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Dynamics and Bifurcation Analysis of a Generalized Three-Dimensional Chaotic Financial System

Anna Levicka ¹  and Inna Samuilik ^{1,2,*} ¹ Institute of Applied Mathematics, Riga Technical University, LV-1048 Riga, Latvia; anna.levicka@rtu.lv² Institute of Life Sciences and Technologies, Daugavpils University, 13 Vienības Street, LV-5401 Daugavpils, Latvia

* Correspondence: inna.samuilika@rtu.lv

Abstract

This paper investigates the dynamics of a three-dimensional nonlinear model of the financial system and the conditions for the emergence of chaotic behavior. The well-known chaotic system with given parameters and initial conditions is considered as a basis. For the initial model, critical points are analyzed, two-dimensional and three-dimensional phase portraits are constructed, and Lyapunov exponents are calculated, which allow confirming the presence of chaos and assessing the degree of sensitivity to initial data. Next, a modification of the system is proposed, consisting of changing the degree of the variable in the second equation. For the group of models obtained, we considered the generalized form of the system, found its critical points, and classified them. At the next stage, a bifurcation analysis was performed: by changing the key parameters of the modified systems, bifurcation diagrams were constructed, and parameter regions corresponding to critical points, periodicity, quasi-periodicity, and chaos were identified. The results demonstrate that the nature of the dynamics depends significantly on both the parameters and the degree of nonlinearity and allow conclusions to be drawn about the mechanisms of chaos in the financial model under consideration.

Keywords: chaos; attractor; Lyapunov exponents; Kaplan–York dimension; chaotic solutions; bifurcation analysis

MSC: 34C25; 34D45

1. Introduction

Chaos theory originated in the natural sciences [1], particularly mathematics and physics [2], where it was developed to analyze complex nonlinear systems that exhibit a strong sensitivity to initial conditions [3,4]. Over time, the fundamental principles of chaos theory have been increasingly applied in economics to investigate disturbances in economic systems and evaluate their potential consequences [5,6]. Chaotic economic [7] and financial market models [8] have long been a central topic of investigation for economists and theoreticians. In recent decades, the increasing instability of financial markets and the increasing role of stochastic factors have further stimulated interest in the application of chaos theory within the financial domain [9]. In this context, a variety of financial models have been extensively examined in the literature with the aim of describing and predicting complex market dynamics (see, for example, [5,10–12]).

In chaos theory, methods such as linear and bifurcation analysis [13–15], the construction of strange attractors [16,17], and the calculation of Lyapunov exponents [18,19], as well



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as numerical methods, are widely used to study nonlinear dynamic systems. The study is based on dynamic systems and their phase spaces [3]. Thanks to these methods, it is possible to most accurately characterize the properties of nonlinear dynamical systems, such as sensitivity to initial conditions, non-periodicity, and unpredictability.

The study focuses on a financial dynamical system described by a special set of ordinary differential equations. The analysis concentrates on the properties of attracting sets and their dependence on the model parameters. Graphs of solutions, phase portraits, bifurcation diagrams, and the dynamics of Lyapunov exponents are constructed to illustrate the evolution of the system and predict its future behavior.

Although a number of nonlinear financial models exhibiting chaotic dynamics have been proposed in the literature, most existing studies are based on quadratic or quartic nonlinearities. However, empirical financial data often exhibit strong nonlinear effects, including leptokurtic distributions and heavy-tailed fluctuations. These features indicate that market dynamics may involve stronger nonlinear feedback mechanisms than those captured by low-order nonlinear terms. In this study, the nonlinear term is generalized to x^k , where $2 \leq k \leq 10$, allowing us to investigate how the degree of nonlinearity affects system stability, bifurcations, and the emergence of chaotic regimes. This approach provides additional insight into the role of higher-order nonlinearities in the generation of complex financial dynamics. Nevertheless, the influence of higher-order nonlinear terms on the dynamical behavior of financial models remains insufficiently investigated. In particular, the role of varying degrees of nonlinearity in shaping the structure of the phase space and the complexity of system dynamics has not been studied in detail. This motivates the present work, where a generalized financial model with a nonlinear term of the form x^k is considered and its dynamical properties are analyzed.

This manuscript is organized as follows. The Introduction provides a brief overview of chaos theory, outlines the current state of research on nonlinear dynamical systems, and emphasizes the relevance of these concepts to financial modeling. Section 2 introduces a three-dimensional mathematical model of the financial system. In Section 3, the dynamic properties of the proposed three-dimensional nonlinear system are investigated. To achieve a more comprehensive analysis, the model is generalized by replacing the nonlinear term with x^k , where $2 \leq k \leq 10$ is an integer. This section includes analysis of critical points, the construction of bifurcation diagrams, the visualization of attractors, the plotting of solution graphs, and the computation of the Lyapunov spectrum and Lyapunov dimension. Finally, the Discussion section summarizes the main results and contributions of this study.

The main contributions of this study can be summarized as follows:

- The critical points of the classical three-dimensional financial system are derived analytically and their existence conditions are determined.
- The local stability of the critical points is investigated by constructing the Jacobian matrix of the system and analyzing the corresponding eigenvalues.
- A generalized nonlinear financial model is proposed by modifying the degree of the variable in the second equation, which leads to a new class of dynamical behavior.
- The critical points of the generalized system are obtained, and their stability properties are analyzed, revealing the conditions under which the system transitions between stable and unstable regimes.
- The global dynamics of the system are investigated using phase portraits, Lyapunov exponents, and bifurcation diagrams.

2. Three-Dimensional Mathematical Model of Financial System

The model (1) describes a three-dimensional financial system where x is the interest rate, y indicates the level of investment demand, and z reflects how prices grow expo-

nentially. In addition, the constant $a \geq 0$ represents the rate of household savings, $b \geq 0$ corresponds to investment costs, and $c \geq 0$ measures the elasticity of demand in commercial markets. The parameter d is a positive scaling parameter [12].

$$\begin{cases} \frac{dx}{dt} = z + (y - a)x, \\ \frac{dy}{dt} = 1 - by - dx^2 - x^4, \\ \frac{dz}{dt} = -x - cz. \end{cases} \quad (1)$$

In system (1), there are two quadratic nonlinearities xy ; x^2 and one quartic nonlinearity, x^4 , which together generate rich and complex dynamics. The three-dimensional system (1) exhibits a chaotic attractor when the parameters are set to

$$a = 7.2, b = 0.1, c = 1, d = 0.1 \quad (2)$$

and the initial conditions are

$$x(0) = 0.5; y(0) = 3; z(0) = -0.4. \quad (3)$$

Definition 1. A continuous map $f : V \rightarrow V$ is chaotic if it is topologically transitive, has dense periodic points, and exhibits sensitive dependence on initial conditions [20].

2.1. Stability and Dynamical Analysis

Dynamical analysis of chaotic systems involves exploring the underlying mechanics and behaviors that lead to chaos [21].

The system is (1) with parameters (2) and initial conditions (3).

Let the third equation of the system (1) be equal to zero: $-x - cz = 0$.

Then

$$z = -\frac{x}{c}. \quad (4)$$

Substitute (4) into the first equation of the system (1)

$$-\frac{x}{c} + (y - a)x = 0. \quad (5)$$

$$x\left(y - a - \frac{1}{c}\right) = 0. \quad (6)$$

Let us consider the first case, where $x = 0$. Then substituting $x = 0$ into Equation (4) to get $z = 0$. After that, substitute $x = 0$ into the second equation of the system (1) and equate to zero

$$1 - by = 0. \quad (7)$$

From Equation (7) get $y = \frac{1}{b}$. Taking into account the parameters (2), the first critical point is obtained: $E_1 = (0, 10, 0)$.

Let us consider the second case, where $y - a - \frac{1}{c} = 0$. Then $y = a + \frac{1}{c}$ and $z = -\frac{x}{c}$. After that, substitute $y = a + \frac{1}{c}$ into the second equation of the system (1) and equate to zero

$$1 - b\left(a + \frac{1}{c}\right) - dx^2 - x^4 = 0. \quad (8)$$

After simplifying (8) get

$$x^2(x^2 + d) = 1 - b\left(a + \frac{1}{c}\right). \quad (9)$$

Taking into account the parameters (2), the two roots of Equation (9) are obtained: $x_{1,2} = \pm 0.6142$. The second and third critical points are $E_2 = (0.6142, 8.2, -0.6142)$ and $E_3 = (-0.6142, 8.2, 0.6142)$.

Once the fixed points are determined, their stability is examined through the computation of the system's Jacobian matrix. Let us build the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial z} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial z} \\ \frac{\partial \dot{z}}{\partial x} & \frac{\partial \dot{z}}{\partial y} & \frac{\partial \dot{z}}{\partial z} \end{pmatrix} = \begin{pmatrix} y-a & x & 1 \\ -2(d+2x^2)x & -b & 0 \\ -1 & 0 & -1 \end{pmatrix}. \quad (10)$$

The Jacobian matrix provides a linear approximation of the system near critical points and allows their type to be determined based on the eigenvalues. This analysis plays a key role in studying the local stability of a dynamical system [22].

The characteristic matrix is

$$J - \lambda I = \begin{pmatrix} y-a-\lambda & x & 1 \\ -2(d+2x^2)x & -b-\lambda & 0 \\ -1 & 0 & -1-\lambda \end{pmatrix} \quad (11)$$

and the characteristic equation is

$$\det(J - \lambda I) = -b - \lambda + (-1 - \lambda)(ab + 4x^4 + 2dx^2 - by + a\lambda + b\lambda - y\lambda + \lambda^2) = 0. \quad (12)$$

Put the values of the first critical point $E_1 = (0, 10, 0)$ in (12) and get three roots

$$\lambda_1 = 2.5156, \quad \lambda_2 = -0.7155, \quad \lambda_3 = -0.1. \quad (13)$$

This result indicates that the critical point $E_1 = (0, 10, 0)$ behaves as a saddle point and is unstable.

Put the values of the second critical point $E_2 = (0.6142, 8.2, -0.6142)$ in (12) and get three roots

$$\lambda_1 = -0.6462, \quad \lambda_{2,3} = 0.2731 \pm 0.9607i. \quad (14)$$

This result indicates that the critical point $E_2 = (0.6142, 8.2, -0.6142)$ behaves as a saddle-focus point and is unstable.

Put the values of the third critical point $E_3 = (-0.6142, 8.2, 0.6142)$ in (12) and get three roots

$$\lambda_1 = -0.6462, \quad \lambda_{2,3} = 0.2731 \pm 0.9607i. \quad (15)$$

This result indicates that the critical point $E_3 = (-0.6142, 8.2, 0.6142)$ behaves as a saddle-focus point and is unstable.

Figure 1 shows the phase plots of system (1), while Figure 2 displays the corresponding solution graphs.

The Lyapunov exponent is a fundamental measure of the sensitivity of a system to initial conditions, which is a defining characteristic of chaotic dynamics [23,24]. The Lyapunov exponents for system (1) with given parameters (2) and initial conditions (3) were calculated as follows:

$$LE_1 = 0.13371; LE_2 = -0.0002; LE_3 = -0.4094. \quad (16)$$

Let us compute the Kaplan–Yorke dimension [25] using the formula

$$D_{KY} = j + \frac{1}{|L_{j+1}|} \sum_{i=1}^j L_i, \quad (17)$$

where j is the largest integer that satisfies

$$\sum_{i=1}^j L_i > 0, \quad \sum_{i=1}^{j+1} L_i < 0.$$

$$D_{KY} = 2 + \frac{0.13371 - 0.0002}{|-0.4094|} = 2.33 \quad (18)$$

A chaotic system has at least one positive Lyapunov exponent, and the more positive the largest Lyapunov exponent, the more unpredictable the system is [26].

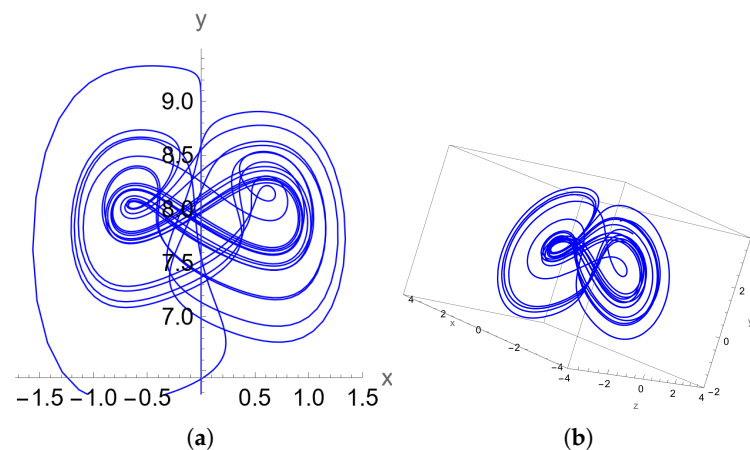


Figure 1. Phase portraits illustrating the dynamics of system (1) in the two-dimensional plane and the three-dimensional state space: (a) Phase trajectories in the $x - y$ plane. (b) Phase portrait in $x - y - z$ state space.

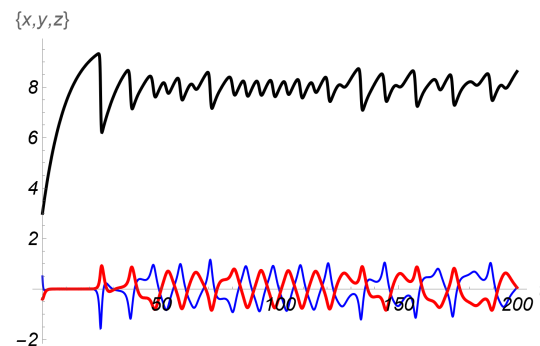


Figure 2. Solutions (x, y, z) of the system (1) with the initial conditions $x(0) = 0.5$; $y(0) = 3$; $z(0) = -0.4$.

The system (1) is dissipative since the sum of the Lyapunov exponents is negative and chaotic since the first Lyapunov exponent is positive, $LE_1 = 0.13371$. Computing the full spectrum of Lyapunov exponents is mathematically challenging. The calculations were carried out using Wolfram Mathematica 13 and Matlab R2025b. Figure 3 displays the Lyapunov characteristic exponents of the system (1).

The system (1) represents the classical three-dimensional financial model that has been widely studied in the literature [12].

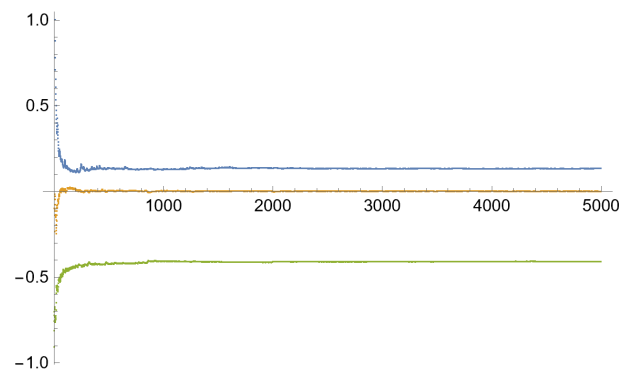


Figure 3. Lyapunov characteristic exponents for the system (1) with initial conditions $x(0) = 0.5$; $y(0) = 3$; $z(0) = -0.4$.

2.2. Bifurcation Analysis

Table 1 presents the analysis of system dynamics for various values of the parameter a .

Table 1. Parameter a analysis.

a	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	0.0009	−0.0103	−0.5290	periodic
1.0	0.0001	−0.0116	−0.5073	periodic
3.0	0.0011	−0.2269	−0.2328	periodic
4.0	0.0009	−0.0379	−0.3876	periodic
4.5	0.0017	−0.0743	−0.3352	periodic
4.7	0.0002	−0.0240	−0.3783	periodic
4.8	0.0240	0.0002	−0.4228	chaotic
5.0	0.0421	−0.0017	−0.4331	chaotic
6.0	0.1248	−0.0006	−0.4616	chaotic
7.2	0.1337	−0.0002	−0.4094	chaotic
8.0	0.0773	−0.0001	−0.3112	chaotic
8.3	0.0280	−0.0004	−0.2637	chaotic
8.4	0.0069	−0.0118	−0.2255	periodic
8.5	0.0120	−0.0497	−0.1811	periodic
9.0	0.0007	−0.0129	−0.0916	periodic
10.0	−0.1123	−0.4897	−0.5018	asymptotically stable

The bifurcation diagram for the parameter a is explored. Figure 4 displays the bifurcation diagram of the system (1), obtained by varying the control parameter a . The parameters b, c , and d remain fixed. The bifurcation diagram is plotted when a is varied between $0 \leq a \leq 10$.

Table 2 presents the analysis of system dynamics for various values of the parameter b .

Table 2. Parameter b analysis.

b	LE_1	LE_2	LE_3	<i>Solution</i>
0.01	0.0024	−0.0065	−0.4841	quasiperiodic
0.10	0.1337	−0.0002	−0.4094	chaotic
0.20	−0.2147	−1.5852	−1.6002	asymptotically stable
0.30	−0.3130	−1.3937	−3.4599	asymptotically stable
0.50	−0.5122	−1.2417	−4.9461	asymptotically stable
1.00	−1.0119	−1.1887	−5.9994	asymptotically stable
3.00	−1.1764	−3.0000	−6.6903	asymptotically stable
7.00	−1.1706	−6.8978	−6.9888	asymptotically stable
10.0	−1.1693	−6.9425	−9.9880	asymptotically stable

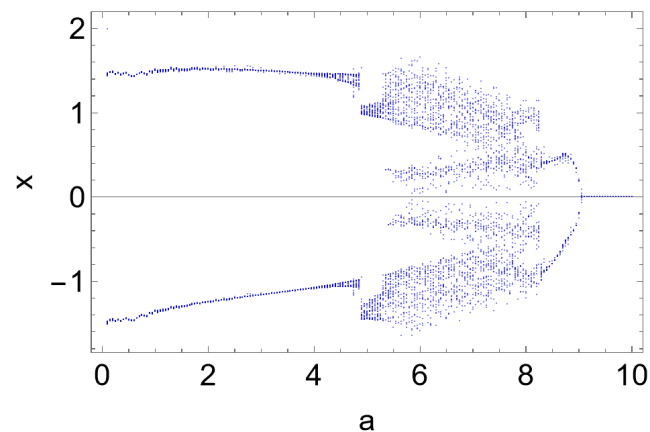


Figure 4. Bifurcation diagram.

The bifurcation diagram for the parameter b is explored. Figure 5 displays the bifurcation diagram of the system (1), obtained by varying the control parameter b . The parameters a , c , and d remain fixed. The bifurcation diagram is plotted when b is varied between $0 \leq b \leq 0.14$.

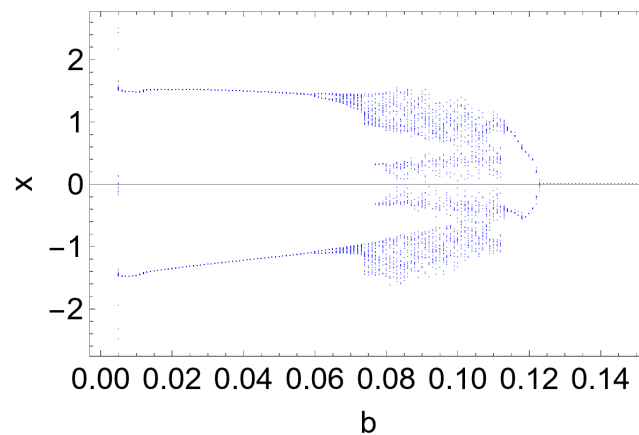


Figure 5. Bifurcation diagram.

Table 3 presents the analysis of system dynamics for various values of the parameter c .

Table 3. Parameter c analysis.

c	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	0.0011	−0.0501	−0.0544	periodic
1.0	0.1337	−0.0002	−0.4094	chaotic
1.1	0.0447	0.0004	−0.4756	chaotic
1.2	0.0022	−0.0101	−0.5598	periodic
1.5	0.0015	−0.1166	−0.8372	periodic
2.0	0.0039	−0.0164	−1.4298	periodic
3.0	−0.0266	−0.0318	−2.7038	asymptotically stable
5.0	−0.0410	−0.0465	−4.8081	asymptotically stable
7.0	−0.0447	−0.0489	−6.8601	asymptotically stable
10.0	−0.0462	−0.0499	−9.9009	asymptotically stable

The bifurcation diagram for the parameter c is explored. Figure 6 displays the bifurcation diagram of the system (1), obtained by varying the control parameter c . The parameters a , b , and d remain fixed. The bifurcation diagram is plotted when c is varied between $0 \leq c \leq 3$.

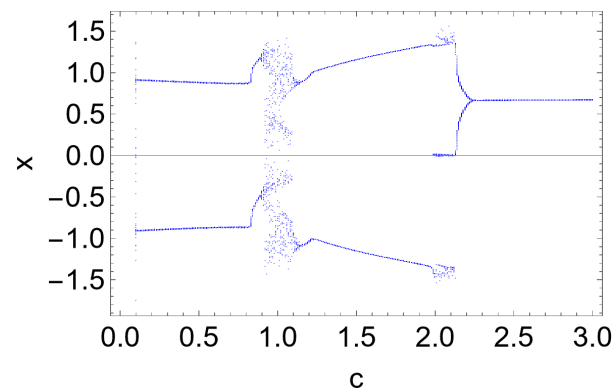


Figure 6. Bifurcation diagram.

Table 4 presents the analysis of system dynamics for various values of the parameter d .

Table 4. Parameter d analysis.

d	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	0.1337	−0.0002	−0.4094	chaotic
0.5	0.0889	−0.0002	−0.3696	chaotic
0.8	0.0802	0.0000	−0.3389	chaotic
0.9	0.0233	0.0000	−0.2710	chaotic
1.0	0.0027	−0.1210	−0.1291	periodic
1.1	0.0539	0.0001	−0.3214	chaotic
1.3	0.0632	0.0002	−0.3386	chaotic
1.5	0.0446	−0.0006	−0.3226	chaotic
2.0	0.0532	0.0006	−0.3296	chaotic
3.0	0.0319	0.0003	−0.3178	chaotic
4.0	0.0233	0.0000	−0.2710	chaotic
4.1	0.0031	−0.0082	−0.2797	quasiperiodic
4.3	0.0029	−0.0032	−0.2848	quasiperiodic
4.5	0.0023	−0.0097	−0.2779	quasiperiodic
5.0	0.0024	−0.0485	−0.2395	periodic
7.0	0.0034	−0.0046	−0.2850	quasiperiodic
8.0	0.0031	−0.0083	−0.2808	quasiperiodic
10.0	0.0032	−0.0127	−0.2765	periodic

The bifurcation diagram for the parameter d is explored. Figure 7 displays the bifurcation diagram of the system (1), obtained by varying the control parameter d . The parameters a , b , and c remain fixed. The bifurcation diagram is plotted when d is varied between $0 \leq d \leq 6$.

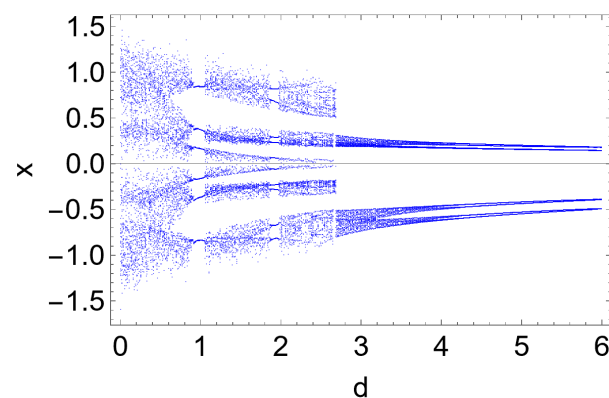


Figure 7. Bifurcation diagram.

In order to investigate the influence of higher-order nonlinearities on the dynamics of financial systems, we consider a generalized version of this model in the next section.

3. Results

In the system of differential Equation (1), the term x^4 introduces a high degree of nonlinearity in the dynamics of the variable y . To perform a more comprehensive analysis, this term is generalized by replacing x^k , where $2 \leq k \leq 10$ is an integer. The system takes the form

$$\begin{cases} \frac{dx}{dt} = z + (y - a)x, \\ \frac{dy}{dt} = 1 - by - dx^2 - x^k, \\ \frac{dz}{dt} = -x - cz. \end{cases} \quad (19)$$

For each specific value of k , a comprehensive analysis will be performed.

The divergence of the vector field of system (19) is given by

$$\nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = y - a - b - c. \quad (20)$$

If the divergence is negative in a region of the phase space, the system is dissipative, and the trajectories converge to a bounded attractor [24].

3.1. Case 1: $k = 2$

3.1.1. Stability and Dynamical Analysis

The system is (19) and $k = 2$ with given parameters (2) and initial conditions (3).

Let the third equation of the system (19) and $k = 2$ be equal to zero: $-x - cz = 0$.

Then

$$z = -\frac{x}{c}. \quad (21)$$

Substitute (21) into the first equation of the system (19)

$$-\frac{x}{c} + (y - a)x = 0. \quad (22)$$

$$x\left(y - a - \frac{1}{c}\right) = 0. \quad (23)$$

Let us consider the first case, where $x = 0$. Then substituting $x = 0$ into Equation (21) to get $z = 0$. After that, substitute $x = 0$ into the second equation of the system (19) and equate to zero

$$1 - by = 0. \quad (24)$$

From Equation (24) get $y = \frac{1}{b}$. Taking into account the parameters (2), the first critical point is obtained: $E_1 = (0, 10, 0)$.

Let us consider the second case, where $y - a - \frac{1}{c} = 0$. Then $y = a + \frac{1}{c}$ and $z = -\frac{x}{c}$. After that, substitute $y = a + \frac{1}{c}$ into the second equation of the system (19) and equate to zero

$$1 - b\left(a + \frac{1}{c}\right) - dx^2 - x^2 = 0. \quad (25)$$

After simplifying (25) get

$$x^2 = \frac{1 - b(a + \frac{1}{c})}{d + 1}. \quad (26)$$

Taking into account the parameters (2), the two roots of Equation (26) are obtained: $x_{1,2} = \pm 0.4045$. The second and third critical points are $E_2 = (0.4045, 8.2, -0.4045)$ and $E_3 = (-0.4045, 8.2, 0.4045)$.

Once the fixed points are determined, their stability is examined through the computation of the system's Jacobian matrix. Let us build the Jakobian matrix

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial z} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial z} \\ \frac{\partial \dot{z}}{\partial x} & \frac{\partial \dot{z}}{\partial y} & \frac{\partial \dot{z}}{\partial z} \end{pmatrix} = \begin{pmatrix} y-a & x & 1 \\ -2(d+1)x & -b & 0 \\ -1 & 0 & -1 \end{pmatrix}. \quad (27)$$

The Jacobian matrix provides a linear approximation of the system near critical points and allows their type to be determined based on the eigenvalues. This analysis plays a key role in studying the local stability of a dynamical system [22].

The characteristic matrix is

$$J - \lambda I = \begin{pmatrix} y-a-\lambda & x & 1 \\ -2(d+1)x & -b-\lambda & 0 \\ -1 & 0 & -1-\lambda \end{pmatrix} \quad (28)$$

and the characteristic equation is

$$\det(J - \lambda I) = -b - \lambda + (-1 - \lambda)(ab + 2x^2 + 2dx^2 - by + a\lambda + b\lambda - y\lambda + \lambda^2) = 0. \quad (29)$$

Put the values of the first critical point $E_1 = (0, 10, 0)$ in (29) and get three roots

$$\lambda_1 = 2.5156, \quad \lambda_2 = -0.7155, \quad \lambda_3 = -0.1. \quad (30)$$

This result indicates that the critical point $E_1 = (0, 10, 0)$ behaves as a saddle point and is unstable.

Put the values of the second critical point $E_2 = (0.4045, 8.2, -0.4045)$ in (29) and get three roots

$$\lambda_1 = -0.5717, \quad \lambda_{2,3} = 0.2359 \pm 0.7577i. \quad (31)$$

This result indicates that the critical point $E_2 = (0.4045, 8.2, -0.4045)$ behaves as a saddle-focus point and is unstable.

Put the values of the third critical point $E_3 = (-0.4045, 8.2, 0.4045)$ in (29) and get three roots

$$\lambda_1 = -0.5717, \quad \lambda_{2,3} = 0.2359 \pm 0.7577i \quad (32)$$

This result indicates that the critical point $E_3 = (-0.4045, 8.2, 0.4045)$ behaves as a saddle-focus point and is unstable.

The stability of the critical points is analyzed using the Jacobian matrix and the corresponding eigenvalues obtained from local linearization of the system. In addition to the Jacobian matrix analysis based on local linearization, global stability of nonlinear systems can also be investigated using the direct Lyapunov method. Such approaches provide complementary tools for studying the stability properties of dynamical systems. The eigenvalues of the Jacobian matrix determine the type and stability of the critical points.

To analyze the possibility of Hopf bifurcation, the Jacobian matrix of system (19) is considered at critical points. A Hopf bifurcation occurs when a pair of complex conjugate eigenvalues of the Jacobian matrix crosses the imaginary axis, while the third eigenvalue remains with a negative real part. Under such conditions, the critical point loses stability, and a periodic orbit may arise.

3.1.2. Bifurcation Analysis

The bifurcation analysis examines the dynamics of the system without regard to parameter interdependence and explores how the behavior of the system changes with different parameter values [22,27].

The bifurcation analysis is carried out by varying the parameter $0.1 \leq a \leq 10$, while keeping the remaining parameters fixed. This approach allows us to investigate the changes in the system dynamics and to identify parameter ranges in which periodic and chaotic regimes occur. Table 5 presents the analysis of system dynamics for various values of the parameter a .

Table 5. Parameter a analysis.

Parameter Value	LE_1	LE_2	LE_3	Behavior
0.1	0.1380	0.0000	−0.5437	chaotic
0.5	0.1381	−0.0001	−0.5292	chaotic
0.6	0.1341	−0.0004	−0.5296	chaotic
0.7	0.1315	0.0003	−0.5256	chaotic
0.8	0.1354	−0.0001	−0.5228	chaotic
0.9	0.1437	−0.0006	−0.5324	chaotic
1.0	0.1322	−0.0006	−0.5275	chaotic
1.5	0.1336	0.0008	−0.5236	chaotic
2.0	0.1336	0.0000	−0.5201	chaotic
3.0	0.1490	−0.0008	−0.5040	chaotic
4.0	0.1545	0.0005	−0.4871	chaotic
5.0	0.1124	0.0011	−0.4441	chaotic
6.0	0.0940	−0.00001	−0.3943	chaotic
7.0	0.0458	−0.0008	−0.3267	chaotic
7.1	0.0240	0.0001	0.0001	non-chaotic
7.2	0.0033	−0.0224	−0.2669	periodic
7.3	0.0031	−0.0216	−0.2687	periodic
7.4	0.0040	−0.0102	−0.2787	periodic
7.5	0.0037	−0.0282	−0.2568	periodic
7.6	0.0042	−0.0542	−0.2260	periodic
7.7	0.0048	−0.1049	−0.1707	periodic
7.8	0.0050	−0.1327	−0.1378	periodic
7.9	0.0059	−0.1294	−0.1352	periodic
8.0	0.0065	−0.1252	−0.1333	periodic
9.0	0.0007	−0.0129	−0.0915	asymptotically stable
9.1	−0.0516	−0.0610	−0.0912	asymptotically stable
9.2	−0.1019	−0.1000	−0.1019	asymptotically stable
10.0	−0.1114	−0.4907	−0.5018	asymptotically stable

The bifurcation analysis performed revealed that the system exhibits chaotic behavior for $0.1 \leq a \leq 7$, but periodic behavior for $7.2 \leq a \leq 8.9$.

Figure 8 shows the phase plots of system (19), $k = 2$, while Figure 9 displays the corresponding solution graphs.

The Lyapunov exponents for system (19), $k = 2$, and parameters $a = 4$, $b = 0.1$, $c = 1$, $d = 0.1$, and initial conditions (3) were calculated as follows:

$$LE_1 = 0.1545; LE_2 = 0.0005; LE_3 = -0.4871. \quad (33)$$

$$D_{KY} = 2 + \frac{0.1545 - 0.0005}{|-0.4871|} = 2.32 \quad (34)$$

The system is (19) and $k = 2$ is dissipative, since the sum of the Lyapunov exponents is negative. Figure 10 displays the Lyapunov characteristic exponents of the system (19), $k = 2$.

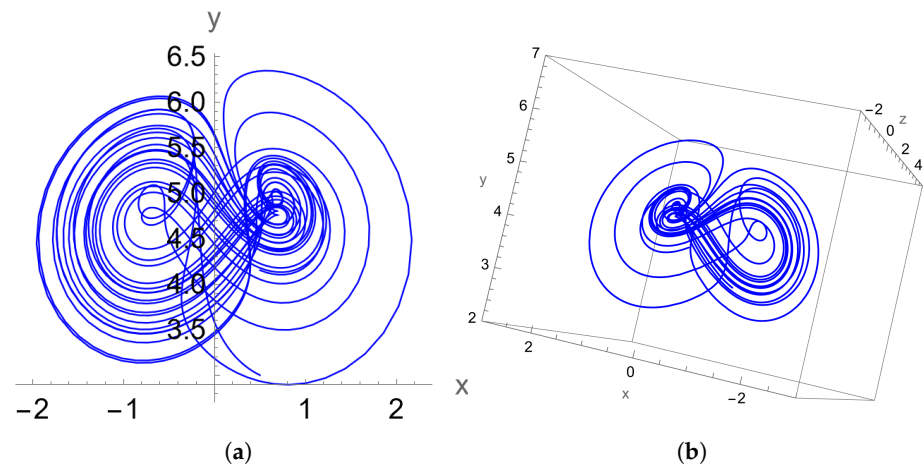


Figure 8. Phase portraits illustrating the dynamics of system (19), $k = 2$ in the two-dimensional planes and the three-dimensional state space: (a) Phase trajectories in the $x - y$ plane, $a = 4$. (b) Phase portrait in $x - y - z$ state space, $a = 4$.

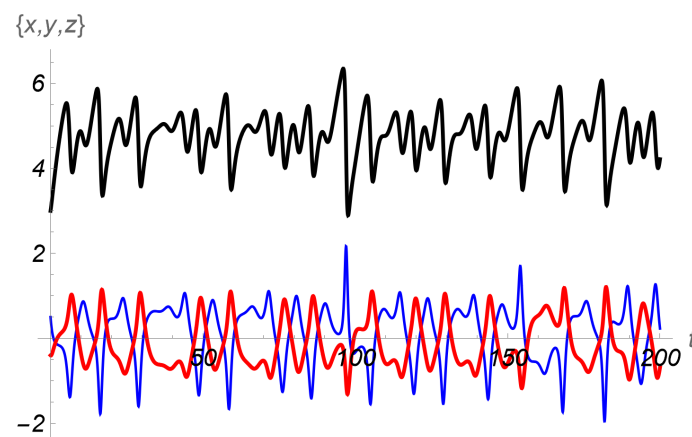


Figure 9. Solutions (x, y, z) of the system (19) with the initial conditions $x(0) = 0.5$; $y(0) = 3$; $z(0) = -0.4$, $a = 4$.

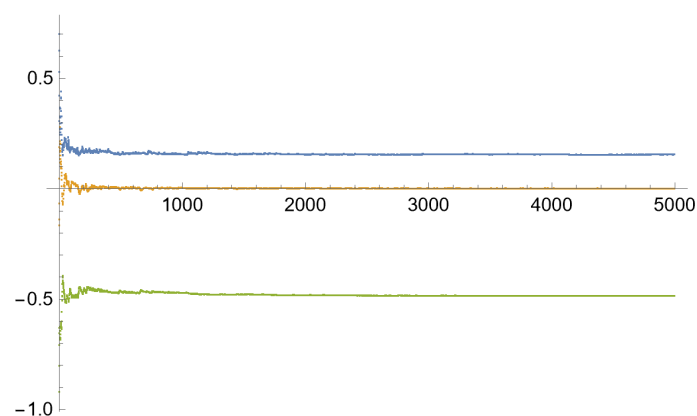


Figure 10. Lyapunov characteristic exponents for the system (19), $k = 2$ and $a = 4$ with initial conditions $x(0) = 0.5$; $y(0) = 3$; $z(0) = -0.4$.

Bifurcation diagrams are analyzed by varying one parameter at a time and keeping the others fixed [28]. The bifurcation diagram for the parameter a is explored. Figure 11 displays the bifurcation diagram of the system (19) for $k = 2$, obtained by varying the control parameter a . The parameters b , c , and d remain fixed. The bifurcation diagram is plotted when a is varied between $0.1 \leq a \leq 10$.

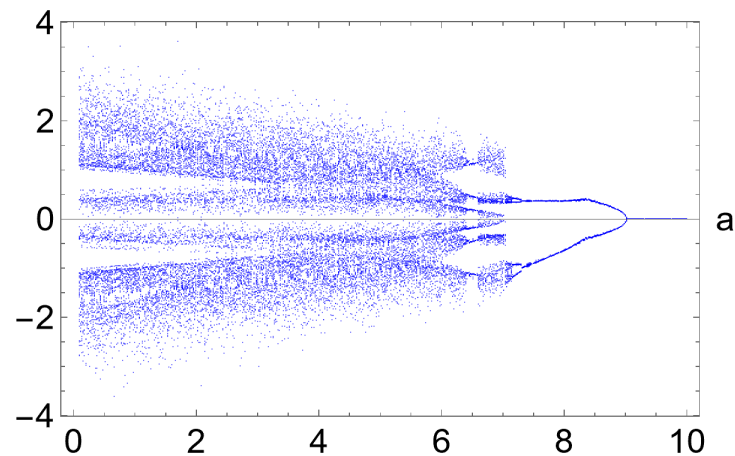


Figure 11. Bifurcation diagram.

For small values of parameter a , the diagram shows a dense cloud of points, indicating chaotic dynamics. As a increases, there is a crisis of chaos, and the attractor shrinks. Periodic windows emerge in intermediate ranges of a , followed by a transition to an asymptotically stable equilibrium for large values of a . The results obtained are in full agreement with the Lyapunov exponent spectrum reported in Table 5.

Figure 12 displays the Poincaré section of the system (19) for $k = 2$.

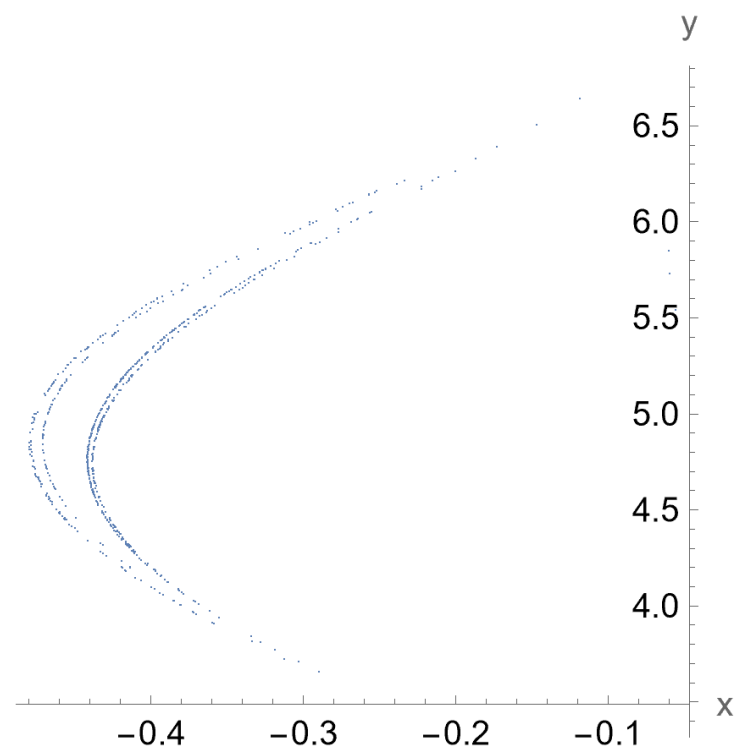


Figure 12. Poincaré section of the system (19) for $k = 2$, $a = 4$

To further confirm the chaotic nature of the proposed system, a Poincaré section was constructed by recording the intersection points of the system trajectories with the plane $z = 0$ under the condition $z' > 0$. The resulting set of points forms a scattered structure, indicating the complex geometry of the attractor. The absence of a closed curve and the irregular distribution of intersection points provide additional evidence of chaotic dynamics in the system. Figure 13 displays the power spectrum of the time series $x(t)$.

The spectrum exhibits a broadband structure without dominant frequency peaks, which is characteristic of chaotic dynamics.

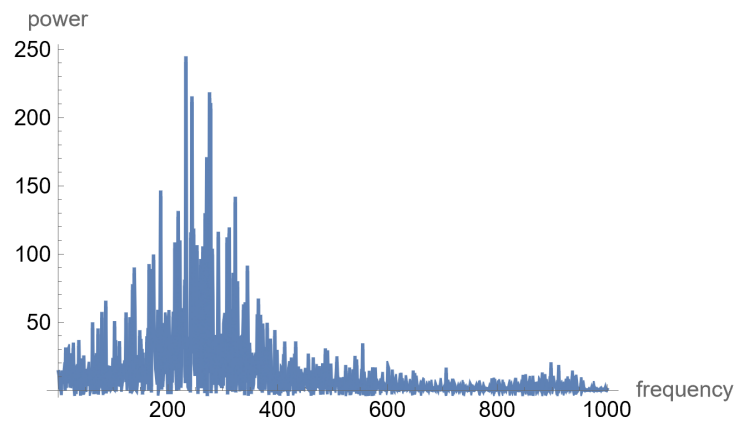


Figure 13. Power spectrum of the time series $x(t)$ for the system (19) for $k = 2$, $a = 4$

Table 6 presents the analysis of system dynamics for various values of the parameter b .

Table 6. Parameter b analysis.

Parameter Value	LE_1	LE_2	LE_3	Behavior
0.005	0.1477	−0.0004	−0.5727	chaotic
0.01	0.1519	−0.0001	−0.5556	chaotic
0.02	0.1617	−0.0007	−0.5396	chaotic
0.05	0.1606	0.0005	−0.4903	chaotic
0.1	0.0033	−0.0224	−0.2669	periodic
0.2	−0.2137	−1.5861	−1.6002	asymptotically stable
0.5	−0.5113	−1.2426	−4.9461	asymptotically stable
1.0	−1.0110	−1.1896	−5.9994	asymptotically stable
2.0	−1.1819	−2.0000	−6.5181	asymptotically stable
4.0	−1.1738	−4.0000	−6.7762	asymptotically stable
9	−1.1696	−6.9309	−8.9882	asymptotically stable

The bifurcation analysis performed revealed that the system exhibits chaotic behavior for $0.005 \leq b \leq 0.05$, but periodic behavior for $b = 0.1$.

Figure 14 shows the phase plots of system (19), $k = 2$, while Figure 15 displays the corresponding solution graphs.

The Lyapunov exponents for system (19), $k = 2$ and parameters are $a = 7.2$, $b = 0.05$, $c = 1$, $d = 0.1$ and initial conditions (3) were calculated as follows:

$$LE_1 = 0.1606; LE_2 = 0.0005; LE_3 = -0.4903. \quad (35)$$

$$D_{KY} = 2 + \frac{0.1606 - 0.0005}{|-0.4903|} = 2.33 \quad (36)$$

Figure 16 displays the Lyapunov characteristic exponents of the system (19), $k = 2$.

The bifurcation diagram for the parameter b is explored. Figure 17 displays the bifurcation diagram of the system (19) for $k = 2$, obtained by varying the control parameter b . The parameters a , c , and d remain fixed. The bifurcation diagram is plotted when b is varied between $0.005 \leq b \leq 0.12$.

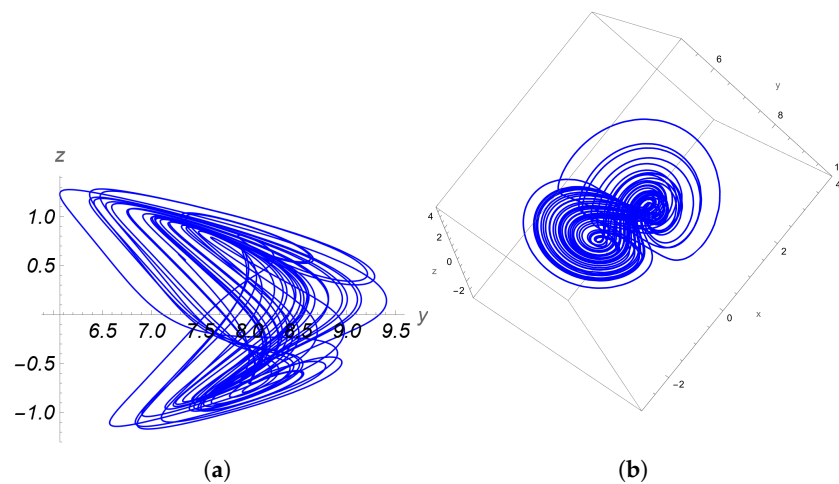


Figure 14. Phase portraits illustrating the dynamics of system (19), $k = 2$ in the two-dimensional planes and the three-dimensional state space: (a) Phase trajectories in the $y - z$ plane, $b = 0.05$. (b) Phase portrait in $x - y - z$ state space, $b = 0.05$.

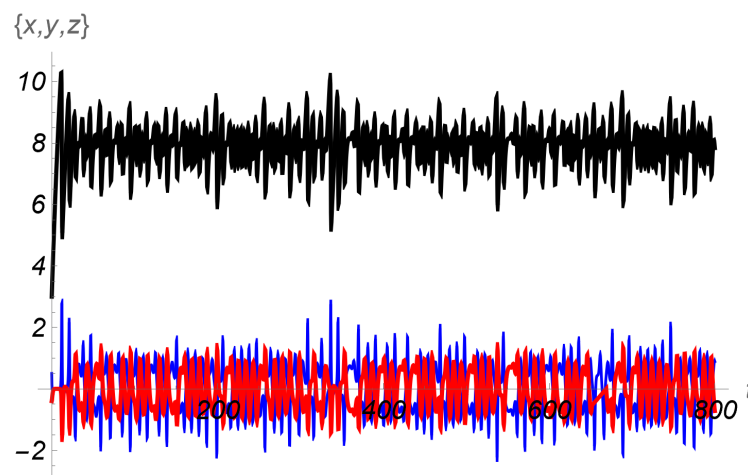


Figure 15. Solutions (x, y, z) of the system (19) with the initial conditions $x(0) = 0.5$; $y(0) = 3$; $z(0) = -0.4$, $b = 0.05$.

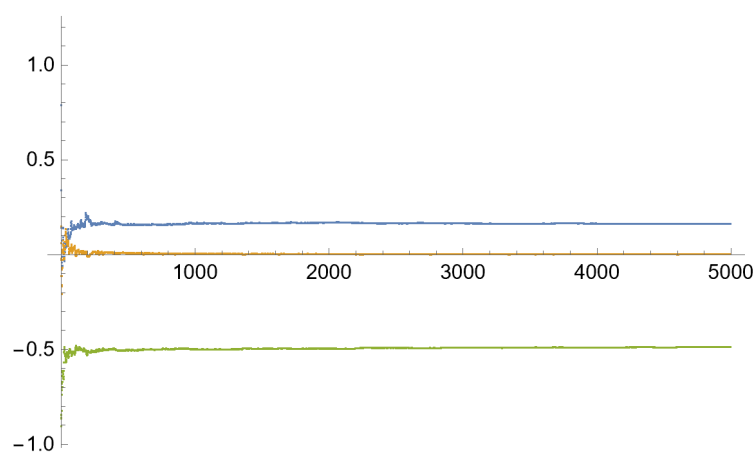


Figure 16. Lyapunov characteristic exponents for the system (19), $k = 2$ and $b = 0.05$ with initial conditions $x(0) = 0.5$; $y(0) = 3$; $z(0) = -0.4$.

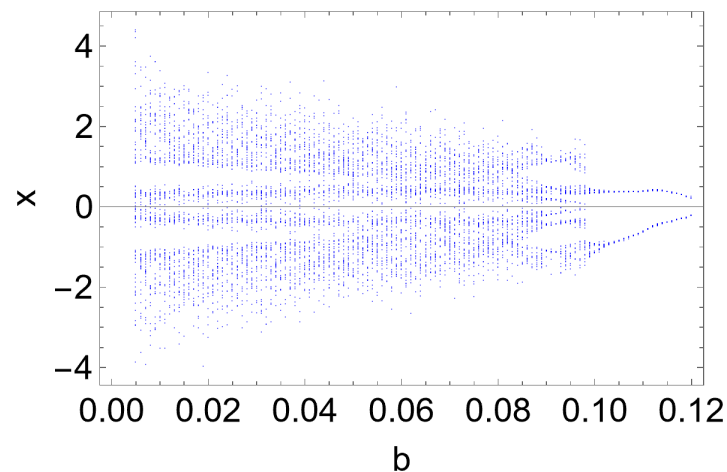


Figure 17. Bifurcation diagram.

Since chaotic behavior occurs only for small values of the parameter b , a zoomed-in bifurcation diagram is presented to highlight the chaotic regime.

Table 7 presents the analysis of system dynamics for various values of the parameter c .

Table 7. Parameter c analysis.

Parameter Value	LE_1	LE_2	LE_3	Behavior
0.1	0.0001	−0.0514	−0.0520	periodic
1.0	0.0033	−0.0224	−0.2669	periodic
1.5	0.0034	−0.0152	−0.9071	periodic
1.6	0.0032	−0.0164	−1.0224	periodic
1.7	0.0030	−0.0094	−1.1402	periodic
1.8	0.0030	−0.0462	−1.1825	periodic
1.85	0.0042	−0.0142	−1.2722	periodic
1.87	0.0047	−0.0205	−1.2862	periodic
1.88	0.0291	−0.00004	−1.3398	chaotic
1.89	0.0196	−0.0005	−1.3454	chaotic
1.9	0.0290	−0.0016	−1.3658	chaotic
2.0	0.0538	0.0002	−1.5031	chaotic
2.01	0.0027	−0.0090	−1.5825	quasiperiodic
2.02	0.0008	−0.0097	−1.5955	quasiperiodic
2.03	−0.0051	−0.0112	−1.6125	asymptotically stable
2.05	−0.0062	−0.0123	−1.6352	asymptotically stable
2.1	−0.0089	−0.0153	−1.6921	asymptotically stable
3.0	−0.0344	−0.0394	−2.6885	asymptotically stable
5.0	−0.0431	−0.0481	−4.8046	asymptotically stable
10.0	−0.0466	−0.0500	−9.9004	asymptotically stable

The bifurcation analysis performed revealed that the system exhibits periodic behavior for $0.1 \leq c \leq 1.87$, chaotic behavior for $1.88 \leq c \leq 2$, quasiperiodic behavior for $2.01 \leq c \leq 2.02$, and asymptotically stable behavior for $2.03 \leq c \leq 10$.

Two zero Lapunov values in 3D systems indicate quasiperiodic dynamic [29]. Quasiperiodicity describes a type of behavior that retains oscillatory features while lacking strict periodic regularity. Such dynamics exhibit patterns that do not repeat with a fixed period and are therefore not exactly periodic [30].

Definition 2. *Quasi-periodic solutions are characterized by a discrete frequency spectrum, which does not consist of integer multiples of one single base frequency. The spectrum consists of linear combinations of frequencies [30,31].*

Figure 18 shows the phase plots of system (19), $k = 2$, while Figure 19 displays the solution graphs.

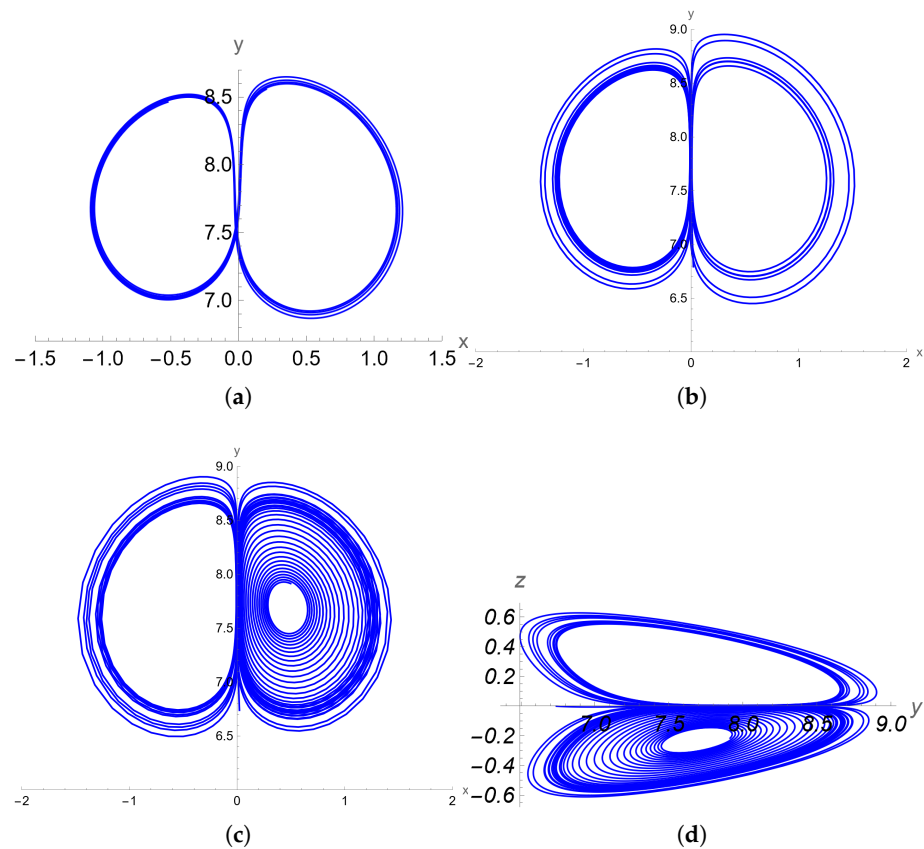


Figure 18. Phase portraits illustrating the dynamics of system (19), $k = 2$ in the two-dimensional planes (a) Phase trajectories in the $x - y$ plane, $c = 1.7$. (b) Phase trajectories in the $x - y$ plane, $c = 2$. (c) Phase trajectories in the $x - y$ plane, $c = 2.02$. (d) Phase portrait in $y - z$ state space, $c = 2.02$.

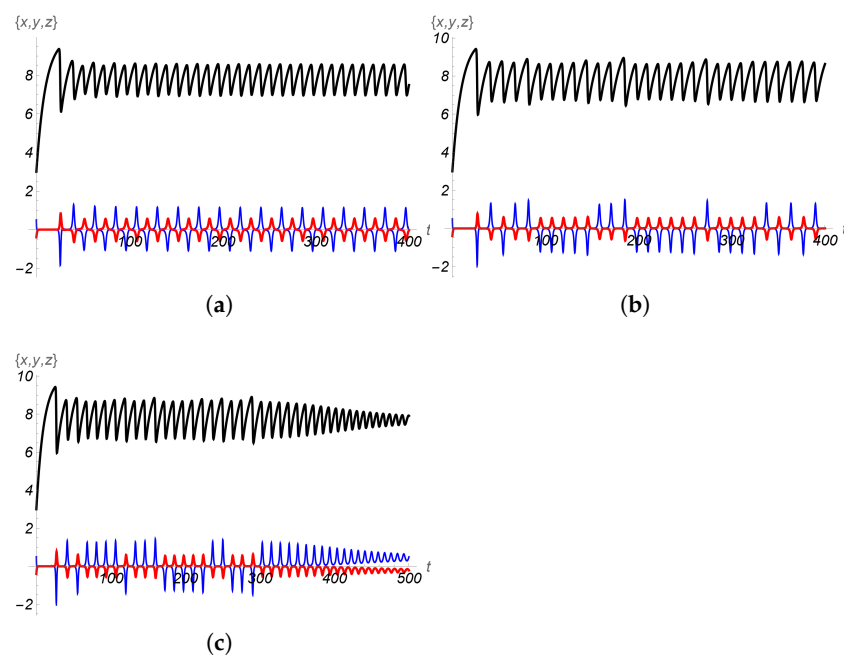


Figure 19. Solutions (x, y, z) of the system (19) with the initial conditions $x(0) = 0.5$; $y(0) = 3$; $z(0) = -0.4$ (a) $c = 1.7$. (b) $c = 2$. (c) $c = 2.02$.

The bifurcation diagram reveals a sequence of qualitative changes in the dynamics of the system as the parameter c varies. For small values of c , the system exhibits stable periodic oscillations, which lose stability and transition to a regime of weak chaotic dynamics. As c increases further, the chaotic attractor transforms into a quasiperiodic regime. The subsequent destruction of quasiperiodic behavior leads to the convergence of system trajectories to a stable critical point, indicating asymptotic stability. Figure 20 displays the bifurcation diagram of the system (19) for $k = 2$, obtained by varying the control parameter c . The parameters a , b , and d remain fixed. The bifurcation diagram is plotted when b is varied between $0.1 \leq c \leq 3$.

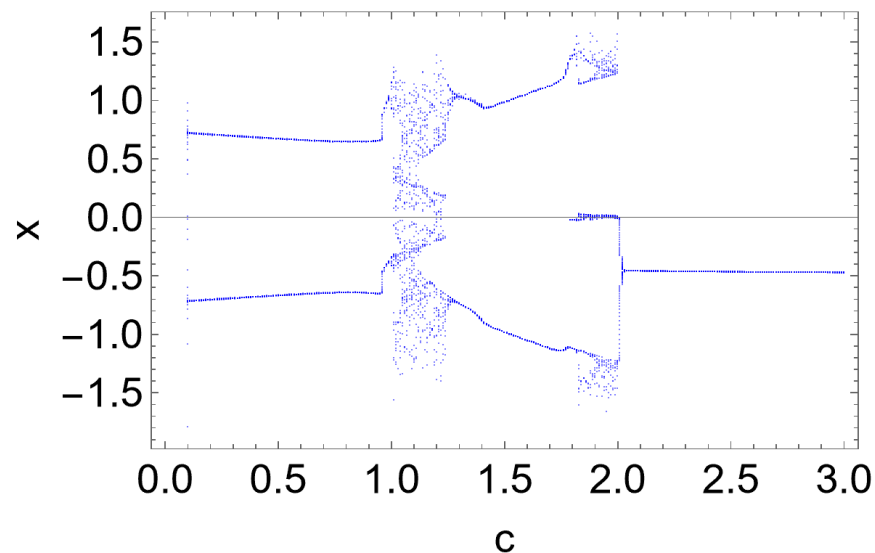


Figure 20. Bifurcation diagram.

Table 8 presents the analysis of system dynamics for various values of the parameter d .

Table 8. Parameter d analysis.

Parameter Value	LE_1	LE_2	LE_3	Behavior
0.005	0.0032	−0.0223	−0.2669	periodic
0.1	0.0033	−0.0224	−0.2669	periodic
0.2	0.0033	−0.0233	−0.2661	periodic
0.5	0.0034	−0.0224	−0.2674	periodic
1.0	0.0036	−0.0219	−0.2686	periodic
5.0	0.0030	−0.0223	−0.2675	periodic
10.0	0.0031	−0.0232	−0.2665	periodic

An analysis of the parameter d shows that the system maintains periodic dynamics throughout the considered range, with no evidence of bifurcations or transitions to more complex regimes.

To analyze the coupling effects between the system parameters, two-parameter bifurcation diagrams were constructed in several parameter planes. The resulting parameter-plane maps are presented in Figure 21. These diagrams provide a global view of the system dynamics and reveal regions corresponding to different dynamical regimes. The results demonstrate that the interaction between parameters plays a significant role in shaping the system behavior and complements the single-parameter bifurcation analysis.

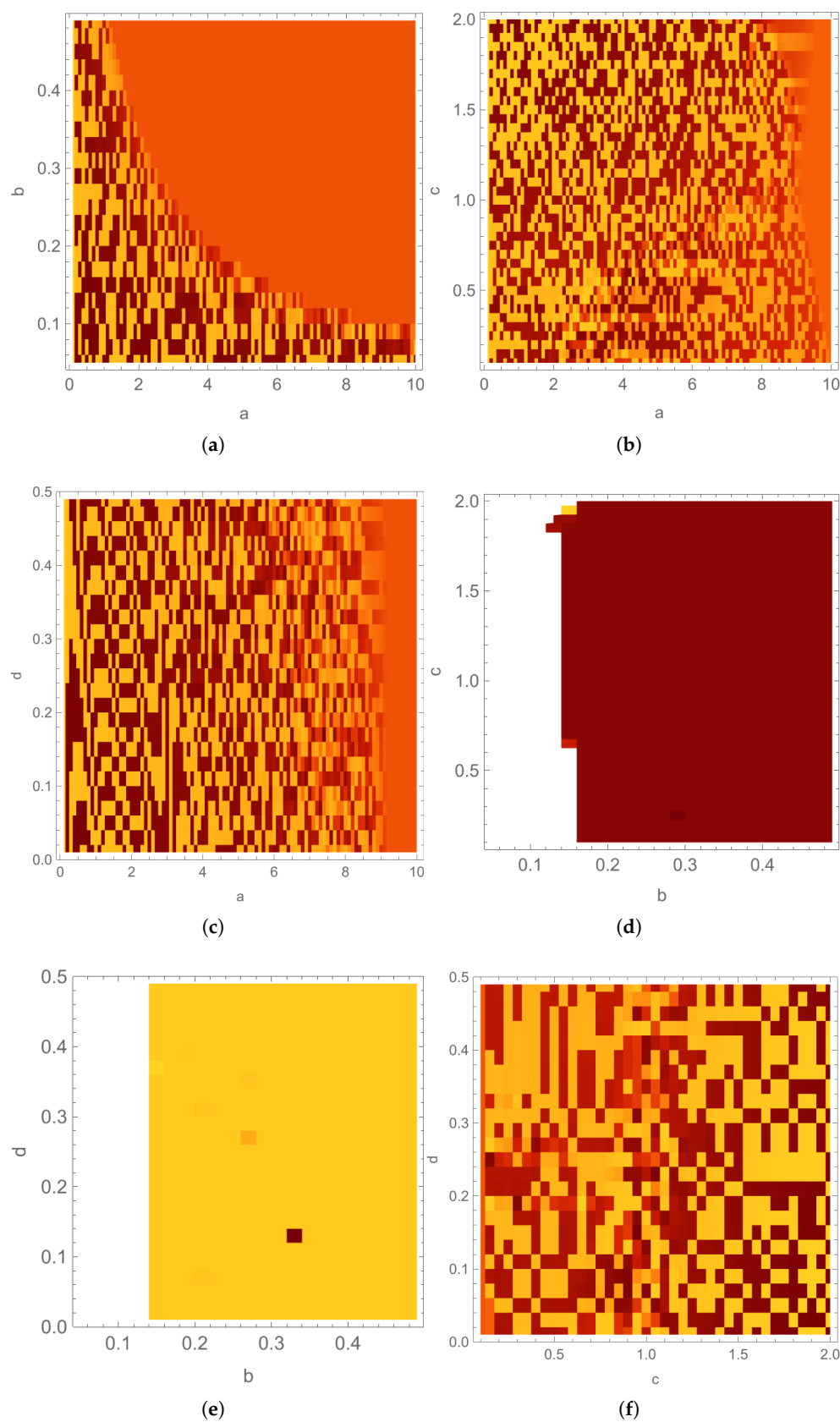


Figure 21. Two-parameter bifurcation diagrams illustrating the coupling effects between system parameters. (a) $a - b$. (b) $a - c$. (c) $a - d$. (d) $b - c$. (e) $b - d$. (f) $c - d$.

3.1.3. Robustness of the System with Respect to Initial Conditions

Figure 22 shows the phase plots of system (19), $k = 2$, while Figure 23 displays the corresponding solution graphs.

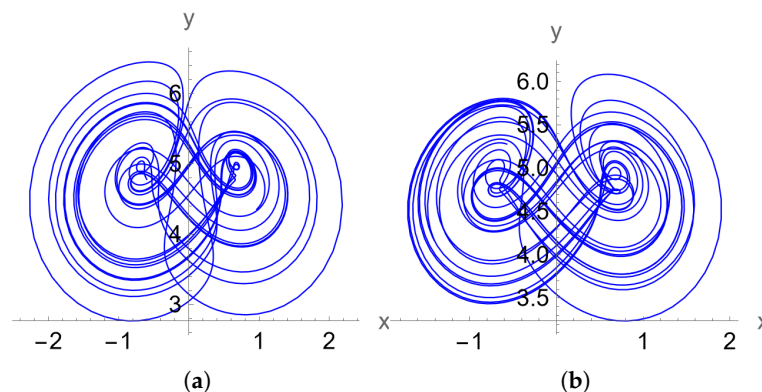


Figure 22. Phase portraits illustrating the dynamics of system (19), $k = 2$ in the two-dimensional planes (a) Phase trajectories in the $x - y$ plane, $a = 4$, $x(0) = 0.1$; $y(0) = 2$; $z(0) = 0.4$. (b) Phase trajectories in the $x - y$ plane, $a = 4$, $x(0) = 0.8$; $y(0) = -2$; $z(0) = -0.1$.

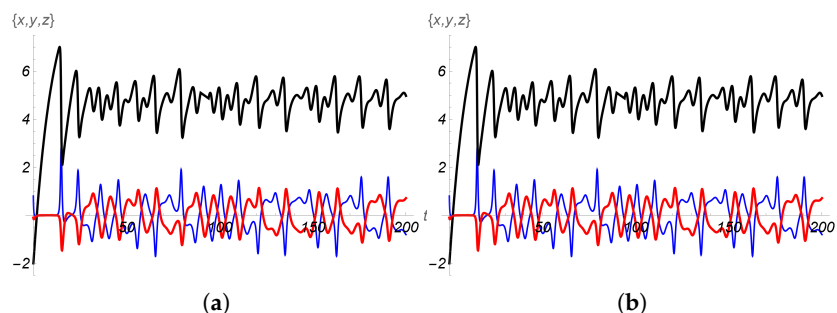


Figure 23. Solutions (x, y, z) of the system (19) with different initial conditions. (a) $a = 4$, $x(0) = 0.1$; $y(0) = 2$; $z(0) = 0.4$. (b) $a = 4$, $x(0) = 0.8$; $y(0) = -2$; $z(0) = -0.1$.

Simulations with different initial conditions lead to the same chaotic attractor, confirming that the dynamics of the system are robust and not dependent on a specific initial state.

3.2. Case 2: $k = 3$

3.2.1. Stability and Dynamical Analysis

The system is (19) and $k = 3$ with given parameters (2) and initial conditions (3). The first critical point $E_1 = (0, 10, 0)$ is obtained similarly as in Case 1.

Let us consider the second case, where $y - a - \frac{1}{c} = 0$. Then $y = a + \frac{1}{c}$ and $z = -\frac{x}{c}$. After that, substitute $y = a + \frac{1}{c}$ into the second equation of the system (19) and equate to zero

$$1 - b\left(a + \frac{1}{c}\right) - dx^2 - x^3 = 0. \quad (37)$$

After simplifying (37) get

$$x^2(x + d) = 1 - b\left(a + \frac{1}{c}\right). \quad (38)$$

Taking into account the parameters (2), one real root of Equation (38) is obtained: $x = 0.5332$. The second critical point is $E_2 = (0.5332, 8.2, -0.5332)$. Let us build the Jakopian matrix

$$J = \begin{pmatrix} y - a & x & 1 \\ -(2d + 3x)x & -b & 0 \\ -1 & 0 & -1 \end{pmatrix}. \quad (39)$$

The characteristic matrix is

$$J - \lambda I = \begin{pmatrix} y - a - \lambda & x & 1 \\ -(2d + 3x)x & -b - \lambda & 0 \\ -1 & 0 & -1 - \lambda \end{pmatrix} \quad (40)$$

and the characteristic equation is

$$\det(J - \lambda I) = -b - \lambda + (-1 - \lambda)(ab + 3x^3 + 2dx^2 - by + a\lambda + b\lambda - y\lambda + \lambda^2) = 0. \quad (41)$$

Put the values of the first critical point $E_1 = (0, 10, 0)$ in (41) and get three roots

$$\lambda_1 = -0.7155, \quad \lambda_2 = -0.1, \quad \lambda_3 = 2.5155. \quad (42)$$

This result indicates that the critical point $E_1 = (0, 10, 0)$ behaves as a saddle point and is unstable.

Put the values of the second critical point $E_2 = (0.5332, 8.2, -0.5332)$ in (41) and get three roots

$$\lambda_1 = -0.6164, \quad \lambda_{2,3} = 0.2582 \pm 0.8736 i. \quad (43)$$

This result indicates that the critical point $E_2 = (0.5332, 8.2, -0.5332)$ behaves as a saddle-focus point and is unstable.

3.2.2. Bifurcation Analysis

Table 9 presents the analysis of system dynamics for various values of the parameter a .

Table 9. Parameter a analysis.

a	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	i	i	i	Indeterminate
1.0	i	i	i	Indeterminate
8.0	i	i	i	Indeterminate
9.0	0.0007	−0.0129	−0.0915	periodic
10.0	−0.1123	−0.4897	−0.5018	asymptotically stable

For small values of the parameter a , the computation of Lyapunov exponents produces indeterminate values. This occurs because the numerical algorithm used to estimate the Lyapunov spectrum does not converge to stable values within the considered integration time. In this parameter range, the system trajectories diverge rapidly in the phase space, which leads to numerical instability in the orthonormalization procedure of the Lyapunov exponent algorithm. As an increase, stable periodic dynamics emerge, followed by convergence to an asymptotically stable critical point.

Table 10 presents the analysis of system dynamics for various values of the parameter b .

Table 10. Parameter b analysis.

b	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	i	i	i	Indeterminate
1.0	−1.0119	−1.1887	−5.9994	asymptotically stable
3.0	−1.1764	−3.0000	−6.6903	asymptotically stable
5.0	−1.1723	−5.0000	−6.8277	asymptotically stable
7.0	−1.1706	−6.8978	−6.9888	asymptotically stable
10.0	−1.1693	−6.9425	−9.9880	asymptotically stable

The analysis of parameter b shows that, except for very small values where the Lyapunov exponents are indeterminate, the system exhibits asymptotically stable behavior throughout the range of parameter b . For $b \geq 0.1$, all Lyapunov exponents are strictly negative, indicating convergence of the trajectories to a stable critical point.

Table 11 presents the analysis of system dynamics for various values of the parameter c .

Table 11. Parameter c analysis.

c	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	i	i	i	Indeterminate
1.0	i	i	i	Indeterminate
5.0	i	i	i	Indeterminate
7.0	−0.0450	−0.0493	−6.8595	asymptotically stable
8.0	−0.0457	−0.0497	−7.8765	asymptotically stable
10.0	−0.0463	−0.0500	−9.9007	asymptotically stable

The analysis of parameter c reveals that for small values of this parameter, the Lyapunov exponents are indeterminate, indicating the absence of a well-defined asymptotic regime. As c increases, all Lyapunov exponents become negative, which implies the asymptotic stability of the system. Thus, the parameter c plays a stabilizing role, leading to suppression of complex dynamics and convergence to a stable critical point.

Table 12 presents the analysis of system dynamics for various values of the parameter d .

Table 12. Parameter d analysis.

d	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	i	i	i	Indeterminate
1.0	i	i	i	Indeterminate
3.0	0.0024	−0.1492	−0.1552	periodic
5.0	0.0020	−0.0219	−0.2774	periodic
6.0	0.0020	−0.0093	−0.2892	quasiperiodic
7.0	0.0012	−0.0017	−0.2954	quasiperiodic
8.0	0.0021	−0.0073	−0.2887	periodic
10.0	0.0022	−0.0279	−0.2663	periodic

For small values of the parameter d , the computation of Lyapunov exponents yields indeterminate results, indicating that the system does not reach a well-defined asymptotic regime within the integration time considered. As d increases, stable periodic dynamics and quasiperiodic dynamics emerge.

3.3. Case 3: $k = 5$

3.3.1. Stability and Dynamical Analysis

The system is (19) and $k = 5$ with given parameters (2) and initial conditions (3). The first critical point $E_1 = (0, 10, 0)$ is obtained similarly as in Case 1.

Let us consider the second case, where $y - a - \frac{1}{c} = 0$. Then $y = a + \frac{1}{c}$ and $z = -\frac{x}{c}$. After that, substitute $y = a + \frac{1}{c}$ into the second equation of the system (19) and equate to zero

$$1 - b\left(a + \frac{1}{c}\right) - dx^2 - x^5 = 0. \quad (44)$$

After simplifying (44) get

$$x^2(x^3 + d) = 1 - b\left(a + \frac{1}{c}\right). \quad (45)$$

Taking into account the parameters (2), one real root of Equation (45) is obtained: $x = 0.6701$. The second critical point is $E_2 = (0.6701, 8.2, -0.6701)$. Let us build the Jakobian matrix

$$J = \begin{pmatrix} y - a & x & 1 \\ -(2d + 5x^3)x & -b & 0 \\ -1 & 0 & -1 \end{pmatrix}. \quad (46)$$

The characteristic matrix is

$$J - \lambda I = \begin{pmatrix} y - a - \lambda & x & 1 \\ -(2d + 5x^3)x & -b - \lambda & 0 \\ -1 & 0 & -1 - \lambda \end{pmatrix} \quad (47)$$

and the characteristic equation is

$$\det(J - \lambda I) = -b - \lambda + (-1 - \lambda)(ab + 5x^5 + 2dx^2 - by + a\lambda + b\lambda - y\lambda + \lambda^2) = 0. \quad (48)$$

Put the values of the first critical point $E_1 = (0, 10, 0)$ in (48) and get three roots

$$\lambda_1 = -0.7155, \quad \lambda_2 = -0.1, \quad \lambda_3 = 2.5155. \quad (49)$$

This result indicates that the critical point $E_1 = (0, 10, 0)$ behaves as a saddle point and is unstable.

Put the values of the second critical point $E_2 = (0.6701, 8.2, -0.6701)$ in (48) and get three roots

$$\lambda_1 = -0.6683, \quad \lambda_{2,3} = 0.2842 \pm 1.0317i. \quad (50)$$

This result indicates that the critical point $E_2 = (0.6701, 8.2, -0.6701)$ behaves as a saddle-focus point and is unstable.

3.3.2. Bifurcation Analysis

Table 13 presents the analysis of system dynamics for various values of the parameter a .

Table 13. Parameter a analysis.

a	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	i	i	i	Indeterminate
1.0	i	i	i	Indeterminate
3.0	i	i	i	Indeterminate
7.2	i	i	i	Indeterminate
9.0	0.0007	−0.0129	−0.0916	periodic
9.5	−0.1113	−0.2406	−0.2519	asymptotically stable
10.0	−0.1123	−0.4897	−0.5018	asymptotically stable

For small values of parameter a , the computation of Lyapunov exponents yields indeterminate results, indicating that the system does not reach a well-defined asymptotic regime within the integration time considered. As the increase, stable periodic dynamics emerge, followed by convergence to an asymptotically stable critical point.

Table 14 presents the analysis of system dynamics for various values of the parameter b .

Table 14. Parameter b analysis.

b	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	i	i	i	Indeterminate
1.0	−1.0119	−1.1887	−5.9994	asymptotically stable
4.0	−1.1738	−4.0000	−6.7762	asymptotically stable
7.0	−1.1706	−6.8978	−6.9888	asymptotically stable
10.0	−1.1693	−6.9425	−9.9880	asymptotically stable

The analysis of parameter b reveals that for small values of this parameter, the Lyapunov exponents are indeterminate, indicating the absence of a well-defined asymptotic regime. As b increases, all Lyapunov exponents become negative, which implies the asymptotic stability of the system.

Table 15 presents the analysis of system dynamics for various values of the parameter c .

Table 15. Parameter c analysis.

c	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	i	i	i	Indeterminate
1.0	i	i	i	Indeterminate
2.0	i	i	i	Indeterminate
5.0	i	i	i	Indeterminate
6.0	−0.0428	−0.0477	−5.8389	asymptotically stable
8.0	−0.0452	−0.0492	−7.8773	asymptotically stable
10.0	−0.0460	−0.0499	−9.9011	asymptotically stable

The analysis of parameter c reveals that for small values of this parameter, the Lyapunov exponents are indeterminate, indicating the absence of a well-defined asymptotic regime. As c increases, all Lyapunov exponents become negative, which implies the asymptotic stability of the system.

Table 16 presents the analysis of system dynamics for various values of the parameter d .

Table 16. Parameter d analysis.

d	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	i	i	i	Indeterminate
1.0	i	i	i	Indeterminate
2.0	i	i	i	Indeterminate
3.0	0.0019	−0.0267	−0.2661	periodic
4.0	0.0019	−0.1386	−0.1515	periodic
5.0	0.0021	−0.0589	−0.2311	periodic
7.0	0.0019	−0.0342	−0.2552	periodic
8.0	0.0023	−0.0303	−0.2593	periodic
10.0	0.0028	−0.0268	−0.2630	periodic

For small values of the parameter d , the computation of Lyapunov exponents yields indeterminate results, indicating that the system does not reach a well-defined asymptotic regime within the integration time considered. As d increases, stable periodic dynamics emerges.

3.4. Case 4: $k = 6$

3.4.1. Stability and Dynamical Analysis

The system is (19) and $k = 6$ with given parameters (2) and initial conditions (3). The first critical point $E_1 = (0, 10, 0)$ is obtained similarly as in Case 1.

Let us consider the second case, where $y - a - \frac{1}{c} = 0$. Then $y = a + \frac{1}{c}$ and $z = -\frac{x}{c}$. After that, substitute $y = a + \frac{1}{c}$ into the second equation of the system (19) and equate to zero

$$1 - b\left(a + \frac{1}{c}\right) - dx^2 - x^6 = 0. \quad (51)$$

After simplifying (51) get

$$x^2(x^4 + d) = 1 - b\left(a + \frac{1}{c}\right). \quad (52)$$

Taking into account the parameters (2), the two roots of Equation (52) are obtained: $x_{1,2} = \pm 0.7112$. The second and third critical points are $E_2 = (0.7112, 8.2, -0.7112)$ and $E_3 = (-0.7112, 8.2, 0.7112)$. Let us build the Jacobian matrix

$$J = \begin{pmatrix} y - a & x & 1 \\ -2(d + 3x^4)x & -b & 0 \\ -1 & 0 & -1 \end{pmatrix}. \quad (53)$$

The characteristic matrix is

$$J - \lambda I = \begin{pmatrix} y - a - \lambda & x & 1 \\ -2(d + 3x^4)x & -b - \lambda & 0 \\ -1 & 0 & -1 - \lambda \end{pmatrix} \quad (54)$$

and the characteristic equation is

$$\det(J - \lambda I) = -b - \lambda + (-1 - \lambda)(ab + 6x^6 + 2dx^2 - by + a\lambda + b\lambda - y\lambda + \lambda^2) = 0. \quad (55)$$

Put the values of the first critical point $E_1 = (0, 10, 0)$ in (55) and get three roots

$$\lambda_1 = -0.7155, \quad \lambda_2 = -0.1, \quad \lambda_3 = 2.5155. \quad (56)$$

This result indicates that the critical point $E_1 = (0, 10, 0)$ behaves as a saddle point and is unstable.

Put the values of the second critical point $E_2 = (0.7112, 8.2, -0.7112)$ in (55) and get three roots

$$\lambda_1 = -0.6859, \quad \lambda_{2,3} = 0.2930 \pm 0.2930i \quad (57)$$

This result indicates that the critical point $E_2 = (0.7112, 8.2, -0.7112)$ behaves as a saddle-focus point and is unstable.

Put the values of the third critical point $E_3 = (-0.7112, 8.2, 0.7112)$ in (55) and get three roots

$$\lambda_1 = -0.6859, \quad \lambda_{2,3} = 0.2930 \pm 0.2930i \quad (58)$$

This result indicates that the critical point $E_3 = (-0.7112, 8.2, 0.7112)$ behaves as a saddle-focus point and is unstable.

3.4.2. Bifurcation Analysis

Table 17 presents the analysis of system dynamics for various values of the parameter a .

Table 17. Parameter a analysis.

a	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	−0.0003	−0.0868	−0.4777	periodic
1.0	−0.0000	−0.0651	−0.4830	periodic
3.0	0.0003	−0.0152	−0.4885	periodic
5.0	0.0010	−0.2156	−0.2179	periodic
6.0	0.0028	−0.0905	−0.2983	periodic
6.1	0.0008	−0.0199	−0.3629	periodic
6.2	0.0025	−0.0087	−0.3708	periodic
6.3	0.0178	−0.0001	−0.3897	chaotic
6.5	0.0493	−0.0027	−0.4143	chaotic
7.0	0.0983	−0.0009	−0.4242	chaotic
7.2	0.1060	−0.0007	−0.4178	chaotic
8.0	0.1061	0.0002	−0.3565	chaotic
8.5	0.0390	0.0002	−0.2574	chaotic
8.6	0.0081	−0.0025	−0.2201	periodic
8.7	0.0074	−0.0996	−0.1077	periodic
8.8	0.0085	−0.0175	−0.1750	periodic
9.0	0.0007	−0.0129	−0.0916	periodic
10.0	−0.1123	−0.4897	−0.5018	asymptotically stable

The bifurcation analysis performed revealed that the system exhibits periodic behavior for $0.1 \leq a \leq 6.2$ and $8.6 \leq a \leq 9$, chaotic behavior for $6.3 \leq a \leq 8.5$, and asymptotically stable for $a > 9$.

Figure 24 shows the phase plots of system (19), $k = 6$, while Figure 25 displays the corresponding solution graphs.

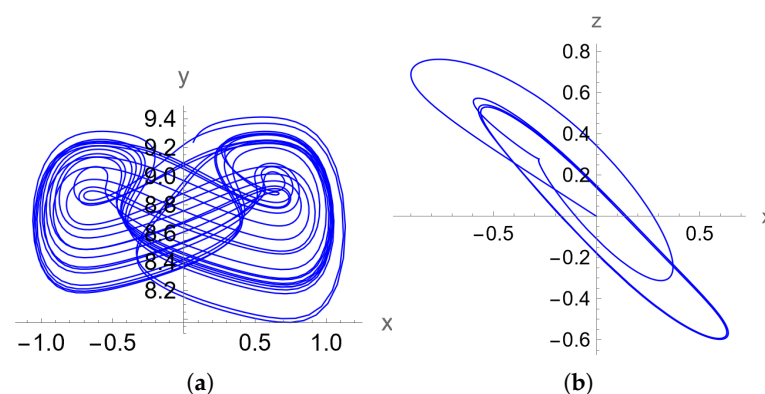


Figure 24. Phase portraits illustrating the dynamics of system (19), $k = 6$ in the two-dimensional planes: (a) Phase trajectories in the $x - y$ plane, $a = 8$. (b) Phase trajectories in the $x - z$ plane, $a = 8.8$.

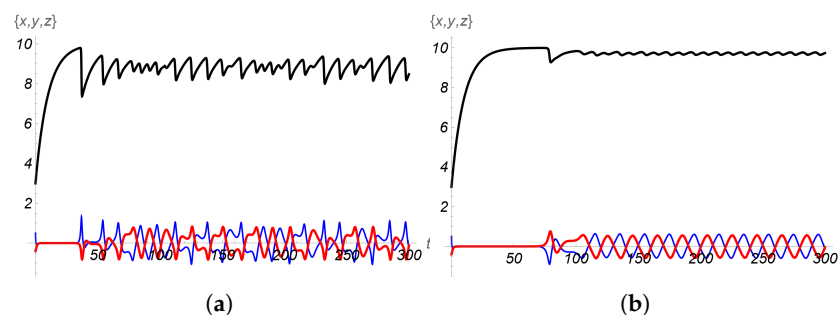


Figure 25. Solutions (x, y, z) of the system (19) with the initial conditions $x(0) = 0.5; y(0) = 3; z(0) = -0.4, k = 6$. (a) Solutions (x, y, z) of the system (19), $a = 8$. (b) Solutions (x, y, z) of the system (19), $a = 8.8$.

Figure 26 displays the bifurcation diagram of the system (19) for $k = 6$, obtained by varying the control parameter a . The parameters b, c , and d remain fixed. The bifurcation diagram is plotted when a is varied between $0.1 \leq a \leq 10$.

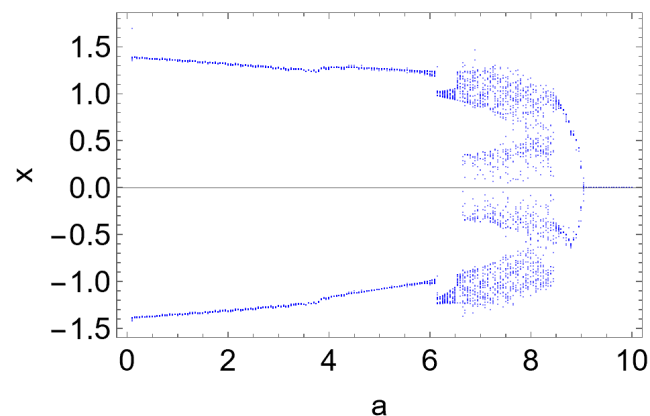


Figure 26. Bifurcation diagram.

Table 18 presents the analysis of system dynamics for various values of the parameter b .

Table 18. Parameter b analysis.

b	LE_1	LE_2	LE_3	<i>Solution</i>
0.05	0.0005	−0.0047	−0.4625	periodic
0.08	0.0029	−0.0218	−0.3770	periodic
0.09	0.0493	0.0002	−0.4161	chaotic
0.10	0.1060	−0.0007	−0.4178	chaotic
0.11	0.1089	0.0006	−0.3624	chaotic
0.12	0.0069	−0.0991	−0.1056	periodic
0.15	−0.1613	−0.7557	−0.7669	asymptotically stable
0.20	−0.2147	−1.5852	−1.6002	asymptotically stable
0.30	−0.3130	−1.3937	−3.4599	asymptotically stable
0.50	−0.5122	−1.2417	−4.9461	asymptotically stable
1.00	−1.0119	−1.1887	−5.9994	asymptotically stable
4.00	−1.1738	−4.0000	−6.7762	asymptotically stable
7.00	−1.1706	−6.8978	−6.9888	asymptotically stable
10.0	−1.1693	−6.9425	−9.9880	asymptotically stable

The bifurcation analysis performed revealed that the system exhibits chaotic behavior for $0.09 \leq b \leq 0.11$, periodic behavior for $0.05 \leq b \leq 0.08$ and $b = 0.12$, and asymptotically stable behavior for $0.15 \leq b \leq 10$. Figure 27 shows the phase plots of system (19), $k = 6$, while Figure 28 displays the corresponding solution graphs.

Figure 29 displays the bifurcation diagram of the system (19) for $k = 6$, obtained by varying the control parameter b . The parameters a, c , and d remain fixed. The bifurcation diagram is plotted when b is varied between $0.05 \leq b \leq 0.15$.

Table 19 presents the analysis of system dynamics for various values of the parameter c .

The bifurcation analysis performed revealed that the system exhibits chaotic behavior for $0.9 \leq c \leq 0.1$, periodic behavior for $0.1 \leq c \leq 0.8$ and $1.1 \leq c \leq 2$, and asymptotically stable behavior for $4 \leq c \leq 10$.

Figure 30 displays the bifurcation diagram of the system (19) for $k = 6$, obtained by varying the control parameter c . The parameters a, b , and d remain fixed. The bifurcation diagram is plotted when b is varied between $0.1 \leq c \leq 4$.

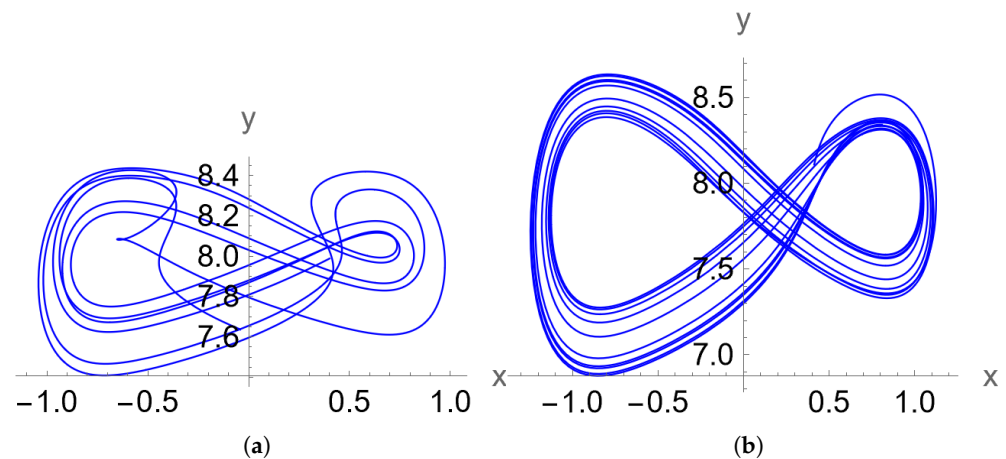


Figure 27. Phase portraits illustrating the dynamics of system (19), $k = 6$ in the two-dimensional planes: (a) Phase trajectories in the $x - y$ plane, $b = 0.11$. (b) Phase trajectories in the $x - y$ plane, $b = 0.08$.

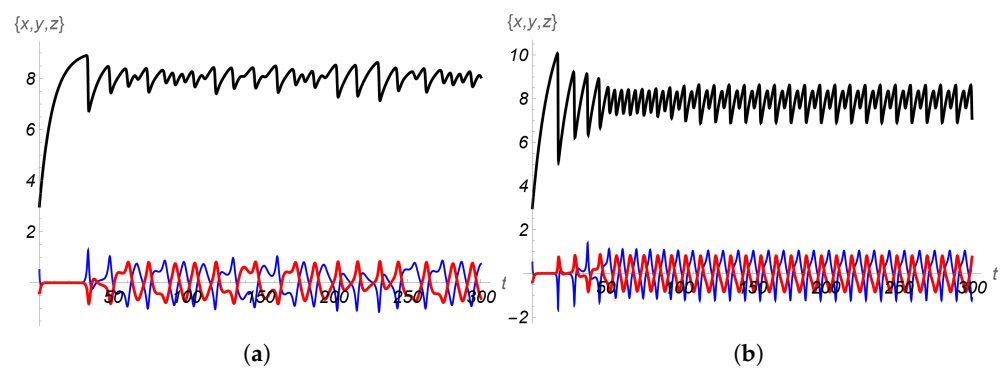


Figure 28. Solutions (x, y, z) of the system (19) with the initial conditions $x(0) = 0.5; y(0) = 3; z(0) = -0.4, k = 6$. (a) Solutions (x, y, z) of the system (19), $b = 0.11$. (b) Solutions (x, y, z) of the system (19), $b = 0.08$.

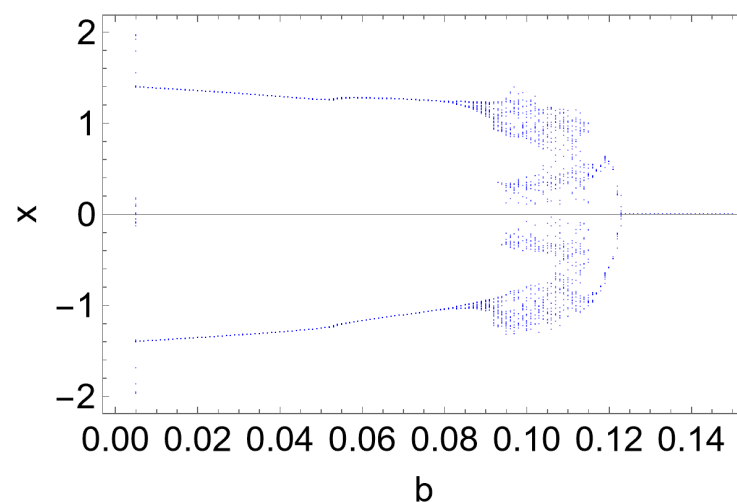


Figure 29. Bifurcation diagram.

Table 19. Parameter c analysis.

c	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	0.0018	−0.0498	−0.0540	periodic
0.8	0.0028	−0.0596	−0.1230	periodic
0.9	0.1159	0.0004	−0.3125	chaotic
1.0	0.1060	−0.0007	−0.4178	chaotic
1.1	0.0029	−0.0910	−0.3618	periodic
1.2	0.0024	−0.0488	−0.5399	periodic
1.5	0.0021	−0.2300	−0.7420	periodic
2.0	0.0044	−0.0168	−1.4747	periodic
4.0	−0.0346	−0.0398	−3.7713	asymptotically stable
7.0	−0.0440	−0.0485	−6.8612	asymptotically stable
10.0	−0.0459	−0.0498	−9.9013	asymptotically stable

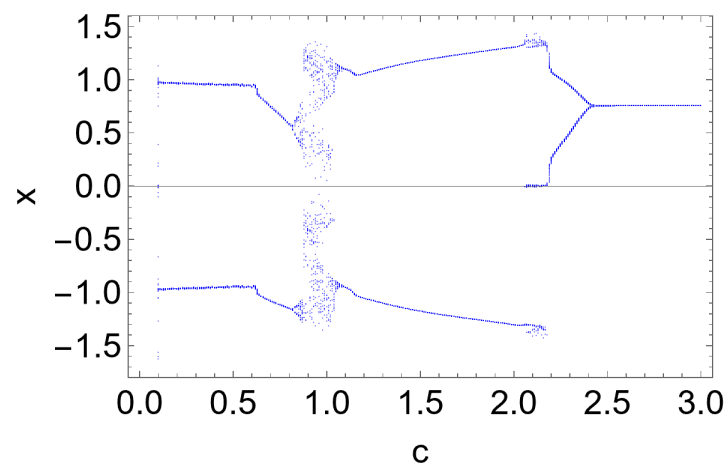
**Figure 30.** Bifurcation diagram.

Table 20 presents the analysis of system dynamics for various values of the parameter b .

Table 20. Parameter d analysis.

d	LE_1	LE_2	LE_3	<i>Solution</i>
0.005	0.0859	−0.0001	−0.4148	chaotic
0.05	0.1002	0.0001	−0.4211	chaotic
0.1	0.1060	−0.0007	−0.4178	chaotic
1.0	0.0406	0.0002	−0.3054	chaotic
2.0	0.0280	0.0007	−0.3152	chaotic
2.2	0.0172	0.0001	−0.3031	chaotic
2.3	0.0019	−0.0066	−0.2803	quasiperiodic
2.4	0.0090	0.0000	−0.2940	quasiperiodic
2.5	0.0025	−0.0133	−0.2746	periodic
3.0	0.0019	−0.0111	−0.2766	periodic
5.0	0.0044	−0.0145	−0.2766	periodic
7.0	0.0031	−0.0195	−0.2702	periodic
10.0	0.0031	−0.0216	−0.2681	periodic

The bifurcation analysis performed revealed that the system exhibits chaotic behavior for $0.05 \leq d \leq 2.2$, quasiperiodic behavior for $2.3 \leq d \leq 2.4$, and periodic behavior for $2.5 \leq d \leq 10$.

Figure 31 displays the bifurcation diagram of the system (19) for $k = 6$, obtained by varying the control parameter d . The parameters a, b , and c remain fixed. The bifurcation diagram is plotted when d is varied between $0.1 \leq d \leq 4$.

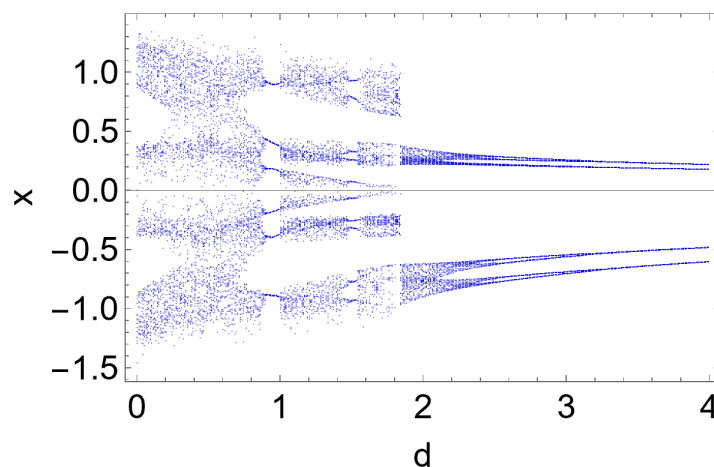


Figure 31. Bifurcation diagram.

3.5. Case 5: $k = 7$

3.5.1. Stability and Dynamical Analysis

The system is (19) and $k = 7$ with given parameters (2) and initial conditions (3). The first critical point $E_1 = (0, 10, 0)$ is obtained similarly as in Case 1. $k = 2$.

Let us consider the second case, where $y - a - \frac{1}{c} = 0$. Then $y = a + \frac{1}{c}$ and $z = -\frac{x}{c}$. After that, substitute $y = a + \frac{1}{c}$ into the second equation of the system (19) and equate to zero

$$1 - b\left(a + \frac{1}{c}\right) - dx^2 - x^7 = 0. \quad (59)$$

After simplifying (25) get

$$x^2(x^5 + d) = 1 - b\left(a + \frac{1}{c}\right). \quad (60)$$

Taking into account the parameters (2), only one real root of Equation (60) is obtained: $x = 0.7428$. The second critical point is $E_2 = (0.7428, 8.2, -0.7428)$. Let us build the Jacobian matrix

$$J = \begin{pmatrix} y - a & x & 1 \\ -(2d + 7x^5)x & -b & 0 \\ -1 & 0 & -1 \end{pmatrix}. \quad (61)$$

The characteristic matrix is

$$J - \lambda I = \begin{pmatrix} y - a - \lambda & x & 1 \\ -(2d + 7x^5)x & -b - \lambda & 0 \\ -1 & 0 & -1 - \lambda \end{pmatrix} \quad (62)$$

and the characteristic equation is

$$\det(J - \lambda I) = -b - \lambda + (-1 - \lambda)(ab + 7x^7 + 2dx^2 - by + a\lambda + b\lambda - y\lambda + \lambda^2) = 0. \quad (63)$$

Put the values of the first critical point $E_1 = (0, 10, 0)$ in (63) and get three roots

$$\lambda_1 = -0.7155, \quad \lambda_2 = -0, 1, \quad \lambda_3 = 2.5155. \quad (64)$$

This result indicates that the critical point $E_1 = (0, 10, 0)$ behaves as a saddle point and is unstable.

Put the values of the second critical point $E_2 = (0.7428, 8.2, -0.7428)$ in (63) and get three roots

$$\lambda_1 = -0.7005, \quad \lambda_{2,3} = 0.3003 \pm 1.1466i \quad (65)$$

This result indicates that the critical point $E_2 = (0.7428, 8.2, -0.7428)$ behaves as a saddle-focus point and is unstable.

3.5.2. Bifurcation Analysis

Table 21 presents the analysis of system dynamics for various values of the parameter a .

Table 21. Parameter a analysis.

a	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	i	i	i	Indeterminate
2.0	i	i	i	Indeterminate
5.0	i	i	i	Indeterminate
8.0	i	i	i	Indeterminate
9.0	0.0007	−0.0129	−0.0916	periodic
9.5	−0.1113	−0.2406	−0.2519	asymptotically stable
10.0	−0.1123	−0.4897	−0.5018	asymptotically stable

For $0.1 \leq a < 9$, the computation of Lyapunov exponents yields indeterminate results, indicating that the system does not reach a well-defined asymptotic regime within the integration time considered. As a increases, stable periodic dynamics emerge.

Table 22 presents the analysis of system dynamics for various values of the parameter b .

Table 22. Parameter b analysis.

b	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	i	i	i	Indeterminate
2.0	−1.1819	−2.0000	−6.5181	asymptotically stable
5.0	−1.1723	−5.0000	−6.8277	asymptotically stable
7.0	−1.1706	−6.8978	−6.9888	asymptotically stable
10.0	−1.1693	−6.9425	−9.9880	asymptotically stable

The bifurcation analysis performed revealed that the system exhibits an asymptotically stable behavior for $2 \leq b \leq 10$.

Table 23 presents the analysis of system dynamics for various values of the parameter c .

Table 23. Parameter c analysis.

c	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	i	i	i	Indeterminate
1.0	i	i	i	Indeterminate
3.0	−0.0178	−0.0233	−2.7209	asymptotically stable
4.0	i	i	i	Indeterminate
5.0	i	i	i	Indeterminate
7.0	−0.0436	−0.0483	−6.8618	asymptotically stable
10.0	−0.0436	−0.0483	−6.8618	asymptotically stable

The bifurcation analysis performed revealed that the system exhibits an asymptotically stable behavior for $c = 3$ and $7 \leq c \leq 10$.

Table 24 presents the analysis of system dynamics for various values of the parameter c .

Table 24. Parameter d analysis.

d	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	i	i	i	Indeterminate
1.0	i	i	i	Indeterminate
3.0	0.0017	−0.0496	−0.2393	periodic
5.0	0.0026	−0.0256	−0.2640	periodic
7.0	0.0029	−0.0234	−0.2663	periodic
9.0	0.0030	−0.0229	−0.2667	periodic
10.0	0.0030	−0.0229	−0.2668	periodic

For small values of the parameter d , the computation of Lyapunov exponents yields indeterminate results, indicating that the system does not reach a well-defined asymptotic regime within the integration time considered. As d increases, stable periodic dynamics emerge.

3.6. Case 6: $k = 8$

3.6.1. Stability and Dynamical Analysis

The system is (19) and $k = 8$ with given parameters (2) and initial conditions (3). The first critical point $E_1 = (0, 10, 0)$ is obtained similarly as in Case 1. $k = 2$.

Let us consider the second case, where $y - a - \frac{1}{c} = 0$. Then $y = a + \frac{1}{c}$ and $z = -\frac{x}{c}$. After that, substitute $y = a + \frac{1}{c}$ into the second equation of the system (19) and equate to zero

$$1 - b\left(a + \frac{1}{c}\right) - dx^2 - x^8 = 0. \quad (66)$$

After simplifying (66) get

$$x^2(x^6 + d) = 1 - b\left(a + \frac{1}{c}\right). \quad (67)$$

Taking into account the parameters (2), the two roots of Equation (67) are obtained: $x_{1,2} = \pm 0.7680$. The second and third critical points are $E_2 = (0.7680, 8.2, -0.7680)$ and $E_3 = (-0.7680, 8.2, 0.7680)$. Let us build the Jacobian matrix

$$J = \begin{pmatrix} y - a & x & 1 \\ -2(d + 4x^6)x & -b & 0 \\ -1 & 0 & -1 \end{pmatrix}. \quad (68)$$

The characteristic matrix is

$$J - \lambda I = \begin{pmatrix} y - a - \lambda & x & 1 \\ -2(d + 4x^6)x & -b - \lambda & 0 \\ -1 & 0 & -1 - \lambda \end{pmatrix} \quad (69)$$

and the characteristic equation is

$$\det(J - \lambda I) = -b - \lambda + (-1 - \lambda)(ab + 7x^7 + 2dx^2 - by + a\lambda + b\lambda - y\lambda + \lambda^2) = 0. \quad (70)$$

Put the values of the first critical point $E_1 = (0, 10, 0)$ in (70) and get three roots

$$\lambda_1 = -0.7155, \quad \lambda_2 = -0.1, \quad \lambda_3 = 2.5155. \quad (71)$$

This result indicates that the critical point $E_1 = (0, 10, 0)$ behaves as a saddle point and is unstable.

Put the values of the second critical point $E_2 = (0.7680, 8.2, -0.7680)$ in (70) and get three roots

$$\lambda_1 = -0.7130, \quad \lambda_{2,3} = 0.3065 \pm 1.1955 i \quad (72)$$

This result indicates that the critical point $E_2 = (0.7680, 8.2, -0.7680)$ behaves as a saddle-focus point and is unstable.

Put the values of the third critical point $E_3 = (-0.7680, 8.2, 0.7680)$ in (70) and get three roots

$$\lambda_1 = -0.7130, \quad \lambda_{2,3} = 0.3065 \pm 1.1955 i \quad (73)$$

This result indicates that the critical point $E_3 = (-0.7680, 8.2, 0.7680)$ behaves as a saddle-focus point and is unstable.

3.6.2. Bifurcation Analysis

Table 25 presents the analysis of system dynamics for various values of the parameter a .

Table 25. Parameter a analysis.

a	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	0.0004	−0.1551	−0.4246	periodic
2.0	0.0001	−0.0922	−0.4501	periodic
5.0	0.0020	−0.0232	−0.4336	periodic
6.0	0.0016	−0.2008	−0.2077	periodic
6.5	0.0025	−0.0179	−0.3643	periodic
6.7	0.0003	−0.0199	−0.3498	periodic
6.8	0.0339	−0.0021	−0.3955	chaotic
7.0	0.0548	0.0001	−0.4090	chaotic
7.2	0.0802	0.0005	−0.4164	chaotic
8.0	0.1124	−0.0001	−0.3688	chaotic
8.5	0.0322	−0.0054	−0.2389	chaotic
8.6	0.0076	−0.0094	−0.2128	quasiperiodic
8.7	0.0077	−0.0377	−0.1711	periodic
9.0	0.0007	−0.0129	−0.0916	periodic
10.0	−0.1123	−0.4897	−0.5018	asymptotically stable

The bifurcation analysis performed revealed that the system exhibits chaotic behavior for $6.8 \leq a \leq 8.5$, periodic behavior for $0.1 \leq a \leq 6.7$ and $8.7 \leq a \leq 9$, quasiperiodic behavior for $a = 8.6$, and asymptotically stable behavior for $a = 10$.

Figure 32 shows the phase plots of system (19), $k = 6$, while Figure 33 displays the corresponding solution graphs.

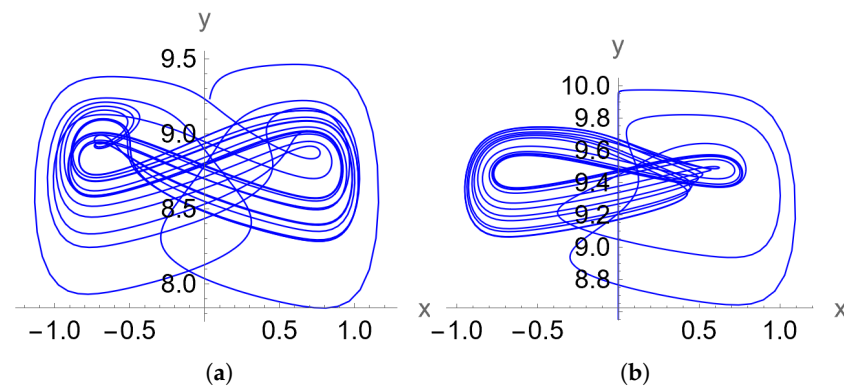


Figure 32. Phase portraits illustrating the dynamics of system (19), $k = 8$ in the two-dimensional planes: (a) Phase trajectories in the $x - y$ plane, $a = 8$. (b) Phase trajectories in the $x - y$ plane, $a = 8.6$.

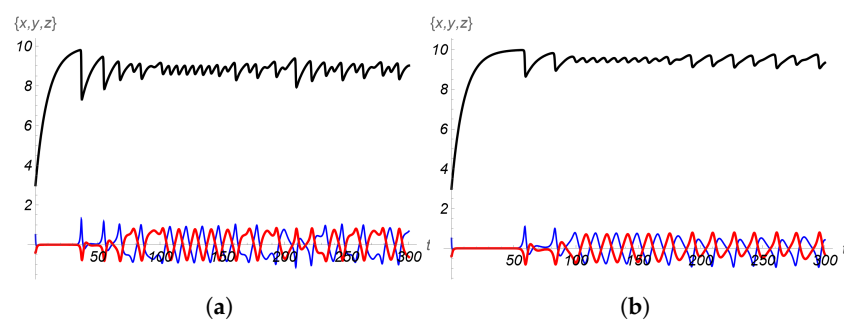


Figure 33. Solutions (x, y, z) of the system (19) with the initial conditions $x(0) = 0.5; y(0) = 3; z(0) = -0.4, k = 8$. (a) Solutions (x, y, z) of the system (19), $a = 8$. (b) Solutions (x, y, z) of the system (19), $a = 8.6$.

Figure 34 displays the bifurcation diagram of the system (19) for $k = 8$, obtained by varying the control parameter a . The parameters b, c and d remain fixed. The bifurcation diagram is plotted when a is varied between $0.1 \leq a \leq 10$.

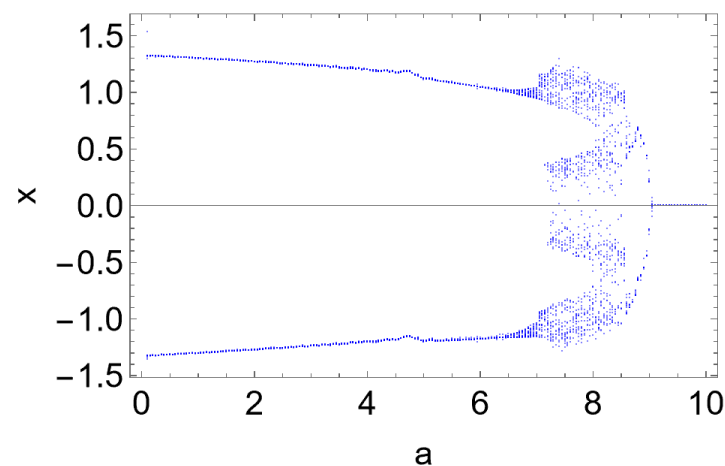


Figure 34. Bifurcation diagram.

Table 26 presents the analysis of system dynamics for various values of the parameter b .

The bifurcation analysis performed revealed that the system exhibits chaotic behavior for $0.10 \leq b \leq 0.11$, periodic behavior for $0.05 \leq b \leq 0.09$ and $b = 0.12$, and asymptotically stable behavior for $0.15 \leq b \leq 10$.

Figure 35 displays the bifurcation diagram of the system (19) for $k = 8$, obtained by varying the control parameter b . The parameters a, c , and d remain fixed. The bifurcation diagram is plotted when b is varied between $0.05 \leq a \leq 0.15$.

Table 27 presents the analysis of system dynamics for various values of the parameter c .

Table 26. Parameter b analysis.

b	LE_1	LE_2	LE_3	<i>Solution</i>
0.05	0.0007	−0.0539	−0.4302	periodic
0.08	0.0033	−0.2003	−0.2184	periodic
0.09	0.0026	−0.0031	−0.3803	periodic
0.10	0.0802	0.0005	−0.4164	chaotic
0.11	0.1179	0.0000	−0.3784	chaotic
0.12	0.0070	−0.0976	−0.1053	periodic
0.15	−0.1613	−0.7557	−0.7669	asymptotically stable
0.20	−0.2147	−1.5852	−1.6002	asymptotically stable
0.30	−0.3130	−1.3937	−3.4599	asymptotically stable
0.50	−0.5122	−1.2417	−4.9461	asymptotically stable
1.00	−1.0119	−1.1887	−5.9994	asymptotically stable
4.00	−1.1738	−4.0000	−6.7762	asymptotically stable
7.00	−1.1706	−6.8978	−6.9888	asymptotically stable
10.0	−1.1693	−6.9425	−9.9880	asymptotically stable

Table 27. Parameter c analysis.

c	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	−0.0003	−0.0296	−0.1160	periodic
0.5	0.0016	−0.0656	−0.0726	periodic
0.7	0.0032	−0.0276	−0.1366	periodic
0.8	0.0307	0.0003	−0.2210	chaotic
0.9	0.1241	0.0002	−0.3347	chaotic
1.0	0.0802	0.0005	−0.4164	chaotic
1.1	0.0032	−0.0372	−0.4297	periodic
1.2	0.0026	−0.0979	−0.5025	periodic
1.5	0.0023	−0.4820	−0.5005	periodic
2.0	0.0036	−0.0353	−1.4654	periodic
3.0	−0.0152	−0.0210	−2.7257	asymptotically stable
5.0	−0.0380	−0.0438	−4.8138	asymptotically stable
7.0	−0.0433	−0.0481	−6.8623	asymptotically stable
10.0	−0.0456	−0.0497	−9.9017	asymptotically stable

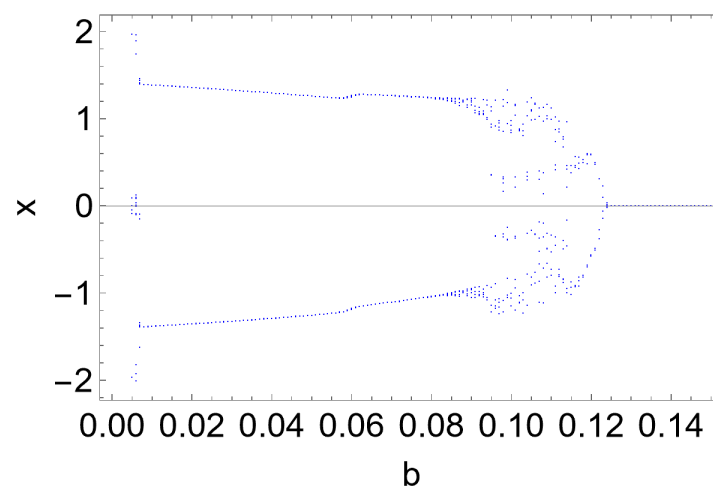


Figure 35. Bifurcation diagram.

Figure 36 displays the bifurcation diagram of the system (19) for $k = 8$, obtained by varying the control parameter c . The parameters a, b and d remain fixed. The bifurcation diagram is plotted when c is varied between $0.05 \leq c \leq 3.5$.

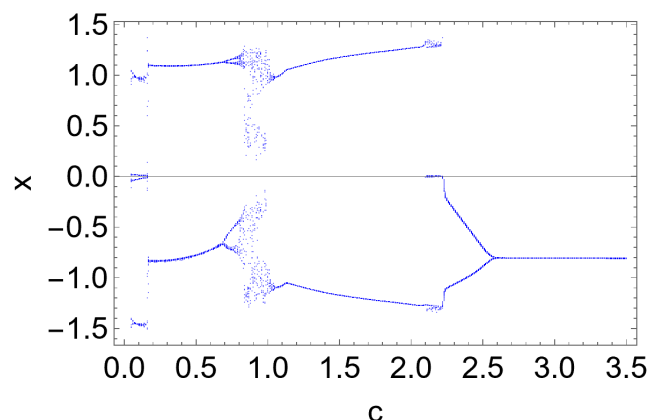


Figure 36. Bifurcation diagram.

Table 28 presents the analysis of system dynamics for various values of the parameter d .

Table 28. Parameter d analysis.

d	LE_1	LE_2	LE_3	<i>Solution</i>
0.05	0.0595	0.0004	−0.4070	chaotic
0.10	0.0802	0.0005	−0.4164	chaotic
1.00	0.0637	0.0002	−0.3433	chaotic
1.50	0.0309	0.0001	−0.3156	chaotic
1.70	0.0177	−0.0001	−0.3047	chaotic
1.80	0.0062	−0.0000	−0.2919	quasiperiodic
1.90	0.0020	−0.0042	−0.2842	quasiperiodic
2.00	0.0018	−0.0399	−0.2482	periodic
5.00	0.0033	−0.0208	−0.2695	periodic
7.00	0.0030	−0.0219	−0.2679	periodic
10.0	0.0030	−0.0239	−0.2658	periodic

Figure 37 displays the bifurcation diagram of the system (19) for $k = 8$, obtained by varying the control parameter d . The parameters a, b , and c remain fixed. The bifurcation diagram is plotted when d is varied between $0.005 \leq d \leq 4$.

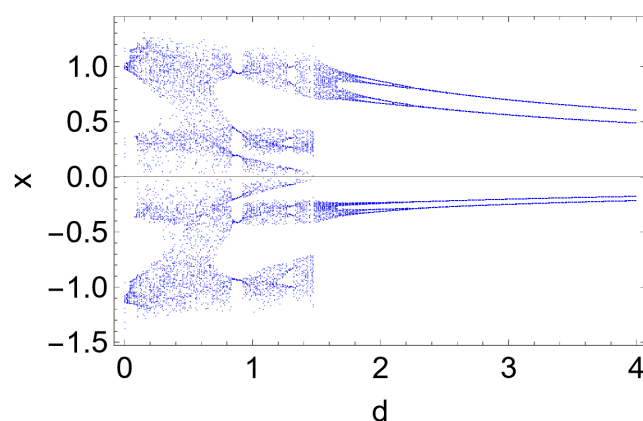


Figure 37. Bifurcation diagram.

3.7. Case 7: $k = 9$

3.7.1. Stability and Dynamical Analysis

The system is (19) and $k = 9$ with given parameters (2) and initial conditions (3). The first critical point $E_1 = (0, 10, 0)$ is obtained similarly as in Case 1. $k = 2$.

Let us consider the second case, where $y - a - \frac{1}{c} = 0$. Then $y = a + \frac{1}{c}$ and $z = -\frac{x}{c}$. After that, substitute $y = a + \frac{1}{c}$ into the second equation of the system (19) and equate to zero

$$1 - b\left(a + \frac{1}{c}\right) - dx^2 - x^9 = 0. \quad (74)$$

After simplifying (74) get

$$x^2(x^7 + d) = 1 - b\left(a + \frac{1}{c}\right). \quad (75)$$

Taking into account the parameters (2), only one real root of Equation (75) is obtained: $x = 0.7885$. The second critical point is $E_2 = (0.7885, 8.2, -0.7885)$. Let us build the Jacobian matrix

$$J = \begin{pmatrix} y - a & x & 1 \\ -(2d + 9x^7)x & -b & 0 \\ -1 & 0 & -1 \end{pmatrix}. \quad (76)$$

The characteristic matrix is

$$J - \lambda I = \begin{pmatrix} y - a - \lambda & x & 1 \\ -(2d + 9x^7)x & -b - \lambda & 0 \\ -1 & 0 & -1 - \lambda \end{pmatrix} \quad (77)$$

and the characteristic equation is

$$\det(J - \lambda I) = -b - \lambda + (-1 - \lambda)(ab + 9x^9 + 2dx^2 - by + a\lambda + b\lambda - y\lambda + \lambda^2) = 0. \quad (78)$$

Put the values of the first critical point $E_1 = (0, 10, 0)$ in (78) and get three roots

$$\lambda_1 = -0.7155, \quad \lambda_2 = -0, 1, \quad \lambda_3 = 2.5155. \quad (79)$$

This result indicates that the critical point $E_1 = (0, 10, 0)$ behaves as a saddle point and is unstable.

Put the values of the second critical point $E_2 = (0.7885, 8.2, -0.7885)$ in (78) and get three roots

$$\lambda_1 = -0.7240, \quad \lambda_{2,3} = 0.3120 \pm 1.2406i \quad (80)$$

This result indicates that the critical point $E_2 = (0.7885, 8.2, -0.7885)$ behaves as a saddle-focus point and is unstable.

3.7.2. Bifurcation Analysis

Table 29 presents the analysis of system dynamics for various values of the parameter a .

For $0.1 \leq a < 8$, the computation of Lyapunov exponents yields indeterminate results, indicating that the system does not reach a well-defined asymptotic regime within the integration time considered. As a increases, stable periodic dynamics emerges.

Table 30 presents the analysis of system dynamics for various values of the parameter b .

The bifurcation analysis performed revealed that the system exhibits an asymptotically stable behavior for $1 \leq b \leq 10$.

Table 31 presents the analysis of system dynamics for various values of the parameter c .

Table 29. Parameter a analysis.

a	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	i	i	i	Indeterminate
2.0	i	i	i	Indeterminate
5.0	i	i	i	Indeterminate
7.2	i	i	i	Indeterminate
8.0	i	i	i	Indeterminate
9.0	0.0007	−0.0129	−0.0916	periodic
9.5	−0.1113	−0.2406	−0.2519	asymptotically stable
10.0	−0.1123	−0.4897	−0.5018	asymptotically stable

Table 30. Parameter b analysis.

b	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	i	i	i	Indeterminate
1.0	−1.0119	−1.1887	−5.9994	asymptotically stable
2.0	−1.1819	−2.0000	−6.5181	asymptotically stable
5.0	−1.1723	−5.0000	−6.8277	asymptotically stable
7.0	−1.1706	−6.8978	−6.9888	asymptotically stable
10.0	−1.1693	−6.9425	−9.9880	asymptotically stable

Table 31. Parameter c analysis.

c	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	i	i	i	Indeterminate
1.0	i	i	i	Indeterminate
2.0	i	i	i	Indeterminate
3.0	−0.0130	−0.0188	−2.7302	asymptotically stable
7.0	−0.0431	−0.0478	−6.8628	asymptotically stable
9.0	−0.0451	−0.0492	−8.8915	asymptotically stable
10.0	−0.0454	−0.0497	−9.9018	asymptotically stable

For $0.1 \leq c < 2$, the computation of Lyapunov exponents yields indeterminate results; the system exhibits an asymptotically stable behavior for $3 \leq c \leq 10$.

Table 32 presents the analysis of system dynamics for various values of the parameter c .

Table 32. Parameter d analysis.

d	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	i	i	i	Indeterminate
1.0	i	i	i	Indeterminate
2.0	i	i	i	Indeterminate
3.0	0.0026	−0.0267	−0.2622	periodic
5.0	0.0028	−0.0227	−0.2668	periodic
7.0	0.0029	−0.0230	−0.2667	periodic
10.0	0.0030	−0.0228	−0.2668	periodic

For $0.1 \leq d < 2$ the computation of Lyapunov exponents yields indeterminate results, the system exhibits periodic behavior for $3 \leq d \leq 10$.

3.8. Case 8: $k = 10$

3.8.1. Stability and Dynamical Analysis

The system is (19) and $k = 10$ with given parameters (2) and initial conditions (3). The first critical point $E_1 = (0, 10, 0)$ is obtained similarly as in Case 1. $k = 2$.

Let us consider the second case, where $y - a - \frac{1}{c} = 0$. Then $y = a + \frac{1}{c}$ and $z = -\frac{x}{c}$. After that, substitute $y = a + \frac{1}{c}$ into the second equation of the system (19) and equate to zero

$$1 - b\left(a + \frac{1}{c}\right) - dx^2 - x^{10} = 0. \quad (81)$$

After simplifying (81) get

$$x^2(x^8 + d) = 1 - b\left(a + \frac{1}{c}\right). \quad (82)$$

Taking into account the parameters (2), the two roots of Equation (82) are obtained: $x_{1,2} = \pm 0.8056$. The second and third critical points are $E_2 = (0.8056, 8.2, -0.8056)$ and $E_3 = (-0.8056, 8.2, 0.8056)$. Let us build the Jakopian matrix

$$J = \begin{pmatrix} y - a & x & 1 \\ -2(d + 5x^8)x & -b & 0 \\ -1 & 0 & -1 \end{pmatrix}. \quad (83)$$

The characteristic matrix is

$$J - \lambda I = \begin{pmatrix} y - a - \lambda & x & 1 \\ -2(d + 5x^8)x & -b - \lambda & 0 \\ -1 & 0 & -1 - \lambda \end{pmatrix} \quad (84)$$

and the characteristic equation is

$$\det(J - \lambda I) = -b - \lambda + (-1 - \lambda)(ab + 10x^{10} + 2dx^2 - by + a\lambda + b\lambda - y\lambda + \lambda^2) = 0. \quad (85)$$

Put the values of the first critical point $E_1 = (0, 10, 0)$ in (85) and get three roots

$$\lambda_1 = -0.7155, \quad \lambda_2 = -0.1, \quad \lambda_3 = 2.5155. \quad (86)$$

This result indicates that the critical point $E_1 = (0, 10, 0)$ behaves as a saddle point and is unstable.

Put the values of the second critical point $E_2 = (0.8056, 8.2, -0.8056)$ in (85) and get three roots

$$\lambda_1 = -0.7337, \quad \lambda_{2,3} = 0.3168 \pm 1.2827i \quad (87)$$

This result indicates that the critical point $E_2 = (0.8056, 8.2, -0.8056)$ behaves as a saddle-focus point and is unstable.

Put the values of the third critical point $E_3 = (-0.8056, 8.2, 0.8056)$ in (85) and get three roots

$$\lambda_1 = -0.7337, \quad \lambda_{2,3} = 0.3168 \pm 1.2827i \quad (88)$$

This result indicates that the critical point $E_3 = (-0.8056, 8.2, 0.8056)$ behaves as a saddle-focus point and is unstable.

3.8.2. Bifurcation Analysis

Table 33 presents the analysis of system dynamics for various values of the parameter a .

Table 33. Parameter a analysis.

a	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	0.0003	−0.2284	−0.3607	periodic
2.0	0.0001	−0.1368	−0.4151	periodic
5.0	0.0010	−0.0128	−0.4538	periodic
6.0	0.0015	−0.1361	−0.2850	periodic
6.5	0.0021	−0.0729	−0.3214	periodic
6.8	0.0024	−0.0369	−0.3402	periodic
6.9	0.0015	−0.0167	−0.3525	periodic
7.0	0.0224	0.0005	−0.3835	chaotic
7.2	0.0628	0.0004	−0.4126	chaotic
8.0	0.1070	−0.0002	−0.3752	chaotic
8.5	0.0549	−0.0004	−0.2770	chaotic
8.6	0.0235	−0.0003	−0.2408	chaotic
8.7	0.0077	−0.0223	−0.1881	periodic
9.0	0.0007	−0.0129	−0.0916	periodic
10.0	−0.1123	−0.4897	−0.5018	asymptotically stable

The bifurcation analysis performed revealed that the system exhibits chaotic behavior for $7 \leq a \leq 8.6$, periodic behavior for $0.1 \leq a \leq 6.9$ and $8.7 \leq a \leq 9$, and asymptotically stable behavior for $a = 10$.

Figure 38 shows the phase plots of system (19), $k = 10$, while Figure 39 displays the corresponding solution graphs.

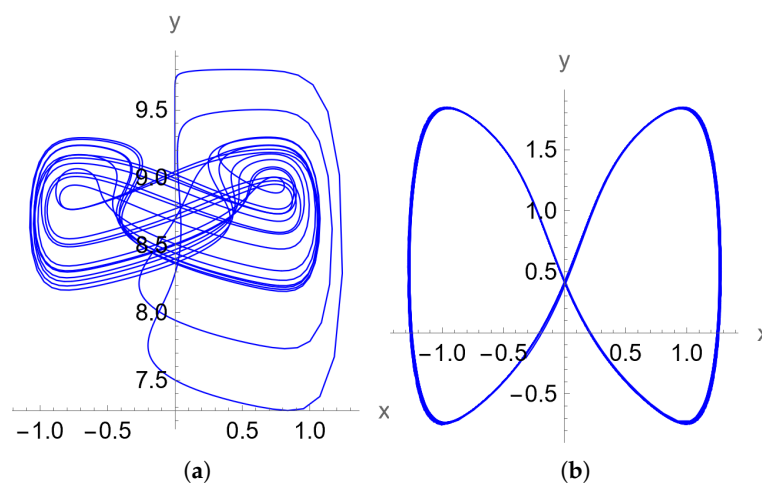


Figure 38. Phase portraits illustrating the dynamics of system (19), $k = 10$ in the two-dimensional planes: (a) Phase trajectories in the $x - y$ plane, $a = 8$. (b) Phase trajectories in the $x - y$ plane, $a = 0.1$.

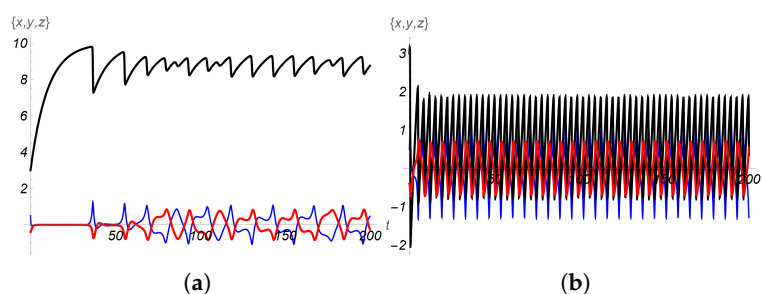


Figure 39. Solutions (x, y, z) of the system (19) with the initial conditions $x(0) = 0.5; y(0) = 3; z(0) = -0.4, k = 10$. (a) Solutions (x, y, z) of the system (19), $a = 8$. (b) Solutions (x, y, z) of the system (19), $a = 0.1$.

Figure 40 displays the bifurcation diagram of the system (19) for $k = 10$, obtained by varying the control parameter a . The parameters b, c and d remain fixed. The bifurcation diagram is plotted when a is varied between $0.1 \leq a \leq 10$.

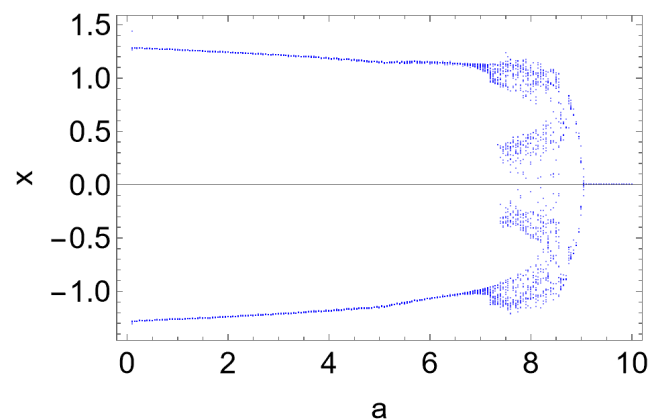


Figure 40. Bifurcation diagram.

Table 34 presents the analysis of system dynamics for various values of the parameter a .

Table 34. Parameter b analysis.

b	LE_1	LE_2	LE_3	<i>Solution</i>
0.05	0.0007	−0.1007	−0.3942	periodic
0.08	0.0021	−0.0492	−0.3795	periodic
0.09	0.0021	−0.0958	−0.2991	periodic
0.10	0.0628	0.0004	−0.4126	chaotic
0.11	0.1065	−0.0003	−0.3774	chaotic
0.12	0.0069	−0.0972	−0.1035	periodic
0.15	−0.1613	−0.7557	−0.7669	asymptotically stable
1.00	−1.0119	−1.1887	−5.9994	asymptotically stable
3.00	−1.1764	−3.0000	−6.6903	asymptotically stable
5.00	−1.1723	−5.0000	−6.8277	asymptotically stable
8.00	−1.1700	−6.9164	−7.9885	asymptotically stable
10.0	−1.1693	−6.9425	−9.9880	asymptotically stable

The bifurcation analysis performed revealed that the system exhibits chaotic behavior for $0.10 \leq b \leq 0.11$, periodic behavior for $0.05 \leq b \leq 0.09$ and $b = 0.12$, and asymptotically stable behavior for $0.15 \leq b \leq 10$. Figure 41 displays the bifurcation diagram of the system (19) for $k = 10$, obtained by varying the control parameter b . The parameters a, c , and d remain fixed. The bifurcation diagram is plotted when b is varied between $0.05 \leq a \leq 0.15$.

Table 35 presents the analysis of system dynamics for various values of the parameter c .

Figure 42 displays the bifurcation diagram of the system (19) for $k = 10$, obtained by varying the control parameter c . The parameters a, b , and d remain fixed. The bifurcation diagram is plotted when c is varied between $0.05 \leq c \leq 3$.

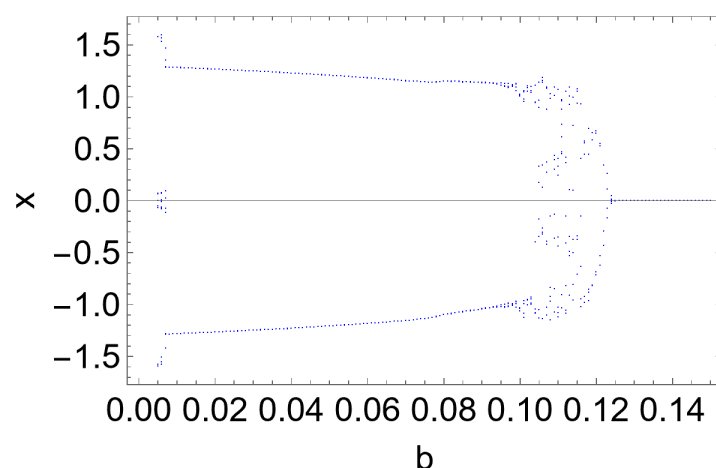


Figure 41. Bifurcation diagram.

Table 35. Parameter c analysis.

c	LE_1	LE_2	LE_3	<i>Solution</i>
0.1	−0.0005	−0.0277	−0.1513	periodic
0.2	0.1056	−0.0009	−0.1960	chaotic
0.3	0.0910	0.0004	−0.1981	chaotic
0.5	0.0939	0.0012	−0.2311	chaotic
1.0	0.0628	0.0004	−0.4126	chaotic
1.1	0.0037	−0.0063	−0.4695	quasiperiodic
1.2	0.0030	−0.1491	−0.4598	periodic
1.5	0.0023	−0.2849	−0.7046	periodic
2.0	0.0037	−0.0466	−1.4598	periodic
3.0	−0.0107	−0.0167	−2.7345	asymptotically stable
5.0	−0.0367	−0.0426	−4.8163	asymptotically stable
8.0	−0.0440	−0.0486	−7.8791	asymptotically stable
10.0	−0.0454	−0.0496	−9.9020	asymptotically stable

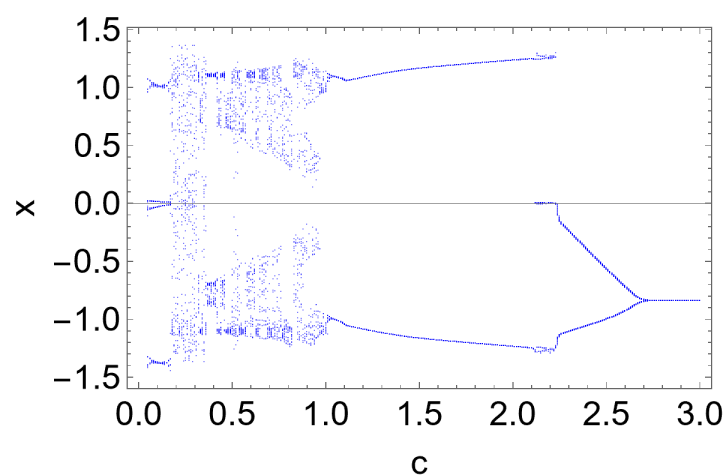


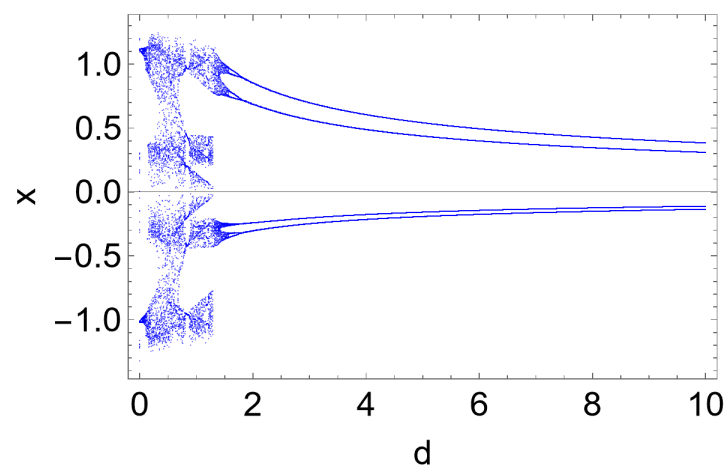
Figure 42. Bifurcation diagram.

Table 36 presents the analysis of system dynamics for various values of the parameter d .

Table 36. Parameter d analysis.

d	LE_1	LE_2	LE_3	<i>Solution</i>
0.01	0.0051	−0.0264	−0.3397	periodic
0.05	0.0158	−0.0003	−0.3698	chaotic
0.1	0.0628	0.0004	−0.4126	chaotic
1.0	0.0582	0.0003	−0.3403	chaotic
1.5	0.0159	0.0004	−0.3038	chaotic
1.6	0.0016	−0.0072	−0.2821	quasiperiodic
1.8	0.0018	−0.0045	−0.2845	quasiperiodic
2.0	0.0019	−0.0114	−0.2772	periodic
5.0	0.0032	−0.0218	−0.2683	periodic
8.0	0.0030	−0.0229	−0.2668	periodic
10.0	0.0030	−0.0231	−0.2665	periodic

Figure 43 displays the bifurcation diagram of the system (19) for $k = 10$, obtained by varying the control parameter d . The parameters a, b , and c remain fixed. The bifurcation diagram is plotted when d is varied between $0.05 \leq d \leq 10$.

**Figure 43.** Bifurcation diagram.

4. Discussion

This paper investigates the dynamics of a three-dimensional nonlinear financial model and analyzes the conditions for the emergence of chaotic behavior. The well-known chaotic system with given parameters and initial conditions was considered as the initial model. Critical points were found for it, two-dimensional and three-dimensional phase portraits were constructed, and Lyapunov exponents were calculated. The results obtained confirmed the presence of chaotic behavior and the high sensitivity of the system to initial conditions.

To deepen the study, a modification of the model was proposed based on changing the degree of the nonlinear term in the second equation and a class of modified systems with nonlinearity of the form x^k , where $2 \leq k \leq 10$. The parameter k plays an important role in controlling the nonlinear interaction between the system variables. As the value of k increases, the strength of the nonlinear terms in the system also increases, which improves the coupling between the state variables. This stronger nonlinear interaction leads to more complex trajectories in phase space and can destabilize regular periodic behavior. As a result, the system undergoes a transition from simple dynamics to more complicated oscillations and chaotic behavior. Therefore, increasing the parameter k contributes to the growth of the dynamical complexity in the proposed system.

Critical points were found and classified for the generalized model, which made it possible to analyze phase spaces at various degrees of nonlinearity.

Next, a bifurcation analysis of the modified systems was performed as the parameters changed. The bifurcation diagrams showed transitions between different dynamical regimes, including equilibrium states, periodic oscillations, quasi-periodic behavior, and chaos. Together with phase portraits and Lyapunov exponent analysis, this shows that both the parameters of the system and the degree of nonlinearity strongly influence the nature of the dynamics of the financial model.

Compared with classical chaotic financial models, the proposed system introduces an additional degree of nonlinearity through the term x^k . Related studies on multidimensional chaotic maps, such as the design of three-dimensional logistic maps and optimization approaches for chaotic systems, also highlight the importance of introducing stronger nonlinear structures to enhance dynamical complexity. In traditional financial models, nonlinear interactions are usually represented by quadratic terms. In contrast, the generalized form considered in this study allows the degree of nonlinearity to vary, providing greater flexibility in describing complex financial dynamics. The numerical analysis shows that increasing the parameter k significantly affects the structure of the phase space and the bifurcation behavior of the system. This shows that the proposed model can capture a wider range of dynamical regimes compared to the existing chaotic financial models.

From a financial interpretation perspective, higher-order nonlinear terms can be associated with stronger nonlinear feedback mechanisms in financial markets. Such nonlinearities may reflect the amplification of market reactions during periods of high volatility, speculative activity, or rapid price adjustments. The presence of strong nonlinear effects is consistent with empirical observations of financial markets. In such markets, price changes often exhibit large irregular fluctuations and occasional extreme events. Although the proposed model represents a stylized dynamical system, the inclusion of higher-order nonlinear terms may provide a possible mathematical mechanism capable of producing large fluctuations and complex irregular dynamics similar to those observed in real financial markets.

High-dimensional chaotic systems have also been widely studied in engineering applications such as image and data encryption. These studies indicate that complex chaotic dynamics can provide useful mechanisms for secure information processing, suggesting that chaotic financial models may also have potential applications in areas such as financial data encryption and risk prediction.

The results obtained deepen our understanding of deterministic chaotic states in financial dynamics and can serve as a basis for further research into stability, management, and forecasting in nonlinear economic and financial models.

Author Contributions: Conceptualization, I.S. and A.L.; methodology, I.S.; software, I.S.; validation, A.L. and I.S.; formal analysis, A.L.; investigation, I.S. and A.L.; resources, I.S. and A.L.; data curation, I.S. and A.L.; writing—original draft preparation, I.S. and A.L.; writing—review and editing, I.S.; visualization, I.S. and A.L.; supervision, I.S.; project administration, I.S. and A.L.; funding acquisition, A.L. All authors have read and agreed to the published version of the manuscript.

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