

Article

A Simplified Approach to Investigating the Oscillatory Behavior of Nonlinear Even-Order Neutral Differential Equations

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Abstract

This article explores the oscillation properties of solutions for higher-order nonlinear neutral functional differential equations. This study sought to establish applicable oscillation conditions for nonlinear NDDEs when $\alpha \neq \beta$. Our analysis focused on identifying the key asymptotic characteristics of positive solutions in the noncanonical case. Based on these properties, we developed a multi-conditional criterion that guarantees oscillation, thereby extending previously established results for second-order equations. Unlike previous results, which involved arbitrary constants and were thus difficult to apply, the criteria derived in this work do not contain such constants, making them directly applicable. Moreover, to overcome the complexity introduced by the multiple constraints of this criterion, we propose a new and previously unreported simplified oscillation condition. The effectiveness of this simplified criterion is demonstrated through an illustrative example.

Keywords: differential equations; oscillatory properties; noncanonical higher-order; neutral delay

MSC: 34C10; 34K11

1. Introduction

Linear approximations fail to provide accurate descriptions. These equations capture behaviors such as bifurcation, chaos, and self-regulation. These behaviors commonly occur in fields such as physics, biology, engineering, and economics. Unlike linear systems, nonlinear differential equations often exhibit multiple equilibria and sensitivity to initial conditions. They also have rich dynamical structures. This makes their analysis both challenging and essential. As a result, they provide a robust mathematical framework for understanding real-world systems where interactions and responses deviate from linearity [1,2].

Neutral differential equations constitute a significant subclass of functional differential equations. In these equations, the derivative of the unknown function is influenced by both its delayed values and delayed derivatives. This formulation enables neutral equations to describe systems in which the present rate of change is affected by previous rates. This is a characteristic frequently encountered in control theory, population dynamics, electrical networks, and mechanical systems with memory-dependent damping. Due to their inherent complexity, neutral differential equations often necessitate advanced



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analytical tools. Their behavior—such as stability, boundedness, and oscillation—differs significantly from that of standard delay equations (see [3,4]).

Higher-order differential equations extend traditional models by incorporating second, third, or even higher derivatives. This allows for the representation of more detailed dynamic behavior. They arise naturally in the study of elastic beams, mechanical vibrations, fluid dynamics, and electromagnetics. In these systems, acceleration, curvature, or higher-level physical responses influence system evolution. Analyzing higher-order equations is essential for capturing complex interactions among multiple physical variables. Their solutions often reveal sophisticated patterns, including resonance, dispersion, and multi-scale dynamics [5,6].

The study of oscillation in differential equations focuses on determining whether solutions exhibit repeating fluctuations or indefinitely changing signs over time. Oscillatory behavior is crucial in modeling periodic motions, wave phenomena, biological rhythms, population cycles, and electrical oscillators. Understanding the conditions that guarantee oscillation—especially in nonlinear, neutral, and higher-order systems—helps researchers characterize system stability. It also helps predict long-term dynamics and design control mechanisms. Consequently, oscillation theory has become a central topic in qualitative analyses. It offers deep insights into the behavior of complex differential systems; see [7].

This work focuses on the analysis of the nonlinear, noncanonical neutral delay differential equation

$$\left(r(u) \left(z^{(n-1)}(u) \right)^\alpha \right)' + q(u) x^\beta(\sigma(u)) = 0, \quad (1)$$

where $u \geq u_0$, $n \geq 4$ is an even integer,

$$z(u) = x(u) + p(u)x(\tau(u)),$$

and we assumed that

- 1 α, β are the ratio of odd positive integers and $\alpha \neq \beta$;
- 2 $r \in C^1([u_0, \infty), [0, \infty))$, $r(u) > 0$, $r'(u) > 0$ and $\eta_0(u) < \infty$, where

$$\eta_0(u) = \int_u^\infty r^{-1/\alpha}(s) ds; \quad (2)$$

- 3 $p, q \in C([u_0, \infty), [0, \infty))$, $0 \leq p(u) < 1$ and $p(u) < \eta(u)/\eta(\tau(u))$;
- 4 $\tau, \sigma \in C([u_0, \infty), \mathbb{R})$, $\tau(u) \leq u$, $\sigma(u) \leq u$, $\sigma'(u) \geq 0$ and $\lim_{u \rightarrow \infty} \tau(u) = \lim_{u \rightarrow \infty} \sigma(u) = \infty$.

A solution to Equation (1) is defined as a real-valued function x that is $(n-1)$ times differentiable and satisfies Equation (1) for sufficiently large values of u_* . In this work, we restricted our attention to solutions to Equation (1) that fulfilled the condition

$$\sup\{|x(u_*)| : u_* \geq u_0\} > 0 \text{ for all } u_0 \geq u_l.$$

Definition 1. A solution to Equation (1) is said to be non-oscillatory if it is eventually positive or eventually negative; otherwise, it is termed oscillatory. The equation itself is called oscillatory if every solution exhibits oscillatory behavior.

Definition 2. A neutral delay differential Equation (1) is called canonical if $\eta_0(u) = \infty$. On the other hand, it is called noncanonical if $\eta_0(u) < \infty$.

Previous Works

Numerous investigations have addressed the analysis of nonlinear differential equations and delay differential equations, with particular emphasis on higher-order cases and oscillatory properties. Early research focused on second-order nonlinear equations, establishing oscillation criteria (see [8–10]). Later, the attention shifted to neutral and delay differential equations, where the presence of delays in the derivatives introduced new challenges in the study of oscillation. In particular, refs. [11–17] provide key contributions on the oscillatory behavior of such equations and the development of related analytical techniques.

Higher-order differential equations have also been extensively investigated due to their applications in mechanical, electrical, and physical systems, leading to the development of multi-conditional oscillation criteria. More recently, studies have emphasized oscillatory behavior in nonlinear, neutral, and higher-order systems, proposing simplified and generalized criteria to guarantee oscillation under various conditions. In particular, refs. [18–22] present key results on simplified oscillation criteria, generalizations to higher-order systems, and applications to practical examples.

In the following, we highlight a number of earlier results that have played a substantial role in advancing the research on differential equations of orders two through four.

In 1997, Agarwal et al. [23] established that the delay differential equation

$$\left(r(u)|z'(u)|^{\alpha-1}z'(u)\right)' + q(u)|x(\sigma(u))|^{\beta-1}x(\sigma(u)) = 0,$$

is oscillatory in the canonical case, provided that $\alpha > \beta$ and

$$\int_{u_0}^{\infty} q(\rho) \left(\int_{u_0}^{\rho} r^{-1/\alpha}(s) ds \right)^{\beta} d\rho = \infty.$$

In the noncanonical case, Baculikova [24] investigated the oscillatory behavior of the DDE

$$\left(r(u)(z'(u))^{\alpha}\right)' + q(u)x^{\beta}(\sigma(u)) = 0,$$

for both $\alpha > \beta$ and $\alpha < \beta$. The results in [24] present several multi-condition criteria that guarantee the oscillation. Many studies in the literature have examined the compulsory nonlinear case involving arbitrary constants; see, for example, refs. [25–27].

Onose [28] examined the oscillatory behavior of certain fourth-order functional differential equations, specifically those given by

$$(rx'')''(u) + f(x(\sigma(u)), u) = 0$$

for $f(u, x) \geq q(u)x^{\beta}$, and

$$(rx'')''(u) + q(u)f(x(\sigma(u))) = \sigma(u),$$

while working under the condition $\int_u^{\infty} r^{-1}(s)ds = \infty$. Subsequent to this study, a considerable research effort has been directed toward the investigation of oscillatory and non-oscillatory properties of fourth-order and higher-order functional differential equations. Using a variety of analytical techniques, the researchers identified several qualitative properties of solutions across different equation classes. For instance, Agarwal and co-authors [29] established notable results for fourth-order equations of the form

$$\left[r_3^{-1} \left(\left[r_2^{-1} \left(\left[r_1^{-1} (x')^{\alpha_1} \right]' \right)^{\alpha_2} \right]' \right)^{\alpha_3} \right]' (u) \pm q(u)f(x(\sigma(u))) = 0,$$

under the condition $\int_0^\infty r_i^{1/\alpha_i}(\theta) d\theta = \infty$, $i = 1, 2, 3$. Grace and collaborators [30] analyzed the oscillatory nature of a fourth-order differential equation given by

$$\left(r(x')^\alpha\right)'''(u) + q(u)f(x(\sigma(u))) = 0,$$

while assuming that condition (2) is satisfied. In related studies, Wu [31] as well as Kamo and Usami [32] investigated the oscillation characteristics of the nonlinear fourth-order equation

$$\left(r|x''|^{\alpha-1}x''\right)''(u) + q|x(u)|^{\beta-1}x(u) = 0,$$

where β and α are positive parameters and the same condition (2) is imposed.

Several works, including [33–38], have contributed to the advancement of oscillation theory for even-order neutral differential equations of the form

$$z^{(n)}(u) + q(u)x(\sigma(u)) = 0.$$

For broader oscillation results concerning neutral equations involving canonical operators, reference may be made to [39–42].

Among the studies examining Equation (1) for all combinations of α and β , one work [43] established a criterion governing the oscillatory behavior of its solutions:

$$\limsup_{u \rightarrow \infty} \int_{u_0}^u \left(\rho(s)q(s)(1 - p(\sigma(s)))^\beta \left(\frac{v_1 \sigma^{n-2}(s)}{(n-2)!} \right)^\beta \eta_0^\alpha(s) - \frac{\alpha^{\alpha+1} r^{-1/\alpha}(s)}{(\alpha+1)^{\alpha+1} \eta_0(s)} \right) ds = \infty,$$

for some constant $v_1 \in (0, 1)$ and for any c_1 and c_2 , which are positive constants, where

$$\rho(u) = \begin{cases} c_1^{\beta-\alpha} & \text{if } \alpha \geq \beta \\ c_2 \eta_0^{\beta-\alpha} & \text{if } \alpha < \beta \end{cases}.$$

It can be noted that arbitrary constants appear in the scenario where $\alpha \neq \beta$. This requires that conditions be satisfied for all positive real values of c_1 and c_2 . Although the original work presents additional oscillation conditions within the same theorem, here we refer to only one of them for illustrative purposes.

In examining the oscillatory behavior of neutral differential equations in the noncanonical case, a central objective is to identify conditions that rule out the existence of positive, monotone-decreasing solutions—namely, solutions satisfying $(-1)^k(x + p \cdot (x \circ \tau))^{(k)}(u) > 0$ for $k = 0, 1, 2, 3$, $u \in [u_0, \infty)$. This direction of research is particularly important because many standard relations used in the canonical case no longer apply here. In a notable contribution, Bohner et al. [44] explored this issue and derived a condition that must be satisfied by any positive decreasing solution x and an associated function, namely

$$x > \left(1 - p(u) \frac{\eta(\tau(u))}{\eta(u)} \right),$$

where the auxiliary function η is defined by $\eta(u) := \int_u^\infty r^{-1/\alpha}(s) ds$. This relation enabled the authors to establish several new criteria that both refined and strengthened earlier findings in the literature. A natural next question arising from their work is how Bohner's results in [44] can be generalized to neutral differential equations of an even order.

The following lemmas are required in order to demonstrate the primary results.

Lemma 1 ([45]). Let $f \in C^n([u_0, \infty), (0, \infty))$. If the derivative $f^{(n)}(u)$ is eventually of one sign for all large u , then there exist a u_x such that $u_x \geq u_0$ and an integer l , $0 \leq l \leq n$, with $n + l$ even for $f^{(n)}(u) \geq 0$ or $n + l$ odd for $f^{(n)}(u) \leq 0$ such that

$$l > 0 \text{ implies } f^{(k)}(u) > 0 \text{ for } u \geq u_x, k = 0, 1, \dots, l - 1,$$

and

$$l \leq n - 1 \text{ implies } (-1)^{l+k} f^{(k)}(u) > 0 \text{ for } u \geq u_x, k = l, l + 1, \dots, n - 1.$$

Lemma 2 ([46]). Suppose that $y \in C^n([u_0, \infty), (0, \infty))$ and $y^{(n)}$ is of one sign for $u \geq u_1 \geq u_0$, $y^{(n)} \neq 0$ on a subray of $[u_1, \infty)$, $y^{(n-1)}(u)y^{(n)}(u) \leq 0$ for $u \geq u_1$, and y does not converge to zero. Then, there is a $u_v \geq u_1$ such that

$$y(u) \geq \frac{v}{(n-1)!} u^{n-1} y^{(n-1)}(u),$$

for $v \in (0, 1)$ and $u \geq u_v$.

The main contributions and novel aspects of this work can be summarized as follows:

- **Extension to Higher-Order Neutral Equations:**
The study generalizes existing oscillation results, such as those of Baculikova [24], from second-order to higher-order neutral functional differential equations.
- **Multi-Conditional Oscillation Criteria:**
New criteria are proposed that involve multiple conditions, allowing for a more comprehensive analysis of oscillatory behavior in nonlinear, neutral, and higher-order systems.
- **Simplified Analytical Approach:**
A simplified oscillation condition is introduced, providing easier verification of oscillation criteria compared to previous general forms.
- **Notably, our results are free of arbitrary constants, in contrast to many previous studies in the literature.**

The paper is organized as follows:

- In Section 1, we present the introduction, where the motivation of the study is discussed and the differential equation under consideration is introduced.
- Section 2 is divided into two subsections. In the first subsection, entitled Auxiliary Lemmas, we collect several lemmas that are essential for the subsequent analysis. The second subsection, Theorems Concerning Oscillations, is devoted to establishing the main oscillation theorems of the paper.
- In Section 3, Examples and Discussion, illustrative examples are provided to demonstrate the applicability of the obtained results, together with relevant discussion.
- Finally, Section 4 concludes the paper with a summary of the main findings and remarks on possible future research directions.

2. Main Results

To facilitate the analysis, we assumed that

$$\eta_{j+1}(u) = \int_u^\infty \eta_j(s) ds,$$

for $j = 0, 2, \dots, n - 3$,

$$\Phi(u) := q(u)(1 - p(\sigma(u)))^\beta,$$

and

$$\hat{\Phi}(u) := q(u) \left(1 - p(\sigma(u)) \frac{\eta_{n-2}(\tau(\sigma(u)))}{\eta_{n-2}(\sigma(u))} \right)^\beta.$$

Remark 1. From the definition of the function $\eta_0(u)$, we observed that it is positive and monotonically decreasing, and

$$\lim_{u \rightarrow \infty} \eta_0(u) = 0.$$

Consequently, there exists $u_0 > 0$ such that $\eta_0(u) < 1$ for all $u \geq u_0$. Therefore, the condition $\eta_0(u) < 1$ is automatically satisfied by the basic properties of η .

2.1. Auxiliary Lemmas

We begin by introducing a set of preliminary results that serve as essential tools for establishing the main results.

Lemma 3. Let $x \in \mathbf{C}([u_0, \infty), (0, \infty))$ be a solution to (1). Then, $z(u) > 0$, $r(u) \left(z^{(n-1)}(u) \right)^\alpha < 0$, and one of the following cases holds for $u \in [u_1, \infty)$, $u_1 \geq u_0$:

- I** $z'(u) > 0$, $z^{(n-1)}(u) > 0$ and $z^{(n)}(u) < 0$;
- II** $z'(u) > 0$, $z^{(n-2)}(u) > 0$ and $z^{(n-1)}(u) < 0$;
- III** $(-1)^i z^{(i)}(u)$ are positive for $i = 1, 2, \dots, n-1$.

Proof. Let x denote a solution of (1) that eventually remains positive. Then, there exists some $u_1 \geq u_0$ for which $x(u)$, $x(\tau(u))$ and $x(\sigma(u))$ remain positive for all $u \geq u_1$. Consequently, $z(u)$ is also positive for every $u \geq u_1$. It then follows from Equation (1) that $\left(r(u) \left(z^{(n-1)}(u) \right)^\alpha \right)' < 0$. Applying Lemma 1 in the case where n is even, the conclusions in cases (I)–(III) follow directly. $\square$

Lemma 4. Suppose that $x \in \mathbf{C}([u_0, \infty), (0, \infty))$ is a solution of (1) and $z(u)$ satisfies case I and II stated in Lemma 3. Under these assumptions, we eventually have

$$\left(r(u) \left(z^{(n-1)}(u) \right)^\alpha \right)' + \Phi(u) z^\beta(\sigma(u)) \leq 0. \quad (3)$$

Proof. Given that x fulfills case II and $\tau(u) < u$, we have $z'(u) > 0$ and $x(\tau(u)) \leq z(\tau(u)) \leq z(u)$. It immediately follows from the definition of z that $x(u) > (1 - p(u))z(u)$. Therefore, Equation (1) becomes as follows:

$$\begin{aligned} \left(r(u) \left(z^{(n-1)}(u) \right)^\alpha \right)' &\leq -q(u)(1 - p(\sigma(u)))^\beta z^\beta(\sigma(u)) \\ &\leq -\Phi(u) z^\beta(\sigma(u)). \end{aligned}$$

$\square$

Lemma 5. Suppose that $x \in \mathbf{C}([u_0, \infty), (0, \infty))$ is a solution to (1) and $z(u)$ satisfies case III stated in Lemma 3. Under these assumptions, we eventually have

$$(-1)^{j+1} z^{(n-j-2)}(u) \leq r^{1/\alpha}(u) z^{(n-1)}(u) \eta_j(u), \quad (4)$$

$$(-1)^j \left(\frac{z^{(n-j-2)}(u)}{\eta_j(u)} \right)' \geq 0, \quad (5)$$

and

$$\left(r(u) \left(z^{(n-1)}(u) \right)^\alpha \right)' + \hat{\Phi}(u) z^\beta(\sigma(u)) \leq 0, \quad (6)$$

for $j = 1, 2, \dots, n - 2$, eventually.

Proof. Suppose that z fulfills case III. From Equation (1), we have $\left(r(u)\left(z^{(n-1)}(u)\right)^\alpha\right)' < 0$, and therefore,

$$\begin{aligned} r^{1/\alpha}(u)z^{(n-1)}(u) \int_u^\infty r^{-1/\alpha}(s)ds &\geq \int_u^\infty r^{-1/\alpha}(s)r^{1/\alpha}(u)z^{(n-1)}(s)ds \\ &= \lim_{s \rightarrow \infty} z^{(n-2)}(s) - z^{(n-2)}(u). \end{aligned} \quad (7)$$

Since $z^{(n-2)}(u)$ is a positive and decreasing function, it converges to a non-negative constant as $u \rightarrow \infty$. Therefore, (7) can be written as

$$-z^{(n-2)}(u) \leq r^{1/\alpha}(u)\eta_0(u)z^{(n-1)}(u). \quad (8)$$

By noting that $(-1)^i z^{(i)}(u)$ for $i = 1, 2, \dots, n - 1$, integrating the inequality in (8) along with the resulting successive inequalities a total of $n - 2$ times from u to ∞ yields

$$(-1)^{j+1}z^{(n-j-2)}(u) \leq r^{1/\alpha}(u)z^{(n-1)}(u)\eta_j(u),$$

for $j = 1, 2, \dots, n - 2$. This directly leads to the result stated in (4). Now, it follows from (8) that

$$\left(\frac{z^{(n-2)}(u)}{\eta_0(u)}\right)' \geq 0, \quad (9)$$

yielding

$$-z^{(n-3)}(u) \geq \int_u^\infty \eta_0(s) \frac{z^{(n-2)}(s)}{\eta_0(s)} ds \geq \frac{z^{(n-2)}(u)}{\eta_0(u)} \eta_1(u).$$

As a result,

$$\left(\frac{z^{(n-3)}(u)}{\eta_1(u)}\right)' \leq 0.$$

By applying the same approach iteratively, we arrive at (5). Using (5) with $j = n - 2$, we obtain

$$z(\tau(u)) \leq \frac{\eta_{n-2}(\tau(u))}{\eta_{n-2}(u)} z(u)$$

Therefore, combining the definition of z with the above inequality yields

$$x(u) \geq \left(1 - p(u) \frac{\eta_{n-2}(\tau(u))}{\eta_{n-2}(u)}\right) z(u).$$

Therefore, Equation (1) becomes as follows:

$$\left(r(u)\left(z^{(n-1)}(u)\right)^\alpha\right)' \leq -\hat{\Phi}z^\beta(\sigma(u)).$$

Hence, the proof is complete. $\square$

Lemma 6. Suppose that $x \in \mathbf{C}([u_0, \infty), (0, \infty))$ is a solution of (1) and $z(u)$ satisfies case II. Then,

(i) $\lim_{u \rightarrow \infty} z^{(n-2)}(u) = 0$ if

$$\int_{u_1}^\infty \left(\frac{1}{r(\rho)} \int_{u_1}^\rho \sigma^{\beta(n-2)}(s)\Phi(s)ds\right)^{1/\alpha} d\rho = \infty; \quad (10)$$

(ii) $\lim_{u \rightarrow \infty} (z^{(n-2)}(u)/\eta_0(u)) = \infty$ if

$$\int_{u_1}^{\infty} \sigma^{\beta(n-2)}(s) \eta_0^{\beta}(\sigma(s)) \Phi(s) ds = \infty. \quad (11)$$

Proof. Suppose that z fulfills case II. Consequently,

$$-z^{(n-2)}(u) \leq \int_u^{\infty} r^{1/\alpha}(s) r^{-1/\alpha}(s) z^{(n-1)}(s) ds \leq r^{1/\alpha}(u) \eta_0(u) z^{(n-1)}(u).$$

Thus,

$$\left(\frac{z^{(n-2)}(u)}{\eta_0(u)} \right)' \geq 0.$$

Now, by applying Lemma 2 with $y = z$ and $n = n - 1$, Equation (1) simplifies to

$$\left(r(u) \left(z^{(n-1)}(u) \right)^{\alpha} \right)' + \frac{v^{\beta}}{((n-2)!)^{\beta}} \sigma^{\beta(n-2)}(u) \left(z^{(n-2)}(\sigma(u)) \right)^{\beta} \Phi(u) \leq 0. \quad (12)$$

By defining $\Psi(u) = z^{(n-2)}(u)$, we see that Ψ is a positive decreasing solution of

$$\left(r(u) (\Psi'(u))^{\alpha} \right)' + \frac{v^{\beta}}{((n-2)!)^{\beta}} \sigma^{\beta(n-2)}(u) \Psi^{\beta}(\sigma(u)) \Phi(u) \leq 0. \quad (13)$$

Because Ψ is positively decreasing, it converges to a non-negative constant ϱ as $u \rightarrow \infty$. Given that $\varrho > 0$, we have $\Psi \geq \varrho$ for $u \geq u_1 \geq u_0$. Applying integration to (13) on the interval $[u_1, u]$ gives

$$\begin{aligned} r(u) (\Psi'(u))^{\alpha} &\leq r(u_1) (\Psi'(u_1))^{\alpha} - \frac{v^{\beta}}{((n-2)!)^{\beta}} \int_{u_1}^u \sigma^{\beta(n-2)}(s) \Psi^{\beta}(\sigma(s)) \Phi(s) ds \\ &\leq - \frac{\varrho^{\beta} v^{\beta}}{((n-2)!)^{\beta}} \int_{u_1}^u \sigma^{\beta(n-2)}(s) \Phi(s) ds. \end{aligned} \quad (14)$$

Thus,

$$\Psi'(u) \leq - \left(\frac{\varrho v}{(n-2)!} \right)^{\beta/\alpha} \left(\frac{1}{r(u)} \int_{u_1}^u \sigma^{\beta(n-2)}(s) \Phi(s) ds \right)^{1/\alpha}. \quad (15)$$

Integration of (15) over the interval $[u_1, \infty)$ gives

$$\Psi(u_1) \leq - \left(\frac{\varrho v}{(n-2)!} \right)^{\beta/\alpha} \int_{u_1}^{\infty} \left(\frac{1}{r(\rho)} \int_{u_1}^{\rho} \sigma^{\beta(n-2)}(s) \Phi(s) ds \right)^{1/\alpha} d\rho.$$

This leads to a contradiction with (10), implying that $\varrho = 0$.

It follows from (9) and (14) that

$$\begin{aligned} r(u) (\Psi'(u))^{\alpha} &\leq \frac{v^{\beta}}{((n-2)!)^{\beta}} \int_{u_1}^u \sigma^{\beta(n-2)}(s) \eta_0^{\beta}(\sigma(s)) \frac{\Psi^{\beta}(\sigma(s))}{\eta_0^{\beta}(\sigma(s))} \Phi(s) ds \\ &\leq \frac{v^{\beta}}{((n-2)!)^{\beta}} \left(\frac{\Psi(\sigma(u_1))}{\eta_0(\sigma(u_1))} \right)^{\beta} \int_{u_1}^u \sigma^{\beta(n-2)}(s) \eta_0^{\beta}(\sigma(s)) \Phi(s) ds, \end{aligned}$$

which, when combined with (11), results in $\lim_{u \rightarrow \infty} (-r(u)(\Psi'(u))^\alpha) = \infty$. With the application of L'Hôpital's rule, we get

$$\lim_{u \rightarrow \infty} \frac{\Psi(u)}{\eta_0(u)} = \lim_{u \rightarrow \infty} \left(-r^{1/\alpha}(u) \Psi'(u) \right) = \infty$$

□

Lemma 7. Suppose that $x \in \mathbf{C}([u_0, \infty), (0, \infty))$ is a solution of (1) and $z(u)$ satisfies case III. Then,

$$(z(u)/\eta_{n-2}(u))' \geq 0$$

holds. Moreover, we obtain the following results:

(i) $\lim_{u \rightarrow \infty} z(u) = \infty$, if

$$\int_{u_1}^{\infty} \eta_{n-3}(\rho) \left(\int_{u_1}^{\rho} \hat{\Phi}(s) ds \right)^{1/\alpha} d\rho = \infty; \quad (16)$$

(ii) $\lim_{u \rightarrow \infty} (z(u)/\eta_{n-2}(u)) = \infty$, if

$$\int_{u_1}^u \eta_{n-2}^\beta(\sigma(s)) \hat{\Phi}(s) ds = \infty. \quad (17)$$

Proof. Suppose that z fulfills case III. Consequently,

$$z^{(n-2)}(u) \geq - \int_u^\infty r^{1/\alpha}(s) r^{-1/\alpha}(s) z^{(n-1)}(s) ds \geq -r^{1/\alpha}(u) \eta_0(u) z^{(n-1)}(u). \quad (18)$$

Thus,

$$\left(\frac{z^{(n-2)}(u)}{\eta_0(u)} \right)' \geq 0. \quad (19)$$

Integrating (18) a total of $n - 2$ times over $[u, \infty)$ yields

$$(-1)^j z^{(n-j-2)}(u) \geq r^{1/\alpha}(u) \eta_j(u) z^{(n-1)}(u), \quad (20)$$

for $j = 1, 2, \dots, n - 2$. It follows from (19) that

$$-z^{(n-3)}(u) \geq \int_u^\infty \left(\frac{z^{(n-2)}(s)}{\eta_0(s)} \right) \eta_0(s) ds \geq \frac{\eta_1(u)}{\eta_0(u)} z^{(n-2)}(u),$$

which leads to $(z^{(n-3)}(u)/\eta_1(u))' \leq 0$. Continuing this process, we obtain

$$\left(\frac{z(u)}{\eta_{n-2}(u)} \right)' \geq 0. \quad (21)$$

Next, observing that $z(u)$ is positively decreasing, we conclude that $\lim_{u \rightarrow \infty} z(u) = \nu \geq 0$. Assuming $\nu > 0$, there is a $u_1 \geq u_0$ such that $z(u) \geq \nu$ for $u \geq u_1$. By integrating (6) from u_1 to u , we arrive at

$$r(u) \left(z^{(n-1)}(u) \right)^\alpha \leq - \int_{u_1}^u \hat{\Phi}(s) z^\beta(\sigma(s)) ds.$$

By employing (20) when $j = n - 3$, we conclude that

$$-\frac{z'(u)}{\eta_{n-3}(u)} \geq -r^{1/\alpha}(u)z^{(n-1)}(u) \geq \left(\int_{u_1}^u \hat{\Phi}(s)z^\beta(\sigma(s))ds\right)^{1/\alpha}. \quad (22)$$

As another possibility,

$$-z'(u) \geq \eta_{n-3}(u)v^{\beta/\alpha} \left(\int_{u_1}^u \hat{\Phi}(s)ds\right)^{1/\alpha}. \quad (23)$$

Performing the integration of the above inequality on $[u_1, \infty)$ gives

$$z(u_1) \geq v^{\beta/\alpha} \int_{u_1}^\infty \eta_{n-3}(\rho) \left(\int_{u_1}^\rho \hat{\Phi}(s)ds\right)^{1/\alpha} d\rho.$$

Together with (16), this shows that $v = 0$.

Next, based on (21) and (23), we derive

$$-\frac{z'(u)}{\eta_{n-3}(u)} \geq \left(\frac{z(\sigma(u_1))}{\eta_{n-2}(\sigma(u_1))}\right)^{\beta/\alpha} \left(\int_{u_1}^u \eta_{n-2}^\beta(\sigma(s))\hat{\Phi}(s)ds\right)^{1/\alpha},$$

which, in conjunction with (17), implies

$$\lim_{u \rightarrow \infty} \frac{z(u)}{\eta_{n-2}(u)} = \lim_{u \rightarrow \infty} \left(-\frac{z'(u)}{\eta_{n-3}(u)}\right) = \infty.$$

This concludes the proof. $\square$

Here, we derive conditions that rule out the existence of positive solutions within certain categories.

Lemma 8. Suppose that $x \in C([u_0, \infty), (0, \infty))$ is a solution to (1) and $z(u)$ satisfies case II. If (i) $\alpha > \beta$, (10) is fulfilled and

$$\limsup_{u \rightarrow \infty} \left(\eta_0^\alpha(u) \int_{u_1}^u \sigma^{\beta(n-2)}(s)\Phi(s)ds\right) > 0. \quad (24)$$

(ii) $\alpha < \beta$, (11) is fulfilled and

$$\limsup_{u \rightarrow \infty} \left(\eta_0^\alpha(u)\eta_0^{\beta-\alpha}(\sigma(u)) \int_{u_1}^u \sigma^{\beta(n-2)}(s)\Phi(s)ds\right) > 0. \quad (25)$$

Then, case II contains no positive solutions.

Proof. Suppose that z fulfills case II. Using the same reasoning as in the proof of Lemma 6, we conclude that (14) holds. As a result,

$$-\Psi'(u) \geq \frac{v^{\beta/\alpha}}{((n-2)!)^{\beta/\alpha} r^{1/\alpha}(u)} \left(\int_{u_1}^u \sigma^{\beta(n-2)}(s)\Psi^\beta(\sigma(s))\Phi(s)ds\right)^{1/\alpha}.$$

Upon integrating the above inequality over $[u, \infty)$, we deduce

$$\begin{aligned} \Psi(u) &\geq \frac{v^{\beta/\alpha}}{((n-2)!)^{\beta/\alpha}} \int_u^\infty \frac{1}{r^{1/\alpha}(\rho)} \left(\int_{u_1}^\rho \sigma^{\beta(n-2)}(s)\Psi^\beta(\sigma(s))\Phi(s)ds\right)^{1/\alpha} d\rho \\ &\geq \frac{v^{\beta/\alpha}}{((n-2)!)^{\beta/\alpha}} \eta_0(u) \left(\int_{u_1}^u \sigma^{\beta(n-2)}(s)\Psi^\beta(\sigma(s))\Phi(s)ds\right)^{1/\alpha}. \end{aligned}$$

Therefore, we find

$$\Psi^\alpha(\sigma(u)) \geq \Psi^\alpha(u) \geq \frac{v^\beta}{((n-2)!)^\beta} \eta_0^\alpha(u) \Psi^\beta(\sigma(u)) \int_{u_1}^u \sigma^{\beta(n-2)}(s) \Phi(s) ds. \quad (26)$$

In the following, we analyze the two cases $\alpha > \beta$ and $\alpha < \beta$ separately.

(i) For $\alpha > \beta$, take into account that $\Psi(u) \rightarrow 0$ and (26) simplifies to

$$\frac{v^\beta}{((n-2)!)^\beta} \eta_0^\alpha(u) \int_{u_1}^u \sigma^{\beta(n-2)}(s) \Phi(s) ds \leq \Psi^{\alpha-\beta}(\sigma(u)) \rightarrow 0 \text{ as } u \rightarrow \infty,$$

which is in contradiction with (24).

(ii) For $\alpha < \beta$, take into account from Lemma 6 that $\Psi/\eta_0 \rightarrow \infty$ and (26) simplifies to

$$\frac{v^\beta}{((n-2)!)^\beta} \eta_0^\alpha(u) \eta_0^{\beta-\alpha}(\sigma(u)) \int_{u_1}^u \sigma^{\beta(n-2)}(s) \Phi(s) ds \leq \left(\frac{\Psi(\sigma(u))}{\eta_0(\sigma(u))} \right)^{\alpha-\beta} \rightarrow 0 \text{ as } u \rightarrow \infty,$$

which is in contradiction with (25). This concludes the proof. $\square$

Lemma 9. Let $x \in \mathbf{C}([u_0, \infty), (0, \infty))$ be a solution to (1) and $z(u)$ satisfy case III. If

(i) $\alpha > \beta$, (16) is fulfilled and

$$\limsup_{u \rightarrow \infty} \left(\eta_{n-2}^\alpha(u) \int_{u_1}^u \hat{\Phi}(s) ds \right) > 0. \quad (27)$$

(ii) $\alpha < \beta$, (17) is fulfilled and

$$\limsup_{u \rightarrow \infty} \left(\eta_{n-2}^\alpha(u) \eta_{n-2}^{\beta-\alpha}(\sigma(u)) \int_{u_1}^u \hat{\Phi}(s) ds \right) > 0. \quad (28)$$

Then, case III contains no positive solutions.

Proof. Suppose that z fulfills case III. By applying the argument presented in Lemma 7, it follows that (22) is satisfied. Integration of (22) from u to ∞ yields

$$\begin{aligned} z(u) &\geq \int_u^\infty \eta_{n-3}(\rho) \left(\int_{u_1}^\rho \hat{\Phi}(s) z^\beta(\sigma(s)) ds \right)^{1/\alpha} d\rho \\ &\geq \eta_{n-2}(u) \left(\int_{u_1}^u \hat{\Phi}(s) z^\beta(\sigma(s)) ds \right)^{1/\alpha}. \end{aligned}$$

Thus,

$$z^\alpha(\sigma(u)) \geq \eta_{n-2}^\alpha(u) z^\beta(\sigma(u)) \int_{u_1}^u \hat{\Phi}(s) ds. \quad (29)$$

In the following, we analyze the two cases $\alpha > \beta$ and $\alpha < \beta$ separately.

(i) For $\alpha > \beta$, noting that $z(u) \rightarrow 0$, (29) simplifies to

$$\frac{z^\alpha(\sigma(u))}{z^\beta(\sigma(u))} \geq \eta_{n-2}^\alpha(u) \int_{u_1}^u \hat{\Phi}(s) ds;$$

hence,

$$\eta_{n-2}^\alpha(u) \int_{u_1}^u \hat{\Phi}(s) ds \leq z^{\alpha-\beta}(\sigma(u)) \rightarrow 0 \text{ as } u \rightarrow \infty,$$

which is in contradiction with (27).

(ii) For $\alpha < \beta$, take into account from Lemma 7 that $z/\eta_{n-2} \rightarrow \infty$ and (29) simplifies to

$$\frac{z^{\alpha-\beta}(\sigma(u))}{\eta_{n-2}^{\alpha-\beta}(\sigma(u))} \geq \eta_{n-2}^{\alpha}(u) \eta_{n-2}^{\beta-\alpha}(\sigma(u)) \int_{u_1}^u \hat{\Phi}(s) ds;$$

hence,

$$\eta_{n-2}^{\alpha}(u) \eta_{n-2}^{\beta-\alpha}(\sigma(u)) \int_{u_1}^u \hat{\Phi}(s) ds \leq \left(\frac{z(\sigma(u))}{\eta_{n-2}(\sigma(u))} \right)^{\alpha-\beta} \rightarrow 0 \text{ as } u \rightarrow \infty,$$

which is in contradiction with (28). This concludes the proof. $\square$

2.2. Theorems Concerning Oscillations

From the preceding results, one can derive criteria ensuring no solutions exist in classes I, II, and III, implying that all solutions to Equation (1) are oscillatory.

Theorem 1. Assume that $\eta_{n-2}(u_0) < \infty$. Then, Equation (1) is oscillatory under each of the following scenarios:

- (i) $\alpha > \beta$ and conditions (10), (16), (24), and (27) hold;
- (ii) $\alpha < \beta$ and conditions (11), (16), (17), (25), and (28) are satisfied.

Proof. Let x denote a solution of (1) that eventually remains positive. Lemma 3 states that one of the three cases I–III must apply to the function z . Lemmas 8 and 9 give us that cases II and III contain no positive solutions. Therefore, the remaining case is case I.

(i) $\alpha > \beta$:

By integrating Equation (3) from u_1 to ∞ and taking into account that z is nondecreasing, we obtain

$$r(u_1) \left(z^{(n-1)}(u_1) \right)^{\alpha} \geq z^{\beta}(\sigma(u)) \int_{u_1}^{\infty} \Phi(s) ds \quad (30)$$

By applying (16) and considering that $\eta_{n-2}(u_0) < \infty$, the right-hand side of (30) becomes unbounded, leading to a contradiction.

(ii) $\alpha < \beta$: The proof proceeds analogously to the case $\alpha > \beta$ and is thus omitted. This concludes the proof. $\square$

Theorem 2. Assume that $\eta_{n-2}(u_0) < \infty$ and that

$$\liminf_{u \rightarrow \infty} \left(r^{1/\alpha}(u) \eta_0^{\kappa+1}(u) \sigma^{\beta(n-2)}(u) \Phi(u) \right) > 0, \quad (31)$$

$$\liminf_{u \rightarrow \infty} \left(\frac{\eta_{n-2}^{\kappa+1}(u)}{\eta_{n-3}(u)} \hat{\Phi}(u) \right) > 0, \quad (32)$$

where $\kappa = \max\{\alpha, \beta\}$. Then, Equation (1) is oscillatory.

Proof. From (31) and (32), it follows that there exist positive constants κ_1 and κ_2 such that

$$r^{1/\alpha}(u) \eta_0^{\kappa}(u) \sigma^{\beta(n-2)}(u) \Phi(u) \geq \kappa \kappa_1, \quad (33)$$

and

$$\frac{\eta_{n-2}^{\kappa+1}(u)}{\eta_{n-3}(u)} \hat{\Phi}(u) \geq \kappa \kappa_2. \quad (34)$$

It follows from (33) that

$$\begin{aligned}\int_{u_1}^u \eta^{\kappa(n-2)}(s) \Phi(s) ds &\geq \int_{u_1}^u \frac{\kappa_1}{r^{1/\alpha}(s) \eta_0^{\kappa+1}(s)} ds \\ &= \kappa_1 [\eta_0^{-\kappa}(u) - \eta_0^{-\kappa}(u_1)].\end{aligned}\quad (35)$$

In view of the fact that $\eta_0(u)$ is decreasing and $\lim_{u \rightarrow \infty} \eta_0(u) = 0$, there exists $u_1 \geq u_0$ such that

$$\eta_0^{-\kappa}(u) - \eta_0^{-\kappa}(u_1) \geq \theta_1 \eta_0^{-\kappa}(u),$$

for all $\theta_1 \in (0, 1)$. Therefore, (35) simplifies to

$$\int_{u_1}^u \eta^{\kappa(n-2)}(s) \Phi(s) ds \geq \kappa_1 \theta_1 \eta_0^{-\kappa}(u). \quad (36)$$

Equation (34) implies that

$$\begin{aligned}\int_{u_1}^u \hat{\Phi}(s) ds &\geq \kappa_2 \int_{u_1}^u \frac{\eta_{n-3}(s)}{\eta_{n-2}^{\kappa+1}(s)} ds \\ &= \kappa_2 [\eta_{n-2}^{-\kappa}(u) - \eta_{n-2}^{-\kappa}(u_1)].\end{aligned}$$

In view of the fact that $\eta_{n-2}(u)$ is decreasing and $\lim_{u \rightarrow \infty} \eta_{n-2}(u) = 0$, there exists $u_1 \geq u_0$ such that

$$\eta_{n-2}^{-\kappa}(u) - \eta_{n-2}^{-\kappa}(u_1) \geq \theta_2 \eta_{n-2}^{-\kappa}(u), \quad (37)$$

for all $\theta_2 \in (0, 1)$. Therefore, (37) simplifies to

$$\int_{u_1}^u \hat{\Phi}(s) ds \geq \kappa_2 \theta_2 \eta_{n-2}^{-\kappa}(u). \quad (38)$$

We shall now demonstrate that all the constraints of Theorem 1 are satisfied. For condition (10), it follows from (36) that

$$\begin{aligned}&\int_{u_1}^{\infty} \left(\frac{1}{r(\rho)} \int_{u_1}^{\rho} \sigma^{\beta(n-2)}(s) \Phi(s) ds \right)^{1/\alpha} d\rho \\ &\geq (\kappa_1 \theta_1)^{1/\alpha} \int_{u_1}^{\infty} \frac{\eta_0^{-\kappa/\alpha}(s)}{r^{1/\alpha}(s)} ds \\ &= \begin{cases} \frac{(\kappa_1 \theta_1)^{1/\alpha}}{\kappa/\alpha - 1} \lim_{u \rightarrow \infty} [\eta_0^{1-\kappa/\alpha}(u) - \eta_0^{1-\kappa/\alpha}(u_1)] & \text{if } \kappa = \beta \\ (\kappa_1 \theta_1)^{1/\alpha} \lim_{u \rightarrow \infty} \left[\ln \frac{\eta_0(u_1)}{\eta_0(u)} \right] & \text{if } \kappa = \alpha \end{cases} \\ &= \infty.\end{aligned}$$

Regarding condition (16), it follows from (38) that

$$\begin{aligned}&\int_{u_1}^{\infty} \eta_{n-3}(\rho) \left(\int_{u_1}^{\rho} \hat{\Phi}(s) ds \right)^{1/\alpha} d\rho \\ &\geq (\kappa_2 \theta_2)^{1/\alpha} \int_{u_1}^{\infty} \frac{\eta_{n-3}(s)}{\eta_{n-2}^{\kappa/\alpha}(s)} ds \\ &= \begin{cases} \frac{(\kappa_2 \theta_2)^{1/\alpha}}{\kappa/\alpha - 1} \lim_{u \rightarrow \infty} [\eta_{n-2}^{1-\kappa/\alpha}(u) - \eta_{n-2}^{1-\kappa/\alpha}(u_1)] & \text{if } \kappa = \beta \\ (\kappa_2 \theta_2)^{1/\alpha} \lim_{u \rightarrow \infty} \left[\ln \frac{\eta_{n-2}(u_1)}{\eta_{n-2}(u)} \right] & \text{if } \kappa = \alpha \end{cases} \\ &= \infty.\end{aligned}$$

Let $\kappa = \alpha > \beta$. By applying (36), condition (24) is automatically satisfied because

$$\limsup_{u \rightarrow \infty} \left(\eta_0^\alpha(u) \int_{u_1}^u \sigma^{\beta(n-2)}(s) \Phi(s) ds \right) \geq \kappa_1 \theta_1 > 0.$$

Moreover, it follows from (38) that

$$\limsup_{u \rightarrow \infty} \left(\eta_{n-2}^\alpha(u) \int_{u_1}^u \hat{\Phi}(s) ds \right) \geq \kappa_2 \theta_2 > 0,$$

which confirms the validity of condition (27). Therefore, invoking Theorem 1 leads to the conclusion that Equation (1) exhibits oscillatory behavior.

Let $\alpha < \beta$; the same argument verifies conditions (11), (17), (25), and (28). This concludes the proof. $\square$

Remark 2. Although Theorem 1 provides sufficient criteria ensuring the oscillation of all solutions, it involves several conditions (four conditions in scenario (i) and five in scenario (ii)). To address this complexity, Theorem 2 presents simplified oscillation criteria. Unlike previous results in the literature, which involved arbitrary constants and were therefore difficult to apply in practice, the criteria derived in this work are free of such constants and hence are directly applicable. Moreover, to overcome the difficulty caused by the multiple constraints in Theorem 1, we propose in Theorem 2 a new and previously unreported simplified oscillation condition that preserves effectiveness while significantly improving applicability.

3. Examples and Discussion

Example 1. Consider the following NDDE:

$$\left(u^\delta \left((x(u) + p_0 x(\lambda u))''' \right)^\alpha \right)' + \frac{q_0}{u^\epsilon} x^\beta(\mu u) = 0, \quad (39)$$

where $u \geq u_0$. Assume that $q_0 > 0$, $\delta > \alpha$, $\lambda \in (0, 1]$, $\epsilon > 0$, $\mu \in (0, 1]$, and $p_0 \in [0, \lambda^{3-\delta/\alpha}]$. Observe that

$$\eta_0(u) = \int_u^\infty r^{-1/\alpha}(s) ds = \int_u^\infty s^{-\delta/\alpha} ds = \frac{\alpha}{\delta - \alpha} u^{1-\delta/\alpha},$$

$$\eta_1(u) = \int_u^\infty \eta_0(s) ds = \frac{\alpha}{\delta - \alpha} \int_u^\infty s^{1-\delta/\alpha} ds = \frac{\alpha^2}{(\delta - \alpha)(\delta - 2\alpha)} u^{2-\delta/\alpha},$$

and

$$\eta_2(u) = \int_u^\infty \eta_1(s) ds = \frac{\alpha^2}{(\delta - \alpha)(\delta - 2\alpha)} \int_u^\infty s^{2-\delta/\alpha} ds = \frac{\alpha^3}{(\delta - \alpha)(\delta - 2\alpha)(\delta - 3\alpha)} u^{3-\delta/\alpha},$$

which ensures that $\eta(u_0) < \infty$. It can also be verified that

$$\Phi(u) := \frac{q_0}{u^\epsilon} (1 - p_0)^\beta,$$

and

$$\hat{\Phi}(u) := \frac{q_0}{u^\epsilon} \left(1 - \lambda^{3-\delta/\alpha} p_0 \right)^\beta.$$

One can easily verify that

$$\begin{aligned} & \liminf_{u \rightarrow \infty} \left(r^{1/\alpha}(u) \eta_0^{\kappa+1}(u) \sigma^{\beta(n-2)}(u) \Phi(u) \right) > 0 \\ & = \liminf_{u \rightarrow \infty} \left(\left(u^\delta \right)^{1/\alpha} \left(\frac{\alpha}{\delta - \alpha} u^{1-\delta/\alpha} \right)^{\kappa+1} (\mu u)^{2\beta} \frac{q_0}{u^\epsilon} (1 - p_0)^\beta \right) > 0; \end{aligned}$$

hence,

$$q_0 \left(\frac{\alpha}{\delta - \alpha} \right)^{\kappa+1} \mu^{2\beta} (1 - p_0)^\beta \liminf_{u \rightarrow \infty} \left(u^{\delta/\alpha} \left(u^{1-\delta/\alpha} \right)^{\kappa+1} u^{2\beta} \frac{1}{u^\varepsilon} \right) > 0,$$

which is fulfilled when

$$\frac{\delta}{\alpha} + (\kappa + 1) \left(1 - \frac{\delta}{\alpha} \right) + 2\beta > \varepsilon. \quad (40)$$

Also,

$$\frac{q_0 \alpha^{3\kappa+1} \left(1 - \lambda^{3-\delta/\alpha} p_0 \right)^\beta}{[(\delta - \alpha)(\delta - 2\alpha)]^{\kappa+1} (\delta - 3\alpha)^{\kappa+1}} \liminf_{u \rightarrow \infty} \left(u^{3\kappa - \kappa\delta/\alpha + 1} \frac{1}{u^\varepsilon} \right) > 0,$$

which is fulfilled when

$$3\kappa - \kappa\delta/\alpha + 1 > \varepsilon. \quad (41)$$

According to Theorem 2, the oscillation of Equation (39) is guaranteed whenever conditions (40) and (41) hold.

Remark 3. Figure 1 shows the parameter region where the oscillation conditions are satisfied. The shaded area represents feasible values of α and β , with the blue boundary indicating the threshold between non-oscillatory and oscillatory behavior. The figure also illustrates that larger δ values require a more restricted combination of α and β to meet the oscillation criteria.

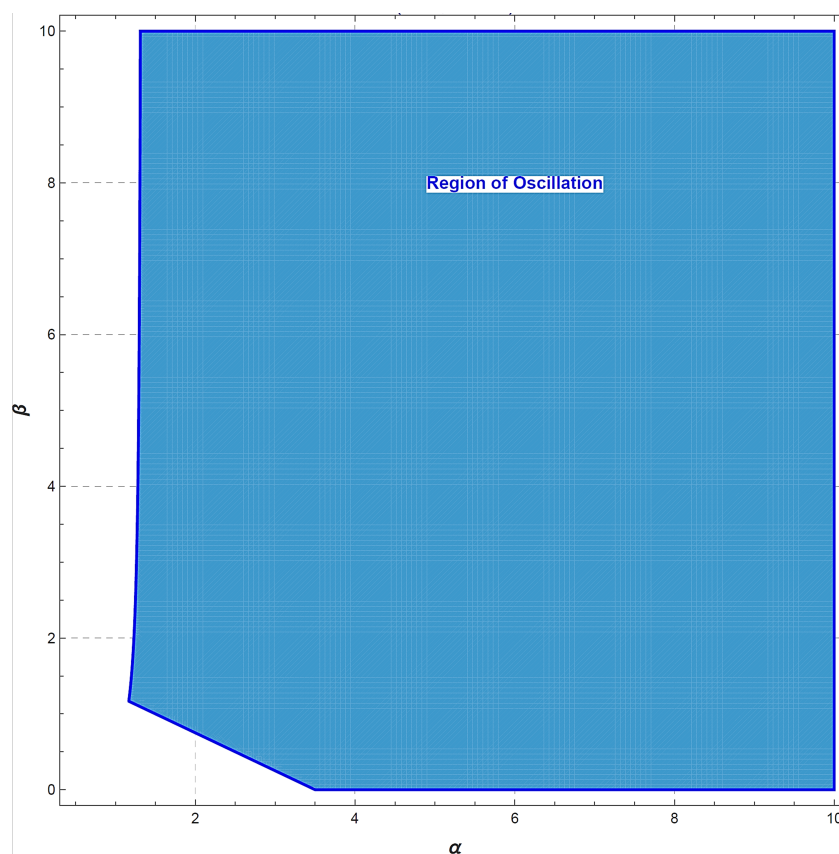


Figure 1. The joint feasible region of parameters (α, β) .

Remark 4. For the particular case $n = 2$, the characterization of positive solutions does not conform to the classification presented in Lemma 3, as none of the solution classes fulfill the requirements of class II. Consequently, the solution set is restricted to classes I and III. Hence, the oscillatory

nature of the considered equation depends solely on the validity of condition (32).

Furthermore, when Equation (39) is examined with $n = 2$ and $\delta = 2$, condition (41) simplifies to

$$\alpha > \varepsilon + 1$$

if $\alpha > \beta$, and

$$\beta < \alpha(1 - \varepsilon)/(2 - \alpha)$$

if $\alpha < \beta$, which coincides with the criteria obtained in [24] (Example 1).

4. Conclusions

This research investigated the asymptotic behavior of positive solutions to higher-order nonlinear neutral functional differential equations in the noncanonical case. It culminated in the development of a multi-condition criterion that ensures oscillation and generalizes classical results to higher-order equations. The reason behind this is the scarcity of results that address this aspect. Most recent studies in oscillation theory have focused on advancing the analysis of linear and half-linear equations rather than the nonlinear case. To overcome the complexity arising from this multi-condition structure, a new simplified oscillation criterion was proposed. It was demonstrated to be effective through an illustrative example. The findings confirm that the proposed criterion offers a valuable and original contribution. It broadens the scope of oscillation theory in nonlinear differential equations. Although the results presented in this paper represent a direct extension of previous findings in the literature, this research area still requires further in-depth study. In particular, additional investigation is needed to address certain parameter constraints more effectively. For instance, in the illustrative example, it would be beneficial to explore the possibility of establishing oscillation conditions in cases where $\delta < \alpha$. Such future work would not only enhance the applicability of the proposed criteria, but also contribute to a more comprehensive understanding of oscillatory behavior in higher-order nonlinear neutral differential equations. Extending these results to third-order equations represents another intriguing direction for further research.

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