mathematics

MDPI

Article

On Subdivision of Cycles with Two Blocks in Chromatic Digraphs Spanned by Hamiltonian Directed Paths

Salman Ghazal * and Steve Karam

College of Engineering and Technology, American University of the Middle East, Egaila 54200, Kuwait; steve.karam@aum.edu.kw

* Correspondence: salman.ghazal@aum.edu.kw

Abstract

Let C be an oriented cycle with two blocks and let n denote the number of its vertices. We show that every $(n+2)$-chromatic digraph spanned by a Hamiltonian directed path contains a subdivision of C. This bound is tight.

Keywords: oriented cycle; Hamiltonian path; chromatic number; subdivision

MSC: 05C15; 05C20; 05C38

1. Introduction

Throughout this paper, unless otherwise stated, considered graphs are finite and simple and digraphs are orientations of such graphs. Therefore, digraphs have neither loops nor parallel arcs.

Let G be a graph. Its vertex set will be denoted by $V(G)$ and its edge set will be denoted by $E(G)$. A graph H is a subgraph of G, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If in addition $V(H) = V(G)$, then the subgraph H is said to be a spanning subgraph of G. Let x be a vertex of G. If xy is an edge of G, then we say that y is a neighbor of x. Also, we say that x and y are neighbors. The degree of x, denoted by $d_G(x)$, is the number of all its neighbors in G. G is said to be m-degenerate, if every subgraph of G has a vertex of degree at most m. A set of vertices $S \subset V(G)$ is said to be a stable set in G, if any two vertices x and y in S are not neighbors in G. The stability of G is $\alpha(G) = min\{|S|; S \text{ is a stable set in } G\}$. The chromatic number of G, denoted by $\chi(G)$, is the smallest k such that $V(G)$ is a union of k stable sets. Equivalently, $\chi(G) = k$ if k is the smallest number such that we can color all the vertices of G by k colors in a way that any two neighbor vertices receive distinct colors. If $\chi(G) = k$, then we say that G is k-chromatic. A complete graph is a graph such that any two distinct vertices are neighbors. A tournament is an orientation of a complete graph.

Let D be a digraph and suppose that D is an orientation of a graph G. We will say that G is the underlying graph of D. The vertex set and arc set of D will be denoted by $V(D)$ and $E(D)$. $D \cup (x,y)$ denotes the digraph with vertex set $V(D) \cup \{x,y\}$ and arc set $E(D) \cup \{(x,y)\}$. If a digraph (resp. graph) H is isomorphic to a subdigraph (resp. graph) of a digraph (resp. subgraph) G, then we say that G contains H. The stability of D and the chromatic number of D are those of its underlying graph G.

A path $P = x_1x_2\cdots x_n$ is a graph with $V(P) = \{x_1, x_2, \cdots, x_n\}$ and $E(P) = \{x_1x_2, x_2x_3, \cdots, x_{n-1}x_n\}$. An oriented path is an orientation of a path. A directed path $P = x_1x_2\cdots x_n$ is an oriented path with $E(P) = \{(x_1,x_2),(x_2,x_3),\cdots,(x_{n-1},x_n)\}$.

Check for updates

Academic Editor: Christian Rubio-Montiel

Received: 16 February 2026
Revised: 4 April 2026
Accepted: 7 April 2026
Published: 9 April 2026

Copyright: © 2026 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license.

The previous path is said to be a directed path from x_1 to x_n. A cycle $C = x_1x_2 \cdots x_nx_1$ is a graph with $V(C) = \{x_1, x_2, \cdots, x_n\}$ and $E(C) = \{x_1x_2, x_2x_3, \cdots, x_{n-1}x_n, x_nx_1\}$. An oriented cycle is an orientation of a cycle. A directed cycle $C = x_1x_2 \cdots x_nx_1$ is an oriented cycle with $E(C) = \{(x_1, x_2), (x_2, x_3), \cdots, (x_{n-1}, x_n), (x_n, x_1)\}$. If $P = x_1x_2 \cdots x_n$ is an oriented path, then for $i \leq j$, $P[x_i, x_j]$ is the oriented path $x_ix_{i+1} \cdots x_j$. The length of a path or cycle (resp. oriented path or cycle) is the number of its edges (resp. arcs). A Hamiltonian oriented path (resp. cycle) P of a digraph H is an oriented path (resp. cycle) contained in H such that $V(P) = V(H)$; that is, it passes through all the vertices of H. A digraph that has a Hamiltonian directed cycle is called a Hamiltonian digraph. A graph is connected if between any two of its vertices there is a path. A digraph is said to be strongly connected if between any two of its vertices there is a directed path. A tree is a connected graph that has no cycle. A rooted tree is a tree in which one vertex has been designated as the root.

Let H be a an oriented path or cycle but not a directed cycle. A block P of H is a maximal directed path contained in H. That is, if P' is also a directed path contained in H and P' contains P, then $P' = P$. Observe that the number of blocks of an oriented cycle which is not directed is always even. An oriented cycle or an oriented path with t blocks is one that has exactly t blocks. A subdivision of a digraph H is a digraph obtained from H by replacing some of its arcs (x, y) by a directed path from x to y. Note that a subdivision of an oriented cycle (resp. oriented path) with t blocks is again an oriented cycle (resp. oriented path) with t blocks, but the length of each block might increase.

It is known that tournaments are "rich" digraphs; that is, they contain many structures such as oriented paths, oriented trees and oriented cycles. For example, with three exceptions only, any tournament on n vertices contains any oriented path on n vertices [1]. Are n-chromatic digraphs also rich? In other words, which digraphs H are sub-digraphs of all digraphs with sufficiently large chromatic number?

We have the following classical result.

Theorem 1 (Roy-Galli [2,3]). *Every digraph with chromatic number at least $n + 1$ contains a directed path of length at least n.*

For non-directed paths, El Joubbeh and Ghazal [4] proved that every $g(t, n)$-chromatic digraph contains every oriented path on n vertices with $t \geq 4$ blocks for some function g. However, the only connected oriented graphs H that are possibly candidates to generalize Theorem 1 are oriented trees. This is due to the following Theorem.

Theorem 2 (Erdős [5]). *For every $k \geq 3$ and $g \geq 3$, there is a graph with chromatic number at least k and girth at least g.*

But for strongly connected digraphs Bondy proved the following.

Theorem 3 (Bondy [6]). *Every strongly connected digraph of chromatic number at least n contains a directed cycle of length at least n.*

By interpreting directed cycles of length at least n as subdivisions of the directed cycle of length n, the preceding Theorem establishes a special case of a conjecture proposed by Cohen et al.:

Conjecture 1 ([7]). *For every oriented cycle C, there is a constant $f(C)$ such that every strongly connected digraph with chromatic number at least $f(C)$ contains a subdivision of C.*

https://doi.org/10.3390/math14081248

Let C be any oriented cycle on n vertices. Note that every Hamiltonian digraph is strongly connected. For the class of Hamiltonian digraphs, El Joubbeh [8] proved that $f(C) \leq 3n$ and Ghazal et al. [9] refined the proof and showed that $f(C) \leq 2n$. For a generalization of these results see [10]. In addition, Al-Mniny and Ghazal [11] used secant edges to show that if C is a cycle with two blocks of respective lengths k and m, say $k \geq m$, then every $3k+1$-chromatic digraph D spanned by a Hamiltonian directed path contains a subdivision of C. The proofs of the results in [8–10] rely on a tool called secant edges, proposed in [11] by Al-Mniny and Ghazal. In this paper, we will also exploit certain secant edges to improve the latter bound $3k+1$ and derive a tight bound.

2. Main Results

The following lemma about the chromatic number is well known in graph theory.

Lemma 1. *Let G be a graph. If G is m-degenerate, then $\chi(G) \leq m+1$.*

Definition 1. *Suppose that $L = x_1x_2...x_N$ is a linear ordering of the vertices of a graph G. Let k be a positive integer. An edge $e = x_ix_j \in E(G)$ is called a k-jump with respect to L, if $j - i \geq k$. It is said to contain an l-jump $x_rx_s \in E(G)$, if $i \leq r < s \leq j$. It is said to be a minimal k-jump if it contains no k-jump other than itself.*

Forbidding k-jumps in a graph yields a bound on its chromatic number, as stated in the following proposition.

Proposition 1. *Let G be a graph that has no k-jump with respect to a linear ordering of its vertices. Then $\chi(G) \leq k$.*

Proof. For each $1 \leq i \leq k$, let $S_i = \{x_{i+\alpha k}; \alpha = 0, 1, 2, \cdots\}$. Assume that S_i is not a stable set for some i. Then there are two non-negative integers $\beta > \alpha$ such that $x_{i+\alpha k}x_{i+\beta k} \in E(G)$. Since $i + \beta k - i + \alpha k = (\beta - \alpha)k \geq k$, then $x_{i+\alpha k}x_{i+\beta k}$ is a k-jump, which is a contradiction. Hence, for every $1 \leq i \leq k$, S_i is a stable set. Since $\bigcup_{i=1}^{k} S_i = V(G)$, then $\chi(G) \leq k$. □

In the next definition and theorem, we consider a more general configuration involving pairs of jumps. Specifically, we introduce the notion of (k, m)-secant edges, which captures a structured interaction between two edges, and examine its implications for the chromatic number.

Definition 2. *Suppose that $L = x_1x_2...x_N$ is a linear ordering of the vertices of a graph G. Let k and m be positive integers. Two edges $e = x_ix_j \in E(G)$ and $e' = x_px_q \in E(G)$ are said to be (k, m)-secant edges if $p - i \geq k$, $j > p$ and $q - j \geq m$.*

Theorem 4. *If a graph has no (k, m)-secant edges with respect to some linear ordering of its vertices, then its chromatic number is at most $k + m + 1$.*

Proof. Let H be a graph that has no (k, m)-secant edges with respect to some linear ordering Θ. We will show that H is $(k + m)$-degenerate. Let G be a subgraph of H and let $L = x_1x_2 \dots x_N$ denote the restriction of Θ to the vertices of G. Clearly, any (k, m)-secant edges of G with respect to L are (k, m)-secant edges of H with respect to Θ. Hence, G has no (k, m)-secant edges with respect to L. We will show that G has a vertex of degree at most $k + m$.

https://doi.org/10.3390/math14081248

Suppose that G has no $k+1$-jump. Then the neighbors of x_1 are in the set $\{x_2, x_3, \cdots, x_{k+1}\}$. Hence $d_G(x_1) \leq k \leq k+m$. Now we suppose that G has a $k+1$-jump. Let x_ix_j be a minimal $k+1$-jump with i being chosen the smallest. That is, if x_px_q is a minimal $k+1$-jump different from x_ix_j, then $i < p$. We will show $d_G(x_{j-1}) \leq k+m$.

Let $p < i$. Assume that $x_px_{j-1} \in E(G)$. Then it is a $k+1$-jump and hence it contains a minimal $k+1$-jump x_rx_s with $r < i$. This contradicts the minimality of i. Hence, $x_px_{j-1} \notin E(G)$. Therefore, x_{j-1} has no neighbor in the set $\{x_1, \cdots, x_{i-1}\}$.

Let $i \leq p < j-1$ and suppose that $x_px_{j-1} \in E(G)$. By minimality of x_ix_j, we get that $(j-1)-p \leq k$. Therefore, x_{j-1} has at most k neighbors in the set $\{x_i, x_{i+1}, \cdots, x_{j-1}\}$. Hence, it has at most $k+1$ neighbors in the set $\{x_i, x_{i+1}, \cdots, x_j\}$.

Let $p \geq j+1$ and suppose that $x_px_{j-1} \in E(G)$. Clearly, if $p-j \leq m$, then x_ix_j and $x_{j-1}x_p$ are (k,m)-secant edges of G, which is a contradiction. Therefore, $p-j \leq m-1$. Therefore, x_{j-1} has at most $m-1$ neighbors in the set $\{x_{j+1}, x_{j+2}, \cdots, x_N\}$.

So, the degree of x_{j-1} in G is $d_G(x_{j-1}) \leq (k+1)+(m-1) = k+m$. This shows that H is $(k+m)$-degenerate. Hence, its chromatic number is at most $k+m+1$. □

The above theorem is a key ingredient in the proof of the main result, namely Theorem 5. Specifically, if the underlying graph of a digraph admitting a Hamiltonian directed path contains no (k,m)-secant edges, then Theorem 4 guarantees that its chromatic number is bounded. On the other hand, if such secant edges exist, they can be utilized to construct the required cycle.

Theorem 5. *Let C be a cycle with two blocks and let n denote the number of its vertices. Then every $(n+2)$-chromatic digraph that has a Hamiltonian directed path contains a subdivision of C.*

Proof. Let C be a cycle with two blocks of respective length k and m. Let n denote the number of its vertices. Then $n = k+m$. Let D be an $(n+2)$-chromatic digraph that has a Hamiltonian directed path $P = x_1x_2\cdots x_N$.

Let G denote the underlying graph of D and we consider the linear order $L = x_1x_2\cdots x_N$ of the vertices of G. Assume that G has no (k,m)-secant edges. Then by Theorem 4, we get that $\chi(G) \leq k+m+1 = n+1$, a contradiction. Therefore, G has (k,m)-secant edges $x_ix_j \in E(G)$ and $x_px_q \in E(G)$ with $p-i \geq k$, $j > p$ and $q-j \geq m$. We have four cases according to the orientations in D of the edges x_ix_j and x_px_q. In each of these case we will construct a subdivision of C.

Suppose that (x_i, x_j) and (x_p, x_q) are in $E(D)$. Then the directed path $P[x_i, x_p] \cup (x_p, x_q)$ and the directed path $(x_i, x_j) \cup P[x_j, x_q]$ form a subdivision of C (see Figure 1).

Suppose that (x_i, x_j) and (x_q, x_p) are in $E(D)$. Then the directed path $P[x_i, x_p]$ and the directed path $(x_i, x_j) \cup P[x_j, x_q] \cup (x_q, x_p)$ form a subdivision of C.

Suppose that (x_j, x_i) and (x_p, x_q) are in $E(D)$. Then the directed path $(x_j, x_i) \cup P[x_i, x_p] \cup (x_p, x_q)$ and the directed path $P[x_j, x_q]$ form a subdivision of C.

Suppose that (x_j, x_i) and (x_q, x_p) are in $E(D)$. Then the directed path $(x_j, x_i) \cup P[x_i, x_p]$ and the directed path $P[x_j, x_q] \cup (x_q, x_p)$ form a subdivision of C. □

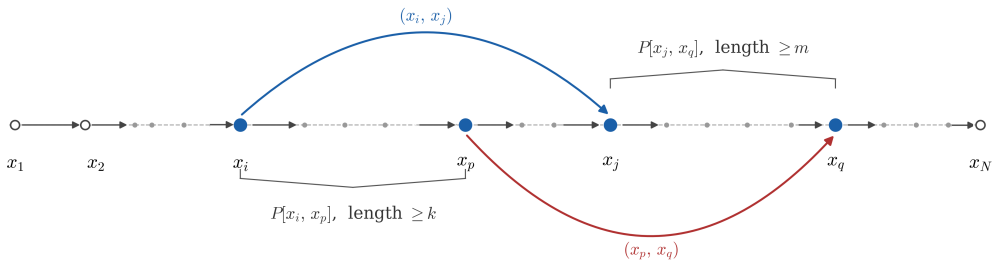


Figure 1. Subdivision of C in the case where (x_i, x_j) and (x_p, x_q) are in $E(D)$.

Let C be a cycle with two blocks of respective length k and m, where $k \geq m$. Then $n := k + m$ is the number of vertices of C. Let D be a digraph spanned by a Hamiltonian directed path. Al-Mniny and Ghazal [11] proved that if $\chi(D) \geq 3k + 1$, then D contains a subdivision of C. Since $n = k + m$ and $k \geq m$, then k can take any integer value $\frac{1}{2}n$ and $n - 1$. So, in terms of n, which is the number of vertices of the cycle, the bound $3k + 1$ obtained in [11] satisfies $\frac{3}{2}n + 1 \leq 3k + 1 \leq 3n - 2$. So, our bound $n + 2$ obtained in Theorem 5 is significantly smaller than $3k + 1$.

In addition, our bound is tight. To see this, let $C = C(1, 1)$ be the oriented cycle of two blocks; each is of length one. That is C consists of only $n = 2$ vertices and only two arcs and these arcs are parallel. Let D be the directed cycle on three vertices. Clearly, $\chi(D) = 3 \geq n + 1$ and D does not contain any subdivision of C.

Theorem 6 ([12]). *The vertex set of any digraph D is the union of the vertex sets of at most $\alpha(D)$ disjoint directed paths.*

Corollary 1. *Let C be a cycle with two blocks and let n denote the number of its vertices. Let D be an $(n + 2)\alpha(D)$-chromatic digraph that has a Hamiltonian directed path. Then D contains a subdivision of C.*

Proof. Let C be a cycle with two blocks and let n denote the number of its vertices. Let D be an $(n + 2)\alpha(D)$-chromatic digraph that has a Hamiltonian directed path. By Theorem 6, the vertex set of any digraph D is the union of the vertex sets of t disjoint directed paths $P_1, P_2, \cdots, P_t$, where $t \leq \alpha(D)$. For each $1 \leq i \leq t$, let D_i be the subdigraph of D induced by the vertices of P_i. Since D is an $(n + 2)\alpha(D)$-chromatic digraph, then there is $1 \leq i \leq t$ such that $\chi(D_i) \geq (n + 2)$. By Theorem 5, D_i, and consequently D, contains a subdivision of C. □

3. Conclusions

In this paper, (k, m)-secant edges were introduced and we have proved that graphs without such edges with respect to some linear ordering have chromatic number at most $k + m + 1$. We then used this to show that any digraph with chromatic number at least $n + 2$ and admitting a Hamiltonian directed path must contain a subdivision of every cycle with two blocks on n vertices. We can generalize this to (k, l, m)-secant edges as follows: Two edges $e = x_i x_j \in E(G)$ and $e' = x_p x_q \in E(G)$ are said to be (k, l, m)-secant edges if $p - i \geq k$, $j - p \geq l$ and $q - j \geq m$. We propose that investigating the chromatic number of graphs without (k, l, m)-secant edges with respect to some linear ordering of its vertices may yield substantive progress on Conjecture 1 in more general settings.

Author Contributions: Conceptualization, S.G.; methodology, S.G.; software, S.G. and S.K.; validation, S.G. and S.K.; formal analysis, S.G.; investigation, S.G. and S.K.; resources, S.G. and S.K.; writing—original draft, S.G.; writing—review and editing, S.G. and S.K.; visualization, S.G. and S.K.; supervision, S.G.; funding acquisition, S.G. and S.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Havet, F.; Thomassé, S. Oriented Hamiltonian Paths in Tournaments: A Proof of Rosenfeld's Conjecture. *J. Combin. Theory Ser. B* **2000**, *78*, 243–273. [CrossRef]
2. Roy, B. Nombre chromatique et plus longs chemins d'un graphe. *Rev. Fr. Inform. Rech. Opér.* **1967**, *1*, 129–132. [CrossRef]

3. Gallai, T. On directed graphs and circuits. In *Theory of Graphs (Proceedings of the Colloquium Held at Tihany 1966)*; Akadémiai Kiadó: Budapest, Hungary, 1968; pp. 115–118.
4. El Joubbeh, M.; Ghazal, S. Existence of paths with t-blocks in $k(t)$-chromatic digraphs. *Discret. Appl. Math.* **2024**, *342*, 381–384. [CrossRef]
5. Erdős, P. Graph theory and probability. *Can. J. Math.* **1959**, *11*, 34–38. [CrossRef]
6. Bondy, J.A. Disconnected orientations and a conjecture of Las Vergnas. *J. Lond. Math. Soc.* **1976**, *14*, 277–282. [CrossRef]
7. Cohen, N.; Havet, F.; Lochet, W.; Nisse, N. Subdivisions of oriented cycles in digraphs with large chromatic number. *J. Graph Theory* **2018**, *89*, 439–456. [CrossRef]
8. El Joubbeh, M. Subdivisions of oriented cycles in Hamiltonian digraphs with small chromatic number. *Discret. Math.* **2023**, *346*, 113209. [CrossRef]
9. Ghazal, S.; Tfaili, S. On subdivisions of oriented cycles in Hamiltonian digraphs with small chromatic number. *Commun. Comb. Optim.* **2026**, *11*, 307–314. [CrossRef]
10. Ghazal, S. Generalized subdivisions in digraphs spanned by subdivision of smaller digraphs and the chromatic number. *Commun. Comb. Optim.* **2025**, *accepted for publication*. [CrossRef]
11. Al-Mniny, D.; Ghazal, S. Secant edges: A tool for Cohen et al.'s conjectures about subdivisions of oriented cycles and bispindles in Hamiltonian digraphs with large chromatic number. *Graphs Combin.* **2025**, *41*, 13. [CrossRef]
12. Gallai, T.; Milgram, A.N. Verallgemeinerung eines graphentheoretischen Satzes von Rédei. *Acta Sci. Math.* **1960**, *21*, 181–186.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.