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Gravity, Electrostatic and Thermal Fields of Slightly Flattened Triaxial Ellipsoidal Conductors via Spherical Harmonic Expansions with Irrational Polar Distance Powers

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Abstract

We present gravitational, electrostatic and thermal field intensities as series of spherical harmonics with the polar distance raised to irrational powers. Theoretical cases involving slightly flattened oblate spheroids and triaxial ellipsoids are considered. For the electrostatic and thermal transfer cases, triaxial ellipsoids are treated as perfect conductors. The resulting Dirichlet boundary value problems, based on the G-modified Helmholtz equation, are solved outside these surfaces. For each geometry, the expansions are constructed explicitly, with a first-order approximation applied for the oblate spheroid and triaxial ellipsoid by retaining only terms containing the first polar and equatorial eccentricities squared. Finally, we determine Earth's disturbing gravity potential as a series of spherical harmonics solving a Dirichlet boundary value problem.

Keywords: gravity; electrostatics; heat transfer; spheroid; triaxial ellipsoid; perfect conductor; Dirichlet problem; spherical harmonics; disturbing potential

MSC: 35A09; 35C10; 35J25; 33C05; 33C75; 35Q99; 35J05; 33E05; 33E10

1. Introduction

Triaxial ellipsoids have become increasingly important in various applications. For example, Earth is slightly flattened at the poles, and its equator is not perfectly circular, resulting in an equatorial flattening that reflects the inequality of the semiaxes. Although the triaxial ellipsoid model has only recently been used for precise geodetic measurements, the concept dates from the 19th century. The first proposal to represent the figure of Earth as an ellipsoid [1] was made by Schubert in Russia in 1859, who determined the values of its principal axes. Later, Clarke of the British Ordnance Survey published revised values for the semiaxes in 1878, with the equator of Clarke's ellipsoid being slightly less elliptical than Schubert's.

In general, three methods are used to determine Earth's triaxial ellipsoid: (a) astro-geodetic, the oldest method; (b) gravimetric; and (c) satellite-based. A recent example of determining Earth's semiaxes using the gravimetric method can be found in [2]. The adoption of a triaxial ellipsoid better accounts for the complexities of Earth's shape and gravitational field. Accurate knowledge of Earth's gravitational field is essential for determining the geoid, which serves as a reference surface for orthometric heights, since all



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points on the geoid have zero orthometric height. Determining the geoid is a fundamental problem in geodesy. Due to the lack of precise data on mass and density distribution [3,4], the geoid is typically computed point by point. The selection of an appropriate rotational ellipsoid is crucial, as the geoid is defined by points with known distances from the ellipsoid's surface, measured along the vertical from each point on the geoid. A triaxial ellipsoid provides more accurate geometric heights, particularly in regions such as coastlines or areas with submarine faults.

The gravitational field of a triaxial ellipsoid is also significant in modeling celestial bodies. An interesting application [5] is the determination of the rigid-body potential, which is necessary for analyzing the orbit-coupled attitude motion of a spacecraft near small celestial bodies. In general, the usefulness of the triaxial ellipsoid is widely recognized across numerous applications. Additionally, the choice of coordinate system plays a significant role in gravity field modeling, particularly in expressing gravity potential. An insightful paper [6] provides an analytical treatment of this topic.

A perfect conductor [7] is a material, liquid or solid that contains an unlimited supply of free charges. Although perfect conductors do not exist, metals approximate this behavior for most practical purposes. Several properties characterize a (perfect) conductor: (a) the electric field inside the conductor (also referred to as the electrostatic intensity vector) is zero, (b) the charge density ρ inside the conductor is zero, (c) any net charge resides on the conductor's surface, (d) the surface of a conductor is an equipotential, and (e) the electric field is perpendicular to the surface just outside the conductor. Conductors are often used as shields to prevent external electromagnetic fields from penetrating their interior.

Electrostatics has numerous practical applications, where oblate spheroids are advantageous in various contexts, such as electrostatic interactions between spheroidal dielectric particles [8,9] and hybrid explicit/implicit solvent biomolecular simulations. Significant progress has also been made in electromagnetic shielding. Electromagnetic interference poses challenges to modern electronic devices and human health [10]. One effective protection method is the reflection loss mechanism, in which charge carriers in the shielding material interact with incident electromagnetic waves and reflect them at the material's surface. It is worth noting that electrostatic interactions between charged, polarizable dielectric spheroids [8] are relevant to the study of abnormally shaped red blood cells, proteins and multicellular tumor spheroidal models.

The study of heat transfer by conduction from oblate spheroidal surfaces into a stationary, infinite medium [11] is valuable for understanding complex practical problems. Conduction from an isolated oblate spheroid can often be solved analytically, providing fundamental insights into the phenomena under consideration. Furthermore, these solutions can be used to develop perturbation solutions for forced convection heat transfer from heated spheroids, which is useful for advanced heat and mass transport studies in fluid mechanics. Heat transfer [12] is a critical aspect of thermal management. In building construction, for example, reducing cooling and heating costs contributes to more sustainable living environments. Additionally, as electronic devices become smaller and more powerful, efficient thermal management is essential to prevent overheating, which can reduce performance and lifespan.

In general, it is worth noting that spheroids and triaxial ellipsoids have increasingly been employed to model physical phenomena. For example, spheroids have been used [13,14] in studies of evolutionary ultrasound and photothermal therapies in three-dimensional breast tumor spheroids. In addition, oblate spheroids are widely utilized in industrial applications [15] owing to their favorable hydrodynamic performance.

In the present work, we provide the solution of a series of well-matched problems in gravity, electrostatics, and heat transfer that involve slightly flattened oblate spheroids

and slightly flattened triaxial ellipsoids. There are two methods for studying these kinds of problems. The first one uses the Laplace equation, and the second uses the G-modified Helmholtz equation. These two approaches are independent of one another. We chose the latter because it is a new method, which enables us to determine the unknown intensity without prior knowledge of the potential. In addition, the resulting solutions are simpler than those obtained using the Laplace equation. Finally, we show that, in certain cases, by applying the G-modified Helmholtz equation, the gravitational, electrostatic, and thermal field intensities can be expressed as series of spherical harmonics with the polar distance r raised to irrational powers.

2. First Case: Gravity

A parameterization for a triaxial ellipsoid [16] with semiaxes a_x , a_y and b is given by the relations

$$x(\varphi, \lambda) = \frac{a_x}{\sqrt{1 - e_x^2 \sin^2 \varphi - e_e^2 \cos^2 \varphi \sin^2 \lambda}} \cos \varphi \cos \lambda, \quad (1)$$

$$y(\varphi, \lambda) = \frac{a_x(1 - e_e^2)}{\sqrt{1 - e_x^2 \sin^2 \varphi - e_e^2 \cos^2 \varphi \sin^2 \lambda}} \cos \varphi \sin \lambda, \quad (2)$$

$$z(\varphi, \lambda) = \frac{a_x(1 - e_x^2)}{\sqrt{1 - e_x^2 \sin^2 \varphi - e_e^2 \cos^2 \varphi \sin^2 \lambda}} \sin \varphi, \quad (3)$$

$$e_x^2 = \frac{a_x^2 - b^2}{a_x^2}, \quad (4)$$

$$e_y^2 = \frac{a_y^2 - b^2}{a_x^2}, \quad (5)$$

$$e_e^2 = \frac{a_x^2 - a_y^2}{a_x^2}, \quad (6)$$

where λ and φ are the geodetic latitude and longitude respectively. In the following, the relations representing the gravitational intensity are expressed in spherical coordinates (r, θ', λ_c) , where r is the radial distance; θ' is the spherical latitude, ranging from $-\pi/2$ to $\pi/2$; and λ_c is the spherical longitude. The necessary equations for this transformation [17] are

$$\sin^2 \lambda = \frac{\sin^2 \lambda_c}{\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c}, \quad (7)$$

$$\cos^2 \lambda = \frac{(1 - e_e^2)^2 \cos^2 \lambda_c}{(1 - e_e^2)^2 \cos^2 \lambda_c + \sin^2 \lambda_c}, \quad (8)$$

$$\sin^2 \varphi = \frac{(1 - e_e^2)^2 \sin^2 \theta'}{(1 - e_x^2)^2 \cos^2 \theta' [\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c] + (1 - e_e^2)^2 \sin^2 \theta'}, \quad (9)$$

$$\cos^2 \varphi = \frac{[\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c] (1 - e_x^2)^2 \cos^2 \theta'}{(1 - e_x^2)^2 \cos^2 \theta' [\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c] + (1 - e_e^2)^2 \sin^2 \theta'}. \quad (10)$$

If the gravitational intensity γ is known on the surface of a slightly flattened triaxial ellipsoid [18] (which does not rotate), it can be determined on the outside of this ellipsoid by solving the following Dirichlet boundary value problem, associated with the G-modified Helmholtz equation:

$$\nabla^2 \gamma - \frac{2}{r^2} \gamma = 0, \quad (11)$$

$$\gamma|_S = \gamma(\varphi, \lambda) = \frac{(a_x \gamma_{a_x} \cos^2 \lambda + a_y \gamma_{a_y} \sin^2 \lambda) \cos^2 \varphi + b \gamma_b \sin^2 \varphi}{\sqrt{(a_x^2 \cos^2 \lambda + a_y^2 \sin^2 \lambda) \cos^2 \varphi + b^2 \sin^2 \varphi}}, \quad (12)$$

where the constants γ_{a_x} , γ_{a_y} and γ_b represent the values of gravity intensity at the end of the respective semiaxes. The solution to this boundary value problem, expressed in spherical coordinates, is given by the following:

$$\begin{aligned} \gamma(r, \theta', \lambda_c) = & \sum_{n=0}^2 \sum_{m=0}^n a_{2n2m} r^{-\frac{1 + \sqrt{9 + 8n(2n+1)}}{2}} [P_{2n2m}(\sin \theta') \\ & + e_x^2 P'_{2n2m}(\sin \theta') \sin \theta' \cos^2 \theta' + e_e^2 P'_{2n2m}(\sin \theta') (\sin^3 \theta' \\ & - \sin \theta' \cos^2 \theta' \sin^2 \lambda_c)] (1 - e_e^2 \sin^2 2m \lambda_c) \cos 2m \lambda_c. \end{aligned} \quad (13)$$

The above result has been provided in [17], together with the coefficients a_{2n2m} , where the prime denotes derivation with respect to the argument $\sin \theta'$. For the case of a slightly flattened oblate spheroid with axes a_x , a_y , and b , this solution becomes

$$\begin{aligned} \gamma(r, \theta') = & \sum_{n=0}^2 a_{2n} r^{-\frac{1 + \sqrt{9 + 8n(2n+1)}}{2}} [P_{2n}(\sin \theta') + e_x^2 P'_{2n}(\sin \theta') \sin \theta' \cos^2 \theta'] \\ = & \sum_{n=0}^2 a_{2n} r^{-\frac{1 + \sqrt{9 + 8n(2n+1)}}{2}} \{P_{2n}(\sin \theta') \\ & + e_x^2 (2n + 1) \sin \theta' [P_{2n}(\sin \theta') \sin \theta' - P_{2n+1}(\sin \theta')]\}. \end{aligned} \quad (14)$$

Owing to Equation (13), the vertical gradient of gravity in spherical approximation for the case of a triaxial ellipsoid is equal to

$$\begin{aligned} \frac{\partial \gamma}{\partial r}(r, \theta', \lambda_c) = & \sum_{n=0}^2 \sum_{m=0}^n -\frac{1 + \sqrt{9 + 8n(2n+1)}}{2} a_{2n2m} r^{-\frac{3 + \sqrt{9 + 8n(2n+1)}}{2}} [P_{2n2m}(\sin \theta') \\ & + e_x^2 P'_{2n2m}(\sin \theta') \sin \theta' \cos^2 \theta' + e_e^2 P'_{2n2m}(\sin \theta') (\sin^3 \theta' \\ & - \sin \theta' \cos^2 \theta' \sin^2 \lambda_c)] (1 - e_e^2 \sin^2 2m \lambda_c) \cos 2m \lambda_c. \end{aligned} \quad (15)$$

Then, the mean curvature of the equipotential surfaces [19] in spherical approximation is given by

$$J(r, \theta', \lambda_c) = -\frac{1}{2\gamma} \frac{\partial \gamma}{\partial r}. \quad (16)$$

For the case of an oblate spheroid, the vertical gradient of normal gravity has the form

$$\begin{aligned} \frac{\partial \gamma}{\partial r}(r, \theta', \lambda_c) = & \sum_{n=0}^2 -\frac{1 + \sqrt{9 + 8n(2n+1)}}{2} a_{2n} r^{-\frac{3 + \sqrt{9 + 8n(2n+1)}}{2}} [P_{2n}(\sin \theta') \\ & + e_x^2 P'_{2n}(\sin \theta') \sin \theta' \cos^2 \theta'] \\ = & \sum_{n=0}^2 -\frac{1 + \sqrt{9 + 8n(2n+1)}}{2} a_{2n} r^{-\frac{3 + \sqrt{9 + 8n(2n+1)}}{2}} \{P_{2n}(\sin \theta') \\ & + e_x^2 (2n + 1) \sin \theta' [P_{2n}(\sin \theta') \sin \theta' - P_{2n+1}(\sin \theta')]\}. \end{aligned} \quad (17)$$

Though it is not a correct definition, we can consider the slightly flattened triaxial ellipsoid as a perfect conductor (for the exterior of the triaxial ellipsoid) since its surface is an equipotential surface and gravity (on its surface) is determined by the formula given by Mineo. This enables us to generalize our study to perfect conductors.

3. Second Case: Electrostatics

The problem here is to determine the electrostatic intensity of a slightly flattened triaxial ellipsoid that is a perfect conductor. Let V_0 be the electrostatic potential on the surface of the conductor, and let Q be the total charge uniformly distributed on its surface. Then, the surface charge density of the conductor is given by the following [20]:

$$\sigma = \frac{Q}{4\pi a_x a_y b} \frac{1}{\sqrt{\frac{x^2}{a_x^4} + \frac{y^2}{a_y^4} + \frac{z^2}{b^4}}} = \frac{Q}{4\pi \sqrt{a_x a_y b}} \sqrt[4]{K_G}, \quad (18)$$

where x , y , and z are functions of the spherical coordinates (r, θ', λ_c) , and K_G is the Gaussian curvature of the triaxial ellipsoid (see Appendix A), while the electrostatic intensity on the surface of the conductor is written as

$$E_S(r, \theta', \lambda_c) = \frac{\sigma}{\epsilon_0} = \frac{Q}{4\pi \epsilon_0 \sqrt{a_x a_y b}} \sqrt[4]{K_G}. \quad (19)$$

Therefore, both the surface charge density and the electrostatic field intensity on the ellipsoidal surface are functions of the Gaussian curvature K_G .

The denominator in Equation (18) must be expressed in spherical coordinates, while only linear terms in the eccentricities are retained. All necessary manipulations are provided in Appendix A. The final result is as follows:

$$E_S(r, \theta', \lambda_c) = \frac{Q}{4\pi \epsilon_0 a_y b} \left(1 + \frac{e_x^2}{2} + \frac{e_e^2}{2} \cos^2 \theta' \sin^2 \lambda_c \right). \quad (20)$$

Hence, the solution of the following Dirichlet boundary value problem will yield the desired solution for E on the outside of the triaxial ellipsoid:

$$\nabla^2 E - \frac{2}{r^2} E = 0, \quad (21)$$

$$E_{\text{triaxial}}(\theta', \lambda_c) = \frac{Q}{4\pi \epsilon_0 a_y b} \left(1 + \frac{e_x^2}{2} + \frac{e_e^2}{2} \cos^2 \theta' \sin^2 \lambda_c \right) = \frac{Q}{4\pi \epsilon_0 \sqrt{a_x a_y b}} \sqrt[4]{K_G}. \quad (22)$$

In the boundary condition given by Equation (22), only square powers of $\cos \theta'$ and $\sin \lambda_c$ appear; hence, considering the relevant arguments presented in [17] regarding the gravitational intensity, the solution here takes an equivalent form:

$$\begin{aligned} E(r, \theta', \lambda_c) = & \sum_{n=0}^1 \sum_{m=0}^n b_{2n2m} r^{-\frac{1 + \sqrt{9 + 8n(2n+1)}}{2}} [P_{2n2m}(\sin \theta') \\ & + e_x^2 P'_{2n2m}(\sin \theta') \sin \theta' \cos^2 \theta' + e_e^2 P'_{2n2m}(\sin \theta') (\sin^3 \theta' \\ & - \sin \theta' \cos^2 \theta' \sin^2 \lambda_c)] (1 - e_e^2 \sin^2 2m \lambda_c) \cos 2m \lambda_c. \end{aligned} \quad (23)$$

where the coefficients b_{2n2m} can be determined in the same way as a_{2n2m} [17]. It should be noted that, owing to the above boundary condition, K_G directly affects the coefficients of the harmonic expansion. As a result, the spatial distribution of the electrostatic field intensity depends not only on the total charge Q but also on the local Gaussian curvature of the surface.

For the case of a slightly flattened oblate spheroid, a similar Dirichlet boundary value problem is formulated. The respective boundary condition is (see Appendix A)

$$E_S(r, \theta', \lambda_c) = \frac{\sigma}{\epsilon_0} = \frac{Q}{4\pi \epsilon_0 a_x b} \left(1 + \frac{1}{2} e_x^2 \sin^2 \theta' \right) = \frac{Q}{4\pi \epsilon_0 a_x \sqrt{b}} \sqrt[4]{K_G} \quad (24)$$

and the solution takes the form

$$\begin{aligned} E(r, \theta') &= \sum_{n=0}^1 b_{2n} r^{-\frac{1 + \sqrt{9 + 8n(2n+1)}}{2}} [P_{2n}(\sin \theta') + e_x^2 P'_{2n}(\sin \theta') \sin \theta' \cos^2 \theta'] \\ &= \sum_{n=0}^1 b_{2n} r^{-\frac{1 + \sqrt{9 + 8n(2n+1)}}{2}} \{P_{2n}(\sin \theta') \\ &\quad + e_x^2 (2n+1) \sin \theta' [P_{2n}(\sin \theta') \sin \theta' - P_{2n+1}(\sin \theta')]\}. \end{aligned} \quad (25)$$

Finally, a similar formula can be determined for the vertical gradient of the electrostatic field intensity and the mean curvature of the equipotential surfaces. It is evident that this vertical gradient is strongly influenced by the Gaussian curvature of the triaxial ellipsoid (or oblate spheroid).

Comment: It is also possible to formulate and solve a similar problem for the case of magnetostatics (absence of currents).

4. Third Case: Heat

In this instance, we define the thermal intensity vector as

$$\bar{h} = \text{grad} T, \quad (26)$$

where T is temperature. To solve a problem equivalent to the one of electrostatics, we suppose that the thermal conductivity k is constant (i.e., uniformly distributed in the surrounding media) and there is no time dependence. For the case of a slightly flattened triaxial ellipsoid that is a perfect thermal conductor, we follow [21] and define the following quantity as the surface density of thermal intensity

$$\sigma_h(r, \theta', \lambda_c) = -k \left(\frac{\partial T}{\partial \theta} \|\text{grad} \theta\| \right)_{\theta=0}, \quad (27)$$

where θ is a variable depending on r , θ' and λ_c . The surface charge density in the case of electrostatics (see Equation (18)) consists of a geometric component and a physical component. The geometric component is determined by the surface, which in both cases—electrostatics and heat transfer—is a slightly flattened triaxial ellipsoid. Moreover, both problems (with heat transfer assumed to be time-independent in the present case) are governed by the Laplace equation. The magnitude of the electrostatic field intensity vector is given by (see Equations (18) and (19))

$$E_S(r, \theta', \lambda_c) = C \frac{1}{\sqrt{\frac{x^2}{a_x^4} + \frac{y^2}{a_y^4} + \frac{z^2}{b^4}}}. \quad (28)$$

The constant C represents a physics-related parameter, while the quotient accounts for the geometry of the surface. Therefore, by analogy, the magnitude of the thermal intensity vector on the surface of the triaxial ellipsoid is given by

$$h_S(r, \theta', \lambda_c) = C_1 \frac{1}{\sqrt{\frac{x^2}{a_x^4} + \frac{y^2}{a_y^4} + \frac{z^2}{b^4}}} = C_2 \sqrt[4]{K_G}. \quad (29)$$

Here C_1 and C_2 are different constants, as we are now considering the heat transfer case, while the surface remains unchanged. It is not necessary to determine the value of C_1 since the level of approximation in our formulas—where only terms multiplied by the

eccentricities e_e^2 and e_x^2 are retained—is influenced solely by the geometrical part of Equation (29). Then, solving a Dirichlet boundary value problem analogous to that of a sphere, with the boundary condition given by Equation (29), the magnitude of the thermal intensity vector on the outside of the triaxial ellipsoid is given by

$$h(r, \theta', \lambda_c) = \sum_{n=0}^1 \sum_{m=0}^n c_{2n2m} r^{-\frac{1 + \sqrt{9 + 8n(2n+1)}}{2}} [P_{2n2m}(\sin \theta') + e_x^2 P'_{2n2m}(\sin \theta') \sin \theta' \cos^2 \theta' + e_e^2 P'_{2n2m}(\sin \theta') (\sin^3 \theta' - \sin \theta' \cos^2 \theta' \sin^2 \lambda_c)] (1 - e_e^2 \sin^2 2m \lambda_c) \cos 2m \lambda_c. \quad (30)$$

Finally, for the case of an oblate spheroid, we have the following relation

$$\begin{aligned} h(r, \theta') &= \sum_{n=0}^1 c_{2n} r^{-\frac{1 + \sqrt{9 + 8n(2n+1)}}{2}} [P_{2n}(\sin \theta') + e_x^2 P'_{2n}(\sin \theta') \sin \theta' \cos^2 \theta'] \\ &= \sum_{n=0}^1 c_{2n} r^{-\frac{1 + \sqrt{9 + 8n(2n+1)}}{2}} \{P_{2n}(\sin \theta') + e_x^2 (2n+1) \sin \theta' [P_{2n}(\sin \theta') \sin \theta' - P_{2n+1}(\sin \theta')]\}. \end{aligned} \quad (31)$$

The vertical gradient of the thermal intensity in spherical approximation is equal to

$$\begin{aligned} \frac{\partial h}{\partial r}(r, \theta', \lambda_c) &= \sum_{n=0}^1 \sum_{m=0}^n -\frac{1 + \sqrt{9 + 8n(2n+1)}}{2} c_{2n2m} r^{-\frac{3 + \sqrt{9 + 8n(2n+1)}}{2}} [P_{2n2m}(\sin \theta') + e_x^2 P'_{2n2m}(\sin \theta') \sin \theta' \cos^2 \theta' + e_e^2 P'_{2n2m}(\sin \theta') (\sin^3 \theta' - \sin \theta' \cos^2 \theta' \sin^2 \lambda_c)] (1 - e_e^2 \sin^2 2m \lambda_c) \cos 2m \lambda_c \end{aligned} \quad (32)$$

and the mean curvature of the isothermal surfaces is given by

$$J_{isothermal}(r, \theta', \lambda_c) = -\frac{1}{2h} \frac{\partial h}{\partial r}. \quad (33)$$

By analogy with the electrostatic case, thermal intensity, its vertical gradient, and the mean curvature of the isothermal surfaces exhibit a strong dependence on the Gaussian curvature of the triaxial ellipsoid (or oblate spheroid).

It is also possible to find the magnitude of the thermal flux vector f through the following relation:

$$\bar{f} = k \text{grad} T. \quad (34)$$

Hence, for the case of a slightly flattened triaxial ellipsoid, we have

$$\begin{aligned} f(r, \theta', \lambda_c) &= k \sum_{n=0}^1 \sum_{m=0}^n c_{2n2m} r^{-\frac{1 + \sqrt{9 + 8n(2n+1)}}{2}} [P_{2n2m}(\sin \theta') + e_x^2 P'_{2n2m}(\sin \theta') \sin \theta' \cos^2 \theta' + e_e^2 P'_{2n2m}(\sin \theta') (\sin^3 \theta' - \sin \theta' \cos^2 \theta' \sin^2 \lambda_c)] (1 - e_e^2 \sin^2 2m \lambda_c) \cos 2m \lambda_c \\ &= \sum_{n=0}^1 \sum_{m=0}^n (k c_{2n2m}) r^{-\frac{1 + \sqrt{9 + 8n(2n+1)}}{2}} [P_{2n2m}(\sin \theta') + e_x^2 P'_{2n2m}(\sin \theta') \sin \theta' \cos^2 \theta' + e_e^2 P'_{2n2m}(\sin \theta') (\sin^3 \theta' - \sin \theta' \cos^2 \theta' \sin^2 \lambda_c)] (1 - e_e^2 \sin^2 2m \lambda_c) \cos 2m \lambda_c. \end{aligned} \quad (35)$$

For the case of a slightly flattened oblate spheroid, we get

$$\begin{aligned} f(r, \theta') &= k \sum_{n=0}^1 c_{2n} r^{-\frac{1 + \sqrt{9 + 8n(2n+1)}}{2}} [P_{2n}(\sin \theta') + e_x^2 P'_{2n}(\sin \theta') \sin \theta' \cos^2 \theta'] \\ &= \sum_{n=0}^1 (kc_{2n}) r^{-\frac{1 + \sqrt{9 + 8n(2n+1)}}{2}} \{P_{2n}(\sin \theta') \\ &\quad + e_x^2 (2n+1) \sin \theta' [P_{2n}(\sin \theta') \sin \theta' - P_{2n+1}(\sin \theta')]\}. \end{aligned} \quad (36)$$

Note that the thermal flux values also depend on the Gaussian curvature of the triaxial ellipsoid (or oblate spheroid).

Finally, to express the magnitude of the thermal intensity vector or the magnitude of the thermal flux vector on the surface of the triaxial ellipsoid as a series of spherical harmonics, we make the following substitution in Equation (30) for the polar distance r [17]

$$\begin{aligned} r_e &= a_x (1 - e_e^2) \sqrt{1 - e_x^2} \{ (1 - e_x^2) \cos^2 \theta' [\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c] + (1 \\ &\quad - e_e^2)^2 \sin^2 \theta' - e_e^2 (1 - e_x^2) \cos^2 \theta' \sin^2 \lambda_c \}^{-\frac{1}{2}}. \end{aligned} \quad (37)$$

For the case of an oblate spheroid, we can do the same using Equation (31) and the relation

$$r_s = \frac{a_x \sqrt{1 - e_x^2}}{\sqrt{1 - e_x^2 \cos^2 \theta'}}. \quad (38)$$

The above substitutions are also valid for the respective expressions of gravitational and electrostatic field intensity, given in Sections 2 and 3.

5. The Indirect Determination of the Disturbing Gravity Potential for a Spherical Earth

Here the physical shape of Earth [19] is represented by an equipotential surface which is called a geoid. Using the G-modified Helmholtz equation whose solution (as an infinite series of spherical harmonics) was introduced in [22], we express gravity disturbance and gravity anomaly as a series of spherical harmonics on and outside of the geoid (R is the mean Earth's radius), solving the following Dirichlet boundary value problems (δg and Δg stand for the gravity disturbance and gravity anomaly respectively)

$$\nabla^2 \delta g - \frac{2}{r^2} \delta g = 0, \quad (39)$$

$$\delta g_{\text{geoid}}(\theta', \lambda) = f_1(\theta', \lambda), \quad f_1 \text{ } C^k \text{ differentiable, } k \geq 2 \quad (40)$$

$$\nabla^2 \Delta g - \frac{2}{r^2} \Delta g = 0, \quad (41)$$

$$\Delta g_{\text{geoid}}(\theta', \lambda) = f_2(\theta', \lambda), \quad f_2 \text{ } C^k \text{ differentiable, } k \geq 2 \quad (42)$$

The solutions to these problems are as follows (see also Appendix A):

$$\begin{aligned} \delta g(r, \theta', \lambda) &= \sum_{n=0}^{+\infty} \sum_{m=0}^n \left(\frac{R}{r}\right)^{\frac{1 + \sqrt{9 + 4n(n+1)}}{2}} \left[A_{nm}^{\delta g} P_{nm}(\sin \theta') \cos m\lambda \right. \\ &\quad \left. + B_{nm}^{\delta g} P_{nm}(\sin \theta') \sin m\lambda \right], \end{aligned} \quad (43)$$

$$\Delta g(r, \theta', \lambda) = \sum_{n=0}^{+\infty} \sum_{m=0}^n \left(\frac{R}{r}\right)^{\frac{1+\sqrt{9+4n(n+1)}}{2}} \left[A_{nm}^{\Delta g} P_{nm}(\sin \theta') \cos m\lambda + B_{nm}^{\Delta g} P_{nm}(\sin \theta') \sin m\lambda \right], \quad (44)$$

where R is Earth's mean radius.

It can be shown that the series in the above solutions converge uniformly. Indeed, let us write the denominator as

$$r^{-\frac{1+\sqrt{9+4n(n+1)}}{2}} = r^{-\frac{1+2n\sqrt{\frac{9}{4n^2} + \frac{1}{n} + 1}}{2}}. \quad (45)$$

If n takes very large values, then the above behaves as

$$r^{-n-\frac{1}{2}} = r^{-(n+1)} r^{\frac{1}{2}}. \quad (46)$$

In this case (for a large $n = N$), series (43) and (44) behave as

$$\begin{aligned} \delta g(r, \theta', \lambda) \approx & \sum_{n=0}^N \sum_{m=0}^n \left(\frac{R}{r}\right)^{\frac{1+\sqrt{9+4n(n+1)}}{2}} \left[A_{nm}^{\delta g} P_{nm}(\sin \theta') \cos m\lambda + B_{nm}^{\delta g} P_{nm}(\sin \theta') \sin m\lambda \right] \\ & + \left(\frac{R}{r}\right)^{-\frac{1}{2}} \sum_{n=N+1}^{+\infty} \sum_{m=N+1}^n \left(\frac{R}{r}\right)^{n+1} \left[A_{nm}^{\delta g} P_{nm}(\sin \theta') \cos m\lambda + B_{nm}^{\delta g} P_{nm}(\sin \theta') \sin m\lambda \right], \end{aligned} \quad (47)$$

$$\begin{aligned} \Delta g(r, \theta', \lambda) \approx & \sum_{n=0}^N \sum_{m=0}^n \left(\frac{R}{r}\right)^{\frac{1+\sqrt{9+4n(n+1)}}{2}} \left[A_{nm}^{\Delta g} P_{nm}(\sin \theta') \cos m\lambda + B_{nm}^{\Delta g} P_{nm}(\sin \theta') \sin m\lambda \right] \\ & + \left(\frac{R}{r}\right)^{-\frac{1}{2}} \sum_{n=N+1}^{+\infty} \sum_{m=N+1}^n \left(\frac{R}{r}\right)^{n+1} \left[A_{nm}^{\Delta g} P_{nm}(\sin \theta') \cos m\lambda + B_{nm}^{\Delta g} P_{nm}(\sin \theta') \sin m\lambda \right]. \end{aligned} \quad (48)$$

Both series are split into two other series. The first one includes differentiable bounded terms; it is finite, and therefore it converges uniformly. The second series is the familiar harmonic series from the Laplace equation multiplied by a factor that does not affect convergence. Therefore, their sum uniformly converges, and the desirable solutions for gravity disturbance and gravity anomaly converge uniformly.

On the surface of the geoid, the following boundary condition [19] holds (T stands for Earth's disturbing gravity potential)

$$\frac{\partial T}{\partial r} + \frac{2}{R} T = -\Delta g. \quad (49)$$

On the geoid, we want the radial derivative of the disturbing potential to be equal to gravity disturbance. Therefore, using the above relation, we form the following Dirichlet boundary value problem for the determination of the disturbing potential

$$\nabla^2 T = 0, \quad (50)$$

$$\begin{aligned}
 T(R, \theta', \lambda) &= \frac{R}{2} [\delta g(R, \theta', \lambda) - \Delta g(R, \theta', \lambda)] \\
 &= \frac{R}{2} \sum_{n=0}^{+\infty} \sum_{m=0}^n \left[(A_{nm}^{\delta g} - A_{nm}^{\Delta g}) P_{nm}(\sin \theta') \cos m\lambda \right. \\
 &\quad \left. + (B_{nm}^{\delta g} - B_{nm}^{\Delta g}) P_{nm}(\sin \theta') \sin m\lambda \right],
 \end{aligned} \tag{51}$$

The above boundary condition is in place due to the G-modified Helmholtz equation. The solution to this problem is as follows

$$T(r, \theta', \lambda) = \sum_{n=0}^{+\infty} \sum_{m=0}^n \left(\frac{R}{r} \right)^{(n+1)} \left[A_{nm}^T P_{nm}(\sin \theta') \cos m\lambda + B_{nm}^T P_{nm}(\sin \theta') \sin m\lambda \right]. \tag{52}$$

The final form of the disturbing potential (see Appendix A) turns out to be very elegant:

$$\begin{aligned}
 T(r, \theta', \lambda) &= \frac{R}{2} \sum_{n=0}^{+\infty} \sum_{m=0}^n \left(\frac{R}{r} \right)^{(n+1)} \left[(A_{nm}^{\delta g} - A_{nm}^{\Delta g}) P_{nm}(\sin \theta') \cos m\lambda + (B_{nm}^{\delta g} \right. \\
 &\quad \left. - B_{nm}^{\Delta g}) P_{nm}(\sin \theta') \sin m\lambda \right],
 \end{aligned} \tag{53}$$

The coefficients of the above series incorporate the spherical harmonic coefficients of the gravity anomaly and gravity disturbance, which are determined through the G-modified Helmholtz equation. Finally, it is worth emphasizing that important quantities, such as geoid undulation and the components of the deflection of the vertical [19], depend exclusively on the coefficients of the gravity anomaly and gravity disturbance.

6. Discussion

The present study concentrates on the theoretical development of the G-modified Helmholtz equation solutions by introducing Dirichlet boundary value problems and providing direct analytical expressions for the determination of gravity, electrostatic and thermal field intensities in the case of a sphere and, more generally, for slightly flattened oblate spheroids and triaxial ellipsoids, thereby constituting the necessary first steps before any large-scale numerical implementation. The development is based on a rigorous analytical treatment of the G-modified Helmholtz equation [22] as a linear elliptic partial differential equation together with the corresponding smooth boundary geometry. The considered fields are strongly related to the relevant potential; however, utilizing the present methodology, we take advantage of boundary conditions that involve measurable quantities instead of the potential, which is not directly measurable.

The solutions are expressed as finite spherical harmonic expansions whose structure follows consistently from a perturbative framework valid for small eccentricities and from the imposed boundary conditions; thus, it is not just a truncated approximation of an infinite harmonic expansion. In particular, for the class of slightly flattened spheroids and triaxial ellipsoids considered here, the boundary conditions were developed consistently up to second order in the eccentricity parameters; hence the solutions which were presented in this work involve only low-degree harmonic expansions ($n = 0, 1, 2$ for the case of gravity and $n = 0, 1$ for the case of electrostatics and heat transfer), embodying associated Legendre functions of the same degree. Consequently, the presented series are finite, with the higher-degree coefficients vanishing identically and the remaining individual terms being smooth and bounded functions. Therefore, there are no issues of convergence, and no truncation error arises in the prescribed order of approximation. The same conclusion applies to the vertical gradient, since it is obtained by the radial differentiation of the finite harmonic expansion and therefore preserves the same finite angular structure.

Note that, in the considered cases of electrostatics and heat transfer, the coefficients of the harmonic expansions are directly affected by the Gaussian curvature of the slightly

triaxial ellipsoid (and similarly for a slightly oblate spheroid), which enters explicitly through the boundary condition. Therefore, not only the calculated intensities but also the vertical gradients are sensitive to the Gaussian curvature of the triaxial ellipsoid. Furthermore, the surface of a triaxial ellipsoid (as well as that of an oblate spheroid) is a smooth closed surface, free of singular or isolated points, while the expression defining the radial distance of points on the surface of the triaxial ellipsoid (see Equation (37)) is itself a bounded and continuously differentiable function. Consequently, it does not affect the boundedness or regularity of the series representation. Hence, each of the series is uniformly convergent for points on and above the triaxial ellipsoid.

On the other hand, the derived irrational radial exponents originate from the solution of the Euler-type ordinary differential equation [22], in analogy with the classical harmonic case. This equation admits two linearly independent solutions, where the physically admissible one is selected by imposing decay at infinity. The presence of irrational exponents does not affect the linear independence of the radial solutions nor the completeness of the angular basis. Moreover, uniqueness is ensured because the G-modified Helmholtz equation is linear and elliptic. In each case, the resulting harmonic representation is uniquely determined by the boundary conditions and remains mathematically well-defined, where the eccentricities quantify the deviation from spherical symmetry and systematically control the structure of the expansion.

For larger eccentricities, that is, if the boundary departs significantly from spherical symmetry (an arbitrary triaxial ellipsoid), the above perturbative approach should be reformulated to include higher-order eccentricity terms and higher-order derivatives of the associated Legendre functions, which would in turn generate additional harmonic terms; the harmonic expansion would still be finite for any prescribed approximation order, though. This would require a new analytical development, which could not be achieved by simply extending the current series. In fact, such an extension would involve additional analytical complexity, substantially more than the one presented analytically in [17] for the case of gravity on the outside of a slightly flattened triaxial ellipsoid. Nevertheless, for any finite degree n , the associated Legendre functions and their derivatives remain smooth and bounded, so no issues of convergence or well-posedness would arise. The increased geometric complexity would affect only the analytical tractability of the expansion, not the mathematical validity of the formulation. The restriction to small eccentricities therefore represents a deliberate modeling choice, ensuring analytical transparency and controlled approximation. Last but not least, in Section 5, regarding the determination of gravity anomaly and gravity disturbance on and outside of a spherical Earth model, it is shown that the infinite spherical harmonic series expansion of the G-modified Helmholtz equation converges uniformly.

As for the assumed Dirichlet-type boundary conditions, it is worth mentioning that, since a slightly flattened triaxial ellipsoid is a smooth closed surface, the imposition of Neumann or Robin boundary conditions also leads to well-posed boundary value problems for the G-modified Helmholtz equation. The analytical structure of the solution remains unchanged, namely a finite spherical harmonic expansion constructed under the same perturbative assumptions (retaining terms up to second order in the eccentricity parameters). Only the determination of the harmonic coefficients is affected, as these must satisfy the modified boundary conditions. Therefore, the investigation of other types of boundary value problems (Neumann or Robin) can also be carried out, provided that the boundary surface is an equipotential surface. Naturally, the determination of the harmonic coefficients is affected, as they must satisfy the modified boundary conditions.

Finally, it should be pointed out that our methodology differs from the classic approach for the determination of the magnitude of gravitational/electrostatic/heat field

intensities, based on the solution of the Laplace equation on the exterior of a slightly triaxial ellipsoid whose surface is an equipotential surface. In that case, the procedure would require appropriate transformations and separation of variables using the corresponding orthogonal curvilinear coordinate system. Then, the resulting solution would be given in terms of special functions that do not generally admit closed-form expressions, while the determination of, e.g., the electrostatic field, would further necessitate the differentiation of the potential (something which is too complicated in itself). Consequently, a complete treatment of this approach would entail disproportionately extensive and technically demanding calculations, as well as the inevitable use of specialized computational techniques for the evaluation of the respective eigenfunctions. However, the above process, supplemented by direct quantitative comparisons with solutions derived via our methodology, can be considered as a future perspective of the present work.

7. Conclusions

We presented an alternative method for determining gravity, electrostatic and thermal intensities as series of spherical harmonic expansions, with the polar distance raised to irrational powers. The problems considered share several common features: (a) they are time-independent, (b) the boundary surfaces are slightly flattened oblate spheroids and slightly flattened triaxial ellipsoids, and (c) the closed surfaces are perfect conductors. These surfaces were chosen because they are broadly used in applications.

The Dirichlet boundary value problems considered are related to the G-modified Helmholtz equation. This approach demonstrates that the selected field intensities can be determined without explicit knowledge of the corresponding potential. The solutions provided offer an alternative method for solving these problems.

The derived solutions for the selected intensities are elegant and considerably simpler than evaluating the magnitude of the gradient of the corresponding potential. Therefore, this approach is preferable for the description of the intensity outside of the selected surfaces. An additional advantage of this method is the direct determination of the vertical gradient of the chosen intensity. In our cases, the vertical gradient (determined in spherical approximation) closely exhibits the maximum variation in the selected intensity and is closely related to the mean curvature of the equipotential surfaces. Furthermore, the Gaussian curvature of the triaxial ellipsoid (or oblate spheroid) exerts a significant influence on the results related to electrostatics and heat transfer, specifically the field intensity, its vertical gradient, and the mean curvature of the equipotential or isothermal surfaces.

In addition, in the last section, we presented an alternative method for determining the disturbing gravity potential for a model of a spherical Earth. Using the G-modified Helmholtz equation, we derived a spherical harmonic expansion representing the disturbing potential on the surface of the geoid. This formulation enabled us to establish a Dirichlet boundary value problem associated with Laplace's equation. The solution is expressed as a spherical harmonic series for the disturbing potential, whose coefficients incorporate the coefficients of gravity disturbance and gravity anomaly. This constitutes a highly significant and original result, as it demonstrates the fundamental importance of the gravity disturbance and gravity anomaly coefficients. Since geoid undulation and deflections of the vertical are directly related to the disturbing potential, their values depend exclusively on the aforementioned coefficients.

The presented method for determining gravity, electrostatic, and thermal intensities is novel and provides practical solutions. It is an efficient alternative for solving the selected boundary value problems, reducing computational effort and time, and merits further study.

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Appendix A

The relations between geodetic latitude and longitude with spherical latitude and longitude [17] are

$$\sin^2 \lambda = \frac{\sin^2 \lambda_c}{\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c}, \quad (\text{A1})$$

$$\cos^2 \lambda = \frac{(1 - e_e^2)^2 \cos^2 \lambda_c}{(1 - e_e^2)^2 \cos^2 \lambda_c + \sin^2 \lambda_c}, \quad (\text{A2})$$

$$\sin^2 \varphi = \frac{(1 - e_e^2)^2 \sin^2 \theta'}{(1 - e_x^2)^2 \cos^2 \theta' [\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c] + (1 - e_e^2)^2 \sin^2 \theta'}, \quad (\text{A3})$$

$$\cos^2 \varphi = \frac{[\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c] (1 - e_x^2)^2 \cos^2 \theta'}{(1 - e_x^2)^2 \cos^2 \theta' [\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c] + (1 - e_e^2)^2 \sin^2 \theta'}. \quad (\text{A4})$$

The squares of the coordinates of the parameterization of the slightly triaxial ellipsoid are

$$x^2(\varphi, \lambda) = \frac{a_x^2}{1 - e_x^2 \sin^2 \varphi - e_e^2 \cos^2 \varphi \sin^2 \lambda} \cos^2 \varphi \cos^2 \lambda, \quad (\text{A5})$$

$$y^2(\varphi, \lambda) = \frac{a_x^2 (1 - e_e^2)^2}{1 - e_x^2 \sin^2 \varphi - e_e^2 \cos^2 \varphi \sin^2 \lambda} \cos^2 \varphi \sin^2 \lambda, \quad (\text{A6})$$

$$z^2(\varphi, \lambda) = \frac{a_x^2 (1 - e_x^2)^2}{1 - e_x^2 \sin^2 \varphi - e_e^2 \cos^2 \varphi \sin^2 \lambda} \sin^2 \varphi. \quad (\text{A7})$$

Replacing Equations (A1)–(A4) in Equation (A5), we have that

$$\begin{aligned} 1 - e_x^2 \sin^2 \varphi - e_e^2 \cos^2 \varphi \sin^2 \lambda &= \left\{ (1 - e_x^2)^2 \cos^2 \theta' [\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c] + (1 - e_e^2)^2 \sin^2 \theta' - e_x^2 (1 - e_e^2)^2 \sin^2 \theta' \right\} \{ (1 - e_x^2)^2 \cos^2 \theta' [\sin^2 \lambda_c \\ &+ (1 - e_e^2)^2 \cos^2 \lambda_c] + (1 - e_e^2)^2 \sin^2 \theta' \}^{-1} \\ &- \{ e_e^2 (1 - e_x^2)^2 \cos^2 \theta' \sin^2 \lambda_c \} \{ (1 - e_x^2)^2 \cos^2 \theta' [\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c] + (1 - e_e^2)^2 \sin^2 \theta' \}^{-1} \\ &= \left\{ (1 - e_x^2)^2 \cos^2 \theta' [\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c] + (1 - e_x^2) (1 - e_e^2)^2 \sin^2 \theta' - e_e^2 (1 - e_x^2)^2 \cos^2 \theta' \sin^2 \lambda_c \right\} \{ (1 - e_x^2)^2 \cos^2 \theta' [\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c] + (1 - e_e^2)^2 \sin^2 \theta' \}^{-1} \end{aligned} \quad (\text{A8})$$

and

$$\begin{aligned}
 a_x^2 \cos^2 \varphi \cos^2 \lambda &= \left\{ a_x^2 \left[\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c \right] (1 - e_x^2)^2 \cos^2 \theta' (1 - e_e^2)^2 \cos^2 \lambda_c \right\} \left\{ \left[(1 - e_x^2)^2 \cos^2 \theta' \left[\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c \right] + (1 - e_e^2)^2 \sin^2 \theta' \right] \left[(1 - e_e^2)^2 \cos^2 \lambda_c + \sin^2 \lambda_c \right] \right\}^{-1} \\
 &= \left[a_x^2 (1 - e_x^2)^2 \cos^2 \theta' (1 - e_e^2)^2 \cos^2 \lambda_c \right] \left\{ (1 - e_x^2)^2 \cos^2 \theta' \left[\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c \right] + (1 - e_e^2)^2 \sin^2 \theta' \right\}^{-1},
 \end{aligned} \tag{A9}$$

hence,

$$\begin{aligned}
 &\frac{a_x^2 \cos^2 \varphi \cos^2 \lambda}{1 - e_x^2 \sin^2 \varphi - e_e^2 \cos^2 \varphi \sin^2 \lambda} \\
 &= \left[a_x^2 (1 - e_x^2)^2 \cos^2 \theta' (1 - e_e^2)^2 \cos^2 \lambda_c \right] \left\{ (1 - e_x^2)^2 \cos^2 \theta' \left[\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c \right] + (1 - e_e^2)^2 \sin^2 \theta' \right\}^{-1} \left\{ (1 - e_x^2)^2 \cos^2 \theta' \left[\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c \right] + (1 - e_e^2)^2 \sin^2 \theta' - e_e^2 (1 - e_x^2)^2 \cos^2 \theta' \sin^2 \lambda_c \right\}^{-1} \\
 &= \left[a_x^2 (1 - e_x^2)^2 (1 - e_e^2)^2 \cos^2 \theta' \cos^2 \lambda_c \right] \left\{ (1 - e_x^2)^2 \cos^2 \theta' \left[\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c \right] + (1 - e_e^2)^2 \sin^2 \theta' \right\}^{-1} \left\{ (1 - e_x^2)^2 \cos^2 \theta' \left[\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c \right] + (1 - e_e^2)^2 \sin^2 \theta' - e_e^2 (1 - e_x^2)^2 \cos^2 \theta' \sin^2 \lambda_c \right\}^{-1}.
 \end{aligned} \tag{A10}$$

We do the same for Equation (A6), obtaining

$$\begin{aligned}
 &a_x^2 (1 - e_e^2)^2 \cos^2 \varphi \sin^2 \lambda \\
 &= \left\{ \left[a_x^2 (1 - e_e^2)^2 \left[\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c \right] (1 - e_x^2)^2 \cos^2 \theta' \right] \sin^2 \lambda_c \right\} \left\{ \left[(1 - e_x^2)^2 \cos^2 \theta' \left[\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c \right] + (1 - e_e^2)^2 \sin^2 \theta' \right] \left[(1 - e_e^2)^2 \cos^2 \lambda_c + \sin^2 \lambda_c \right] \right\}^{-1} \\
 &= \left\{ \left[a_x^2 (1 - e_e^2)^2 (1 - e_x^2)^2 \cos^2 \theta' \right] \sin^2 \lambda_c \right\} \left\{ (1 - e_x^2)^2 \cos^2 \theta' \left[\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c \right] + (1 - e_e^2)^2 \sin^2 \theta' \right\}^{-1},
 \end{aligned} \tag{A11}$$

and

$$\begin{aligned}
& \frac{a_x^2(1-e_e^2)^2}{1-e_x^2\sin^2\varphi-e_e^2\cos^2\varphi\sin^2\lambda} \cos^2\varphi\sin^2\lambda \\
&= \left\{ \left[a_x^2(1-e_e^2)^2(1-e_x^2)^2\cos^2\theta' \right] \sin^2\lambda_c \right\} \{ (1 \\
&\quad -e_x^2)^2\cos^2\theta' [\sin^2\lambda_c + (1-e_e^2)^2\cos^2\lambda_c] + (1 \\
&\quad -e_e^2)^2\sin^2\theta' \}^{-1} \left\{ (1-e_x^2)^2\cos^2\theta' [\sin^2\lambda_c + (1-e_e^2)^2\cos^2\lambda_c] \right. \\
&\quad \left. + (1-e_x^2)(1-e_e^2)^2\sin^2\theta' - e_e^2(1-e_x^2)^2\cos^2\theta'\sin^2\lambda_c \right\}^{-1} \{ (1 \\
&\quad -e_x^2)^2\cos^2\theta' [\sin^2\lambda_c + (1-e_e^2)^2\cos^2\lambda_c] + (1-e_e^2)^2\sin^2\theta' \} \\
&= \left\{ a_x^2(1-e_e^2)^2(1-e_x^2)^2\cos^2\theta'\sin^2\lambda_c \right\} \{ (1 \\
&\quad -e_x^2)^2\cos^2\theta' [\sin^2\lambda_c + (1-e_e^2)^2\cos^2\lambda_c] + (1-e_x^2)(1 \\
&\quad -e_e^2)^2\sin^2\theta' - e_e^2(1-e_x^2)^2\cos^2\theta'\sin^2\lambda_c \}^{-1} \\
&= \left\{ a_x^2(1-e_e^2)^2(1 \right. \\
&\quad \left. -e_x^2)\cos^2\theta'\sin^2\lambda_c \right\} \{ (1-e_x^2)\cos^2\theta'[\sin^2\lambda_c \\
&\quad + (1-e_e^2)^2\cos^2\lambda_c] + (1-e_e^2)^2\sin^2\theta' - e_e^2(1 \\
&\quad -e_x^2)\cos^2\theta'\sin^2\lambda_c \}^{-1}.
\end{aligned} \tag{A12}$$

For Equation (A7), we have

$$\begin{aligned}
& a_x^2(1-e_x^2)^2\sin^2\varphi \\
&= \frac{a_x^2(1-e_x^2)^2(1-e_e^2)^2\sin^2\theta'}{(1-e_x^2)^2\cos^2\theta'[\sin^2\lambda_c + (1-e_e^2)^2\cos^2\lambda_c] + (1-e_e^2)^2\sin^2\theta'}.
\end{aligned} \tag{A13}$$

Therefore,

$$\begin{aligned}
& \frac{a_x^2(1-e_x^2)^2\sin^2\varphi}{1-e_x^2\sin^2\varphi-e_e^2\cos^2\varphi\sin^2\lambda} \\
&= \left\{ a_x^2(1-e_x^2)^2(1 \right. \\
&\quad \left. -e_e^2)^2\sin^2\theta' \right\} \left\{ (1-e_x^2)^2\cos^2\theta' [\sin^2\lambda_c + (1-e_e^2)^2\cos^2\lambda_c] \right. \\
&\quad \left. + (1-e_e^2)^2\sin^2\theta' \right\}^{-1} \left\{ (1-e_x^2)^2\cos^2\theta' [\sin^2\lambda_c + (1 \right. \\
&\quad \left. -e_e^2)^2\cos^2\lambda_c] + (1-e_x^2)(1-e_e^2)^2\sin^2\theta' - e_e^2(1 \right. \\
&\quad \left. -e_x^2)^2\cos^2\theta'\sin^2\lambda_c \right\}^{-1} \left\{ (1-e_x^2)^2\cos^2\theta' [\sin^2\lambda_c + (1 \right. \\
&\quad \left. -e_e^2)^2\cos^2\lambda_c] + (1-e_e^2)^2\sin^2\theta' \right\} \\
&= \left\{ a_x^2(1-e_x^2)(1 \right. \\
&\quad \left. -e_e^2)^2\sin^2\theta' \right\} \left\{ (1-e_x^2)\cos^2\theta' [\sin^2\lambda_c + (1-e_e^2)^2\cos^2\lambda_c] + (1 \right. \\
&\quad \left. -e_e^2)^2\sin^2\theta' - e_e^2(1-e_x^2)\cos^2\theta'\sin^2\lambda_c \right\}^{-1}.
\end{aligned} \tag{A14}$$

In addition,

$$a_y^4 = a_x^4(1-e_e^2)^2, \tag{A15}$$

$$b^4 = a_x^4(1-e_x^2)^2. \tag{A16}$$

Then, the relative quotients become

$$\frac{x^2}{a_x^4} = (1 - e_x^2)(1 - e_e^2)^2 \cos^2 \theta' \cos^2 \lambda_c a_x^{-2} \{ (1 - e_x^2) \cos^2 \theta' [\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c] + (1 - e_e^2)^2 \sin^2 \theta' - e_e^2 (1 - e_x^2) \cos^2 \theta' \sin^2 \lambda_c \}^{-1}, \quad (\text{A17})$$

$$\begin{aligned} \frac{y^2}{a_y^4} &= \frac{y^2}{a_x^4 (1 - e_e^2)^2} \\ &= a_x^{-4} (1 - e_e^2)^{-2} [a_x^2 (1 - e_e^2)^2 (1 - e_x^2) \cos^2 \theta' \sin^2 \lambda_c] \{ (1 - e_x^2) \cos^2 \theta' [\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c] + (1 - e_e^2)^2 \sin^2 \theta' - e_e^2 (1 - e_x^2) \cos^2 \theta' \sin^2 \lambda_c \}^{-1} \\ &= (1 - e_x^2) \cos^2 \theta' \sin^2 \lambda_c a_x^{-2} \{ (1 - e_x^2) \cos^2 \theta' [\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c] + (1 - e_e^2)^2 \sin^2 \theta' - e_e^2 (1 - e_x^2) \cos^2 \theta' \sin^2 \lambda_c \}^{-1}, \end{aligned} \quad (\text{A18})$$

$$\begin{aligned} \frac{z^2}{b^4} &= \frac{z^2}{a_x^4 (1 - e_x^2)^2} \\ &= a_x^{-4} (1 - e_x^2)^{-2} [a_x^2 (1 - e_x^2) (1 - e_e^2)^2 \sin^2 \theta'] \{ (1 - e_x^2) \cos^2 \theta' [\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c] + (1 - e_e^2)^2 \sin^2 \theta' - e_e^2 (1 - e_x^2) \cos^2 \theta' \sin^2 \lambda_c \}^{-1} \\ &= (1 - e_e^2)^2 \sin^2 \theta' a_x^{-2} (1 - e_x^2)^{-1} \{ (1 - e_x^2) \cos^2 \theta' [\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c] + (1 - e_e^2)^2 \sin^2 \theta' - e_e^2 (1 - e_x^2) \cos^2 \theta' \sin^2 \lambda_c \}^{-1}, \end{aligned} \quad (\text{A19})$$

and the summation of the above three relations gives

$$\begin{aligned} \frac{x^2}{a_x^4} + \frac{y^2}{a_y^4} + \frac{z^2}{b^4} &= \left[(1 - e_x^2)^2 (1 - e_e^2)^2 \cos^2 \theta' \cos^2 \lambda_c + (1 - e_x^2)^2 \cos^2 \theta' \sin^2 \lambda_c + (1 - e_e^2)^2 \sin^2 \theta' \right] a_x^{-2} (1 - e_x^2)^{-1} \{ (1 - e_x^2) \cos^2 \theta' [\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c] + (1 - e_e^2)^2 \sin^2 \theta' - e_e^2 (1 - e_x^2) \cos^2 \theta' \sin^2 \lambda_c \}^{-1}, \end{aligned} \quad (\text{A20})$$

$$\begin{aligned}
\frac{1}{\sqrt{\frac{x^2}{a_x^4} + \frac{y^2}{a_y^4} + \frac{z^2}{b^4}}} &= \left[(1 - e_x^2)^2 (1 - e_e^2)^2 \cos^2 \theta' \cos^2 \lambda_c + (1 - e_x^2)^2 \cos^2 \theta' \sin^2 \lambda_c \right. \\
&\quad \left. + (1 - e_e^2)^2 \sin^2 \theta' \right]^{-\frac{1}{2}} \{ a_x^2 (1 - e_x^2) \left[(1 - e_x^2) \cos^2 \theta' \left[\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c \right] + (1 - e_e^2)^2 \sin^2 \theta' - e_e^2 (1 - e_x^2) \cos^2 \theta' \sin^2 \lambda_c \right] \}^{\frac{1}{2}} \\
&= a_x \left[(1 - e_x^2) (1 - e_e^2)^2 \cos^2 \theta' \cos^2 \lambda_c + (1 - e_x^2) \cos^2 \theta' \sin^2 \lambda_c \right. \\
&\quad \left. + (1 - e_e^2)^2 \sin^2 \theta' \right]^{\frac{1}{2}} \{ (1 - e_x^2) \left[(1 - e_x^2) \cos^2 \theta' \left[\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c \right] + (1 - e_e^2)^2 \sin^2 \theta' - e_e^2 (1 - e_x^2) \cos^2 \theta' \sin^2 \lambda_c \right] \}^{-\frac{1}{2}}.
\end{aligned} \tag{A21}$$

We analyze the nominator in terms of the order of flattening:

$$e_e^2 (1 - e_x^2)^2 \cos^2 \theta' \sin^2 \lambda_c = e_e^2 \cos^2 \theta' \sin^2 \lambda_c, \tag{A22}$$

$$\begin{aligned}
(1 - e_x^2) \left((1 - e_e^2)^2 \sin^2 \theta' - (1 - e_x^2) (\sin^2 \theta' - 2e_e^2 \sin^2 \theta') \right) \\
= \sin^2 \theta - 2e_e^2 \sin^2 \theta' - e_x^2 \sin^2 \theta',
\end{aligned} \tag{A23}$$

$$[\sin^2 \lambda_c + (1 - e_e^2)^2 \cos^2 \lambda_c] = (1 - 2e_e^2 \cos^2 \lambda_c) \tag{A24}$$

$$\begin{aligned}
(1 - e_x^2)^2 \cos^2 \theta' (1 - 2e_e^2 \cos^2 \lambda_c) &= (\cos^2 \theta' - 2e_x^2 \cos^2 \theta') (1 - 2e_e^2 \cos^2 \lambda_c) \\
&= \cos^2 \theta' - 2e_x^2 \cos^2 \theta' - 2e_e^2 \cos^2 \theta' \cos^2 \lambda_c.
\end{aligned} \tag{A25}$$

Thus, the nominator is written as

$$\begin{aligned}
\cos^2 \theta' - 2e_x^2 \cos^2 \theta' - 2e_e^2 \cos^2 \theta' \cos^2 \lambda_c + \sin^2 \theta - 2e_e^2 \sin^2 \theta' - e_x^2 \sin^2 \theta' \\
- e_e^2 \cos^2 \theta' \sin^2 \lambda_c \\
= 1 - 2e_x^2 \cos^2 \theta' - 2e_e^2 \cos^2 \theta' \cos^2 \lambda_c \sin^2 \theta - 2e_e^2 \sin^2 \theta' \\
- e_x^2 \sin^2 \theta' - e_e^2 \cos^2 \theta' \sin^2 \lambda_c
\end{aligned} \tag{A26}$$

and doing the same for the denominator yields the following:

$$\begin{aligned}
(1 - e_x^2) (1 - e_e^2)^2 \cos^2 \theta' \cos^2 \lambda_c + (1 - e_x^2) \cos^2 \theta' \sin^2 \lambda_c + (1 - e_e^2)^2 \sin^2 \theta' \\
= (1 - e_x^2) (1 - 2e_e^2) \cos^2 \theta' \cos^2 \lambda_c + \cos^2 \theta' \sin^2 \lambda_c \\
- e_x^2 \cos^2 \theta' \sin^2 \lambda_c + \sin^2 \theta' - 2e_e^2 \sin^2 \theta' \\
= \cos^2 \theta' \cos^2 \lambda_c - e_x^2 \cos^2 \theta' \cos^2 \lambda_c - 2e_e^2 \cos^2 \theta' \cos^2 \lambda_c \\
+ \cos^2 \theta' \sin^2 \lambda_c - e_x^2 \cos^2 \theta' \sin^2 \lambda_c + \sin^2 \theta' - 2e_e^2 \sin^2 \theta' \\
= 1 - e_x^2 \cos^2 \theta' - 2e_e^2 \sin^2 \theta' - 2e_e^2 \cos^2 \theta' \cos^2 \lambda_c.
\end{aligned} \tag{A27}$$

Therefore, we have

$$\begin{aligned}
\frac{1}{\sqrt{\frac{x^2}{a_x^4} + \frac{y^2}{a_y^4} + \frac{z^2}{b^4}}} &= a_x (1 - e_x^2 \cos^2 \theta' - 2e_e^2 \sin^2 \theta' - 2e_e^2 \cos^2 \theta' \cos^2 \lambda_c)^{\frac{1}{2}} (1 \\
&\quad - 2e_e^2 \cos^2 \theta' \cos^2 \lambda_c - 2e_x^2 \cos^2 \theta' - 2e_e^2 \sin^2 \theta' - e_x^2 \sin^2 \theta' \\
&\quad - e_e^2 \cos^2 \theta' \sin^2 \lambda_c)^{-\frac{1}{2}}.
\end{aligned} \tag{A28}$$

But

$$\begin{aligned}
 & (1 - 2e_e^2 \cos^2 \theta' \cos^2 \lambda_c - 2e_x^2 \cos^2 \theta' - 2e_e^2 \sin^2 \theta' - e_x^2 \cos^2 \theta' \\
 & \quad - e_e^2 \cos^2 \theta' \sin^2 \lambda_c)^{-\frac{1}{2}} \\
 & = 1 + e_e^2 \cos^2 \theta' \cos^2 \lambda_c + e_x^2 \cos^2 \theta' + e_e^2 \sin^2 \theta' + \frac{1}{2} e_x^2 \sin^2 \theta' \\
 & \quad + \frac{1}{2} e_e^2 \cos^2 \theta' \sin^2 \lambda_c (1 - e_x^2 \cos^2 \theta' - 2e_e^2 \sin^2 \theta' \\
 & \quad - 2e_e^2 \cos^2 \theta' \cos^2 \lambda_c)^{\frac{1}{2}} \\
 & = 1 - \frac{1}{2} e_x^2 \cos^2 \theta' - e_e^2 \sin^2 \theta' - e_e^2 \cos^2 \theta' \cos^2 \lambda_c.
 \end{aligned} \tag{A29}$$

In addition,

$$\begin{aligned}
 & (1 + e_e^2 \cos^2 \theta' \cos^2 \lambda_c + e_x^2 \cos^2 \theta' + e_e^2 \sin^2 \theta' + \frac{1}{2} e_x^2 \sin^2 \theta' \\
 & \quad + \frac{1}{2} e_e^2 \cos^2 \theta' \sin^2 \lambda_c) \left(1 - \frac{1}{2} e_x^2 \cos^2 \theta' - e_e^2 \sin^2 \theta' \right. \\
 & \quad \left. - e_e^2 \cos^2 \theta' \cos^2 \lambda_c \right) \\
 & = 1 - \frac{1}{2} e_x^2 \cos^2 \theta' - e_e^2 \sin^2 \theta' - e_e^2 \cos^2 \theta' \cos^2 \lambda_c \\
 & \quad + e_e^2 \cos^2 \theta' \cos^2 \lambda_c + e_x^2 \cos^2 \theta' + e_e^2 \sin^2 \theta' + \frac{1}{2} e_x^2 \sin^2 \theta' \\
 & \quad + \frac{1}{2} e_e^2 \cos^2 \theta' \sin^2 \lambda_c = 1 + \frac{e_x^2}{2} + \frac{e_e^2}{2} \cos^2 \theta' \sin^2 \lambda_c.
 \end{aligned} \tag{A30}$$

Hence, the final result is

$$\frac{1}{\sqrt{\frac{x^2}{a_x^4} + \frac{y^2}{a_y^4} + \frac{z^2}{b^4}}} = a_x \left(1 + \frac{e_x^2}{2} + \frac{e_e^2}{2} \cos^2 \theta' \sin^2 \lambda_c \right). \tag{A31}$$

For the case of an oblate spheroid, we have

$$x^2(\varphi, \lambda) = \frac{a_x^2}{1 - e_x^2 \sin^2 \varphi} \cos^2 \varphi \cos^2 \lambda, \tag{A32}$$

$$y^2(\varphi, \lambda) = \frac{a_x^2}{1 - e_x^2 \sin^2 \varphi} \cos^2 \varphi \sin^2 \lambda, \tag{A33}$$

$$z^2(\varphi, \lambda) = \frac{a_x^2 (1 - e_x^2)^2}{1 - e_x^2 \sin^2 \varphi} \sin^2 \varphi \tag{A34}$$

and

$$\begin{aligned}
 & \frac{x^2}{a_x^4} + \frac{y^2}{a_x^4} + \frac{z^2}{a_x^4 (1 - e_x^2)^2} \\
 & = \frac{\frac{a_x^2}{1 - e_x^2 \sin^2 \varphi} \cos^2 \varphi \cos^2 \lambda}{a_x^4} + \frac{\frac{a_x^2}{1 - e_x^2 \sin^2 \varphi} \cos^2 \varphi \sin^2 \lambda}{a_x^4} \\
 & \quad + \frac{\frac{a_x^2 (1 - e_x^2)^2}{1 - e_x^2 \sin^2 \varphi} \sin^2 \varphi}{a_x^4 (1 - e_x^2)^2} \\
 & = \frac{\cos^2 \varphi \cos^2 \lambda}{a_x^2 (1 - e_x^2 \sin^2 \varphi)} + \frac{\cos^2 \varphi \sin^2 \lambda}{a_x^2 (1 - e_x^2 \sin^2 \varphi)} + \frac{\sin^2 \varphi}{a_x^2 (1 - e_x^2 \sin^2 \varphi)} \\
 & = \frac{1}{a_x^2 (1 - e_x^2 \sin^2 \varphi)}.
 \end{aligned} \tag{A35}$$

In this case,

$$\sin^2 \varphi = \frac{\sin^2 \theta'}{(1 - e_x^2) \cos^2 \theta' + \sin^2 \theta'} \tag{A36}$$

and hence,

$$\begin{aligned}
 1 - e_x^2 \sin^2 \varphi &= 1 - \frac{e_x^2 \sin^2 \theta'}{(1 - e_x^2)^2 \cos^2 \theta' + \sin^2 \theta'} \\
 &= \frac{(1 - e_x^2)^2 \cos^2 \theta' + \sin^2 \theta' - e_x^2 \sin^2 \theta'}{(1 - e_x^2)^2 \cos^2 \theta' + \sin^2 \theta'} \\
 &= \frac{\cos^2 \theta' - 2e_x^2 \cos^2 \theta' + \sin^2 \theta' - e_x^2 \sin^2 \theta'}{\cos^2 \theta' - 2e_x^2 \cos^2 \theta' + \sin^2 \theta'} \\
 &= \frac{1 - 2e_x^2 \cos^2 \theta' - e_x^2 \sin^2 \theta'}{1 - 2e_x^2 \cos^2 \theta'}.
 \end{aligned} \tag{A37}$$

Therefore, in spherical coordinates, we arrive at

$$\begin{aligned}
 \frac{1}{\sqrt{\frac{x^2}{a_x^4} + \frac{y^2}{a_x^4} + \frac{z^2}{a_x^4(1 - e_x^2)^2}}} &= \frac{1}{\sqrt{\frac{1}{a_x^2} \frac{1 - 2e_x^2 \cos^2 \theta' - e_x^2 \sin^2 \theta'}{1 - 2e_x^2 \cos^2 \theta'}}} \\
 &= a_x \left(\frac{1 - 2e_x^2 \cos^2 \theta'}{1 - 2e_x^2 \cos^2 \theta' - e_x^2 \sin^2 \theta'} \right)^{\frac{1}{2}} \\
 &= a_x (1 - e_x^2 \cos^2 \theta') \left(1 + e_x^2 \cos^2 \theta' + \frac{1}{2} e_x^2 \sin^2 \theta' \right) \\
 &= a_x \left(1 - e_x^2 \cos^2 \theta' + e_x^2 \cos^2 \theta' + \frac{1}{2} e_x^2 \sin^2 \theta' \right) \\
 &= a_x \left(1 + \frac{1}{2} e_x^2 \sin^2 \theta' \right).
 \end{aligned} \tag{A38}$$

Next, for the coefficients of the harmonic series of gravity disturbance and gravity anomaly [19,22], we have that

$$A_{nm}^{\delta g} = R^{-\frac{1+\sqrt{9+4n(n+1)}}{2}} a_{nm}^{\delta g}, \tag{A39}$$

$$B_{nm}^{\delta g} = R^{-\frac{1+\sqrt{9+4n(n+1)}}{2}} b_{nm}^{\delta g}, \tag{A40}$$

$$A_{nm}^{\Delta g} = R^{-\frac{1+\sqrt{9+4n(n+1)}}{2}} a_{nm}^{\Delta g}, \tag{A41}$$

$$B_{nm}^{\Delta g} = R^{-\frac{1+\sqrt{9+4n(n+1)}}{2}} b_{nm}^{\Delta g}. \tag{A42}$$

The harmonic series of the disturbing potential converges uniformly. The coefficients of this series are given by

$$\begin{aligned}
 A_{j0}^T &= \frac{2j+1}{4\pi} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} T(R, \theta', \lambda) P_j(\sin \theta') \cos \theta' d\theta' d\lambda \\
 &= \frac{(2j+1)R}{8\pi} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sum_{n=0}^{+\infty} \sum_{m=0}^n \left[(A_{nm}^{\delta g} - A_{nm}^{\Delta g}) P_{nm}(\sin \theta') \cos m\lambda \right. \\
 &\quad \left. + (B_{nm}^{\delta g} - B_{nm}^{\Delta g}) P_{nm}(\sin \theta') \sin m\lambda \right] P_j(\sin \theta') \cos \theta' d\theta' d\lambda \\
 &= \frac{(2n+1)R}{8\pi} \sum_{n=0}^{+\infty} \int_0^{2\pi} \int_{-1}^1 (A_{n0}^{\delta g} - A_{n0}^{\Delta g}) P_n^2(x) \cos m\lambda dx d\lambda \\
 &\quad + \frac{(2n+1)R}{8\pi} \sum_{n=0}^{+\infty} \int_0^{2\pi} \int_{-1}^1 (B_{n0}^{\delta g} - A_{n0}^{\Delta g}) P_n^2 \sin m\lambda dx d\lambda.
 \end{aligned} \tag{A43}$$

The second equality holds for $j = n$ and $m = 0$. The series can be integrated since it converges uniformly. But [23]

$$\int_{-1}^1 P_n^2(x) dx = \frac{2}{2n+1}. \quad (\text{A44})$$

In addition, for $m = 0$, we have $\sin m\lambda = 0$ and $\cos m\lambda = 1$; thus for a chosen n ,

$$A_{n0}^T = \frac{R}{2} (A_{n0}^{\delta g} - A_{n0}^{\Delta g}). \quad (\text{A45})$$

For $m \neq 0$, we have that [19]

$$\begin{aligned} A_{jk}^T &= \frac{(2j+1)}{2\pi} \frac{(j-k)!}{(j+k)!} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} T(R, \theta', \lambda) P_{jk}(\sin \theta') \cos m\lambda \cos \theta' d\theta' d\lambda \\ &= \frac{R(2n+1)}{4\pi} \frac{(n-m)!}{(n+m)!} \int_0^{2\pi} \int_{-1}^1 \sum_{n=0}^{+\infty} \sum_{m=0}^n \left[(A_{nm}^{\delta g} - A_{nm}^{\Delta g}) P_{nm}^2(x) \cos^2 m\lambda dx d\lambda \right. \\ &\quad \left. + \frac{R(2n+1)}{4\pi} \frac{(n-m)!}{(n+m)!} \int_0^{2\pi} \int_{-1}^1 \sum_{n=0}^{+\infty} \sum_{m=0}^n (B_{nm}^{\delta g} - B_{nm}^{\Delta g}) P_{nm}^2(x) \sin m\lambda \cos m\lambda dx d\lambda \right]. \end{aligned} \quad (\text{A46})$$

The first equality on the above relation holds for $j = n$ and $m = k$. Also

$$\int_{-1}^1 P_{nm}^2(x) dx = \frac{2(n+m)!}{(2n+1)[(n-m)]!}, \quad (\text{A47})$$

and [24]

$$\int_0^{2\pi} \cos^2 m\lambda d\lambda = \frac{1}{2} \left(\lambda + \frac{1}{2m} \sin 2m\lambda \right)_0^{2\pi} = \pi, \quad (\text{A48})$$

$$\int_0^{2\pi} \sin^2 m\lambda d\lambda = \left(\frac{\lambda}{2} - \frac{1}{4m} \sin 2m\lambda \right)_0^{2\pi} = \pi, \quad (\text{A49})$$

$$\int_0^{2\pi} \sin m\lambda \cos m\lambda d\lambda = \frac{1}{4m} (\sin^2 m\lambda - \cos^2 m\lambda)_0^{2\pi} = 0. \quad (\text{A50})$$

Substituting Equations (A47), (A48) and (A50) into Equation (A46) for a chosen n and m , we obtain

$$A_{nm}^T = \frac{R}{2} (A_{nm}^{\delta g} - A_{nm}^{\Delta g}). \quad (\text{A51})$$

Finally,

$$\begin{aligned}
 B_{jk}^T &= \frac{(2j+1)}{2\pi} \frac{(j-k)!}{(j+k)!} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} T(R, \theta', \lambda) P_{jk}(\sin \theta') \sin m\lambda \cos \theta' d\theta' d\lambda \\
 &= \frac{R(2n+1)}{4\pi} \frac{(n-m)!}{(n+m)!} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sum_{n=0}^{+\infty} \sum_{m=0}^n \left(A_{nm}^{\delta g} - A_{nm}^{\Delta g} \right) P_{nm}^2(x) \sin m\lambda \cos m\lambda dx d\lambda \\
 &\quad + \frac{R(2n+1)}{4\pi} \frac{(n-m)!}{(n+m)!} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sum_{n=0}^{+\infty} \sum_{m=0}^n \left(B_{nm}^{\delta g} - B_{nm}^{\Delta g} \right) \cos^2 m\lambda dx d\lambda.
 \end{aligned} \tag{A52}$$

Substituting Equations (A47), (A49) and (A50) into Equation (A52) (for chosen n and m), we have that

$$B_{nm}^T = \frac{R}{2} \left(B_{nm}^{\delta g} - B_{nm}^{\Delta g} \right). \tag{A53}$$

Therefore, the final formula for the disturbing potential is as follows

$$\begin{aligned}
 T(r, \theta', \lambda) &= \frac{R}{2} \sum_{n=0}^{+\infty} \sum_{m=0}^n \left(\frac{R}{r} \right)^{(n+1)} \left[\left(A_{nm}^{\delta g} - A_{nm}^{\Delta g} \right) P_{nm}(\sin \theta') \cos m\lambda \right. \\
 &\quad \left. + \left(B_{nm}^{\delta g} - B_{nm}^{\Delta g} \right) P_{nm}(\sin \theta') \sin m\lambda \right].
 \end{aligned} \tag{A54}$$

Additionally, the Gaussian curvature of a triaxial ellipsoid [25] is written as

$$K_G(x, y, z) = \frac{1}{a_x^2 a_y^2 b^2} \left(\frac{x^2}{a_x^4} + \frac{y^2}{a_y^4} + \frac{z^2}{b^4} \right)^{-2}, \tag{A55}$$

and thus

$$\sqrt[4]{K_G} = \frac{1}{\sqrt{a_x a_y b}} \frac{1}{\sqrt{\frac{x^2}{a_x^4} + \frac{y^2}{a_y^4} + \frac{z^2}{b^4}}}. \tag{A56}$$

The surface charge density is related to K_G (see Equation (18)), since

$$\begin{aligned}
 \sigma &= \frac{Q}{4\pi a_x a_y b} \frac{1}{\sqrt{\frac{x^2}{a_x^4} + \frac{y^2}{a_y^4} + \frac{z^2}{b^4}}} = \frac{Q}{4\pi \sqrt{a_x a_y b}} \frac{1}{\sqrt{a_x a_y b}} \frac{1}{\sqrt{\frac{x^2}{a_x^4} + \frac{y^2}{a_y^4} + \frac{z^2}{b^4}}} \\
 &= \frac{Q}{4\pi \sqrt{a_x a_y b}} \sqrt[4]{K_G}.
 \end{aligned} \tag{A57}$$

Using Equation (A31), the above relation becomes

$$\begin{aligned}
 \sigma &= \frac{Q}{4\pi a_x a_y b} \frac{1}{\sqrt{\frac{x^2}{a_x^4} + \frac{y^2}{a_y^4} + \frac{z^2}{b^4}}} = \frac{Q}{4\pi a_x a_y b} a_x \left(1 + \frac{e_x^2}{2} + \frac{e_c^2}{2} \cos^2 \theta' \sin^2 \lambda_c \right) \\
 &= \frac{Q}{4\pi \sqrt{a_x a_y b}} \sqrt[4]{K_G},
 \end{aligned} \tag{A58}$$

where

$$\sqrt[4]{K_G} = \frac{1}{\sqrt{a_x a_y b}} a_x \left(1 + \frac{e_x^2}{2} + \frac{e_c^2}{2} \cos^2 \theta' \sin^2 \lambda_c \right). \tag{A59}$$

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