recognize Equation (33) as generating functions for $\sigma_i(s)$, where $i = 1, 2$ and 3 , leading to the identity:

$$\sum_{s=0}^{\infty} \left(\sigma_1(s) q^s + 2\sigma_2(s) q^{s+1} - \sigma_3(s) q^s \right) = 0.$$

Setting $\sigma_i(0) = 1$, where $i = 1, 2$ and 3 , we extract coefficients of q^s to obtain the desired result. $\square$

Example 1. For $s = 1$, the following Table 1 verifies the above theorem, where the subscripts r and g denote the colors red and green, respectively.

Table 1. Verification of Theorem 2 for $s = 1$.

Function	Partitions
$\sigma_1(1) = 0$	
$\sigma_2(0) = 1$	
$\sigma_3(1) = 2$	$1_r, 1_g$

Theorem 3. Let $m, n \in \mathbb{N}$.

- $\sigma_1(m)$: part $16n + c$, with 10 distinct colors;
- $\sigma_2(m)$: parts $16n + c$, $16n + d$, with 6, and 8 colors, respectively;
- $\sigma_3(m)$: parts $16n + b$, $16n + d$, each in 4 colors.

For $m \geq 2$:

$$\sigma_1(m) + 4\sigma_2(m-2) - \sigma_3(m) = 0.$$

Proof. By substituting (3) into (17), we obtain:

$$f_2^4 f_8^{14} + 4q^2 f_2^4 f_4^4 f_8^2 f_{16}^8 - f_4^{14} f_{16}^4 = 0. \quad (34)$$

Dividing Equation (34) by f_1^{18} we obtain:

$$\frac{f_2^4 f_8^{14}}{f_1^{18}} + 4q^2 \frac{f_2^4 f_4^4 f_8^2 f_{16}^8}{f_1^{18}} - \frac{f_4^{14} f_{16}^4}{f_1^{18}} = 0. \quad (35)$$

By applying Equations (29) and (30)–(35), we obtain:

$$\frac{1}{(q_{18}^a, q_{14}^b, q_{14}^c, q_0^d; q^{16})_\infty} + 4 \frac{q^2}{(q_{18}^a, q_{14}^b, q_{10}^c, q_8^d; q^{16})_\infty} - \frac{1}{(q_{18}^a, q_{18}^b, q_4^c, q_4^d; q^{16})_\infty} = 0.$$

Multiplying by $(q_{18}^a, q_{14}^b, q_4^c, q_0^d; q^{16})_\infty$ yields:

$$\frac{1}{(q_0^a, q_0^b, q_{10}^c, q_0^d; q^{16})_\infty} + 4 \frac{q^2}{(q_0^a, q_0^b, q_6^c, q_8^d; q^{16})_\infty} - \frac{1}{(q_0^a, q_4^b, q_0^c, q_4^d; q^{16})_\infty} = 0. \quad (36)$$

We recognize Equation (36) as generating functions for $\sigma_i(s)$, where $i = 1, 2$ and 3 , leading to the identity:

$$\sum_{s=0}^{\infty} \left(\sigma_1(s) q^s + 4 \sum_{s=0}^{\infty} \sigma_2(s) q^{s+2} - \sigma_3(s) q^s \right) = 0.$$

Setting $\sigma_1(0) = 1$, where $i = 1, 2$ and 3 , we extract coefficients of q^s to obtain the desired result. $\square$

Example 2. For $s = 2$, the following Table 2 verifies the above theorem, where subscripts “r”, “g”, “y”, and “b” denote the colors red, green, yellow, and blue, respectively.

Table 2. Verification of Theorem 3 for $s = 2$.

Function	Partitions
$\sigma_1(2) = 0$	
$\sigma_2(0) = 1$	
$\sigma_3(2) = 4$	$2_r, 2_g, 2_y, 2_b$

Theorem 4. Let $m, n \in \mathbb{N}$.

$\sigma_1(m)$: parts $16n + a, 16n + b, 16n + c$, with 2, 6 and 6 colors, respectively;

$\sigma_2(m)$: parts $16n + b, 16n + c, 16n + d$, with 5, 1, and 6 colors, respectively;

$\sigma_3(m)$: parts $16n + a, 16n + b, 16n + d$, with 2, 2 and 4 colors, respectively;

$\sigma_4(m)$: parts $16n + a, 16n + d$, with 6 and 4 colors, respectively.

For $m \geq 1$:

$$4\sigma_1(m) + 4\sigma_2(m-1) - 3\sigma_3(m) - \sigma_4(m) = 0.$$

Proof. By substituting (3) into (18), we obtain:

$$4f_1^4 f_2 f_8^{11} + 4q f_1^6 f_4^4 f_{16}^6 - 3f_1^4 f_2^5 f_4^2 f_8 f_{16}^4 - f_2^{11} f_8 f_{16}^4 = 0. \quad (37)$$

The result follows by dividing Equation (37) by f_1^{16} :

$$4 \frac{f_1^4 f_2 f_8^{11}}{f_1^{16}} + 4q \frac{f_1^6 f_4 f_{16}^6}{f_1^{16}} - 3 \frac{f_1^4 f_2^5 f_4 f_8 f_{16}^4}{f_1^{16}} - \frac{f_2^{11} f_8 f_{16}^4}{f_1^{16}} = 0. \quad (38)$$

By applying Equations (29) and (30)–(38), we obtain:

$$4 \frac{1}{(q_{12}^a, q_{11}^b, q_{11}^c, q_0^d, q^{16})_\infty} + 4 \frac{q}{(q_{10}^a, q_{10}^b, q_6^c, q_6^d, q^{16})_\infty} - 3 \frac{1}{(q_{12}^a, q_7^b, q_5^c, q_4^d, q^{16})_\infty} - \frac{1}{(q_{16}^a, q_5^b, q_5^c, q_4^d, q^{16})_\infty} = 0.$$

Multiplying by $(q_{10}^a, q_5^b, q_5^c, q_0^d, q^{16})_\infty$ yields:

$$4 \frac{1}{(q_2^a, q_6^b, q_6^c, q_0^d, q^{16})_\infty} + 4 \frac{q}{(q_0^a, q_5^b, q_1^c, q_6^d, q^{16})_\infty} - 3 \frac{1}{(q_2^a, q_2^b, q_0^c, q_4^d, q^{16})_\infty} - \frac{1}{(q_6^a, q_0^b, q_0^c, q_4^d, q^{16})_\infty} = 0. \quad (39)$$

We recognize Equation (39) as generating functions for $\sigma_i(s)$, where $i = 1, 2, 3$ and 4, leading to the identity:

$$\sum_{s=0}^{\infty} \left(4\sigma_1(s)q^s + 4\sigma_2(s)q^{s+1} - 3\sigma_3(s)q^s - \sigma_4(s)q^s \right) = 0.$$

Setting $\sigma_1(0) = 1$, where $i = 1, 2, 3$ and 4, we extract coefficients of q^s to obtain the desired result. $\square$

Theorem 5. Let $m, n \in \mathbb{N}$.

$\sigma_1(m)$: parts $16n + a$, $16n + c$, $16n + d$, with 6, 4 and 2 colors, respectively;

$\sigma_2(m)$: parts $16n + b$, $16n + c$, with 5 and 7 colors, respectively;

$\sigma_3(m)$: parts $16n + a$, $16n + b$, $16n + d$, with 2, 10 and 2 colors, respectively;

$\sigma_4(m)$: parts $16n + b$, $16n + c$, $16n + d$, with 5, 5 and 4 colors, respectively.

For $m \geq 1$:

$$\sigma_1(m) + \sigma_2(m) - 2\sigma_3(m) - 2\sigma_4(m-1) = 0.$$

Proof. By substituting (3) into (19), we obtain:

$$f_2^{14} f_8^4 f_{16}^2 + f_1^6 f_2^3 f_4^2 f_8^9 - 2f_1^4 f_4^{14} f_{16}^2 - 2q f_1^6 f_2^3 f_4^4 f_8^3 f_{16}^4 = 0. \quad (40)$$

The result follows by dividing Equation (40) by f_1^{20} :

$$\frac{f_2^{14} f_8^4 f_{16}^2}{f_1^{20}} + \frac{f_1^6 f_2^3 f_4^2 f_8^9}{f_1^{20}} - 2 \frac{f_1^4 f_4^{14} f_{16}^2}{f_1^{20}} - 2q \frac{f_1^6 f_2^3 f_4^4 f_8^3 f_{16}^4}{f_1^{20}} = 0. \quad (41)$$

By applying Equations (29) and (30)–(41) and then multiplying by $(q_{14}^a, q_6^b, q_2^c, q_0^d; q^{16})_\infty$ yields:

$$4 \frac{1}{(q_6^a, q_0^b, q_4^c, q_2^d; q^{16})_\infty} + 4 \frac{q}{(q_0^a, q_5^b, q_7^c, q_0^d; q^{16})_\infty} - 3 \frac{1}{(q_2^a, q_{10}^b, q_0^c, q_2^d; q^{16})_\infty} - \frac{1}{(q_0^a, q_5^b, q_5^c, q_4^d; q^{16})_\infty} = 0. \quad (42)$$

We recognize Equation (42) as generating functions for $\sigma_i(s)$, where $i = 1, 2, 3$ and 4, leading to the identity:

$$\sum_{s=0}^{\infty} \left(\sigma_1(s)q^s + \sigma_2(s)q^s - 2\sigma_3(s)q^s - 2\sigma_4(s)q^{s+1} \right) = 0.$$

Setting $\sigma_i(0) = 1$, where $i = 1, 2, 3$ and 4, we extract coefficients of q^s to obtain the desired result. $\square$

4. Discussion

The present study develops new TFI of level 16 using classical q -series techniques and demonstrates their combinatorial significance through applications to colored partition functions. In comparison with earlier works on lower levels, our results extend the theory by revealing richer algebraic structures and more intricate relationships between $\varphi(q)$, $\psi(q)$, and their higher-level analogues. The resulting identities are known to have obvious symmetries, which are also naturally converted into linear relationships between colored partitions with certain congruence and coloring conditions, further supporting the relationship between analytic q -series and this theory of partitions.

Furthermore, the strategy followed in the present paper demonstrates that elementary techniques are still useful in obtaining meaningful and non-trivial outcomes, as well as offering a structure that can be rewritten, possibly, into Dedekind η -functions to obtain deeper modular interpretations. These findings not only conform to and build upon the existing literature but also provide some directions that future research ought to take.

Specifically, the techniques developed here can be used to produce functions related to MTFs, wherein coefficients along appropriate arithmetic sequences may be interpreted to derive identities analogous to those obtained in this paper. Such a strategy leads to further relations involving MTFs and associated partition interpretations. Further directions include the investigation of more complex theta identities and arithmetic properties, including congruences of the corresponding partition functions. For a more detailed treatment of the relationship between mock theta functions and colored partition structures, the interested reader is referred to [11], where such connections are systematically developed through coefficient extraction techniques and their subsequent partition-theoretic reinterpretations.

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