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On Sequential Coupled Caputo-Type Fractional Differential Inclusions with Coupled Boundary Conditions: A Multivalued Fixed-Point Approach

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Abstract

This study addresses the existence of solutions for a class of coupled fractional differential inclusions subject to coupled boundary conditions. The analysis is developed within the framework of nonlinear functional analysis by employing Carathéodory-type assumptions, the Leray–Schauder nonlinear alternative, and the Covitz–Nadler fixed-point theorem for multivalued mappings. The proposed approach accommodates both convex and non-convex set-valued nonlinearities, thereby broadening the scope of the results. Under suitable restrictions on the problem parameters, several corollaries are established as direct consequences of the main findings. An example is included to demonstrate the practical applicability and validity of the theoretical results.

Keywords: fractional differential system; sequential; inclusion; boundary conditions; fixed point

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1. Introduction

Fractional differential equations (FDEs) provide an effective framework for modeling systems with memory and hereditary effects, with wide applications in science and engineering. Their nonlocal nature makes them suitable for studying existence, uniqueness, and stability under various boundary conditions [1–4]. Coupled systems of FDEs further extend this framework by describing interacting processes, though they introduce significant analytical challenges, particularly under nonlinear and nonlocal boundary conditions [5–9].

A natural generalization is given by fractional differential inclusions, which incorporate set-valued mappings to model uncertainty and multivalued dynamics. Existence results for such problems have been widely studied using multivalued fixed-point techniques [10–14]. In particular, coupled fractional differential inclusions have been inves-

tigated under various boundary conditions [15–18]. Recently, sequential fractional differential equations and inclusions have gained attention due to their ability to capture higher-order memory effects. Although several studies address their solvability [19,20], analysis of coupled sequential fractional differential inclusions with nonlocal multipoint boundary conditions is required. More recently, attention has been directed toward sequential fractional differential equations and inclusions, which involve compositions of fractional operators and provide enhanced flexibility in describing complex dynamical behavior [20–23]. Motivated by the above developments, we consider the coupled system of fractional differential inclusions

$$\begin{cases} ({}^{c}\mathcal{D}^{\alpha} + \lambda_1 \, {}^{c}\mathcal{D}^{\alpha-1})\mathcal{X}(t) \in \mathcal{F}(t, \mathcal{X}(t), \mathcal{Y}(t)), \\ ({}^{c}\mathcal{D}^{\beta} + \lambda_2 \, {}^{c}\mathcal{D}^{\beta-1})\mathcal{Y}(t) \in \mathcal{G}(t, \mathcal{X}(t), \mathcal{Y}(t)), \end{cases} \tag{1}$$

subject to nonlocal coupled boundary conditions

$$\begin{cases} \mathcal{X}(\mathcal{T}) = r_1 \mathcal{Y}(k_1) + r_2 \mathcal{T}\mathcal{Y}'(k_2), \\ \mathcal{T}\mathcal{X}'(\mathcal{T}) = s_1 \mathcal{Y}(u_1) + s_2 \mathcal{T}\mathcal{Y}'(u_2), \\ \mathcal{Y}(\mathcal{T}) = p_1 \mathcal{X}(\zeta_1) + p_2 \mathcal{T}\mathcal{X}'(\zeta_2), \\ \mathcal{T}\mathcal{Y}'(\mathcal{T}) = q_1 \mathcal{X}(v_1) + q_2 \mathcal{T}\mathcal{Y}'(v_2), \end{cases} \tag{2}$$

where ${}^{c}\mathcal{D}^{\alpha}$ and ${}^{c}\mathcal{D}^{\beta}$ denote the right Caputo fractional derivatives of orders α and $\beta \in (1,2]$, respectively. The multifunctions $\mathcal{F}, \mathcal{G} : [0,\mathcal{T}] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ are given set-valued maps, where $\mathcal{P}(\mathbb{R})$ denotes the family of all non-empty subsets of $\mathbb{R}$. The constants $\lambda_1, \lambda_2, r_1, r_2, s_1, s_2, p_1, p_2, q_1, q_2 \in \mathbb{R}$ are real parameters, while $k_1, k_2, u_1, u_2, v_1, v_2, \zeta_1, \zeta_2 \in [0,\mathcal{T}]$ are arbitrary points in the interval.

The boundary conditions (2) represent nonlocal multipoint constraints, where the values of the unknown functions and their derivatives at the right endpoint $\mathcal{T}$ are expressed as linear combinations of their values at several interior points of the interval. Such conditions significantly generalize classical two-point boundary conditions and provide a more flexible framework for modeling real-world phenomena involving memory and spatial interactions. This nonlocal structure plays a crucial role in capturing complex dynamical behavior within the context of fractional differential inclusions.

The main objective of this paper is to establish existence results for the above system under suitable assumptions on the multivalued maps. The analysis is carried out using techniques from nonlinear functional analysis, including the Leray–Schauder nonlinear alternative and the Covitz–Nadler fixed-point theorem for multivalued operators. The obtained results cover both convex and non-convex valued multifunctions, thereby extending several existing results in the literature.

The paper is organized as follows. In Section 2, we recall some basic definitions and preliminary results from fractional calculus and multivalued analysis. Section 3 is devoted to the main results, which are established using fixed-point techniques for multivalued operators. In Section 4, illustrative examples are presented to demonstrate the applicability of the theoretical findings. Finally, concluding remarks are provided in the last section.

2. Preliminaries

In this section, we recall some basic concepts from multivalued analysis [1–3].

Let $(X, \|\cdot\|)$ be a normed space. Denote

$$\mathcal{P}_{cl}(X) = \{A \in X : A \text{ is closed}\}, \quad \mathcal{P}_{cp,c}(X) = \{A \in X : A \text{ is compact and convex}\}.$$

A multivalued map $G : X \to \mathcal{P}(X)$ is said to be:

(i) Convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$;
(ii) Upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$, $G(x_0)$ is a non-empty closed subset of X, and for every open set N containing $G(x_0)$, there exists a neighborhood U of x_0 such that $G(U) \subseteq N$;
(iii) Lower semicontinuous (l.s.c.) if for every open set $B \subset X$, the set

$$\{y \in X : G(y) \cap B \neq \emptyset\}$$

is open in X;
(iv) Completely continuous if $G(B)$ is relatively compact for every bounded set $B \subset X$.

Remark 1 ([24,25]). *If the multivalued map G is completely continuous with non-empty compact values, then G is u.s.c. if and only if G has a closed graph; i.e., $x_n \to x$, $y_n \to y$, $y_n \in G(x_n)$ imply $y \in G(x)$; the set*

$$Gr(G) = \{(x, y) \in X \times Y : y \in G(x)\}$$

defines the graph of G.

Definition 1 ([26,27]). *A multivalued map $G : [a, b] \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if the function*

$$t \mapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable for every $y \in \mathbb{R}$.

Definition 2 ([24,27,28]). *A multivalued map $F : [a, b] \times \mathbb{R}^2 \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if:*

1. *$t \mapsto F(t, x, y)$ is measurable for all $x, y \in \mathbb{R}$;*
2. *$(x, y) \mapsto F(t, x, y)$ is upper semicontinuous for almost all $t \in [a, b]$.*

Definition 3 ([24,28]). *A Carathéodory multifunction F is called L^1-Carathéodory if for each $\rho > 0$, there exists $\varphi_\rho \in L^1([a, b], \mathbb{R}^+)$ such that*

$$\|F(t, x, y)\| = \sup\{|v| : v \in F(t, x, y)\} \leq \varphi_\rho(t)$$

for all $x, y \in \mathbb{R}$ with $\|x\|, \|y\| \leq \rho$ and for a.e. $t \in [a, b]$.

Definition 4 ([2]). *Let $\alpha_1 > 0$ and $f : [0, \infty) \to \mathbb{R}$ be a measurable function. The Riemann–Liouville fractional integral of order α_1 is defined by*

$$I^{\alpha_1} f(t) = \frac{1}{\Gamma(\alpha_1)} \int_0^t (t - s)^{\alpha_1 - 1} f(s), ds, \quad t > 0,$$

provided the integral exists. Here, the Gamma function is given by

$$\Gamma(\alpha_1) = \int_0^\infty e^{-t} t^{\alpha_1 - 1}, dt.$$

Definition 5 ([2]). *Let $\alpha_1 > 0$ with $n - 1 < \alpha_1 < n$, $n \in \mathbb{N}$. The Riemann–Liouville fractional derivative of order α_1 is defined by*

$$D_{0^+}^{\alpha_1} f(t) = \frac{1}{\Gamma(n - \alpha_1)} \left(\frac{d}{dt}\right)^n \int_0^t (t - s)^{n - \alpha_1 - 1} f(s), ds,$$

provided that the right-hand side exists.

Definition 6 ([2]). *Let $\alpha_1 > 0$ with $n-1 < \alpha_1 < n$. The Caputo fractional derivative of order α_1 is defined by*

$$ {}^{C}D_{0^+}^{\alpha_1} f(t) = \frac{1}{\Gamma(n-\alpha_1)} \int_0^t (t-s)^{n-\alpha_1-1} f^{(n)}(s), ds, \quad t > 0, $$

whenever $f \in C^n[0,\infty)$.

Definition 7. *A pair of functions $(\mathcal{X}, \mathcal{Y}) \in AC([0,\mathcal{T}],\mathbb{R}) \times AC([0,\mathcal{T}],\mathbb{R})$ is said to be a solution of problem (1) and (2) if there exist functions*

$$ f, g \in L^1([0,\mathcal{T}],\mathbb{R}) $$

such that

$$ f(t) \in \mathcal{F}(t,\mathcal{X}(t),\mathcal{Y}(t)), \quad g(t) \in \mathcal{G}(t,\mathcal{X}(t),\mathcal{Y}(t)) \quad \text{for a.e. } t \in [0,\mathcal{T}], $$

and

$$ {}^{C}D^{\alpha}\mathcal{X}(t) + \lambda_1 {}^{C}D^{\alpha-1}\mathcal{X}(t) = f(t), $$
$$ {}^{C}D^{\beta}\mathcal{Y}(t) + \lambda_2 {}^{C}D^{\beta-1}\mathcal{Y}(t) = g(t), $$

hold for almost every $t \in [0,\mathcal{T}]$, together with the boundary conditions (2).

Lemma 1 (Proposition 1.2 in [24]). *Let $G : X \to \mathcal{P}_{cl}(Y)$ be upper semicontinuous. Then, $\mathrm{Gr}(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{(x_n, y_n)\} \subset \mathrm{Gr}(G)$ with $x_n \to x$ and $y_n \to y$, we have $y \in G(x)$. Conversely, if G is completely continuous and has compact values, then it is upper semicontinuous.*

Lemma 2. *Let X be a separable Banach space. Let $G : [0,T] \times \mathbb{R}^2 \to \mathcal{P}_{cp}(\mathbb{R})$ be an L^1-Carathéodory multivalued map, and let χ be a linear continuous mapping from $L^1([0,T],\mathbb{R})$ to $C([0,T],\mathbb{R})$. Then the operator*

$$ \chi \circ S_{G,x,y} : C([0,T],\mathbb{R}) \to \mathcal{P}_{cp}(C([0,T],\mathbb{R})), $$

defined by $x \mapsto \chi(S_{G,x,y})$, is a closed graph operator in $C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R})$.

Lemma 3 (Nonlinear alternative of Kakutani map [25]). *Let E_1 be a closed convex subset of a Banach space E, and let U be an open subset of E with $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{cp}(E_1)$ is an upper semicontinuous compact map. Then either:*

(i) F has a fixed point in $\overline{U}$;
(ii) There exist $u \in \partial U$ and $\lambda \in (0,1)$ such that $u \in \lambda F(u)$.

Definition 8. *A multivalued operator $\mathcal{G} : X \to \mathcal{P}_{cl}(X)$ is called* (i) γ-Lipschitz *if and only if there exists $\gamma > 0$ such that*

$$ H_d(\mathcal{G}(x),\mathcal{G}(y)) \le \gamma d(x,y), \quad \text{for each } x, y \in X; $$

and (ii) a contraction *if and only if it is γ-Lipschitz with $\gamma < 1$.*

Lemma 4 (Covitz and Nadler [29]). *Let (X,d) be a complete metric space. If $\mathcal{G} : X \to \mathcal{P}_{cl}(X)$ is a contraction, then the set of fixed points of $\mathcal{G}$ is not empty.*

Auxiliary Results

This section examines a system of linear fractional differential equations, analyzing their solutions, boundary conditions, and existence-uniqueness properties,

$$\begin{cases} ({}^c\mathcal{D}^{\alpha} + \lambda_1 {}^c\mathcal{D}^{\alpha-1})\mathcal{X}(t) = \mathcal{H}_1(t), \\ ({}^c\mathcal{D}^{\beta} + \lambda_2 {}^c\mathcal{D}^{\beta-1})\mathcal{Y}(t) = \mathcal{H}_2(t), \\ \mathcal{X}(\mathcal{T}) = r_1\mathcal{Y}(k_1) + r_2\mathcal{T}\mathcal{Y}'(k_2), \qquad \mathcal{T}\mathcal{X}'(\mathcal{T}) = s_1\mathcal{Y}(u_1) + s_2\mathcal{T}\mathcal{Y}'(u_2), \\ \mathcal{Y}(\mathcal{T}) = p_1\mathcal{X}(\zeta_1) + p_2\mathcal{T}\mathcal{X}'(\zeta_2), \qquad \mathcal{T}\mathcal{Y}'(\mathcal{T}) = q_1\mathcal{X}(v_1) + q_2\mathcal{T}\mathcal{Y}'(v_2), \end{cases} \tag{3}$$

supplemented with the boundary conditions (2), with $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{C}[0, \mathcal{T}]$. We denote this by

$$w_1 = e^{-\lambda_1\mathcal{T}}, w_2 = \frac{(1-e^{-\lambda_1\mathcal{T}})}{\lambda_1}, w_3 = \left\{-r_1e^{-\lambda_2k_1} + r_2\mathcal{T}\lambda_2e^{-\lambda_2k_2}\right\}$$
$$w_4 = \left\{-\frac{(1-e^{-\lambda_2k_1})}{\lambda_2} - r_2\mathcal{T}\frac{(\lambda_2e^{-\lambda_2k_2})}{\lambda_2}\right\}, \ w_5 = \mathcal{T}(-\lambda_1e^{-\lambda_1\mathcal{T}}), w_6 = (\lambda_1e^{-\lambda_1\mathcal{T}})$$
$$w_7 = \left\{-s_1e^{-\lambda_2u_1} + s_2\mathcal{T}\lambda_2e^{-\lambda_2u_2}\right\}, w_8 = \left\{-\frac{(1-e^{-\lambda_2u_1})}{\lambda_2} - s_2\mathcal{T}\frac{(\lambda_2e^{-\lambda_2u_2})}{\lambda_2}\right\}$$
$$w_9 = \left[-p_1e^{-\lambda_1\zeta_1} + p_2\mathcal{T}\lambda_1e^{-\lambda_1\zeta_2}\right], w_{10} = \left(-p_1\frac{1}{\lambda_1}(1-e^{-\lambda_1\zeta_1}) - p_2\mathcal{T}\frac{1}{\lambda_1}(\lambda_1e^{-\lambda_1\zeta_2})\right)$$
$$w_{11} = e^{-\lambda_2\mathcal{T}}, w_{12} = \frac{1}{\lambda_2}(1-e^{-\lambda_2\mathcal{T}}), w_{13} = \left[-q_1e^{-\lambda_1(v_1)} + q_2\mathcal{T}e^{-\lambda_1v_2}\right]$$
$$w_{14} = \left[-q_1\frac{1}{\lambda_1}(1-e^{-\lambda_1v}) - q_2\frac{\mathcal{T}}{\lambda_1}(\lambda_1e^{-\lambda_1v_2})\right], w_{15} = \left[-\lambda_2\mathcal{T}e^{-\lambda_2\mathcal{T}}\right], w_{16} = \frac{\mathcal{T}}{\lambda_2}(\lambda_2e^{-\lambda_2\mathcal{T}}),$$

and

$$\begin{aligned}
\mathcal{E}_1 =& r_1\int_0^{k_1} e^{-\lambda_1(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \lambda_2r_2\mathcal{T}\int_0^{t} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds \\
&+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds \\
\mathcal{E}_2 =& s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \lambda_2s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds \\
&+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\mathcal{T}), \\
\mathcal{E}_3 =& p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \lambda_1p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds \\
&+ p_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_1(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds \\
\mathcal{E}_4 =& q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \lambda_1q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds \\
&+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(\mathcal{T}),
\end{aligned}$$

$$\begin{aligned}
\Delta_1 =& (w_4w_7 - w_3w_8)(w_{10}w_{15} - w_{11}w_{14})(w_1w_7 - w_3w_5)(w_{12}w_{15} - w_{11}w_{16}) \\
&- (w_2w_7 - w_3w_6)(w_{12}w_{15} - w_{11}w_{16})(w_9w_{15} - w_{11}w_{13})(w_4w_7 - w_3w_8), \\
\Delta_2 =& (w_4w_5 - w_1w_8)(w_{10}w_{13} - w_9w_{14})(w_{11}w_{13} - w_9w_{15})(w_2w_5 - w_1w_6) \\
&- (w_{12}w_{13} - w_9w_{16})(w_2w_5 - w_1w_6)(w_3w_5 - w_1w_7)(w_{10}w_{13} - w_9w_{14}).
\end{aligned} \tag{4}$$

Lemma 5. *If $\Delta_1, \Delta_2 \neq 0$, then the solution $(\mathcal{X}, \mathcal{Y}) \in \mathcal{C}([0, \mathcal{T}], \mathbb{R})$ of the boundary value problem (1) and (2) is given by*

$$\begin{aligned}\mathcal{X}(t) =& \frac{e^{-\lambda_1 t}}{\Delta_1}\Big[\Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds \\ &+ p_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_1(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\Big\}w_{15} - \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds \\ &- \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds + q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(\mathcal{T})\Big\}w_{11}\Big) \\ &\times (w_4w_7 - w_3w_8)(w_2w_7 - w_3w_6)(w_{12}w_{15} - w_{11}w_{16}) - \Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_1(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds \\ &- \lambda_2 r_2\mathcal{T}\int_0^{t} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds + r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\Big\}w_7 \\ &- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds \\ &+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\mathcal{T})\Big\}w_3\Big) \\ &\times (w_{12}w_{15} - w_{11}w_{16})(w_4w_7 - w_3w_8)(w_{10}w_{15} - w_{11}w_{16})\Big] \\ &+ \frac{1}{\lambda_1}(1-e^{-\lambda_1 t})\frac{1}{\Delta_1}\Big[\Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds \\ &+ p_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_1(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\Big\}w_{15} - \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds \\ &- \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds + q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(\mathcal{T})\Big\}w_{11}\Big) \\ &\times (w_4w_7 - w_3w_8)(w_1w_7 - w_3w_5)(w_{12}w_{15} - w_{11}w_{16}) \\ &- \Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_1(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \lambda_2 r_2\mathcal{T}\int_0^{t} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds \\ &+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\Big\}w_7 \\ &- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds \\ &+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\mathcal{T})\Big\}w_3\Big) \\ &\times (w_{12}w_{15} - w_{11}w_{16})(w_9w_{15} - w_{11}w_{13})(w_4w_7 - w_3w_8)\Big] + \int_0^t e^{-\lambda_1(t-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds, \end{aligned} \tag{5}$$

and

$$\begin{aligned}\mathcal{Y}(t) =& \frac{e^{-\lambda_2 t}}{\Delta_2}\Big[\Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_1(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \lambda_2 r_2\mathcal{T}\int_0^{t} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds \\ &+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\Big\}w_5 \\ &- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds \\ &+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\mathcal{T})\Big\}w_1\Big) \\ &\times (w_{10}w_{13} - w_9w_{14})(w_{12}w_{13} - w_9w_{16})(w_2w_5 - w_1w_6) \\ &- \Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds \\ &+ p_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_1(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\Big\}w_{13} \end{aligned} \tag{6}$$

$$
\begin{aligned}
&- \Big\{ q_1 \int_0^{v_1} e^{-\lambda_1(v_1-s)} \mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \lambda_1 q_2 \mathcal{T} \int_0^{v_2} e^{-\lambda_1(v_2-s)} \mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds \\
&+ q_2 \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(v_2) + \lambda_2 \mathcal{T} \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)} \mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(\mathcal{T}) \Big\} w_9 \Big) \\
&\times (w_2w_5 - w_1w_6)(w_4w_5 - w_1w_8)(w_{10}w_{13} - w_9w_{14}) \Big] \\
&+ \frac{1}{\lambda_2}(1 - e^{-\lambda_2 t}) \frac{1}{\Delta_2} \Big[\Big(\Big\{ r_1 \int_0^{k_1} e^{-\lambda_1(k_1-s)} \mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \lambda_2 r_2 \mathcal{T} \int_0^{t} e^{-\lambda_2(k_2-s)} \mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds \\
&+ r_2 \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)} \mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds \Big\} w_5 \\
&- \Big\{ s_1 \int_0^{u_1} e^{-\lambda_2(u_1-s)} \mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \lambda_2 s_2 \mathcal{T} \int_0^{u_2} e^{-\lambda_2(u_2-s)} \mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds \\
&+ s_2 \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(u_2) + \lambda_1 \mathcal{T} \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)} \mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\mathcal{T}) \Big\} w_1 \Big) \\
&\times (w_{10}w_{13} - w_9w_{14})(w_{11}w_{13} - w_9w_{15})(w_2w_5 - w_1w_6) \\
&- \Big(\Big\{ p_1 \int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)} \mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \lambda_1 p_2 \mathcal{T} \int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)} \mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds \\
&+ p_2 \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_1(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)} \mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds \Big\} w_{13} \\
&- \Big\{ q_1 \int_0^{v_1} e^{-\lambda_1(v_1-s)} \mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \lambda_1 q_2 \mathcal{T} \int_0^{v_2} e^{-\lambda_1(v_2-s)} \mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds \\
&+ q_2 \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(v_2) + \lambda_2 \mathcal{T} \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)} \mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(\mathcal{T}) \Big\} w_9 \Big) \\
&\times (w_2w_5 - w_1w_6)(w_3w_5 - w_1w_7)(w_{10}w_{13} - w_9w_{14}) \Big] + \int_0^t e^{-\lambda_2(t-s)} \mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds.
\end{aligned}
$$

Proof. The system (3) can be equivalently written as the following system

$$
\begin{cases}
{}^{C}D^{\alpha}(\mathcal{X}(t) + \lambda_1 {}^{C}D^{-1}\mathcal{X}(t)) = \mathcal{H}_1(t), \\
{}^{C}D^{\beta}(\mathcal{X}(t) + \lambda_1 {}^{C}D^{-1}\mathcal{X}(t)) = \mathcal{H}_2(t),
\end{cases} \tag{7}
$$

where ${}^{C}D^{-1}$ is the integral operator I. The general solution of system (7) is

$$
\begin{cases}
(\mathcal{X}(t) + \lambda_1 {}^{C}D^{-1}\mathcal{X}(t)) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}\mathcal{H}_1(s)\,ds + c_0 + c_1 t, \\
(\mathcal{X}(t) + \lambda_1 {}^{C}D^{-1}\mathcal{X}(t)) = \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (t-s)^{\beta-1}\mathcal{H}_2(s)\,ds + d_0 + d_1 t.
\end{cases}
$$

Or equivalently,

$$
\begin{cases}
\mathcal{X}(t) = -\lambda_1 \displaystyle\int_0^t \mathcal{X}(s)\,ds + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}\mathcal{H}_1(s)\,ds + c_0 + c_1 t, \\
\mathcal{Y}(t) = -\lambda_2 \displaystyle\int_0^t y(s)\,ds + \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (t-s)^{\beta-1}\mathcal{H}_2(s)\,ds + d_0 + d_1 t.
\end{cases}
$$

With $c_i, d_i \in \mathbb{R}$, $i = 0, 1$. We differentiate the above relations and find

$$
\begin{cases}
x'(t) = -\lambda_1 \mathcal{X}(t) + \dfrac{1}{\Gamma(\alpha-1)} \displaystyle\int_0^t (t-s)^{\alpha-2}\mathcal{H}_1(s)\,ds + c_1, \\
y'(t) = -\lambda_2 \mathcal{Y}(t) + \dfrac{1}{\Gamma(\beta-1)} \displaystyle\int_0^t (t-s)^{\beta-2}\mathcal{H}_2(s)\,ds + d_1.
\end{cases}
$$

And then

$$\begin{cases} \left(e^{\lambda_1 t}\mathcal{X}(t)\right)' = \dfrac{e^{\lambda_1 t}}{\Gamma(\alpha-1)}\displaystyle\int_0^t (t-s)^{\alpha-2}\mathcal{H}_1(s)\,ds + c_1 e^{\lambda_1 t} + c_0, \\ \left(e^{\lambda_2 t}\mathcal{Y}(t)\right)' = \dfrac{e^{\lambda_2 t}}{\Gamma(\beta-1)}\displaystyle\int_0^t (t-s)^{\beta-2}\mathcal{H}_2(s)\,ds + d_1 e^{\lambda_2 t} + d_0. \end{cases}$$

We integrate now these relations and we obtain

$$\begin{cases} \mathcal{X}(t) = c_0 e^{-\lambda_1 t} + \dfrac{c_1}{\lambda_1}(1-e^{-\lambda_1 t}) + \displaystyle\int_0^t e^{-\lambda_1(t-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds, \\ \mathcal{Y}(t) = d_0 e^{-\lambda_2 t} + \dfrac{d_1}{\lambda_2}(1-e^{-\lambda_2 t}) + \displaystyle\int_0^t e^{-\lambda_2(t-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds, \end{cases} \tag{8}$$

We differentiate the system (8) and we find

$$\begin{cases} \mathcal{X}'(t) = -\lambda_1 c_0 e^{-\lambda_1 t} + \dfrac{c_1}{\lambda_1}(\lambda_1 e^{-\lambda_1 t}) - \lambda_1\displaystyle\int_0^t e^{-\lambda_1(t-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds + \mathcal{I}^{\alpha-1}\mathcal{H}_1(t), \\ \mathcal{Y}'(t) = -\lambda_2 d_0 e^{-\lambda_2 t} + \dfrac{d_1}{\lambda_2}(\lambda_2 e^{-\lambda_2 t}) - \lambda_2\displaystyle\int_0^t e^{-\lambda_2(t-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds + \mathcal{I}^{\beta-1}\mathcal{H}_2(t), \end{cases} \tag{9}$$

where c_0, c_1, d_0 and d_1 are unknown arbitrary constants. Using the nonlocal closed boundary conditions, $\mathcal{X}(\mathcal{T}) = r_1\mathcal{Y}(k_1) + r_2\mathcal{T}\mathcal{Y}'(k_2)$, $\mathcal{T}\mathcal{X}'(\mathcal{T}) = s_1\mathcal{Y}(u_1) + s_2\mathcal{T}\mathcal{Y}'(u_2)$, $\mathcal{Y}(\mathcal{T}) = p_1\mathcal{X}(\zeta_1) + p_2\mathcal{T}\mathcal{X}'(\zeta_2)$, and $\mathcal{T}\mathcal{Y}'(\mathcal{T}) = q_1\mathcal{X}(v_1) + q_2\mathcal{T}\mathcal{Y}'(v_2)$, we find a system of algebraic equations in c_0, c_1, d_0 and d_1:

$$\begin{cases} w_1\mathcal{C}_0 + w_2\mathcal{C}_1 + w_3\mathcal{D}_0 + w_4\mathcal{D}_1 - \mathcal{E}_1 = 0, \\ w_5\mathcal{C}_0 + w_6\mathcal{C}_1 + w_7\mathcal{D}_0 + w_8\mathcal{D}_1 - \mathcal{E}_2 = 0, \\ w_9\mathcal{C}_0 + w_{10}\mathcal{C}_1 + w_{11}\mathcal{D}_0 + w_{12}\mathcal{D}_1 - \mathcal{E}_3 = 0, \\ w_{13}\mathcal{C}_0 + w_{14}\mathcal{C}_1 + w_{15}\mathcal{D}_0 + w_{16}\mathcal{D}_1 - \mathcal{E}_4 = 0 \end{cases} \tag{10}$$

$$\begin{aligned} c_0 =& \frac{1}{\Delta_1}\Big((\mathcal{E}_3 w_{15} - \mathcal{E}_4 w_{11})(w_4 w_7 - w_3 w_8)(w_2 w_7 - w_3 w_6)(w_{12} w_{15} - w_{11} w_{16}) \\ & - (\mathcal{E}_1 w_7 - \mathcal{E}_2 w_3)(w_{12} w_{15} - w_{11} w_{16})(w_4 w_7 - w_3 w_8)(w_{10} w_{15} - w_{11} w_{16})\Big) \\ c_1 =& \frac{1}{\Delta_1}\Big((\mathcal{E}_3 w_{15} - \mathcal{E}_4 w_{11})(w_4 w_7 - w_3 w_8)(w_1 w_7 - w_3 w_5)(w_{12} w_{15} - w_{11} w_{16}) \\ & - (\mathcal{E}_1 w_7 - \mathcal{E}_2 w_3)(w_{12} w_{15} - w_{11} w_{16})(w_9 w_{15} - w_{11} w_{13})(w_4 w_7 - w_3 w_8)\Big) \\ d_0 =& \frac{1}{\Delta_2}\Big((\mathcal{E}_1 w_5 - \mathcal{E}_2 w_1)(w_{10} w_{13} - w_9 w_{14})(w_{12} w_{13} - w_9 w_{16})(w_2 w_5 - w_1 w_6) \\ & - (\mathcal{E}_3 w_{13} - \mathcal{E}_4 w_9)(w_2 w_5 - w_1 w_6)(w_4 w_5 - w_1 w_8)(w_{10} w_{13} - w_9 w_{14})\Big) \\ d_1 =& \frac{1}{\Delta_2}\Big((\mathcal{E}_1 w_5 - \mathcal{E}_2 w_1)(w_{10} w_{13} - w_9 w_{14})(w_{11} w_{13} - w_9 w_{15})(w_2 w_5 - w_1 w_6) \\ & - (\mathcal{E}_3 w_{13} - \mathcal{E}_4 w_9)(w_2 w_5 - w_1 w_6)(w_3 w_5 - w_1 w_7)(w_{10} w_{13} - w_9 w_{14})\Big). \end{aligned}$$

Substituting the values of c_0, c_1, d_0, and d_1 in (8) yields the solution (5) and (6). This completes the proof. □

3. Main Results

Let $\mathcal{T} > 0$ be given and denote by $\mathcal{C}([0,\mathcal{T}])$ the Banach space of all continuous functions defined on $[0,\mathcal{T}]$ with values in $\mathbb{R}$, equipped with the supremum norm.

Define the spaces

$$\Theta_1 = \{\mathcal{X}(t) \ : \ \mathcal{X} \in \mathcal{C}([0,\mathcal{T}])\}, \quad \Theta_2 = \{\mathcal{Y}(t) \ : \ \mathcal{Y} \in \mathcal{C}([0,\mathcal{T}])\},$$

endowed with the norms

$$\|\mathcal{X}\| = \max_{t\in[0,\mathcal{T}]} |\mathcal{X}(t)|, \quad \|\mathcal{Y}\| = \max_{t\in[0,\mathcal{T}]} |\mathcal{Y}(t)|.$$

It is well known that $(\Theta_1, \|\cdot\|)$ and $(\Theta_2, \|\cdot\|)$ are Banach spaces.

Consequently, the product space $(\Theta_1 \times \Theta_2, \|\cdot\|)$ is also a Banach space when equipped with the norm

$$\|(\mathcal{X},\mathcal{Y})\| = \|\mathcal{X}\| + \|\mathcal{Y}\|.$$

For each pair $(\mathcal{X},\mathcal{Y}) \in \Theta_1 \times \Theta_2$, we define the set of selections for the functions $\mathcal{F}$ and $\mathcal{G}$ by

$$\mathcal{S}_{\mathcal{F},(\mathcal{X},\mathcal{Y})} = \{\mathcal{H}_1 \in \mathcal{L}^1(\mathcal{J},\mathbb{R}) : \mathcal{H}_1(t) \in \mathcal{F}(t,\mathcal{X}(t),\mathcal{Y}(t)),\ for\ a.e\ t \in \mathcal{J}\}$$
$$\mathcal{S}_{\mathcal{G},(\mathcal{X},\mathcal{Y})} = \{\mathcal{H}_2 \in \mathcal{L}^1(\mathcal{J},\mathbb{R}) : \mathcal{H}_2(t) \in \mathcal{G}(t,\mathcal{X}(t),\mathcal{Y}(t)),\ for\ a.e\ t \in \mathcal{J}\}$$

which describe the sets of $\mathcal{F},\mathcal{G}$ selections for each $(\mathcal{X},\mathcal{Y}) \in \Theta_1 \times \Theta_2$.

By implementing Lemma 5, the following operators $\mathcal{K}_1, \mathcal{K}_2 : \Theta_1 \times \Theta_2 \to \mathcal{P}(\Theta_1 \times \Theta_2)$ are given by

$$\mathcal{K}_1(\mathcal{X},\mathcal{Y})(t) = \{g_1 \in \Theta_1 \times \Theta_2 : \exists\ \mathcal{H}_1 \in \mathcal{S}_{\mathcal{F},(\mathcal{X},\mathcal{Y})}, \mathcal{H}_2 \in \mathcal{S}_{\mathcal{G},(\mathcal{X},\mathcal{Y})} \ \ni \ g_1(\mathcal{X},\mathcal{Y})(t) = P_1(\mathcal{X},\mathcal{Y})(t),\ \forall\ t \in \mathcal{J}\}, \tag{11}$$
$$\mathcal{K}_2(\mathcal{X},\mathcal{Y})(t) = \{g_2 \in \Theta_1 \times \Theta_2 : \exists\ \mathcal{H}_1 \in \mathcal{S}_{\mathcal{F},(\mathcal{X},\mathcal{Y})}, \mathcal{H}_2 \in \mathcal{S}_{\mathcal{G},(\mathcal{X},\mathcal{Y})} \ \ni \ g_2(\mathcal{X},\mathcal{Y})(t) = P_2(\mathcal{X},\mathcal{Y})(t),\ \forall\ t \in \mathcal{J}\}, \tag{12}$$

Following that, the operator $\Lambda : \Theta_1 \times \Theta_2 \to \mathcal{P}(\Theta_1 \times \Theta_2)$ is described by

$$\Lambda(\mathcal{X},\mathcal{Y})(t) = \begin{bmatrix} \mathcal{K}_1(\mathcal{X},\mathcal{Y})(t) \\ \mathcal{K}_2(\mathcal{X},\mathcal{Y})(t) \end{bmatrix},$$

where $\mathcal{K}_1$ and $\mathcal{K}_2$ are defined in (11) and (3), respectively.

$$\begin{aligned}
&P_1(\mathcal{X},\mathcal{Y})(t) \qquad\qquad (13)\\
=&\frac{e^{-\lambda_1 t}}{\Delta_1}\Big[\Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\\
&+ p_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_1(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\Big\}w_{15} - \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\\
&- \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds + q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(\mathcal{T})\Big\}w_{11}\Big)\\
&\times(w_4w_7 - w_3w_8)(w_2w_7 - w_3w_6)(w_{12}w_{15} - w_{11}w_{16}) - \Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_1(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\\
&- \lambda_2 r_2\mathcal{T}\int_0^{t} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds + r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\Big\}w_7\\
&- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\\
&+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\mathcal{T})\Big\}w_3\Big)\\
&\times(w_{12}w_{15} - w_{11}w_{16})(w_4w_7 - w_3w_8)(w_{10}w_{15} - w_{11}w_{16})\Big]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\lambda_1}(1-e^{-\lambda_1 t})\frac{1}{\Delta_1}\Big[\Big(\Big\{p_1\int_0^{\zeta_1}e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds-\lambda_1 p_2\mathcal{T}\int_0^{\zeta_2}e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\\
&+p_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_1(\zeta_2)-\int_0^{\mathcal{T}}e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\Big\}w_{15}-\Big\{q_1\int_0^{v_1}e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\\
&-\lambda_1 q_2\mathcal{T}\int_0^{v_2}e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds+q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(v_2)+\lambda_2\mathcal{T}\int_0^{\mathcal{T}}e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds-\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(\mathcal{T})\Big\}w_{11}\Big)\\
&\times(w_4w_7-w_3w_8)(w_1w_7-w_3w_5)(w_{12}w_{15}-w_{11}w_{16})\\
&-\Big(\Big\{r_1\int_0^{k_1}e^{-\lambda_1(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds-\lambda_2 r_2\mathcal{T}\int_0^{t}e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\\
&+r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(k_2)-\int_0^{\mathcal{T}}e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\Big\}w_7\\
&-\Big\{s_1\int_0^{u_1}e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds-\lambda_2 s_2\mathcal{T}\int_0^{u_2}e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\\
&+s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(u_2)+\lambda_1\mathcal{T}\int_0^{\mathcal{T}}e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds-\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\mathcal{T})\Big\}w_3\Big)\\
&\times(w_{12}w_{15}-w_{11}w_{16})(w_9w_{15}-w_{11}w_{13})(w_4w_7-w_3w_8)\Big]+\int_0^{t}e^{-\lambda_1(t-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds,
\end{aligned}
$$

$$
\begin{aligned}
&P_2(\mathcal{X},\mathcal{Y})(t) \qquad (14)\\
&=\frac{e^{-\lambda_2 t}}{\Delta_2}\Big[\Big(\Big\{r_1\int_0^{k_1}e^{-\lambda_1(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds-\lambda_2 r_2\mathcal{T}\int_0^{t}e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\\
&+r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(k_2)-\int_0^{\mathcal{T}}e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\Big\}w_5\\
&-\Big\{s_1\int_0^{u_1}e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds-\lambda_2 s_2\mathcal{T}\int_0^{u_2}e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\\
&+s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(u_2)+\lambda_1\mathcal{T}\int_0^{\mathcal{T}}e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds-\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\mathcal{T})\Big\}w_1\Big)\\
&\times(w_{10}w_{13}-w_9w_{14})(w_{12}w_{13}-w_9w_{16})(w_2w_5-w_1w_6)\\
&-\Big(\Big\{p_1\int_0^{\zeta_1}e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds-\lambda_1 p_2\mathcal{T}\int_0^{\zeta_2}e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\\
&+p_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_1(\zeta_2)-\int_0^{\mathcal{T}}e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\Big\}w_{13}\\
&-\Big\{q_1\int_0^{v_1}e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds-\lambda_1 q_2\mathcal{T}\int_0^{v_2}e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\\
&+q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(v_2)+\lambda_2\mathcal{T}\int_0^{\mathcal{T}}e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds-\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(\mathcal{T})\Big\}w_9\Big)\\
&\times(w_2w_5-w_1w_6)(w_4w_5-w_1w_8)(w_{10}w_{13}-w_9w_{14})\Big]\\
&+\frac{1}{\lambda_2}(1-e^{-\lambda_2 t})\frac{1}{\Delta_2}\Big[\Big(\Big\{r_1\int_0^{k_1}e^{-\lambda_1(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds-\lambda_2 r_2\mathcal{T}\int_0^{t}e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\\
&+r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(k_2)-\int_0^{\mathcal{T}}e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\Big\}w_5\\
&-\Big\{s_1\int_0^{u_1}e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds-\lambda_2 s_2\mathcal{T}\int_0^{u_2}e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\\
&+s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(u_2)+\lambda_1\mathcal{T}\int_0^{\mathcal{T}}e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds-\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\mathcal{T})\Big\}w_1\Big)\\
&\times(w_{10}w_{13}-w_9w_{14})(w_{11}w_{13}-w_9w_{15})(w_2w_5-w_1w_6)\\
&-\Big(\Big\{p_1\int_0^{\zeta_1}e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds-\lambda_1 p_2\mathcal{T}\int_0^{\zeta_2}e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds
\end{aligned}
$$

$$+ p_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_1(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\Big\}w_{13}$$
$$- \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds$$
$$+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(\mathcal{T})\Big\}w_9\Big)$$
$$\times (w_2w_5 - w_1w_6)(w_3w_5 - w_1w_7)(w_{10}w_{13} - w_9w_{14})\Big] + \int_0^t e^{-\lambda_2(t-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds.$$

For the sake of computational convenience, we set

$$\begin{aligned}
\Upsilon_1 =& \frac{e^{-\lambda_1 t}}{\Delta_1}\Big\{\Big(\Big[p_1\frac{\zeta_1^\alpha}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1\zeta_1}) + \lambda_1 p_2\mathcal{T}\frac{\zeta_2^\alpha}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1\zeta_2}) + p_2\mathcal{T}\frac{\zeta_2^{\alpha-1}}{\Gamma(\alpha)}\Big]w_2\\
&+ \Big[q_1\frac{v_1^\alpha}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1 v_1}) + \lambda_1 q_2\mathcal{T}\frac{v_2^\alpha}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1 v_2}) + q_2\mathcal{T}\frac{v_2^{\alpha-1}}{\lambda_1\Gamma(\alpha)}\Big]w_{11}\Big)\\
&\times\Big[(w_4w_7 - w_3w_8)(w_2w_7 - w_3w_6)(w_{12}w_{15} - w_{11}w_{16})\Big]\\
&+\Big(\Big[\frac{\mathcal{T}^\alpha}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1\mathcal{T}})\Big]w_7 + \Big[\lambda_1\mathcal{T}\frac{\mathcal{T}^\alpha}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1\mathcal{T}}) + \mathcal{T}\frac{\mathcal{T}^{\alpha-1}}{\Gamma(\alpha)}\Big]w_3\Big)\\
&\times\Big[(w_{12}w_{15} - w_{11}w_{16})(w_4w_7 - w_3w_8)(w_{10}w_{15} - w_{11}w_{16})\Big]\Big\}\\
&+\frac{1}{\lambda_1}(1-e^{-\lambda_1 t})\frac{1}{\Delta_1}\Big\{\Big(\Big[p_1\frac{\zeta_1^\alpha}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1\zeta_1}) + \lambda_1 p_2\mathcal{T}\frac{\zeta_2^\alpha}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1\zeta_2}) + p_2\mathcal{T}\frac{\zeta_2^{\alpha-1}}{\Gamma(\alpha)}\Big]w_2\\
&+ \Big[q_1\frac{v_1^\alpha}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1 v_1}) + \lambda_1 q_2\mathcal{T}\frac{v_2^\alpha}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1 v_2}) + q_2\mathcal{T}\frac{v_2^{\alpha-1}}{\lambda_1\Gamma(\alpha)}\Big]w_{11}\Big)\\
&\times\Big[(w_4w_7 - w_3w_8)(w_1w_7 - w_3w_5)(w_{12}w_{15} - w_{11}w_{16})\Big]\\
&+\Big(\Big[\frac{\mathcal{T}^\alpha}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1\mathcal{T}})\Big]w_7 + \Big[\lambda_1\mathcal{T}\frac{\mathcal{T}^\alpha}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1\mathcal{T}}) + \mathcal{T}\frac{\mathcal{T}^{\alpha-1}}{\Gamma(\alpha)}\Big]w_3\Big)\\
&\times\Big[(w_{12}w_{15} - w_{11}w_{16})(w_9w_{15} - w_{11}w_{13})(w_4w_7 - w_3w_8)\Big]\Big\} + \frac{1}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1}),
\end{aligned}$$

$$\begin{aligned}
\Upsilon_2 =& \frac{e^{-\lambda_1 t}}{\Delta_1}\Big\{\Big(\Big[\frac{\mathcal{T}^\beta}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2\mathcal{T}})\Big]w_2 + \Big[\lambda_2\mathcal{T}\frac{\mathcal{T}^\beta}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2\mathcal{T}}) + \mathcal{T}\frac{\mathcal{T}^{\beta-1}}{\Gamma(\beta)}\Big]w_{11}\Big)\\
&\times\Big[(w_4w_7 - w_3w_8)(w_2w_7 - w_3w_6)(w_{12}w_{15} - w_{11}w_{16})\Big]\\
&+\Big(\Big[r_1\frac{k_1^\beta}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2 k_1}) + \lambda_2 r_2\mathcal{T}\frac{k_2^\beta}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2 k_2}) + r_2\mathcal{T}\frac{k_2^{\beta-1}}{\Gamma(\beta)}\Big]w_7\\
&+\Big[s_1\frac{u_1^\beta}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2 u_1}) + \lambda_2 s_2\mathcal{T}\frac{u_2^\beta}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2 u_2}) + s_2\mathcal{T}\frac{u_2^{\beta-1}}{\Gamma(\beta)}\Big]w_3\Big)\\
&\times\Big[(w_{12}w_{15} - w_{11}w_{16})(w_4w_7 - w_3w_8)(w_{10}w_{15} - w_{11}w_{16})\Big]\Big\}\\
&+\frac{1}{\lambda_1}(1-e^{-\lambda_1 t})\frac{1}{\Delta_1}\Big\{\Big(\Big[\frac{\mathcal{T}^\beta}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2\mathcal{T}})\Big]w_2 + \Big[\lambda_2\mathcal{T}\frac{\mathcal{T}^\beta}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2\mathcal{T}}) + \mathcal{T}\frac{\mathcal{T}^{\beta-1}}{\Gamma(\beta)}\Big]w_{11}\Big)\\
&\times\Big[(w_4w_7 - w_3w_8)(w_1w_7 - w_3w_5)(w_{12}w_{15} - w_{11}w_{16})\Big]\\
&+\Big(\Big[r_1\frac{k_1^\beta}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2 k_1}) + \lambda_1 r_2\mathcal{T}\frac{k_2^\beta}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2 k_2}) + r_2\mathcal{T}\frac{k_2^{\beta-1}}{\Gamma(\beta)}\Big]w_7\\
&+\Big[s_1\frac{u_1^\beta}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2 u_1}) + \lambda_2 s_2\mathcal{T}\frac{u_2^\beta}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2 u_2}) + s_2\mathcal{T}\frac{u_2^{\beta-1}}{\Gamma(\beta)}\Big]w_3\Big)\\
&\times\Big[(w_{12}w_{15} - w_{11}w_{16})(w_9w_{15} - w_{11}w_{13})(w_4w_7 - w_3w_8)\Big]\Big\}
\end{aligned}$$

$$
\begin{aligned}
\Upsilon_3 =& \frac{e^{-\lambda_2 t}}{\Delta_2}\Big\{\Big(\Big[\frac{\mathcal{T}^{\alpha}}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1\mathcal{T}})\Big]w_5+\Big[\lambda_1\mathcal{T}\frac{\mathcal{T}^{\alpha}}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1\mathcal{T}})+\mathcal{T}\frac{\mathcal{T}^{\alpha-1}}{\Gamma(\alpha)}\Big]w_1\Big)\\
&\times\Big[(w_{10}w_{13}-w_9w_{14})(w_{12}w_{13}-w_9w_{16})(w_2w_5-w_1w_6)\Big]\\
&+\Big(\Big[p_1\frac{\zeta_1^{\alpha}}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1\zeta_1})+\lambda_1p_2\mathcal{T}\frac{\zeta_2^{\alpha}}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1\zeta_2})+p_2\mathcal{T}\frac{\zeta_2^{\alpha-1}}{\Gamma(\alpha)}\Big]w_{13}\\
&+\Big[q_1\frac{v_1^{\alpha}}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1v_1})+\lambda_1q_2\mathcal{T}\frac{v_2^{\alpha}}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1v_2})+q_2\mathcal{T}\frac{v_2^{\alpha-1}}{\lambda_1\Gamma(\alpha)}\Big]w_9\Big)\\
&\times\Big[(w_2w_5-w_1w_6)(w_4w_5-w_1w_8)(w_{10}w_{13}-w_9w_{14})\Big]\Big\}\\
&+\frac{1}{\lambda_1}(1-e^{-\lambda_1t})\frac{1}{\Delta_1}\Big\{\Big(\Big[\frac{\mathcal{T}^{\alpha}}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1\mathcal{T}})\Big]w_5+\Big[\lambda_1\mathcal{T}\frac{\mathcal{T}^{\alpha}}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1\mathcal{T}})+\mathcal{T}\frac{\mathcal{T}^{\alpha-1}}{\Gamma(\alpha)}\Big]w_1\Big)\\
&\times\Big[(w_{10}w_{13}-w_9w_{14})(w_{11}w_{13}-w_9w_{15})(w_2w_5-w_1w_6)\Big]\\
&+\Big(\Big[p_1\frac{\zeta_1^{\alpha}}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1\zeta_1})+\lambda_1p_2\mathcal{T}\frac{\zeta_2^{\alpha}}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1\zeta_2})+p_2\mathcal{T}\frac{\zeta_2^{\alpha-1}}{\Gamma(\alpha)}\Big]w_{13}\\
&+\Big[q_1\frac{v_1^{\alpha}}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1v_1})+\lambda_1q_2\mathcal{T}\frac{v_2^{\alpha}}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1v_2})+q_2\mathcal{T}\frac{v_2^{\alpha-1}}{\lambda_1\Gamma(\alpha)}\Big]w_9\Big)\\
&\times\Big[(w_2w_5-w_1w_6)(w_3w_5-w_1w_7)(w_{10}w_{13}-w_9w_{14})\Big]\Big\}+\frac{1}{\lambda_1\Gamma(\alpha)}(1-e^{-\lambda_1}),
\end{aligned}
$$

$$
\begin{aligned}
\Upsilon_4 =& \frac{e^{-\lambda_2 t}}{\Delta_2}\Big\{\Big(\Big[r_1\frac{k_1^{\beta}}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2k_1})+\lambda_1r_2\mathcal{T}\frac{k_2^{\beta}}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2k_2})+r_2\mathcal{T}\frac{k_2^{\beta-1}}{\Gamma(\beta)}\Big]w_5\\
&+\Big[s_1\frac{u_1^{\beta}}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2u_1})+\lambda_2s_2\mathcal{T}\frac{u_2^{\beta}}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2u_2})+s_2\mathcal{T}\frac{u_2^{\beta-1}}{\Gamma(\beta)}\Big]w_1\Big)\\
&\times\Big[(w_{10}w_{13}-w_9w_{14})(w_{12}w_{13}-w_9w_{16})(w_2w_5-w_1w_6)\Big]\\
&+\Big(\Big[\frac{\mathcal{T}^{\beta}}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2\mathcal{T}})\Big]w_{13}+\Big[\lambda_2\mathcal{T}\frac{\mathcal{T}^{\beta}}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2\mathcal{T}})+\mathcal{T}\frac{\mathcal{T}^{\beta-1}}{\Gamma(\beta)}\Big]w_9\Big)\\
&\times\Big[(w_2w_5-w_1w_6)(w_4w_5-w_1w_8)(w_{10}w_{13}-w_9w_{14})\Big]\Big\}\\
&+\frac{1}{\lambda_1}(1-e^{-\lambda_1t})\frac{1}{\Delta_1}\Big\{\Big(\Big[r_1\frac{k_1^{\beta}}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2k_1})+\lambda_1r_2\mathcal{T}\frac{k_2^{\beta}}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2k_2})+r_2\mathcal{T}\frac{k_2^{\beta-1}}{\Gamma(\beta)}\Big]w_5\\
&+\Big[s_1\frac{u_1^{\beta}}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2u_1})+\lambda_2s_2\mathcal{T}\frac{u_2^{\beta}}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2u_2})+s_2\mathcal{T}\frac{u_2^{\beta-1}}{\Gamma(\beta)}\Big]w_1\Big)\\
&\times\Big[(w_{10}w_{13}-w_9w_{14})(w_{11}w_{13}-w_9w_{15})(w_2w_5-w_1w_6)\Big]\\
&+\Big(\Big[\frac{\mathcal{T}^{\beta}}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2\mathcal{T}})\Big]w_{13}+\Big[\lambda_2\mathcal{T}\frac{\mathcal{T}^{\beta}}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2\mathcal{T}})+\mathcal{T}\frac{\mathcal{T}^{\beta-1}}{\Gamma(\beta)}\Big]w_9\Big)\\
&\times\Big[(w_2w_5-w_1w_6)(w_3w_5-w_1w_7)(w_{10}w_{13}-w_9w_{14})\Big]\Big\}+\frac{1}{\lambda_2\Gamma(\beta)}(1-e^{-\lambda_2}).
\end{aligned}
\tag{15}
$$

3.1. The Carathéodory Case

Our first result dealing with convex values $\mathcal{F}$ and $\mathcal{G}$ is proved via the Leray–Schauder nonlinear alternative for multivalued maps [25]

$(\mathcal{W}_1)$ The maps $\mathcal{F},\mathcal{G}:\mathcal{J}\times\mathbb{R}^2\to\mathcal{P}(\mathbb{R})$ are $\mathcal{L}^1$-Carathéodory and have convex values;

$(\mathcal{W}_2)$ There exist continuous nondecreasing functions $\varpi_1,\varpi_2,\varsigma_1,\varsigma_2:[0,\infty)\to[0,\infty)$ and functions $l_1,l_2\in\mathcal{C}(\mathcal{J},\mathbb{R}^+)$, such that

$$\|\mathcal{F}(t,\mathcal{X},\mathcal{Y})\|_{\mathcal{P}} := \sup\{|\mathcal{H}_1| : \mathcal{H}_1 \in \mathcal{F}(t,\mathcal{X},\mathcal{Y})\} \leq l_1(t)[\varpi_1(\|\mathcal{X}\|) + \varsigma_1(\|\mathcal{Y}\|)] \quad \text{for each } (t,\mathcal{X},\mathcal{Y}) \in \mathcal{J}\times\mathbb{R}^2,$$

and

$$\|\mathcal{G}(t,\mathcal{X},\mathcal{Y})\|_{\mathcal{P}} := \sup\{|\mathcal{H}_2| : \mathcal{H}_2 \in \mathcal{G}(t,\mathcal{X},\mathcal{Y})\} \leq l_2(t)[\varpi_2(\|\mathcal{X}\|) + \varsigma_2(\|\mathcal{Y}\|)] \quad \text{for each } (t,\mathcal{X},\mathcal{Y}) \in \mathcal{J}\times\mathbb{R}^2.$$

$(\mathcal{W}_3)$ There exists a positive number $\mathcal{N}$ such that

$$\frac{\mathcal{N}}{(\Upsilon_1+\Upsilon_3)\|l_1\|(\varpi_1(\mathcal{N})+\varsigma_1(\mathcal{N})) + (\Upsilon_2+\Upsilon_4)\|l_2\|(\varpi_2(\mathcal{N})+\varsigma_2(\mathcal{N}))} > 1,$$

where $\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4$ are defined by (15).

$(\mathcal{W}_4)$ $\mathcal{F},\mathcal{G} : \mathcal{J}\times\mathbb{R}^2 \to \mathcal{P}_{cp}(\mathbb{R})$ are such that $\mathcal{F}(\cdot,\mathcal{X},\mathcal{Y}) : \mathcal{J} \to \mathcal{P}_{cp}(\mathbb{R}^2)$ and $\mathcal{G}(\cdot,\mathcal{X},\mathcal{Y}) : \mathcal{J} \to \mathcal{P}_{cp}(\mathbb{R}^2)$ are measurable for each $\mathcal{X},\mathcal{Y} \in \mathbb{R}$.

$(\mathcal{W}_5)$

$$\Pi_d(\mathcal{F}(t,\mathcal{X},\mathcal{Y}),\mathcal{F}(t,\hat{\mathcal{X}},\hat{\mathcal{Y}})) \leq s_1(t)(|\mathcal{X}-\hat{\mathcal{X}}| + |\mathcal{Y}-\hat{\mathcal{Y}}|),$$
$$\Pi_d(\mathcal{G}(t,\mathcal{X},\mathcal{Y}),\mathcal{G}(t,\hat{\mathcal{X}},\hat{\mathcal{Y}})) \leq s_2(t)(|\mathcal{X}-\hat{\mathcal{X}}| + |\mathcal{Y}-\hat{\mathcal{Y}}|),$$

where $\forall\ t \in \mathcal{J}$ and $\mathcal{X},\mathcal{Y},\hat{\mathcal{X}},\hat{\mathcal{Y}} \in \mathbb{R}$ with $s_1, s_2 \in \mathcal{C}(\mathcal{J},\mathbb{R}^+)$ and $d(0,\mathcal{F}(t,0,0)) \leq s_1(t)$, $d(0,\mathcal{G}(t,0,0)) \leq s_2(t)\ \forall\ t \in \mathcal{J}$.

Theorem 1. *Under the assumptions* $(\mathcal{W}_1)$, $(\mathcal{W}_2)$, *and* $(\mathcal{W}_3)$, *the coupled system* (1) *and* (2) *has at least one solution on* $\mathcal{J}$.

Proof. Consider $\mathcal{K}_1,\mathcal{K}_2 : \Theta_1\times\Theta_2 \to \mathcal{P}(\Theta_1\times\Theta_2)$ defined by (11) and (3). From the assumption $(\mathcal{W}_1)$, it follows that the sets $\mathcal{S}_{\mathcal{F},(\mathcal{X},\mathcal{Y})}$ and $\mathcal{S}_{\mathcal{G},(\mathcal{X},\mathcal{Y})}$ are non-empty for each $(\mathcal{X},\mathcal{Y}) \in \Theta_1\times\Theta_2$. Then, for $\mathcal{H}_1 \in \mathcal{S}_{\mathcal{F},(\mathcal{X},\mathcal{Y})}$, $\mathcal{H}_2 \in \mathcal{S}_{\mathcal{G},(\mathcal{X},\mathcal{Y})}$ for $(\mathcal{X},\mathcal{Y}) \in \Theta_1\times\Theta_2$, we have

$$\begin{aligned}
&g_1(\mathcal{X},\mathcal{Y})(t) \qquad (16)\\
=&\frac{e^{-\lambda_1 t}}{\Delta_1}\Big(\Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}(\mathcal{F}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds - \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}(\mathcal{F}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds\\
&+ p_2\mathcal{T}\mathcal{I}^{\alpha-1}(\mathcal{F}(s,\mathcal{X}(s),\mathcal{Y}(s)))(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}(\mathcal{G}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds\Big\}w_{15}\\
&- \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}(\mathcal{F}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds - \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}(\mathcal{F}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds\\
&+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}(\mathcal{F}(s,\mathcal{X}(s),\mathcal{Y}(s)))(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}(\mathcal{G}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds\\
&- \mathcal{T}\mathcal{I}^{\beta-1}(\mathcal{G}(s,\mathcal{X}(s),\mathcal{Y}(s)))(\mathcal{T})\Big\}w_{11}\Big)(w_4w_7 - w_3w_8)(w_2w_7 - w_3w_6)(w_{12}w_{15} - w_{11}w_{16})\\
&- \Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}(\mathcal{G}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds - \lambda_2 r_2\mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}(\mathcal{G}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds
\end{aligned}$$

$$+ r_2\mathcal{T}\mathcal{I}^{\beta-1}(\mathcal{G}(s,\mathcal{X}(s),\mathcal{Y}(s)))(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}(\mathcal{F}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds\Big\}w_7$$
$$- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}(\mathcal{G}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}(\mathcal{G}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds$$
$$+ s_2\mathcal{T}\mathcal{I}^{\beta-1}(\mathcal{G}(s,\mathcal{X}(s),\mathcal{Y}(s)))(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}(\mathcal{F}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds$$
$$- \mathcal{T}\mathcal{I}^{\alpha-1}(\mathcal{F}(s,\mathcal{X}(s),\mathcal{Y}(s)))(\mathcal{T})\Big\}w_3\Big)(w_{12}w_{15}-w_{11}w_{16})(w_4w_7-w_3w_8)(w_{10}w_{15}-w_{11}w_{16})\Big)$$
$$+ \frac{1}{\lambda_1}(1-e^{-\lambda_1 t})\frac{1}{\Delta_1}\Big(\Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}(\mathcal{F}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds$$
$$- \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}(\mathcal{F}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds$$
$$+ p_2\mathcal{T}\mathcal{I}^{\alpha-1}(\mathcal{F}(s,\mathcal{X}(s),\mathcal{Y}(s)))(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}(\mathcal{G}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds\Big\}w_{15}$$
$$- \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}(\mathcal{F}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds - \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}(\mathcal{F}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds$$
$$+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}(\mathcal{F}(s,\mathcal{X}(s),\mathcal{Y}(s)))(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}(\mathcal{G}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds$$
$$- \mathcal{T}\mathcal{I}^{\beta-1}(\mathcal{G}(s,\mathcal{X}(s),\mathcal{Y}(s)))(\mathcal{T})\Big\}w_{11}\Big)(w_4w_7-w_3w_8)(w_1w_7-w_3w_5)(w_{12}w_{15}-w_{11}w_{16})$$
$$- \Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}(\mathcal{G}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds - \lambda_2 r_2\mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}(\mathcal{G}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds$$
$$+ r_2\mathcal{T}\mathcal{I}^{\beta-1}(\mathcal{G}(s,\mathcal{X}(s),\mathcal{Y}(s)))(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}(\mathcal{F}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds\Big\}w_7$$
$$- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}(\mathcal{G}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}(\mathcal{G}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds$$
$$+ s_2\mathcal{T}\mathcal{I}^{\beta-1}(\mathcal{G}(s,\mathcal{X}(s),\mathcal{Y}(s)))(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}(\mathcal{F}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds$$
$$- \mathcal{T}\mathcal{I}^{\alpha-1}(\mathcal{F}(s,\mathcal{X}(s),\mathcal{Y}(s)))(\mathcal{T})\Big\}w_3\Big)(w_{12}w_{15}-w_{11}w_{16})(w_9w_{15}-w_{11}w_{13})(w_4w_7-w_3w_8)\Big)$$
$$+ \int_0^t e^{-\lambda_1(t-s)}\mathcal{I}^{\alpha-1}(\mathcal{F}(s,\mathcal{X}(s),\mathcal{Y}(s)))ds,$$

where $g_1 \in \mathcal{K}_1(\mathcal{X},\mathcal{Y})$, and similarly, we get $g_2 \in \mathcal{K}_2(\mathcal{X},\mathcal{Y})$, and so $(g_1,g_2) \in \mathcal{K}(\mathcal{X},\mathcal{Y})$.

We will establish that the operator $\mathcal{K}$ satisfies the Leray–Schauder nonlinear alternative through a series of steps. Initially, we will demonstrate that $\mathcal{K}(\mathcal{X},\mathcal{Y})$ exhibits a convex valued property. Let $(g_i,\hat{g}_i) \in (\mathcal{K}_1,\mathcal{K}_2)$, $i=1,2$. Then $\exists\ \mathcal{H}_{1i} \in \mathcal{S}_{\mathcal{F},(\mathcal{X},\mathcal{Y})}$, $\mathcal{H}_{2i} \in \mathcal{S}_{\mathcal{G},(\mathcal{X},\mathcal{Y})}$, $i=1,2$, $\ni$ for each $t \in \mathcal{J}$, and we achieve

$$g_i(t) = \frac{e^{-\lambda_1 t}}{\Delta_1}\Big(\Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds - \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds$$
$$+ p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\Big\}w_{15}$$
$$- \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds - \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}[\mathcal{H}_{1i}(s)]ds$$
$$+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds$$
$$- \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(\mathcal{T})\Big\}w_{11}\Big)(w_4w_7-w_3w_8)(w_2w_7-w_3w_6)(w_{12}w_{15}-w_{11}w_{16})$$
$$- \Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds - \lambda_2 r_2\mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds$$

$$
\begin{aligned}
&+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\Big\}w_7\\
&- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&- \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\mathcal{T})\Big\}w_3\Big)(w_{12}w_{15}-w_{11}w_{16})(w_4w_7-w_3w_8)(w_{10}w_{15}-w_{11}w_{16})\Big)\\
&+ \frac{1}{\lambda_1}(1-e^{-\lambda_1 t})\frac{1}{\Delta_1}\Big(\Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds - \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&+ p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\Big\}w_{15}\\
&- \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds - \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&- \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(\mathcal{T})\Big\}w_{11}\Big)(w_4w_7-w_3w_8)(w_1w_7-w_3w_5)(w_{12}w_{15}-w_{11}w_{16})\\
&- \Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds - \lambda_2 r_2\mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\Big\}w_7\\
&- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&- \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\mathcal{T})\Big\}w_3\Big)(w_{12}w_{15}-w_{11}w_{16})(w_9w_{15}-w_{11}w_{13})(w_4w_7-w_3w_8)\Big)\\
&+ \int_0^t e^{-\lambda_1(t-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{g}_i(t) =& \frac{e^{-\lambda_2 t}}{\Delta_2}\Big(\Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds - \lambda_2 r_2\mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\Big\}w_5\\
&- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds - \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\mathcal{T})\Big\}w_1\Big)\\
&\times (w_{10}w_{13}-w_9w_{14})(w_{12}w_{13}-w_9w_{16})(w_2w_5-w_1w_6)\\
&- \Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds - \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&+ p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\Big\}w_{13}\\
&- \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds - \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds - \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(\mathcal{T})\Big\}w_9\Big)\\
&\times (w_2w_5-w_1w_6)(w_4w_5-w_1w_8)(w_{10}w_{13}-w_9w_{14})\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\lambda_2}(1-e^{-\lambda_2 t})\frac{1}{\Delta_2}\Big(\Big(\Big\{r_1\int_0^{k_1}e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds-\lambda_2 r_2\mathcal{T}\int_0^{k_2}e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&+r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(k_2)-\int_0^{\mathcal{T}}e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\Big\}w_5\\
&-\Big\{s_1\int_0^{u_1}e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds-\lambda_2 s_2\mathcal{T}\int_0^{u_2}e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&+s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(u_2)+\lambda_1\mathcal{T}\int_0^{\mathcal{T}}e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds-\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\mathcal{T})\Big\}w_1\Big)\\
&\times(w_{10}w_{13}-w_9w_{14})(w_{11}w_{13}-w_9w_{15})(w_2w_5-w_1w_6)\\
&-\Big(\Big\{p_1\int_0^{\zeta_1}e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds-\lambda_1 p_2\mathcal{T}\int_0^{\zeta_2}e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&+p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\zeta_2)-\int_0^{\mathcal{T}}e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\Big\}w_{13}\\
&-\Big\{q_1\int_0^{v_1}e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds-\lambda_1 q_2\mathcal{T}\int_0^{v_2}e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&+q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(v_2)+\lambda_2\mathcal{T}\int_0^{\mathcal{T}}e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&-\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(\mathcal{T})\Big\}w_9\Big)\times(w_2w_5-w_1w_6)(w_3w_5-w_1w_7)(w_{10}w_{13}-w_9w_{14})\Big)\\
&+\int_0^t e^{-\lambda_2(t-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds.
\end{aligned}
$$

Let $0\le\nu\le 1$. Then, for each $t\in\mathcal{J}$, we arrive at

$$
\begin{aligned}
&\Big[\nu g_1+(1-\nu)g_2\Big](t)\\
&=\frac{e^{-\lambda_1 t}}{\Delta_1}\Big(\Big(\Big\{p_1\int_0^{\zeta_1}e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(s)+(1-\nu)\mathcal{H}_{12}(s)]ds-\lambda_1 p_2\mathcal{T}\int_0^{\zeta_2}e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(s)+(1-\nu)\mathcal{H}_{12}(s)]ds\\
&+p_2\mathcal{T}\mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(s)+(1-\nu)\mathcal{H}_{12}(s)](\zeta_2)-\int_0^{\mathcal{T}}e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}[\nu\mathcal{H}_{21}(s)+(1-\nu)\mathcal{H}_{22}(s)]ds\Big\}w_{15}\\
&-\Big\{q_1\int_0^{v_1}e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(s)+(1-\nu)\mathcal{H}_{12}(s)]ds-\lambda_1 q_2\mathcal{T}\int_0^{v_2}e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(s)+(1-\nu)\mathcal{H}_{12}(s)]ds\\
&+q_2\mathcal{T}\mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(s)+(1-\nu)\mathcal{H}_{12}(s)](v_2)+\lambda_2\mathcal{T}\int_0^{\mathcal{T}}e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}[\nu\mathcal{H}_{21}(s)+(1-\nu)\mathcal{H}_{22}(s)]ds\\
&-\mathcal{T}\mathcal{I}^{\beta-1}[\nu\mathcal{H}_{21}(s)+(1-\nu)\mathcal{H}_{22}(s)](\mathcal{T})\Big\}w_{11}\Big)(w_4w_7-w_3w_8)(w_2w_7-w_3w_6)(w_{12}w_{15}-w_{11}w_{16})\\
&-\Big(\Big\{r_1\int_0^{k_1}e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}[\nu\mathcal{H}_{21}(s)+(1-\nu)\mathcal{H}_{22}(s)]ds-\lambda_2 r_2\mathcal{T}\int_0^{k_2}e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}[\nu\mathcal{H}_{21}(s)+(1-\nu)\mathcal{H}_{22}(s)]ds\\
&+r_2\mathcal{T}\mathcal{I}^{\beta-1}[\nu\mathcal{H}_{21}(s)+(1-\nu)\mathcal{H}_{22}(s)](k_2)-\int_0^{\mathcal{T}}e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(s)+(1-\nu)\mathcal{H}_{12}(s)]ds\Big\}w_7\\
&-\Big\{s_1\int_0^{u_1}e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}[\nu\mathcal{H}_{21}(s)+(1-\nu)\mathcal{H}_{22}(s)]ds-\lambda_2 s_2\mathcal{T}\int_0^{u_2}e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}[\nu\mathcal{H}_{21}(s)+(1-\nu)\mathcal{H}_{22}(s)]ds\\
&+s_2\mathcal{T}\mathcal{I}^{\beta-1}[\nu\mathcal{H}_{21}(s)+(1-\nu)\mathcal{H}_{22}(s)](u_2)+\lambda_1\mathcal{T}\int_0^{\mathcal{T}}e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(s)+(1-\nu)\mathcal{H}_{12}(s)]ds\\
&-\mathcal{T}\mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(s)+(1-\nu)\mathcal{H}_{12}(s)](\mathcal{T})\Big\}w_3\Big)(w_{12}w_{15}-w_{11}w_{16})(w_4w_7-w_3w_8)(w_{10}w_{15}-w_{11}w_{16})\Big)\\
&+\frac{1}{\lambda_1}(1-e^{-\lambda_1 t})\frac{1}{\Delta_1}\Big(\Big(\Big\{p_1\int_0^{\zeta_1}e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(s)+(1-\nu)\mathcal{H}_{12}(s)]ds
\end{aligned}
$$

$$
\begin{aligned}
&- \lambda_1 p_2 \mathcal{T} \int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)} \mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(s) + (1-\nu)\mathcal{H}_{12}(s)]ds \\
&+ p_2 \mathcal{T}\mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(s) + (1-\nu)\mathcal{H}_{12}(s)](\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)} \mathcal{I}^{\beta-1}[\nu\mathcal{H}_{21}(s) + (1-\nu)\mathcal{H}_{22}(s)]ds \Big\} w_{15} \\
&- \Big\{ q_1 \int_0^{v_1} e^{-\lambda_1(v_1-s)} \mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(s) + (1-\nu)\mathcal{H}_{12}(s)]ds - \lambda_1 q_2 \mathcal{T} \int_0^{v_2} e^{-\lambda_1(v_2-s)} \mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(s) + (1-\nu)\mathcal{H}_{12}(s)]ds \\
&+ q_2 \mathcal{T}\mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(s) + (1-\nu)\mathcal{H}_{12}(s)](v_2) + \lambda_2 \mathcal{T} \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)} \mathcal{I}^{\beta-1}[\nu\mathcal{H}_{21}(s) + (1-\nu)\mathcal{H}_{22}(s)]ds \\
&- \mathcal{T}\mathcal{I}^{\beta-1}[\nu\mathcal{H}_{21}(s) + (1-\nu)\mathcal{H}_{22}(s)](\mathcal{T}) \Big\} w_{11} \Big) (w_4 w_7 - w_3 w_8)(w_1 w_7 - w_3 w_5)(w_{12} w_{15} - w_{11} w_{16}) \\
&- \Big(\Big\{ r_1 \int_0^{k_1} e^{-\lambda_2(k_1-s)} \mathcal{I}^{\beta-1}[\nu\mathcal{H}_{21}(s) + (1-\nu)\mathcal{H}_{22}(s)]ds - \lambda_2 r_2 \mathcal{T} \int_0^{k_2} e^{-\lambda_2(k_2-s)} \mathcal{I}^{\beta-1}[\nu\mathcal{H}_{21}(s) + (1-\nu)\mathcal{H}_{22}(s)]ds \\
&+ r_2 \mathcal{T}\mathcal{I}^{\beta-1}[\nu\mathcal{H}_{21}(s) + (1-\nu)\mathcal{H}_{22}(s)](k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)} \mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(s) + (1-\nu)\mathcal{H}_{12}(s)]ds \Big\} w_7 \\
&- \Big\{ s_1 \int_0^{u_1} e^{-\lambda_2(u_1-s)} \mathcal{I}^{\beta-1}[\nu\mathcal{H}_{21}(s) + (1-\nu)\mathcal{H}_{22}(s)]ds - \lambda_2 s_2 \mathcal{T} \int_0^{u_2} e^{-\lambda_2(u_2-s)} \mathcal{I}^{\beta-1}[\nu\mathcal{H}_{21}(s) + (1-\nu)\mathcal{H}_{22}(s)]ds \\
&+ s_2 \mathcal{T}\mathcal{I}^{\beta-1}[\nu\mathcal{H}_{21}(s) + (1-\nu)\mathcal{H}_{22}(s)](u_2) + \lambda_1 \mathcal{T} \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)} \mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(s) + (1-\nu)\mathcal{H}_{12}(s)]ds \\
&- \mathcal{T}\mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(s) + (1-\nu)\mathcal{H}_{12}(s)](\mathcal{T}) \Big\} w_3 \Big) (w_{12} w_{15} - w_{11} w_{16})(w_9 w_{15} - w_{11} w_{13})(w_4 w_7 - w_3 w_8) \Big) \\
&+ \int_0^t e^{-\lambda_1(t-s)} \mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(s) + (1-\nu)\mathcal{H}_{12}(s)]ds,
\end{aligned}
$$

and since $\mathcal{F}, \mathcal{G}$ are convex valued, we deduce that $\mathcal{S}_{\mathcal{F},(\mathcal{X},\mathcal{Y})}$ and $\mathcal{S}_{\mathcal{G},(\mathcal{X},\mathcal{Y})}$ are convex valued. Obviously, $\nu g_1 + (1-\nu) g_2 \in \mathcal{K}_1$, $\nu \hat{g_1} + (1-\nu)\hat{g_2} \in \mathcal{K}_2$, and hence $\nu(g_1, \hat{g_1}) + (1-\nu)(g_2, \hat{g_2}) \in \mathcal{K}$.

Now we show that $\mathcal{K}$ maps bounded sets into bounded sets in $\Theta_1 \times \Theta_2$. For a non-negative number r, let $\mathcal{B}_r = \{(\mathcal{X}, \mathcal{Y}) \in \Theta_1 \times \Theta_2 : \|\mathcal{X}, \mathcal{Y}\| \leq r\}$ be a bounded set in $\Theta_1 \times \Theta_2$. Then, there exist $\mathcal{H}_1 \in \mathcal{S}_{\mathcal{F},(\mathcal{X},\mathcal{Y})}$, $\mathcal{H}_2 \in \mathcal{S}_{\mathcal{G},(\mathcal{X},\mathcal{Y})}$, and we obtain

$$
\begin{aligned}
&|g_1(\mathcal{X}, \mathcal{Y})(t)| \qquad (17) \\
&= \frac{e^{-\lambda_1 t}}{\Delta_1} \Big(\Big(\Big\{ p_1 \int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)} \mathcal{I}^{\alpha-1} \|l_1\|(\gamma_1(r) + \varsigma_1(r))ds + \lambda_1 p_2 \mathcal{T} \int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)} \mathcal{I}^{\alpha-1} \|l_1\|(\gamma_1(r) + \varsigma_1(r))ds \\
&+ p_2 \mathcal{T}\mathcal{I}^{\alpha-1} \|l_1\|(\gamma_1(r) + \varsigma_1(r))(\zeta_2) + \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)} \mathcal{I}^{\beta-1} \|l_2\|(\gamma_2(r) + \varsigma_2(r))ds \Big\} w_{15} \\
&+ \Big\{ q_1 \int_0^{v_1} e^{-\lambda_1(v_1-s)} \mathcal{I}^{\alpha-1} \|l_1\|(\gamma_1(r) + \varsigma_1(r))ds + \lambda_1 q_2 \mathcal{T} \int_0^{v_2} e^{-\lambda_1(v_2-s)} \mathcal{I}^{\alpha-1} \|l_1\|(\gamma_1(r) + \varsigma_1(r))ds \\
&+ q_2 \mathcal{T}\mathcal{I}^{\alpha-1} \|l_1\|(\gamma_1(r) + \varsigma_1(r))(v_2) + \lambda_2 \mathcal{T} \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)} \mathcal{I}^{\beta-1} \|l_2\|(\gamma_2(r) + \varsigma_2(r))ds \\
&+ \mathcal{T}\mathcal{I}^{\beta-1} \|l_2\|(\gamma_2(r) + \varsigma_2(r))(\mathcal{T}) \Big\} w_{11} \Big) (w_4 w_7 - w_3 w_8)(w_2 w_7 - w_3 w_6)(w_{12} w_{15} - w_{11} w_{16}) \\
&+ \Big(\Big\{ r_1 \int_0^{k_1} e^{-\lambda_2(k_1-s)} \mathcal{I}^{\beta-1} \|l_2\|(\gamma_2(r) + \varsigma_2(r))ds + \lambda_2 r_2 \mathcal{T} \int_0^{k_2} e^{-\lambda_2(k_2-s)} \mathcal{I}^{\beta-1} \|l_2\|(\gamma_2(r) + \varsigma_2(r))ds \\
&+ r_2 \mathcal{T}\mathcal{I}^{\beta-1} \|l_2\|(\gamma_2(r) + \varsigma_2(r))(k_2) + \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)} \mathcal{I}^{\alpha-1} \|l_1\|(\gamma_1(r) + \varsigma_1(r))ds \Big\} w_7 \\
&+ \Big\{ s_1 \int_0^{u_1} e^{-\lambda_2(u_1-s)} \mathcal{I}^{\beta-1} \|l_2\|(\gamma_2(r) + \varsigma_2(r))ds + \lambda_2 s_2 \mathcal{T} \int_0^{u_2} e^{-\lambda_2(u_2-s)} \mathcal{I}^{\beta-1} \|l_2\|(\gamma_2(r) + \varsigma_2(r))ds \\
&+ s_2 \mathcal{T}\mathcal{I}^{\beta-1} \|l_2\|(\gamma_2(r) + \varsigma_2(r))(u_2) + \lambda_1 \mathcal{T} \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)} \mathcal{I}^{\alpha-1} \|l_1\|(\gamma_1(r) + \varsigma_1(r))ds \\
&+ \mathcal{T}\mathcal{I}^{\alpha-1} \|l_1\|(\gamma_1(r) + \varsigma_1(r))(\mathcal{T}) \Big\} w_3 \Big) (w_{12} w_{15} - w_{11} w_{16})(w_4 w_7 - w_3 w_8)(w_{10} w_{15} - w_{11} w_{16}) \Big) \\
&+ \frac{1}{\lambda_1}(1 - e^{-\lambda_1 t}) \frac{1}{\Delta_1} \Big(\Big(\Big\{ p_1 \int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)} \mathcal{I}^{\alpha-1} \|l_1\|(\gamma_1(r) + \varsigma_1(r))ds
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\|l_1\|(\gamma_1(r)+\varsigma_1(r))ds\\
&+p_2\mathcal{T}\mathcal{I}^{\alpha-1}\|l_1\|(\gamma_1(r)+\varsigma_1(r))(\zeta_2)+\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\|l_2\|(\gamma_2(r)+\varsigma_2(r))ds\Big\}w_{15}\\
&+\Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\|l_1\|(\gamma_1(r)+\varsigma_1(r))ds+\lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\|l_1\|(\gamma_1(r)+\varsigma_1(r))ds\\
&+q_2\mathcal{T}\mathcal{I}^{\alpha-1}\|l_1\|(\gamma_1(r)+\varsigma_1(r))(v_2)+\lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\|l_2\|(\gamma_2(r)+\varsigma_2(r))ds\\
&+\mathcal{T}\mathcal{I}^{\beta-1}\|l_2\|(\gamma_2(r)+\varsigma_2(r))(\mathcal{T})\Big\}w_{11}\Big)(w_4w_7-w_3w_8)(w_1w_7-w_3w_5)(w_{12}w_{15}-w_{11}w_{16})\\
&+\Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\|l_2\|(\gamma_2(r)+\varsigma_2(r))ds+\lambda_2 r_2\mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\|l_2\|(\gamma_2(r)+\varsigma_2(r))ds\\
&+r_2\mathcal{T}\mathcal{I}^{\beta-1}\|l_2\|(\gamma_2(r)+\varsigma_2(r))(k_2)+\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\|l_1\|(\gamma_1(r)+\varsigma_1(r))ds\Big\}w_7\\
&+\Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\|l_2\|(\gamma_2(r)+\varsigma_2(r))ds+\lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\|l_2\|(\gamma_2(r)+\varsigma_2(r))ds\\
&+s_2\mathcal{T}\mathcal{I}^{\beta-1}\|l_2\|(\gamma_2(r)+\varsigma_2(r))(u_2)+\lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\|l_1\|(\gamma_1(r)+\varsigma_1(r))ds\\
&+\mathcal{T}\mathcal{I}^{\alpha-1}\|l_1\|(\gamma_1(r)+\varsigma_1(r))(\mathcal{T})\Big\}w_3\Big)(w_{12}w_{15}-w_{11}w_{16})(w_9w_{15}-w_{11}w_{13})(w_4w_7-w_3w_8)\Big)\\
&+\int_0^{t} e^{-\lambda_1(t-s)}\mathcal{I}^{\alpha-1}\|l_1\|(\gamma_1(r)+\varsigma_1(r))ds\\
=&\Upsilon_1\|l_1\|(\gamma_1(r)+\varsigma_1(r))+\Upsilon_2\|l_2\|(\gamma_2(r)+\varsigma_2(r)),
\end{aligned}
$$

and similarly, we get the value of

$$|g_2(\mathcal{X},\mathcal{Y})(t)|\le\Upsilon_1\|l_1\|(\gamma_1(r)+\varsigma_1(r))+\Upsilon_2\|l_2\|(\gamma_2(r)+\varsigma_2(r)).$$

Thus,

$$||g_1(\mathcal{X},\mathcal{Y})(t)||\le\Upsilon_3\|l_1\|(\varpi_1(r)+\varsigma_1(r))+\Upsilon_4\|l_2\|(\varpi_2(r)+\varsigma_2(r)),$$

and

$$||g_2(\mathcal{X},\mathcal{Y})(t)||\le\Upsilon_3\|l_1\|(\varpi_1(r)+\varsigma_1(r))+\Upsilon_4\|l_2\|(\varpi_2(r)+\varsigma_2(r)).$$

Hence, we obtain

$$
\begin{aligned}
\|g_1,g_2\|&=\|g_1(\mathcal{X},\mathcal{Y})\|+\|g_2(\mathcal{X},\mathcal{Y})\|\\
&\le(\Upsilon_1+\Upsilon_3)\|l_1\|(\varpi_1(r)+\varsigma_1(r))+(\Upsilon_2+\Upsilon_4)\|l_2\|(\varpi_2(r)+\varsigma_2(r)).
\end{aligned}
$$

Next, we show that $\mathcal{K}$ is equi-continuous. Let $t_1,t_2\in\mathcal{J}$ with $t_1<t_2$. Then there exists $\mathcal{H}_1\in\mathcal{S}_{\mathcal{F},(\mathcal{X},\mathcal{Y})}$, $\mathcal{H}_2\in\mathcal{S}_{\mathcal{G},(\mathcal{X},\mathcal{Y})}$ such that

$$
\begin{aligned}
&|g_1(\mathcal{X},\mathcal{Y})(t_2)-g_1(\mathcal{X},\mathcal{Y})(t_1)| \qquad (18)\\
\le&\left|\frac{e^{-\lambda_1 t_2}-e^{-\lambda_1 t_1}}{\Delta_1}\right|\Bigg[\Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds+\lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\\
&+p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)(\zeta_2)+\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\Big\}w_{15}
\end{aligned}
$$

$$
\begin{aligned}
&+\Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds+\lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}[\mathcal{H}_1(s)]ds\\
&+q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)(v_2)+\lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\\
&+\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)(\mathcal{T})\Big\}w_{11}\Big)(w_4w_7-w_3w_8)(w_2w_7-w_3w_6)(w_{12}w_{15}-w_{11}w_{16})\\
&+\Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds-\lambda_2 r_2\mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\\
&+r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)(k_2)+\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\Big\}w_7\\
&+\Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds+\lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\\
&+s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)(u_2)+\lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\\
&+\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)(\mathcal{T})\Big\}w_3\Big)(w_{12}w_{15}-w_{11}w_{16})(w_4w_7-w_3w_8)(w_{10}w_{15}-w_{11}w_{16})\Big]\Big|\\
&+\Big|\frac{1}{\lambda_1}[(1-e^{-\lambda_1 t_2})-(1-e^{-\lambda_1 t_1})]\frac{1}{\Delta_1}\Big[\Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\\
&+\lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\\
&+p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)(\zeta_2)+\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\Big\}w_{15}\\
&+\Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds+\lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\\
&+q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)(v_2)+\lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\\
&+\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)(\mathcal{T})\Big\}w_{11}\Big)(w_4w_7-w_3w_8)(w_1w_7-w_3w_5)(w_{12}w_{15}-w_{11}w_{16})\\
&+\Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds+\lambda_2 r_2\mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\\
&+r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)(k_2)+\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\Big\}w_7\\
&+\Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds+\lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\\
&+s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)(u_2)+\lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\\
&+\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)(\mathcal{T})\Big\}w_3\Big)(w_{12}w_{15}-w_{11}w_{16})(w_9w_{15}-w_{11}w_{13})(w_4w_7-w_3w_8)\Big]\Big|\\
&+\Big|\int_0^{t_2} e^{-\lambda_1(t_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds-\int_0^{t_1} e^{-\lambda_1(t_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\Big|\to 0,
\end{aligned}
$$

we deduce that $|g_1(\mathcal{X},\mathcal{Y})(t_2)-g_1(\mathcal{X},\mathcal{Y})(t_1)|\to 0$, as $t_2\to t_1$. Similarly, we conclude that $|g_2(\mathcal{X},\mathcal{Y})(t_2)-g_2(\mathcal{X},\mathcal{Y})(t_1)|\to 0$ as $t_2\to t_1$.

We deduce that g_1 and g_2 are equi-continuous operators and so the operator g is also equi-continuous. From the foregoing arguments, it follows that the operator $\mathcal{K}(\mathcal{X},\mathcal{Y})$ is equi-continuous, and so completely by the Ascoli–Arzelá theorem.

In our next step, we show that the operator $\mathcal{K}(\mathcal{X},\mathcal{Y})$ can be classified as a completely continuous operator. Let $(\mathcal{X}_n,\mathcal{Y}_n)\to(\mathcal{X}_*,\mathcal{Y}_*)$, $(g_n,\hat{g_n})\in\mathcal{K}(\mathcal{X}_n,\mathcal{Y}_n)$ and $(g_n,\hat{g_n})\to(g_*,\hat{g_*})$; then, we need to show $(g_*,\hat{g_*})\in\mathcal{K}(\mathcal{X}_*,\mathcal{Y}_*)$.

Observe that $(g_n,\hat{g_n})\in\mathcal{K}(\mathcal{X}_n,\mathcal{Y}_n)$ implies that there exist $\mathcal{H}_{1n}\in\mathcal{S}_{\mathcal{F},(\mathcal{X}_n,\mathcal{Y}_n)}$ and $\mathcal{H}_{2n}\in\mathcal{S}_{\mathcal{G},(\mathcal{X}_n,\mathcal{Y}_n)}$ such that

$$
\begin{aligned}
g_n(\mathcal{X}_n,\mathcal{Y}_n)(t) =& \frac{e^{-\lambda_1 t}}{\Delta_1}\Big(\Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds - \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&+ p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds\Big\}w_{15} \\
&- \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds - \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}[\mathcal{H}_{1n}(s)]ds \\
&+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&- \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(\mathcal{T})\Big\}w_{11}\Big)(w_4w_7 - w_3w_8)(w_2w_7 - w_3w_6)(w_{12}w_{15} - w_{11}w_{16}) \\
&- \Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds - \lambda_2 r_2\mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds\Big\}w_7 \\
&- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&- \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(\mathcal{T})\Big\}w_3\Big)(w_{12}w_{15} - w_{11}w_{16})(w_4w_7 - w_3w_8)(w_{10}w_{15} - w_{11}w_{16})\Big) \\
&+ \frac{1}{\lambda_1}(1-e^{-\lambda_1 t})\frac{1}{\Delta_1}\Big(\Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&- \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&+ p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds\Big\}w_{15} \\
&- \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds - \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&- \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(\mathcal{T})\Big\}w_{11}\Big)(w_4w_7 - w_3w_8)(w_1w_7 - w_3w_5)(w_{12}w_{15} - w_{11}w_{16}) \\
&- \Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds - \lambda_2 r_2\mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds\Big\}w_7 \\
&- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&- \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(\mathcal{T})\Big\}w_3\Big)(w_{12}w_{15} - w_{11}w_{16})(w_9w_{15} - w_{11}w_{13})(w_4w_7 - w_3w_8)\Big) \\
&+ \int_0^t e^{-\lambda_1(t-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{g}_n(\mathcal{X}_n,\mathcal{Y}_n)(t) =& \frac{e^{-\lambda_2 t}}{\Delta_2}\Big(\Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds - \lambda_2 r_2\mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds\Big\}w_5 \\
&- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&- \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(\mathcal{T})\Big\}w_1\Big) \times (w_{10}w_{13} - w_9w_{14})(w_{12}w_{13} - w_9w_{16})(w_2w_5 - w_1w_6) \\
&- \Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds - \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&+ p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds\Big\}w_{13} \\
&- \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds - \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&- \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(\mathcal{T})\Big\}w_9\Big) \times (w_2w_5 - w_1w_6)(w_4w_5 - w_1w_8)(w_{10}w_{13} - w_9w_{14})\Big) \\
&+ \frac{1}{\lambda_2}(1-e^{-\lambda_2 t})\frac{1}{\Delta_2}\Big(\Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&- \lambda_2 r_2\mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds\Big\}w_5 \\
&- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&- \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(\mathcal{T})\Big\}w_1\Big) \times (w_{10}w_{13} - w_9w_{14})(w_{11}w_{13} - w_9w_{15})(w_2w_5 - w_1w_6) \\
&- \Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds - \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&+ p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds\Big\}w_{13} \\
&- \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds - \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&- \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(\mathcal{T})\Big\}w_9\Big) \times (w_2w_5 - w_1w_6)(w_3w_5 - w_1w_7)(w_{10}w_{13} - w_9w_{14})\Big) \\
&+ \int_0^t e^{-\lambda_2(t-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds.
\end{aligned}
$$

Let us consider the continuous linear operators $\Psi_1, \Psi_2 : \mathcal{L}^1([0,\mathcal{T}],\Theta_1\times\Theta_2) \to \mathcal{C}([0,\mathcal{T}],\Theta_1\times\Theta_2)$ given by

$$\Psi_1(\mathcal{X},\mathcal{Y})(t) \tag{19}$$
$$=\frac{e^{-\lambda_1 t}}{\Delta_1}\Big[\Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds-\lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds$$
$$+p_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_1(\zeta_2)-\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\Big\}w_{15}-\Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds$$
$$-\lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds+q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(v_2)+\lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds-\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(\mathcal{T})\Big\}w_{11}\Big)$$
$$\times(w_4w_7-w_3w_8)(w_2w_7-w_3w_6)(w_{12}w_{15}-w_{11}w_{16})-\Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_1(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds$$
$$-\lambda_2 r_2\mathcal{T}\int_0^{t} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds+r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(k_2)-\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\Big\}w_7$$
$$-\Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds-\lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds$$
$$+s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(u_2)+\lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds-\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\mathcal{T})\Big\}w_3\Big)$$
$$\times(w_{12}w_{15}-w_{11}w_{16})(w_4w_7-w_3w_8)(w_{10}w_{15}-w_{11}w_{16})\Big]$$
$$+\frac{1}{\lambda_1}(1-e^{-\lambda_1 t})\frac{1}{\Delta_1}\Big[\Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds-\lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds$$
$$+p_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_1(\zeta_2)-\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\Big\}w_{15}-\Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds$$
$$-\lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds+q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(v_2)+\lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds-\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(\mathcal{T})\Big\}w_{11}\Big)$$
$$\times(w_4w_7-w_3w_8)(w_1w_7-w_3w_5)(w_{12}w_{15}-w_{11}w_{16})$$
$$-\Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_1(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds-\lambda_2 r_2\mathcal{T}\int_0^{t} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds$$
$$+r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(k_2)-\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\Big\}w_7$$
$$-\Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds-\lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds$$
$$+s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(u_2)+\lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds-\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\mathcal{T})\Big\}w_3\Big)$$
$$\times(w_{12}w_{15}-w_{11}w_{16})(w_9w_{15}-w_{11}w_{13})(w_4w_7-w_3w_8)\Big]+\int_0^t e^{-\lambda_1(t-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds,$$

$$\Psi_2(\mathcal{X},\mathcal{Y})(t) \tag{20}$$
$$=\frac{e^{-\lambda_2 t}}{\Delta_2}\Big[\Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_1(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds-\lambda_2 r_2\mathcal{T}\int_0^{t} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds$$
$$+r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(k_2)-\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\Big\}w_5$$
$$-\Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds-\lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds$$
$$+s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(u_2)+\lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds-\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\mathcal{T})\Big\}w_1\Big)$$
$$\times(w_{10}w_{13}-w_9w_{14})(w_{12}w_{13}-w_9w_{16})(w_2w_5-w_1w_6)$$
$$-\Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds-\lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds$$
$$+p_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_1(\zeta_2)-\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\Big\}w_{13}$$

$$
\begin{aligned}
&- \Big\{q_1 \int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \lambda_1 q_2 \mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds \\
&+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(\mathcal{T})\Big\}w_9\Big) \\
&\times (w_2w_5 - w_1w_6)(w_4w_5 - w_1w_8)(w_{10}w_{13} - w_9w_{14})\Big] \\
&+ \frac{1}{\lambda_2}(1-e^{-\lambda_2 t})\frac{1}{\Delta_2}\Big[\Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_1(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \lambda_2 r_2\mathcal{T}\int_0^{t} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds \\
&+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\Big\}w_5 \\
&- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds \\
&+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\mathcal{T})\Big\}w_1\Big) \\
&\times (w_{10}w_{13} - w_9w_{14})(w_{11}w_{13} - w_9w_{15})(w_2w_5 - w_1w_6) \\
&- \Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds \\
&+ p_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_1(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\Big\}w_{13} \\
&- \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds \\
&+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(\mathcal{T})\Big\}w_9\Big) \\
&\times (w_2w_5 - w_1w_6)(w_3w_5 - w_1w_7)(w_{10}w_{13} - w_9w_{14})\Big] + \int_0^{t} e^{-\lambda_2(t-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds.
\end{aligned}
$$

From [30], we know that $(\Psi_1, \Psi_2) \circ (\mathcal{S}_{\mathcal{F}}, \mathcal{S}_{\mathcal{G}})$ is a closed graph operator.

Furthermore, we have $(g_n, \hat{g_n}) \in (\Psi_1, \Psi_2) \circ (\mathcal{S}_{\mathcal{F},(\mathcal{X}_n,\mathcal{Y}_n)}, \mathcal{S}_{\mathcal{G},(\mathcal{X}_n,\mathcal{Y}_n)})$ for all n. Since $(\mathcal{X}_n, \mathcal{Y}_n) \to (\mathcal{X}_*, \mathcal{Y}_*), (g_n, \hat{g_n}) \to (g_*, \hat{g}_*)$, it follows that $\mathcal{H}_{1n} \in \mathcal{S}_{\mathcal{F},(\mathcal{X},\mathcal{Y})}$ and $\mathcal{H}_{2n} \in \mathcal{S}_{\mathcal{G},(\mathcal{X},\mathcal{Y})}$ such that

$$
\begin{aligned}
g_*(\mathcal{X}_*, \mathcal{Y}_*)(t) =& \frac{e^{-\lambda_1 t}}{\Delta_1}\Big(\Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)ds - \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)ds \\
&+ p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)ds\Big\}w_{15} \\
&- \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)ds - \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}[\mathcal{H}_{1*}(s)]ds \\
&+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)ds \\
&- \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)(\mathcal{T})\Big\}w_{11}\Big)(w_4w_7 - w_3w_8)(w_2w_7 - w_3w_6)(w_{12}w_{15} - w_{11}w_{16}) \\
&- \Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)ds - \lambda_2 r_2\mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)ds \\
&+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)ds\Big\}w_7 \\
&- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)ds \\
&+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)ds \\
&- \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)(\mathcal{T})\Big\}w_3\Big)(w_{12}w_{15} - w_{11}w_{16})(w_4w_7 - w_3w_8)(w_{10}w_{15} - w_{11}w_{16})\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\lambda_1}(1-e^{-\lambda_1 t})\frac{1}{\Delta_1}\Big(\Big(\Big\{p_1\int_0^{\zeta_1}e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)ds\\
&-\lambda_1 p_2\mathcal{T}\int_0^{\zeta_2}e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)ds\\
&+p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)(\zeta_2)-\int_0^{\mathcal{T}}e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)ds\Big\}w_{15}\\
&-\Big\{q_1\int_0^{v_1}e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)ds-\lambda_1 q_2\mathcal{T}\int_0^{v_2}e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)ds\\
&+q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)(v_2)+\lambda_2\mathcal{T}\int_0^{\mathcal{T}}e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)ds\\
&-\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)(\mathcal{T})\Big\}w_{11}\Big)(w_4w_7-w_3w_8)(w_1w_7-w_3w_5)(w_{12}w_{15}-w_{11}w_{16})\\
&-\Big(\Big\{r_1\int_0^{k_1}e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)ds-\lambda_2 r_2\mathcal{T}\int_0^{k_2}e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)ds\\
&+r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)(k_2)-\int_0^{\mathcal{T}}e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)ds\Big\}w_7\\
&-\Big\{s_1\int_0^{u_1}e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)ds-\lambda_2 s_2\mathcal{T}\int_0^{u_2}e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)ds\\
&+s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)(u_2)+\lambda_1\mathcal{T}\int_0^{\mathcal{T}}e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)ds\\
&-\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)(\mathcal{T})\Big\}w_3\Big)(w_{12}w_{15}-w_{11}w_{16})(w_9w_{15}-w_{11}w_{13})(w_4w_7-w_3w_8)\Big)\\
&+\int_0^t e^{-\lambda_1(t-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{g}_*(\mathcal{X}_*,\mathcal{Y}_*)(t)=&\frac{e^{-\lambda_2 t}}{\Delta_2}\Big(\Big(\Big\{r_1\int_0^{k_1}e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)ds-\lambda_2 r_2\mathcal{T}\int_0^{k_2}e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)ds\\
&+r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)(k_2)-\int_0^{\mathcal{T}}e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)ds\Big\}w_5\\
&-\Big\{s_1\int_0^{u_1}e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)ds-\lambda_2 s_2\mathcal{T}\int_0^{u_2}e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)ds\\
&+s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)(u_2)+\lambda_1\mathcal{T}\int_0^{\mathcal{T}}e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)ds\\
&-\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)(\mathcal{T})\Big\}w_1\Big)\times(w_{10}w_{13}-w_9w_{14})(w_{12}w_{13}-w_9w_{16})(w_2w_5-w_1w_6)\\
&-\Big(\Big\{p_1\int_0^{\zeta_1}e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)ds-\lambda_1 p_2\mathcal{T}\int_0^{\zeta_2}e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)ds\\
&+p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)(\zeta_2)-\int_0^{\mathcal{T}}e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)ds\Big\}w_{13}\\
&-\Big\{q_1\int_0^{v_1}e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)ds-\lambda_1 q_2\mathcal{T}\int_0^{v_2}e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)ds\\
&+q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(s)(v_2)+\lambda_2\mathcal{T}\int_0^{\mathcal{T}}e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)ds\\
&-\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)(\mathcal{T})\Big\}w_9\Big)\times(w_2w_5-w_1w_6)(w_4w_5-w_1w_8)(w_{10}w_{13}-w_9w_{14})\Big)\\
&+\frac{1}{\lambda_2}(1-e^{-\lambda_2 t})\frac{1}{\Delta_2}\Big(\Big(\Big\{r_1\int_0^{k_1}e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&- \lambda_2 r_2 \mathcal{T} \int_0^{k_2} e^{-\lambda_2(k_2-s)} \mathcal{I}^{\beta-1} \mathcal{H}_{2*}(s) ds \\
&+ r_2 \mathcal{T} \mathcal{I}^{\beta-1} \mathcal{H}_{2*}(s)(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)} \mathcal{I}^{\alpha-1} \mathcal{H}_{1*}(s) ds \Big\} w_5 \\
&- \Big\{ s_1 \int_0^{u_1} e^{-\lambda_2(u_1-s)} \mathcal{I}^{\beta-1} \mathcal{H}_{2*}(s) ds - \lambda_2 s_2 \mathcal{T} \int_0^{u_2} e^{-\lambda_2(u_2-s)} \mathcal{I}^{\beta-1} \mathcal{H}_{2*}(s) ds \\
&+ s_2 \mathcal{T} \mathcal{I}^{\beta-1} \mathcal{H}_{2*}(s)(u_2) + \lambda_1 \mathcal{T} \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)} \mathcal{I}^{\alpha-1} \mathcal{H}_{1*}(s) ds \\
&- \mathcal{T} \mathcal{I}^{\alpha-1} \mathcal{H}_{1*}(s)(\mathcal{T}) \Big\} w_1 \Big) \times (w_{10} w_{13} - w_9 w_{14})(w_{11} w_{13} - w_9 w_{15})(w_2 w_5 - w_1 w_6) \\
&- \Big(\Big\{ p_1 \int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)} \mathcal{I}^{\alpha-1} \mathcal{H}_{1*}(s) ds - \lambda_1 p_2 \mathcal{T} \int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)} \mathcal{I}^{\alpha-1} \mathcal{H}_{1*}(s) ds \\
&+ p_2 \mathcal{T} \mathcal{I}^{\alpha-1} \mathcal{H}_{1*}(s)(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)} \mathcal{I}^{\beta-1} \mathcal{H}_{2*}(s) ds \Big\} w_{13} \\
&- \Big\{ q_1 \int_0^{v_1} e^{-\lambda_1(v_1-s)} \mathcal{I}^{\alpha-1} \mathcal{H}_{1*}(s) ds - \lambda_1 q_2 \mathcal{T} \int_0^{v_2} e^{-\lambda_1(v_2-s)} \mathcal{I}^{\alpha-1} \mathcal{H}_{1*}(s) ds \\
&+ q_2 \mathcal{T} \mathcal{I}^{\alpha-1} \mathcal{H}_{1*}(s)(v_2) + \lambda_2 \mathcal{T} \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)} \mathcal{I}^{\beta-1} \mathcal{H}_{2*}(s) ds \\
&- \mathcal{T} \mathcal{I}^{\beta-1} \mathcal{H}_{2*}(s)(\mathcal{T}) \Big\} w_9 \Big) \times (w_2 w_5 - w_1 w_6)(w_3 w_5 - w_1 w_7)(w_{10} w_{13} - w_9 w_{14}) \Big) \\
&+ \int_0^t e^{-\lambda_2(t-s)} \mathcal{I}^{\beta-1} \mathcal{H}_{2*}(s) ds,
\end{aligned}
$$

that is, $(g_n, \hat{g_n}) \in \mathcal{K}(\mathcal{X}_*, \mathcal{Y}_*)$.

Finally, we discuss a priori solutions. Let $(\mathcal{X}, \mathcal{Y}) \in v\mathcal{K}(\mathcal{X}, \mathcal{Y})$ for $v \in [0, \mathcal{T}]$. Then $\exists\, \mathcal{H}_1 \in \mathcal{S}_{\mathcal{F},(\mathcal{X},\mathcal{Y})}$, $\mathcal{H}_2 \in \mathcal{S}_{\mathcal{G},(\mathcal{X},\mathcal{Y})}$ such that

$$
\begin{aligned}
\mathcal{X}(t) = &\frac{e^{-\lambda_1 t}}{\Delta_1} v \Bigg[\Big(\Big\{ p_1 \int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)} \mathcal{I}^{\alpha-1} \mathcal{H}_1(s) ds - \lambda_1 p_2 \mathcal{T} \int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)} \mathcal{I}^{\alpha-1} \mathcal{H}_1(s) ds \\
&+ p_2 \mathcal{T} \mathcal{I}^{\beta-1} \mathcal{H}_1(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)} \mathcal{I}^{\beta-1} \mathcal{H}_2(s) ds \Big\} w_{15} - \Big\{ q_1 \int_0^{v_1} e^{-\lambda_1(v_1-s)} \mathcal{I}^{\alpha-1} \mathcal{H}_1(s) ds \\
&- \lambda_1 q_2 \mathcal{T} \int_0^{v_2} e^{-\lambda_1(v_2-s)} \mathcal{I}^{\alpha-1} \mathcal{H}_1(s) ds + q_2 \mathcal{T} \mathcal{I}^{\alpha-1} \mathcal{H}_1(v_2) + \lambda_2 \mathcal{T} \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)} \mathcal{I}^{\beta-1} \mathcal{H}_2(s) ds - \mathcal{T} \mathcal{I}^{\beta-1} \mathcal{H}_2(\mathcal{T}) \Big\} w_{11} \Big) \\
&\times (w_4 w_7 - w_3 w_8)(w_2 w_7 - w_3 w_6)(w_{12} w_{15} - w_{11} w_{16}) - \Big(\Big\{ r_1 \int_0^{k_1} e^{-\lambda_1(k_1-s)} \mathcal{I}^{\beta-1} \mathcal{H}_2(s) ds \\
&- \lambda_2 r_2 \mathcal{T} \int_0^t e^{-\lambda_2(k_2-s)} \mathcal{I}^{\beta-1} \mathcal{H}_2(s) ds + r_2 \mathcal{T} \mathcal{I}^{\beta-1} \mathcal{H}_2(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)} \mathcal{I}^{\alpha-1} \mathcal{H}_1(s) ds \Big\} w_7 \\
&- \Big\{ s_1 \int_0^{u_1} e^{-\lambda_2(u_1-s)} \mathcal{I}^{\beta-1} \mathcal{H}_2(s) ds - \lambda_2 s_2 \mathcal{T} \int_0^{u_2} e^{-\lambda_2(u_2-s)} \mathcal{I}^{\beta-1} \mathcal{H}_2(s) ds \\
&+ s_2 \mathcal{T} \mathcal{I}^{\beta-1} \mathcal{H}_2(u_2) + \lambda_1 \mathcal{T} \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)} \mathcal{I}^{\alpha-1} \mathcal{H}_1(s) ds - \mathcal{T} \mathcal{I}^{\alpha-1} \mathcal{H}_1(\mathcal{T}) \Big\} w_3 \Big) \\
&\times (w_{12} w_{15} - w_{11} w_{16})(w_4 w_7 - w_3 w_8)(w_{10} w_{15} - w_{11} w_{16}) \Bigg] \\
&+ \frac{1}{\lambda_1}(1 - e^{-\lambda_1 t}) \frac{1}{\Delta_1} v \Bigg[\Big(\Big\{ p_1 \int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)} \mathcal{I}^{\alpha-1} \mathcal{H}_1(s) ds - \lambda_1 p_2 \mathcal{T} \int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)} \mathcal{I}^{\alpha-1} \mathcal{H}_1(s) ds \\
&+ p_2 \mathcal{T} \mathcal{I}^{\beta-1} \mathcal{H}_1(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)} \mathcal{I}^{\beta-1} \mathcal{H}_2(s) ds \Big\} w_{15} - \Big\{ q_1 \int_0^{v_1} e^{-\lambda_1(v_1-s)} \mathcal{I}^{\alpha-1} \mathcal{H}_1(s) ds \\
&- \lambda_1 q_2 \mathcal{T} \int_0^{v_2} e^{-\lambda_1(v_2-s)} \mathcal{I}^{\alpha-1} \mathcal{H}_1(s) ds + q_2 \mathcal{T} \mathcal{I}^{\alpha-1} \mathcal{H}_1(v_2) + \lambda_2 \mathcal{T} \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)} \mathcal{I}^{\beta-1} \mathcal{H}_2(s) ds - \mathcal{T} \mathcal{I}^{\beta-1} \mathcal{H}_2(\mathcal{T}) \Big\} w_{11} \Big) \\
&\times (w_4 w_7 - w_3 w_8)(w_1 w_7 - w_3 w_5)(w_{12} w_{15} - w_{11} w_{16}) \\
&- \Big(\Big\{ r_1 \int_0^{k_1} e^{-\lambda_1(k_1-s)} \mathcal{I}^{\beta-1} \mathcal{H}_2(s) ds - \lambda_2 r_2 \mathcal{T} \int_0^t e^{-\lambda_2(k_2-s)} \mathcal{I}^{\beta-1} \mathcal{H}_2(s) ds
\end{aligned}
\tag{21}
$$

$$+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\Big\}w_7$$
$$- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds$$
$$+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\mathcal{T})\Big\}w_3\Big)$$
$$\times (w_{12}w_{15} - w_{11}w_{16})(w_9w_{15} - w_{11}w_{13})(w_4w_7 - w_3w_8)\Big] + v\int_0^t e^{-\lambda_1(t-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds,$$

$$\mathcal{Y}(t) = \frac{e^{-\lambda_2 t}}{\Delta_2}v\Big[\Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_1(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \lambda_2 r_2\mathcal{T}\int_0^t e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds \tag{22}$$
$$+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\Big\}w_5$$
$$- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds$$
$$+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\mathcal{T})\Big\}w_1\Big)$$
$$\times (w_{10}w_{13} - w_9w_{14})(w_{12}w_{13} - w_9w_{16})(w_2w_5 - w_1w_6)$$
$$- \Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds$$
$$+ p_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_1(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\Big\}w_{13}$$
$$- \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds$$
$$+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(\mathcal{T})\Big\}w_9\Big)$$
$$\times (w_2w_5 - w_1w_6)(w_4w_5 - w_1w_8)(w_{10}w_{13} - w_9w_{14})\Big]$$
$$+ \frac{1}{\lambda_2}(1 - e^{-\lambda_2 t})\frac{1}{\Delta_2}v\Big[\Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_1(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \lambda_2 r_2\mathcal{T}\int_0^t e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds$$
$$+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds\Big\}w_5$$
$$- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds$$
$$+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\mathcal{T})\Big\}w_1\Big)$$
$$\times (w_{10}w_{13} - w_9w_{14})(w_{11}w_{13} - w_9w_{15})(w_2w_5 - w_1w_6)$$
$$- \Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds$$
$$+ p_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_1(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds\Big\}w_{13}$$
$$- \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds - \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(s)ds$$
$$+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_1(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds - \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_2(\mathcal{T})\Big\}w_9\Big)$$
$$\times (w_2w_5 - w_1w_6)(w_3w_5 - w_1w_7)(w_{10}w_{13} - w_9w_{14})\Big] + v\int_0^t e^{-\lambda_2(t-s)}\mathcal{I}^{\beta-1}\mathcal{H}_2(s)ds.$$

and for each $t \in [0, \mathcal{T}]$, we obtain

$$||\mathcal{X}|| \leq \Upsilon_1||l_1||(\varpi_1(\|\mathcal{X}\|) + \varsigma_1(\|\mathcal{Y}\|)) + \Upsilon_1||l_2||(\varpi_2(\|\mathcal{X}\|) + \varsigma_2(\|\mathcal{Y}\|)),$$

and

$$||\mathcal{Y}|| \leq \Upsilon_3||l_1||(\varpi_1(\|\mathcal{X}\|) + \varsigma_1(\|\mathcal{Y}\|)) + \Upsilon_4||l_2||(\varpi_2(\|\mathcal{X}\|) + \varsigma_2(\|\mathcal{Y}\|)).$$

following the same arguments as in the second step.

Thus

$$\begin{aligned}\|\mathcal{X}, \mathcal{Y}\| &= \|\mathcal{X}\| + \|\mathcal{Y}\| \\ &\leq (\Upsilon_1 + \Upsilon_3)||l_1||(\varpi_1(\|\mathcal{X}\|) + \varsigma_1(\|\mathcal{Y}\|)) + (\Upsilon_2 + \Upsilon_4)||l_2||(\varpi_2(\|\mathcal{X}\|) + \varsigma_2(\|\mathcal{Y}\|)),\end{aligned}$$

which implies that

$$\frac{\|(\mathcal{X}, \mathcal{Y})\|}{(\Upsilon_1 + \Upsilon_3)||l_1||(\varpi_1(\|\mathcal{X}\|) + \varsigma_1(\|\mathcal{Y}\|)) + (\Upsilon_2 + \Upsilon_4)||l_2||(\varpi_2(\|\mathcal{X}\|) + \varsigma_2(\|\mathcal{Y}\|))} \leq 1.$$

According to $(\mathcal{W}_3)$, $\mathcal{N}$ exists such that $\|(\mathcal{X}, \mathcal{Y})\| \neq \mathcal{N}$. Let us set

$$\mathcal{E} = \{(\mathcal{X}, \mathcal{Y}) \in \Theta_1 \times \Theta_2 : \|(\mathcal{X}, \mathcal{Y})\| < \mathcal{N}\}.$$

Note that the operator $\mathcal{K} : \overline{\mathcal{E}} \to \mathcal{P}_{cv,cp}(\Theta_1) \times \mathcal{P}_{cv,cp}(\Theta_2)$ is completely continuous and upper semicontinuous. From the choice of $\mathcal{E}$, there is no $(\mathcal{X}, \mathcal{Y}) \in \partial\mathcal{E}$ such that $(\mathcal{X}, \mathcal{Y}) \in v\mathcal{K}(\mathcal{X}, \mathcal{Y})$ for some $v \in (0, 1)$. Consequently, by the nonlinear alternative of Leray–Schauder type [25], we deduce that Λ has a fixed point $(\mathcal{X}, \mathcal{Y}) \in \overline{\mathcal{E}}$, which is a solution of the problem (2). This completes the proof. □

3.2. The Lipschitz Case

This subsection is concerned with the case when the multivalued maps in the system (1) have non-convex values. Let us first recall some auxiliary material.

Let (X, d) be a metric space induced from the normed space $(X; \|\cdot\|)$ and let $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ be defined by

$$H_d(U, V) = \max\left\{\sup_{u \in U} d(u, V), \sup_{v \in V} d(v, U)\right\},$$

where

$$d(U, v) = \inf_{u \in U} d(u, v), \quad \text{and} \quad d(u, V) = \inf_{v \in V} d(u, v).$$

Then $(\mathcal{P}_{cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [28]).

In the forthcoming result, we make use of the fixed-point theorem for multivalued maps due to Convitz and Nadler [29].

Theorem 2. *Assume that the following conditions hold* $(\mathcal{W}_4)$ *and* $(\mathcal{W}_5)$. *Then the problem of system* (1) *and* (2) *has at least one solution on* $\mathcal{J}$ *if*

$$(\Upsilon_1 + \Upsilon_3)\|s_1\| + (\Upsilon_2 + \Upsilon_4)\|s_2\| < 1, \tag{23}$$

where $\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4$ *are given (15).*

Proof. Assuming condition $(\mathcal{W}_4)$, where the sets $\mathcal{S}_{\mathcal{F},(\mathcal{X},\mathcal{Y})}$ and $\mathcal{S}_{\mathcal{G},(\mathcal{X},\mathcal{Y})}$ are non-empty for each $(\mathcal{X},\mathcal{Y}) \in \Theta_1 \times \Theta_2$, and that $\mathcal{F}$ and $\mathcal{G}$ have measurable selections (as established in Theorem III.6 in [31]), we now proceed to demonstrate that the operator Λ satisfies the conditions of the Covitz and Nadler theorem [29].

First, we illustrate that $\mathcal{K}(\mathcal{X},\mathcal{Y}) \in \mathcal{P}_{cl}(\Theta_1) \times \mathcal{P}_{cl}(\Theta_1)$ for each $(\mathcal{X},\mathcal{Y}) \in \Theta_1 \times \Theta_2$. Let $(g_n,\hat{g}_n) \in \mathcal{K}(\mathcal{X}_n,\mathcal{Y}_n)$ such that $(g_n,\hat{g}_n) \to (g,\hat{g})$ in $\Theta_1 \times \Theta_2$. Then $(g,\hat{g}) \in \Theta_1 \times \Theta_2$ and $\exists\ \mathcal{H}_{1n} \in \mathcal{S}_{\mathcal{F},(\mathcal{X}_n,\mathcal{Y}_n)}$ and $\mathcal{H}_{2n} \in \mathcal{S}_{\mathcal{F},(\mathcal{X}_n,\mathcal{Y}_n)}$.

$$
\begin{aligned}
g_n(\mathcal{X}_n,\mathcal{Y}_n)(t) =& \frac{e^{-\lambda_1 t}}{\Delta_1}\Big(\Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds - \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&+ p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds\Big\}w_{15} \\
&- \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds - \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}[\mathcal{H}_{1n}(s)]ds \\
&+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&- \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(\mathcal{T})\Big\}w_{11}\Big)(w_4w_7 - w_3w_8)(w_2w_7 - w_3w_6)(w_{12}w_{15} - w_{11}w_{16}) \\
&- \Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds - \lambda_2 r_2\mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds\Big\}w_7 \\
&- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&- \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(\mathcal{T})\Big\}w_3\Big)(w_{12}w_{15} - w_{11}w_{16})(w_4w_7 - w_3w_8)(w_{10}w_{15} - w_{11}w_{16})\Big) \\
&+ \frac{1}{\lambda_1}(1-e^{-\lambda_1 t})\frac{1}{\Delta_1}\Big(\Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&- \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&+ p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds\Big\}w_{15} \\
&- \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds - \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&- \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(\mathcal{T})\Big\}w_{11}\Big)(w_4w_7 - w_3w_8)(w_1w_7 - w_3w_5)(w_{12}w_{15} - w_{11}w_{16}) \\
&- \Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds - \lambda_2 r_2\mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds\Big\}w_7 \\
&- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&- \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(\mathcal{T})\Big\}w_3\Big)(w_{12}w_{15} - w_{11}w_{16})(w_9w_{15} - w_{11}w_{13})(w_4w_7 - w_3w_8)\Big) \\
&+ \int_0^t e^{-\lambda_1(t-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{g}_n(\mathcal{X}_n,\mathcal{Y}_n)(t) =& \frac{e^{-\lambda_2 t}}{\Delta_2}\Big(\Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds - \lambda_2 r_2\mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds\Big\}w_5 \\
&- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&- \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(\mathcal{T})\Big\}w_1\Big) \times (w_{10}w_{13} - w_9w_{14})(w_{12}w_{13} - w_9w_{16})(w_2w_5 - w_1w_6) \\
&- \Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds - \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&+ p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds\Big\}w_{13} \\
&- \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds - \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&- \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(\mathcal{T})\Big\}w_9\Big) \times (w_2w_5 - w_1w_6)(w_4w_5 - w_1w_8)(w_{10}w_{13} - w_9w_{14})\Big) \\
&+ \frac{1}{\lambda_2}(1-e^{-\lambda_2 t})\frac{1}{\Delta_2}\Big(\Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&- \lambda_2 r_2\mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds\Big\}w_5 \\
&- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&- \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(\mathcal{T})\Big\}w_1\Big) \times (w_{10}w_{13} - w_9w_{14})(w_{11}w_{13} - w_9w_{15})(w_2w_5 - w_1w_6) \\
&- \Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds - \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&+ p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds\Big\}w_{13} \\
&- \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds - \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)ds \\
&+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(s)(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds \\
&- \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)(\mathcal{T})\Big\}w_9\Big) \times (w_2w_5 - w_1w_6)(w_3w_5 - w_1w_7)(w_{10}w_{13} - w_9w_{14})\Big) \\
&+ \int_0^t e^{-\lambda_2(t-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(s)ds.
\end{aligned}
$$

Since both $\mathcal{F}$ and $\mathcal{G}$ have compact values, we consider subsequences (which we denote as sequences) such that $\mathcal{H}_{1n}$ and $\mathcal{H}_{2n}$ converge to $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, in the space $\mathcal{L}^1(\mathcal{J},\mathbb{R})$. Thus $\mathcal{H}_1 \in \mathcal{S}_{\mathcal{F},(\mathcal{X},\mathcal{Y})}$ and $\mathcal{H}_2 \in \mathcal{S}_{\mathcal{F},(\mathcal{X},\mathcal{Y})}$ for each $t \in \mathcal{J}$ and that

$$
\begin{aligned}
& g_n(\mathcal{X}_n, \mathcal{Y}_n)(t) \to g(\mathcal{X}, \mathcal{Y})(t) \qquad (24)\\
=& \frac{e^{-\lambda_1 t}}{\Delta_1}\Big(\Big(\Big\{p_1 \int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds - \lambda_1 p_2 \mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&+ p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\Big\}w_{15}\\
&- \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds - \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}[\mathcal{H}_{1i}(s)]ds\\
&+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&- \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(\mathcal{T})\Big\}w_{11}\Big)(w_4w_7 - w_3w_8)(w_2w_7 - w_3w_6)(w_{12}w_{15} - w_{11}w_{16})\\
&- \Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds - \lambda_2 r_2\mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\Big\}w_7\\
&- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&- \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\mathcal{T})\Big\}w_3\Big)(w_{12}w_{15} - w_{11}w_{16})(w_4w_7 - w_3w_8)(w_{10}w_{15} - w_{11}w_{16})\Big)\\
&+ \frac{1}{\lambda_1}(1-e^{-\lambda_1 t})\frac{1}{\Delta_1}\Big(\Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&- \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&+ p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\Big\}w_{15}\\
&- \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds - \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&- \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(\mathcal{T})\Big\}w_{11}\Big)(w_4w_7 - w_3w_8)(w_1w_7 - w_3w_5)(w_{12}w_{15} - w_{11}w_{16})\\
&- \Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds - \lambda_2 r_2\mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\Big\}w_7\\
&- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&- \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\mathcal{T})\Big\}w_3\Big)(w_{12}w_{15} - w_{11}w_{16})(w_9w_{15} - w_{11}w_{13})(w_4w_7 - w_3w_8)\Big)\\
&+ \int_0^t e^{-\lambda_1(t-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds,
\end{aligned}
$$

and

$$\hat{g}_n(\mathcal{X}_n,\mathcal{Y}_n)(t)\to\hat{g}(\mathcal{X},\mathcal{Y})(t) \tag{25}$$

$$\begin{aligned}
=&\frac{e^{-\lambda_2 t}}{\Delta_2}\Big(\Big(\Big\{r_1\int_0^{k_1}e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds-\lambda_2 r_2\mathcal{T}\int_0^{k_2}e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&+r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(k_2)-\int_0^{\mathcal{T}}e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\Big\}w_5\\
&-\Big\{s_1\int_0^{u_1}e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds-\lambda_2 s_2\mathcal{T}\int_0^{u_2}e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&+s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(u_2)+\lambda_1\mathcal{T}\int_0^{\mathcal{T}}e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&-\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\mathcal{T})\Big\}w_1\Big)\times(w_{10}w_{13}-w_9w_{14})(w_{12}w_{13}-w_9w_{16})(w_2w_5-w_1w_6)\\
&-\Big(\Big\{p_1\int_0^{\zeta_1}e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds-\lambda_1 p_2\mathcal{T}\int_0^{\zeta_2}e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&+p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\zeta_2)-\int_0^{\mathcal{T}}e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\Big\}w_{13}\\
&-\Big\{q_1\int_0^{v_1}e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds-\lambda_1 q_2\mathcal{T}\int_0^{v_2}e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&+q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(v_2)+\lambda_2\mathcal{T}\int_0^{\mathcal{T}}e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&-\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(\mathcal{T})\Big\}w_9\Big)\times(w_2w_5-w_1w_6)(w_4w_5-w_1w_8)(w_{10}w_{13}-w_9w_{14})\Big)\\
&+\frac{1}{\lambda_2}(1-e^{-\lambda_2 t})\frac{1}{\Delta_2}\Big(\Big(\Big\{r_1\int_0^{k_1}e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&-\lambda_2 r_2\mathcal{T}\int_0^{k_2}e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&+r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(k_2)-\int_0^{\mathcal{T}}e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\Big\}w_5\\
&-\Big\{s_1\int_0^{u_1}e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds-\lambda_2 s_2\mathcal{T}\int_0^{u_2}e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&+s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(u_2)+\lambda_1\mathcal{T}\int_0^{\mathcal{T}}e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&-\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\mathcal{T})\Big\}w_1\Big)\times(w_{10}w_{13}-w_9w_{14})(w_{11}w_{13}-w_9w_{15})(w_2w_5-w_1w_6)\\
&-\Big(\Big\{p_1\int_0^{\zeta_1}e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds-\lambda_1 p_2\mathcal{T}\int_0^{\zeta_2}e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&+p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\zeta_2)-\int_0^{\mathcal{T}}e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\Big\}w_{13}\\
&-\Big\{q_1\int_0^{v_1}e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds-\lambda_1 q_2\mathcal{T}\int_0^{v_2}e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&+q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(v_2)+\lambda_2\mathcal{T}\int_0^{\mathcal{T}}e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&-\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(\mathcal{T})\Big\}w_9\Big)\times(w_2w_5-w_1w_6)(w_3w_5-w_1w_7)(w_{10}w_{13}-w_9w_{14})\Big)\\
&+\int_0^t e^{-\lambda_2(t-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds.
\end{aligned}$$

Hence, $(g,\hat{g})\in\mathcal{K}$, which implies that Λ is closed. Next, we show that there exists $\hat{\rho}<1$ such that

$$\Pi_d(\mathcal{K}(\mathcal{X},\mathcal{Y}),\mathcal{K}(\hat{\mathcal{X}},\hat{\mathcal{Y}}))\le\hat{\rho}(\|\mathcal{X}-\hat{\mathcal{X}}\|+\|\mathcal{Y}-\hat{\mathcal{Y}}\|)\text{ for each }\mathcal{X},\hat{\mathcal{X}},\mathcal{Y},\hat{\mathcal{Y}}\in\Theta_1.$$

Let $(\mathcal{X}, \hat{\mathcal{X}}), (\mathcal{Y}, \hat{\mathcal{Y}}) \in \Theta_1 \times \Theta_2$ and $(g_1, \hat{g_1}) \in \mathcal{K}(\mathcal{X}, \mathcal{Y})$. Then there exists $\mathcal{H}_{11} \in \mathcal{S}_{\mathcal{F},(\mathcal{X},\mathcal{Y})}$ and $\mathcal{H}_{21} \in \mathcal{S}_{\mathcal{G},(\mathcal{X},\mathcal{Y})}$ such that, for each $t \in \mathcal{J}$, we have

$$
\begin{aligned}
g_1(\mathcal{X}_n, \mathcal{Y}_n)(t) =& \frac{e^{-\lambda_1 t}}{\Delta_1}\Big(\Big(\Big\{p_1 \int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds - \lambda_1 p_2 \mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds \\
&+ p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\Big\}w_{15} \\
&- \Big\{q_1 \int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds - \lambda_1 q_2 \mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}[\mathcal{H}_{1i}(s)]ds \\
&+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(v_2) + \lambda_2 \mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds \\
&- \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(\mathcal{T})\Big\}w_{11}\Big)(w_4w_7 - w_3w_8)(w_2w_7 - w_3w_6)(w_{12}w_{15} - w_{11}w_{16}) \\
&- \Big(\Big\{r_1 \int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds - \lambda_2 r_2 \mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds \\
&+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\Big\}w_7 \\
&- \Big\{s_1 \int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds - \lambda_2 s_2 \mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds \\
&+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(u_2) + \lambda_1 \mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds \\
&- \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\mathcal{T})\Big\}w_3\Big)(w_{12}w_{15} - w_{11}w_{16})(w_4w_7 - w_3w_8)(w_{10}w_{15} - w_{11}w_{16})\Big) \\
&+ \frac{1}{\lambda_1}(1 - e^{-\lambda_1 t})\frac{1}{\Delta_1}\Big(\Big(\Big\{p_1 \int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds \\
&- \lambda_1 p_2 \mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds \\
&+ p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\Big\}w_{15} \\
&- \Big\{q_1 \int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds - \lambda_1 q_2 \mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds \\
&+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(v_2) + \lambda_2 \mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds \\
&- \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(\mathcal{T})\Big\}w_{11}\Big)(w_4w_7 - w_3w_8)(w_1w_7 - w_3w_5)(w_{12}w_{15} - w_{11}w_{16}) \\
&- \Big(\Big\{r_1 \int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds - \lambda_2 r_2 \mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds \\
&+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\Big\}w_7 \\
&- \Big\{s_1 \int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds - \lambda_2 s_2 \mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds \\
&+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(u_2) + \lambda_1 \mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds \\
&- \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\mathcal{T})\Big\}w_3\Big)(w_{12}w_{15} - w_{11}w_{16})(w_9w_{15} - w_{11}w_{13})(w_4w_7 - w_3w_8)\Big) \\
&+ \int_0^t e^{-\lambda_1(t-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{g}_1(\mathcal{X}_n,\mathcal{Y}_n)(t) =& \frac{e^{-\lambda_2 t}}{\Delta_2}\Big(\Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds - \lambda_2 r_2\mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\Big\}w_5\\
&- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&- \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\mathcal{T})\Big\}w_1\Big) \times (w_{10}w_{13}-w_9w_{14})(w_{12}w_{13}-w_9w_{16})(w_2w_5-w_1w_6)\\
&- \Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds - \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&+ p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\Big\}w_{13}\\
&- \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds - \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&- \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(\mathcal{T})\Big\}w_9\Big) \times (w_2w_5-w_1w_6)(w_4w_5-w_1w_8)(w_{10}w_{13}-w_9w_{14})\Big)\\
&+ \frac{1}{\lambda_2}(1-e^{-\lambda_2 t})\frac{1}{\Delta_2}\Big(\Big(\Big\{r_1\int_0^{k_1} e^{-\lambda_2(k_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&- \lambda_2 r_2\mathcal{T}\int_0^{k_2} e^{-\lambda_2(k_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&+ r_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(k_2) - \int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\Big\}w_5\\
&- \Big\{s_1\int_0^{u_1} e^{-\lambda_2(u_1-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds - \lambda_2 s_2\mathcal{T}\int_0^{u_2} e^{-\lambda_2(u_2-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&+ s_2\mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(u_2) + \lambda_1\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_1(\mathcal{T}-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&- \mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\mathcal{T})\Big\}w_1\Big) \times (w_{10}w_{13}-w_9w_{14})(w_{11}w_{13}-w_9w_{15})(w_2w_5-w_1w_6)\\
&- \Big(\Big\{p_1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds - \lambda_1 p_2\mathcal{T}\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&+ p_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(\zeta_2) - \int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\Big\}w_{13}\\
&- \Big\{q_1\int_0^{v_1} e^{-\lambda_1(v_1-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds - \lambda_1 q_2\mathcal{T}\int_0^{v_2} e^{-\lambda_1(v_2-s)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)ds\\
&+ q_2\mathcal{T}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(s)(v_2) + \lambda_2\mathcal{T}\int_0^{\mathcal{T}} e^{-\lambda_2(\mathcal{T}-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds\\
&- \mathcal{T}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)(\mathcal{T})\Big\}w_9\Big) \times (w_2w_5-w_1w_6)(w_3w_5-w_1w_7)(w_{10}w_{13}-w_9w_{14})\Big)\\
&+ \int_0^t e^{-\lambda_2(t-s)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(s)ds,
\end{aligned}
$$

Utilizing $(\mathcal{W}_5)$, we have

$$
\begin{aligned}
\Pi_d(\mathcal{F}(t,\mathcal{X},\mathcal{Y}),\mathcal{F}(t,\hat{\mathcal{X}},\hat{\mathcal{Y}})) &\le s_1(t)(|\mathcal{X}(t)-\hat{\mathcal{X}}(t)|+|\mathcal{Y}(t)-\hat{\mathcal{Y}}(t)|),\\
\Pi_d(\mathcal{G}(t,\mathcal{X},\mathcal{Y}),\mathcal{G}(t,\hat{\mathcal{X}},\hat{\mathcal{Y}})) &\le s_2(t)(|\mathcal{X}(t)-\hat{\mathcal{X}}(t)|+|\mathcal{Y}(t)-\hat{\mathcal{Y}}(t)|).
\end{aligned}
$$

So, $\exists\, \mathcal{H}_1 \in \mathcal{F}(t, \mathcal{X}(t), \mathcal{Y}(t))$ and $\mathcal{H}_2 \in \mathcal{G}(t, \mathcal{X}(t), \mathcal{Y}(t))$ such that

$$|\mathcal{H}_{11}(t) - \mathcal{H}_1| \leq s_1(t)(|\mathcal{X}(t) - \hat{\mathcal{X}}(t)| + |\mathcal{Y}(t) - \hat{\mathcal{Y}}(t)|),$$
$$|\mathcal{H}_{21}(t) - \mathcal{H}_2| \leq s_2(t)(|\mathcal{X}(t) - \hat{\mathcal{X}}(t)| + |\mathcal{Y}(t) - \hat{\mathcal{Y}}(t)|).$$

Define $\mathcal{Q}_1, \mathcal{Q}_2 : \mathcal{J} \to \mathcal{P}(\mathbb{R})$ by

$$\mathcal{Q}_1(t) = \{\mathcal{H}_1 \in \mathcal{L}^1(\mathcal{J}, \mathbb{R}) : s_1(t)(|\mathcal{X}(t) - \hat{\mathcal{X}}(t)| + |\mathcal{Y}(t) - \hat{\mathcal{Y}}(t)|)\},$$
$$\mathcal{Q}_2(t) = \{\mathcal{H}_2 \in \mathcal{L}^1(\mathcal{J}, \mathbb{R}) : s_2(t)(|\mathcal{X}(t) - \hat{\mathcal{X}}(t)| + |\mathcal{Y}(t) - \hat{\mathcal{Y}}(t)|)\}.$$

Since the multivalued operators $\mathcal{Q}_1 \cap \mathcal{F}(t, \mathcal{X}(t), \mathcal{Y}_1(t))$ and $\mathcal{Q}_2 \cap \mathcal{G}(t, \mathcal{X}(t), \mathcal{Y}_1(t))$ are measurable (Proposition III.4 in [31]), there exist functions for $\mathcal{H}_{12}(t)$,$\mathcal{H}_{22}(t)$ which are measurable selections for $\mathcal{Q}_1$ and $\mathcal{Q}_2$, and $\mathcal{H}_{12}(t) \in \mathcal{F}(t, \mathcal{X}(t), \mathcal{Y}_1(t))$, $\mathcal{H}_{22}(t) \in \mathcal{G}(t, \mathcal{X}(t), \mathcal{Y}_1(t))$ such that $\forall\, t \in \mathcal{J}$. We have

$$|\mathcal{H}_{11}(t) - \mathcal{H}_{12}(t)| \leq s_1(t)(|\mathcal{X}(t) - \hat{\mathcal{X}}(t)| + |\mathcal{Y}(t) - \hat{\mathcal{Y}}(t)|)$$
$$|\mathcal{H}_{21}(t) - \mathcal{H}_{22}(t)| \leq s_2(t)(|\mathcal{X}(t) - \hat{\mathcal{X}}(t)| + |\mathcal{Y}(t) - \hat{\mathcal{Y}}(t)|).$$

Thus,

$$\|g_1(\mathcal{X}, \mathcal{Y}) - g_2(\mathcal{X}, \mathcal{Y})\| \leq \Upsilon_1 \|s_1\|(\|\mathcal{X} - \hat{\mathcal{X}}\| + \|\mathcal{Y} - \hat{\mathcal{Y}}\|) + \Upsilon_2 \|s_2\|(\|\mathcal{X} - \hat{\mathcal{X}}\| + \|\mathcal{Y} - \hat{\mathcal{Y}}\|).$$

Similarly, we can define that

$$\|\hat{g}_1(\mathcal{X}, \mathcal{Y}) - \hat{g}_2(\mathcal{X}, \mathcal{Y})\| \leq \Upsilon_3 \|s_1\|(\|\mathcal{X} - \hat{\mathcal{X}}\| + \|\mathcal{Y} - \hat{\mathcal{Y}}\|) + \Upsilon_4 \|s_2\|(\|\mathcal{X} - \hat{\mathcal{X}}\| + \|\mathcal{Y} - \hat{\mathcal{Y}}\|).$$

Therefore

$$\|(g_1, \hat{\mathcal{H}}_1), (g_2, \hat{\mathcal{H}}_2)\| \leq (\Upsilon_1 + \Upsilon_3)\|s_1\|(\|\mathcal{X} - \hat{\mathcal{X}}\| + \|\mathcal{Y} - \hat{\mathcal{Y}}\|) + (\Upsilon_2 + \Upsilon_4)\|s_2\|(\|\mathcal{X} - \hat{\mathcal{X}}\| + \|\mathcal{Y} - \hat{\mathcal{Y}}\|).$$

Likewise, by interchanging the positions of $(\mathcal{X}, \mathcal{Y})$ and $(\hat{\mathcal{X}}, \hat{\mathcal{Y}})$, we can acquire

$$\|\Pi_d(\mathcal{K}(\mathcal{X}, \mathcal{Y}), \mathcal{K}(\hat{\mathcal{X}}, \hat{\mathcal{Y}}))\| \leq (\Upsilon_1 + \Upsilon_3)\|s_1\|(\|\mathcal{X} - \hat{\mathcal{X}}\| + \|\mathcal{Y} - \hat{\mathcal{Y}}\|) + (\Upsilon_2 + \Upsilon_4)\|s_2\|(\|\mathcal{X} - \hat{\mathcal{X}}\| + \|\mathcal{Y} - \hat{\mathcal{Y}}\|).$$

Given the provided assumption, it can be established that $\mathcal{K}$ satisfies the contraction condition (23). Consequently, by virtue of Covitz and Nadler's fixed point [29], it follows that $\mathcal{K}$ possesses a fixed point $(\mathcal{X}, \mathcal{Y})$, which serves as a solution to the system (2). □

4. An Application

In this section, we illustrate the applicability of the obtained theoretical results through concrete examples.

Example 1. *Consider the coupled system of fractional differential inclusions*

$$\begin{cases} ({}^c\mathcal{D}^{\alpha} + \lambda_1 {}^c\mathcal{D}^{\alpha-1})\mathcal{X}(t) \in \mathcal{F}(t, \mathcal{X}(t), \mathcal{Y}(t)), \\ ({}^c\mathcal{D}^{\beta} + \lambda_2 {}^c\mathcal{D}^{\beta-1})\mathcal{Y}(t) \in \mathcal{G}(t, \mathcal{X}(t), \mathcal{Y}(t)), \\ \mathcal{X}(\mathcal{T}) = r_1\mathcal{Y}(k_1) + r_2\mathcal{T}\mathcal{Y}'(k_2), \\ \mathcal{T}\mathcal{X}'(\mathcal{T}) = s_1\mathcal{Y}(u_1) + s_2\mathcal{T}\mathcal{Y}'(u_2), \\ \mathcal{Y}(\mathcal{T}) = p_1\mathcal{X}(\zeta_1) + p_2\mathcal{T}\mathcal{X}'(\zeta_2), \\ \mathcal{T}\mathcal{Y}'(\mathcal{T}) = q_1\mathcal{X}(v_1) + q_2\mathcal{T}\mathcal{Y}'(v_2), \end{cases} \tag{26}$$

where the parameters are chosen as

$$\begin{aligned}
&\mathcal{T} = 0.9, \quad \alpha = \tfrac{9}{2}, \quad \beta = \tfrac{7}{2}, \quad \lambda_1 = 0.5, \quad \lambda_2 = 0.3,\\
&\zeta_1 = 0.7, \quad \zeta_2 = 0.6, \quad v_1 = 0.5, \quad v_2 = 0.4,\\
&k_1 = 0.6, \quad k_2 = 0.7, \quad s_1 = 0.8, \quad s_2 = 0.9,\\
&r_1 = 0.3, \quad r_2 = 0.4, \quad p_1 = 0.1, \quad p_2 = 0.2,\\
&q_1 = 0.15, \quad q_2 = 0.25, \quad u_1 = 0.5, \quad u_2 = 0.4,\\
&\Delta_1 = 0.0118796, \quad \Delta_2 = 0.000123519,\\
&\mathrm{Y}_1 = 0.155716, \quad \mathrm{Y}_2 = 1.02615, \quad \mathrm{Y}_3 = 0.134608, \quad \mathrm{Y}_4 = 0.268368.
\end{aligned}$$

The multivalued mappings are defined by

$$\mathcal{H}_1(t,\mathcal{X},\mathcal{Y}) = \left[\frac{-1}{16}\frac{|\mathcal{X}|}{1+|\mathcal{X}|}, 0\right] \cup \left[0, \frac{1}{16}\frac{|\sin(\mathcal{Y})|}{1+|\sin(\mathcal{Y})|}\right],$$

$$\mathcal{H}_2(t,\mathcal{X},\mathcal{Y}) = \left[\frac{-1}{16}\frac{|\mathcal{Y}|}{1+|\mathcal{Y}|}, 0\right] \cup \left[0, \frac{1}{16}\frac{|\cos(\mathcal{X})|}{1+|\cos(\mathcal{X})|}\right].$$

It is straightforward to verify that the multifunctions $\mathcal{F}$ and $\mathcal{G}$ satisfy the required Lipschitz-type conditions. Moreover,

$$(\mathrm{Y}_1 + \mathrm{Y}_3)\|s_1\| + (\mathrm{Y}_2 + \mathrm{Y}_4)\|s_2\| \approx 0.09905 < 1.$$

Hence, all the assumptions of Theorem 2 are satisfied. Therefore, the system (26) admits at least one solution on $[0,\mathcal{T}]$.

Remark 2. *The nonlinear structure of the multivalued mappings illustrates the combined influence of fractional operators and nonlinear terms, where memory effects and oscillatory behavior coexist (Figure 1).*

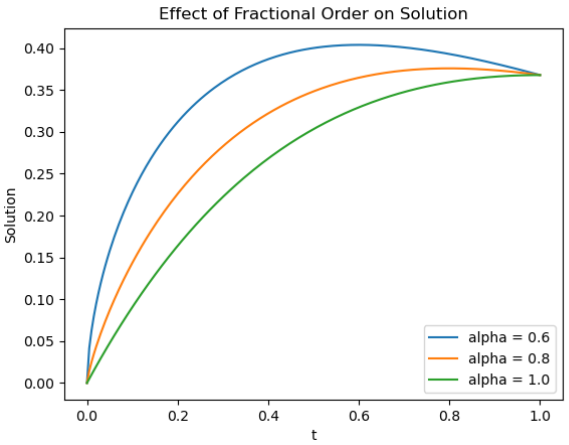


Figure 1. Fractional order effect on the solution behavior.

Example 2. *Consider the multivalued mappings $\mathcal{F},\mathcal{G} : [0,\mathcal{T}] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ satisfying*

$$\|\mathcal{F}(t,\mathcal{X},\mathcal{Y})\|_{\mathcal{P}} \le l_1(t)\left[\varpi_1(\|\mathcal{X}\|) + \varsigma_1(\|\mathcal{Y}\|)\right],$$

$$\|\mathcal{G}(t,\mathcal{X},\mathcal{Y})\|_{\mathcal{P}} \le l_2(t)\left[\varpi_2(\|\mathcal{X}\|) + \varsigma_2(\|\mathcal{Y}\|)\right],$$

where $l_1(t) = l_2(t) = \frac{1}{16}$ for all $t \in [0,\mathcal{T}]$.

Then

$$\|l_1\| = \|l_2\| = \frac{1}{16}.$$

Let

$$\varpi_i(r) = r, \quad \varsigma_i(r) = r, \quad i = 1, 2.$$

Thus,

$$\varpi_i(N) + \varsigma_i(N) = 2N.$$

Using the constants (same as Example 1):

$$\Upsilon_1 = 0.155716, \quad \Upsilon_2 = 1.02615, \quad \Upsilon_3 = 0.134608, \quad \Upsilon_4 = 0.268368,$$

we compute

$$(\Upsilon_1 + \Upsilon_3) + (\Upsilon_2 + \Upsilon_4) = 1.584842.$$

Hence,

$$\frac{N}{(\Upsilon_1 + \Upsilon_3)\|l_1\|(2N) + (\Upsilon_2 + \Upsilon_4)\|l_2\|(2N)} = \frac{8}{1.584842} \approx 5.05 > 1.$$

Therefore, condition $(\mathcal{W}_3)$ *is satisfied.*

Since all assumptions of Theorem 1 are fulfilled, the system defined in Example 1 admits at least one solution on $[0, \mathcal{T}]$.

Remark 3. *This example demonstrates that the boundedness of the multivalued maps, together with suitably chosen control functions and parameters, ensures the applicability of the fixed-point theorem. The small constant* $\frac{1}{16}$ *plays a crucial role in satisfying the key inequality in* $(\mathcal{W}_3)$.

5. Conclusions

In this paper, we have investigated the existence of solutions for a class of coupled fractional differential inclusions subject to coupled boundary conditions. The analysis was carried out within the framework of nonlinear functional analysis, employing tools such as the Leray–Schauder nonlinear alternative and the Covitz–Nadler fixed-point theorem for multivalued operators.

The proposed results cover both convex and non-convex valued multifunctions, thereby extending several existing works in the literature. The formulation allows for a broad class of nonlinearities and boundary conditions, making the results applicable to a wide range of problems arising in applied sciences.

Furthermore, suitable growth conditions were established to guarantee the existence of solutions. The theoretical findings were supported by illustrative examples, which demonstrate the effectiveness of the obtained results and highlight the role of fractional operators in capturing memory and hereditary properties of dynamical systems.

Overall, the developed approach provides a flexible and robust framework for analyzing coupled fractional inclusion systems. Future work may focus on the investigation of uniqueness, stability analysis, and the development of numerical schemes for solving such problems, as well as extending the results to more general fractional operators and higher-dimensional systems.

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