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Stability of Sets for Ebola Virus Disease Models Through Impulsive Conformable Approach

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Abstract

In this paper, we extend some existing models of the Ebola virus disease through a hybrid impulsive conformable approach. The base of the introduced model is a class of partial differential equations that incorporate diffusion terms to describe the development of the Ebola virus disease in time and space. In the extended model, we have considered impulsive effects at fixed moments of time, which is of high significance in investigating opportunities for impulsive vaccination strategies and impulsive control drug treatment on disease evolution. In addition, conformable setting is proposed, which provides modeling flexibility without the complications inherent in classical fractional derivatives. Instead of studying the global stability of an equilibrium, the more general notion of stability of sets is introduced and analyzed. The main stability of sets results are obtained by using the impulsive conformable Lyapunov technique and comparison principle. The proposed framework, concepts and techniques may serve as effective tools for analyzing numerous phenomena in medicine and biology.

Keywords: Ebola virus disease models; conformable derivatives; impulses; reaction–diffusion terms; stability of sets

MSC: 26A33; 34A08; 34A37; 34D35

1. Introduction

The Ebola virus that caused Ebola virus disease (EVD) is a member of the Filoviridae family and can be a source of severe hemorrhagic fever in humans and nonhuman primates. An EVD was first located in 1976 near the Ebola River (Africa) in the Democratic Republic of the Congo [1,2]. Since then, several Ebola outbreaks have affected West African countries [3]. This is a disease that has been proven fatal. The high mortality rate caused by the EVD makes the Ebola virus a major problem for public health. This motivates a relevant attention of the researchers in the study of adequate EVD mathematical models and methods.

In fact, mathematical models play an essential role in understanding the behavior of numerous problems in virology [4]. Most of the existing EVD models are in the form of ordinary differential equations (ODEs). These models commonly categorize populations into classes such as susceptible, infected, recovered, dead, and some others. For example, the paper [3] proposed a mathematical model with ODEs that included susceptible, infectious, recovered, dead, and environmental classes. The authors of [5] proposed a two-host

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model for Ebola with indirect transmission in the format of an ODE system in order to study the impact of the spillover event on the dynamics of EVD. The consideration of an environmental compartment is initiated by considering a second mode of transmission from a dead person to a living person [6]. The authors in [7] investigated a model in the form of a system of ODEs with six subpopulations named susceptible, exposed (infected but not yet infective), symptomatic infectious, recovered, asymptomatic infectious (characterized by small or null level of infection, with acquired immunity, but can transmit the infective disease to others) and dead-infective. The research [8] introduced a common SIR (susceptible-infectious-recovery) epidemic model that describes the 2014 Ebola outbreak in Liberia.

In all models mentioned above, since the authors used ODEs, the spatiotemporal dynamics of EVD has been ignored. However, due to different problems connected with mobility of the population and losing the control of the spread of EVD it is necessary to consider the evolution of EVD in space. Assuming that populations follow Fickian diffusion, the authors in [3] introduced a model using a system of PDEs

$$\begin{cases} \dfrac{\partial \zeta_1(t,z)}{\partial t} = d_1\Delta\zeta_1 + A - \mu\zeta_1(t,z) - f(\zeta_1(t,z),\zeta_2(t,z))\zeta_2(t,z) \\ \qquad\qquad -g(\zeta_1(t,z),\zeta_3(t,z))\zeta_3(t,z) - h(\zeta_1(t,z),\zeta_4(t,z))\zeta_4(t,z), \\ \dfrac{\partial \zeta_2(t,z)}{\partial t} = d_2\Delta\zeta_2 + f(\zeta_1(t,z),\zeta_2(t,z))\zeta_2(t,z) + g(\zeta_1(t,z),\zeta_3(t,z))\zeta_3(t,z) \\ \qquad\qquad +h(\zeta_1(t,z),\zeta_4(t,z))\zeta_4(t,z) - (\mu + d + r)\zeta_2(t,z), \\ \dfrac{\partial \zeta_3(t,z)}{\partial t} = d_3\Delta\zeta_3 + (\mu + d)\zeta_2(t,z) - b\zeta_3(t,z), \\ \dfrac{\partial \zeta_4(t,z)}{\partial t} = d_4\Delta\zeta_4 + \sigma\zeta_2(t,z) + \gamma\zeta_3(t,z) - \eta\zeta_4(t,z), \end{cases} \tag{1}$$

where $\zeta_1, \zeta_2, \zeta_3$ and ζ_4 are susceptible, infectious, dead, and environmental classes, respectively, at time t and location $z \in \Omega$. The time $t \in \mathbb{R}_+$ and the location $\Omega \subseteq \mathbb{R}^n$, and it was assumed that the domain Ω is bounded with a smooth boundary $\partial\Omega$ not crossed by the four populations moves. Here, $\mathbb{R}_+ = [0, \infty)$, and $\mathbb{R}^n$ is the Euclidean n-dimensional space with elements $z = (z_1, z_2, \ldots, z_n)^T \in \mathbb{R}^n$. The constants $d_j,\ j = 1, 2, 3, 4$ are the diffusion coefficients for the four classes and Δ is the Laplace's operator. A susceptible individual may become infected by contact with an infectious individual at a rate $f(\zeta_1, \zeta_2)$, contact with a deceased individual at a rate $g(\zeta_1, \zeta_3)$, or by exposure to a contaminated environment at a rate $h(\zeta_1, \zeta_4)$. The constant A represents the growth rate of susceptible individuals, and the positive constant μ denotes the natural death rate. The population of infectious individuals increases with the total Ebola infection rate $f(\zeta_1, \zeta_2)\zeta_2 + g(\zeta_1, \zeta_3))\zeta_3 + h(\zeta_1, \zeta_4)\zeta_4$ and decreases due to the natural death rate μ or due to the death rate of EVD d. r is the natural mortality rate of recovered individuals. The constant b denotes the rate of buried deceased individuals who are carriers of the Ebola virus. The constants σ and γ represent the rates at which the virus is generated in the environment by infected and deceased people, respectively, and η is the decay rate.

It is important to note that considering diffusion effects in processes studied by virology is biologically meaningful. That is why numerous diffused models have been introduced and studied in virology, bioinformatics, computational biology, and immunology [9,10]. However, the existing literature on the topic does not offer enough EVD models with diffusion coefficients [11,12]. Hence, this area is subject to development.

Fractional-order modeling is another effective mathematical approach that has recently been applied in biology and medicine. The use of this modeling approach is motivated by its ability to capture the phenomenon of memory effects and long-term predictions. The flexibility of the proposed models that improves the reliability of the integer-order models and allows a better understanding of the complex dynamics of viruses and diseases is also a well known advantage [13,14]. To create efficient fractional-order models, fractional-order derivatives of Caputo, Riemann–Liouville, and other types are mainly used [15]. Fractional-order EVD models are also proposed in few published works [16,17]. Such models use the non-integer-order derivatives to improve upon classical integer-order models, providing better fit to real-world outbreak data by capturing memory effects and nonlocal dynamics. Reaction–diffusion terms have also been considered in some existing fractional-order models proposed in biology and virotherapy [18]. However, fractional-order diffusion EVD models have not yet been established, which is a challenging open problem.

In addition, since the application of the classical fractional calculus leads to some difficulties in the qualitative analysis of the corresponding models, many researchers proposed new fractional operators in their models [19,20]. Some of the difficulties affected the use of the Lyapunov function method as one of the most powerful qualitative methods, mainly due to the complex chain rule.

One of the directions in which fractional operators have been extended is related to the introduction of the local type of conformal operator [21,22]. Since the definition of the conformable derivative is limit based, it avoids many of the complications in the use of the classical fractional-order derivatives. As a result, the recently introduced conformable calculus has attracted significant research attention [23–25]. Some researchers initially used the name "fractional-like" instead of "conformable" for the new operator. See, for example, [26], where a physical interpretation of this derivative is presented. Furthermore, the benefits of conformal derivatives have expanded their applicability in mathematical modeling, including epidemic models [27,28]. The existence of a few conformable mathematical models of EVD [29–31] introduced very recently demonstrates the importance of this modeling approach in studying virus and disease behavior, as well as in virology and epidemic modeling in general. In fact, conformable derivatives enable growth rates and population interactions to be influenced by past states, which is essential for accurately modeling biological systems. Particularly in epidemiological modeling, conformable derivatives are employed to represent the transmission dynamics of infectious diseases, capturing the complex interactions among the defined subpopulations.

Another effective approach used in the mathematical modeling of biological and medical phenomena is the incorporation of discrete, sudden changes (impulses) into the models and the study of so-called impulsive models. It is well known that short-term impulsive perturbations are very common in the evolution of processes investigated in ecology, epidemiology, biology, and medicine. Their reflection made impulsive models more adequate, and therefore they gained increasing attention [32–34], including impulsive models with reaction–diffusion terms [35]. Due to the creation of impulsive models in science, medicine, and engineering, impulsive differential equations play an important role [36].

In addition, impulsive models can be effectively used to simulate impulsive control strategies [37,38], as discrete short-term impulsive perturbations can often influence the behavior of a real process. This motivates the existence of the large number of results on appropriate treatment strategies based on impulse control theory [39,40]. In [41], an impulsive control strategy in the form of impulsive immigration of infectives is proposed for an EVD model without using conformal calculus techniques. In order to refine the mathematical modeling in biology, neurobiology and medicine, impulsive control strategies

have also been proposed to fractional-order models [42,43], as well as to some conformable biological models proposed very recently [44–46]. However, to the best of our knowledge, the impulsive conformable modeling approach has not been used for the construction of EVD models, which is one of the aims of this research.

The study of the stability of mathematical models in biology and medicine has important practical implications. This is due to the fact that stability is fundamental to the efficiency of models', which explains the existence of the large number of studies devoted to the stability of biological, virological and epidemic models [4,6–8,14,17–19,40,42,43,45,46]

However, in most existing studies, researchers focus on the stability of single states of the model. The single-state stability strategies cannot be applied in the case of multistable systems. For such systems, an extended notion of stability, namely, set stability, is more appropriate. The concept is introduced in [47] and generalizes several particular stability cases when the sets are in the forms of integral manifolds, invariant sets, or include just individual states. The practical importance of the stability of sets concept motivates its application to numerous systems [48–50], including virotherapy models [51]. The observation that this concept has not been studied for impulsive conformable EVD models prompted our investigation and serves as a central objective of this study.

Motivated by the foregoing considerations, in this paper, we introduce and analyze a class of EVD models using the impulsive conformable approach.

The main novelty of the paper is in the following points:

(i) The conformable caluclus modeling approach is applied to introduce an EVD model, which extends the existing integer-order models studied in [3,6–8,11,12]. The use of this approach is motivated by its ability to incorporate memory effects, maintain useful properties of classical calculus, improve modeling flexibility, and facilitate analytical and stability analysis of the epidemic dynamics.
(ii) The model further integrates impulsive controllers to capture sudden health interventions and epidemic control policies, thus providing a more realistic and practical framework for the analysis and control of Ebola epidemic dynamics.
(iii) Unlike the existing conformable EVD models reported in [29–31], diffusion terms are considered to account for the natural dependence of system dynamics on both temporal and spatial variables.
(iv) Different from previous stability results for EVD models, this study introduces the concept of set stability and develops efficient criteria based on the impulsive conformable Lyapunov method.

The rest of this paper is structured as follows. In the preliminary Section 2, we recall basic definitions and facts related to the impulsive conformable calculus. The new EVD model with diffusion terms, impulsive perturbations, and conformable derivatives is proposed. The stability of sets definition is adopted to the newly introduced model. Some fundamental definitions and lemmas from the conformable Lyapunov technique are also given. Section 3 is devoted to the main stability of sets results. The conditions obtained are proved using the conformable impulsive Lyapunov function method. In Section 4, a discussion is proposed and an example is presented to demonstrate the validity and efficiency of the new results obtained. Finally, Section 5 contains the concluding remarks and some future directions.

2. Preliminaries

2.1. Impulsive Conformable Approach

Some notation and properties of the conformable derivatives will be given in this subsection. Let $a \in \mathbb{R}_+$, $t \in (a, \infty)$ and $\alpha \in (0, 1]$.

Definition 1 ([21,22]). *The conformable derivative of order α at the point t for a function $f : (a, \infty) \to \mathbb{R}$ is defined by*

$$\mathcal{T}_a^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon(t-a)^{1-\alpha}) - f(t)}{\varepsilon}, \ t > a. \tag{2}$$

The point a is called the lower limit [21,22] or the lower terminal [24] of the conformable derivative (2).

In the case where the conformable derivative of f of order α exists in some interval $(a, a+\theta)$, then [24]

$$\mathcal{T}_a^\alpha f(a) = \lim_{t \to a^+} \mathcal{T}_a^\alpha f(t).$$

If the conformable derivative (2) for a function f of order α exists at $t \in [a, \infty)$, we will say that f is α conformable differentiable on this interval, which will be denoted by $f \in C_a^\alpha[[a, \infty), \mathbb{R}]$.

Definition 2 ([24,44,46,51]). *For $t > a$, the conformable integral of order $0 < \alpha \leq 1$ with a lower limit $a \in \mathbb{R}_+$ of the function $f(t)$ is given by*

$$I_a^\alpha f(t) = \int_a^t (s-a)^{\alpha-1} f(s) ds, \ t > a.$$

We will also recall the following properties of the conformable derivatives.

Lemma 1 ([21,24,44,46,51]). *Let the function $f : [a, \infty) \to \mathbb{R}$, $f \in C_a^\alpha[(a, \infty), \mathbb{R}]$, and $0 < \alpha \leq 1$. Then, we have:*

(i) $I_a^\alpha(\mathcal{T}_a^\alpha f(t)) = f(t) - \lim_{t \to a^+} f(t)$;

(ii) If $u(f(t)) : [a, \infty) \to \mathbb{R}$ is differentiable with respect to $f(t)$, then for any $t \in (a, \infty)$ and $f(t) \neq 0$, we have

$$\mathcal{T}_a^\alpha u(f(t)) = u'(f(t)) \mathcal{T}_a^\alpha f(t),$$

where u' is the derivative of $u(\cdot)$;

(iii) If f is differentiable, then $\mathcal{T}_a^\alpha f(t) = (t-a)^{1-\alpha} f'(t)$.

Note that if the function $f(t)$ is continuous from the right at a, then [24]

$$I_a^\alpha(\mathcal{T}_a^\alpha f(t)) = f(t) - f(a), \ t > a.$$

In order to incorporate the impulsive control approach, we consider the discrete points

$$0 < \eta_1 < \eta_2 < \cdots < \eta_k < \ldots, \tag{3}$$

such that

$$\lim_{k \to \infty} \eta_k = \infty.$$

We have [44,46,51],

$$\mathcal{T}_{\eta_k}^\alpha f(\eta_k) = \lim_{t \to \eta_k^+} \mathcal{T}_{\eta_k}^\alpha f(t),$$

for a function $f : [\eta_k, \eta_{k+1}) \to \mathbb{R}$, $k = 1, 2, \ldots$.

Next, for a function $\zeta : [a, \infty) \times \Omega \to \mathbb{R}$, $\zeta = \zeta(t, z)$, the conformable derivative with respect to time is defined as follows.

Definition 3 ([44,51]). *The limit*

$$\mathcal{T}_a^{\alpha}\zeta(t,z) = \lim_{\varepsilon\to 0}\frac{\zeta(t+\varepsilon(t-a)^{1-\alpha},z)-\zeta(t,z)}{\varepsilon},\ t>a,\ z\in\Omega$$

is the conformable derivative of the function $\zeta=\zeta(t,z)$ along t of order α, $0<\alpha\leq 1$.

If $a=\eta_k$, $k=1,2,\ldots$, then

$$\mathcal{T}_{\eta_k}^{\alpha}\zeta(\eta_k,z) = \lim_{t\to\eta_k^+}\mathcal{T}_{\eta_k}^{\alpha}\zeta(t,z).$$

Remark 1. *For the time conformable derivative defined by Definition 3, we have the same properties as the properties given in Lemma 1 for α-conformable differentiable functions [44,46,51].*

Definition 4 ([21,22]). *For $0<\alpha\leq 1$, the conformable exponential function $E_\alpha(\xi,\tau)$ is defined by*

$$E_\alpha(\xi,\tau) = \exp\left(\xi\frac{\tau^\alpha}{\alpha}\right),\ \xi\in\mathbb{R},\ \tau\in\mathbb{R}_+.$$

Remark 2. *More results of the conformable calculus can be found in [21–23,25,26], and for more details related to the impulsive conformable calculus, we refer to [44,46,51] and the references cited therein.*

2.2. Model Formulation

In this section, we will formulate an extended EVD model using the hybrid impulsive conformable approach. The impulses will be considered at the discrete instances (3). The model will generalize the equation introduced in [3] model (1) and will incorporate diffusion effects and concentration gradients of the subpopulations. Consider the following impulsive conformable EVD model:

$$\begin{cases}
\mathcal{T}_{\eta_k}^{\alpha}\zeta_1(t,z) = d_1\Delta\zeta_1 + A - \mu\zeta_1(t,z) - f(\zeta_1(t,z),\zeta_2(t,z))\zeta_2(t,z) \\
\qquad\qquad -g(\zeta_1(t,z),\zeta_3(t,z))\zeta_3(t,z) - h(\zeta_1(t,z),\zeta_4(t,z))\zeta_4(t,z),\ t\neq\eta_k, \\
\mathcal{T}_{\eta_k}^{\alpha}\zeta_2(t,z) = d_2\Delta\zeta_2 + f(\zeta_1(t,z),\zeta_2(t,z))\zeta_2(t,z) + g(\zeta_1(t,z),\zeta_3(t,z))\zeta_3(t,z) \\
\qquad\qquad +h(\zeta_1(t,z),\zeta_4(t,z))\zeta_4(t,z) - (\mu+d+r)\zeta_2(t,z),\ t\neq\eta_k, \\
\mathcal{T}_{\eta_k}^{\alpha}\zeta_3(t,z) = d_3\Delta\zeta_3 + (\mu+d)\zeta_2(t,z) - b\zeta_3(t,z),\ t\neq\eta_k, \\
\mathcal{T}_{\eta_k}^{\alpha}\zeta_4(t,z) = d_4\Delta\zeta_4 + \sigma\zeta_2(t,z) + \gamma\zeta_3(t,z) - \eta\zeta_4(t,z),\ t\neq\eta_k, \\
\zeta_j(\eta_k^+,z) = \zeta_j(\eta_k,z) + J_{jk}(\zeta_j(\eta_k,z)),\ j=1,4,\ k=1,2,\ldots, \\
\zeta_j(\eta_k^+,z) = \zeta_j(\eta_k,z),\ j=2,3,\ k=1,2,\ldots,
\end{cases} \tag{4}$$

where $0<\alpha<1$, $\zeta_1(t,z)$, $\zeta_2(t,z)$, $\zeta_3(t,z)$, and $\zeta_4(t,z)$ are the same subpopulations as in (1) for $t\in\mathbb{R}_+$, $z\in\Omega$. The model's parameters d_j, $\Delta\zeta_j$, $j=1,2,3,4$, A, μ, f, g, h, d, r, b, σ, γ, and η, will be used with the same meanings as in (1), too. Impulsive perturbations in instances $\{\eta_k\}$, $k=1,2,\ldots$ are considered only for the susceptible and environmental classes. In these instances, impulsive control functions J_{jk}, $j=1,4$, $k=1,2,\ldots$ are applied, so that abrupt changes of the states $\zeta_j(t,z)$, $j=1,4$ from the states $\zeta_j(\eta_k^-,z)=\zeta_j(\eta_k,z)$ to

the states $\zeta_j(\eta_k^+, z)$ are observed. The functions J_{jk}, regulate the controlled outputs $\zeta_j(\eta_k^+, z)$, $j = 1, 4$, $k = 1, 2, \ldots$.

To help readers, Table 1 provides descriptions of the parameters and operators used in the proposed model (4).

Table 1. Descriptions of the parameters and operators in model (4).

Description of the parameters and operators in the introduced model (4).	
ζ_1	The class of susceptible individuals
ζ_2	The class of infectious individuals
ζ_3	The class of deceased individuals
ζ_4	Environmental class
A	The growth rate of susceptible individuals
d_1, d_2, d_3, d_4	Diffusion coefficients
Δ	Laplace's operator
$f(\zeta_1, \zeta_2)$	The rate of contact (effective) of infected individual
$g(\zeta_1, \zeta_3)$	Rate of contact (effective) of deceased individual
$h(\zeta_1, \zeta_4)$	Rate of contact (effective) of Ebola virus in the environment
μ	Natural death rate
d	Rate of deaths of human individuals due to EVD
r	The natural mortality rate of recovered individuals
b	The rate of buried deceased individuals who are carriers of the Ebola virus
σ	The rates at which the virus is generated in the environment by infected individuals
γ	the rates at which the virus is generated in the environment by deceased individuals
η	The decay rate
η_k	The impulsive control instances
J_{jk}	The impulsive control functions
$\zeta_j(t, z)$,	The states before an impulsive perturbation
$\zeta_j(t^+, z)$	The states after an impulsive perturbation

Remark 3. *The proposed model (4) extends the model (1) studied in [3] considering conformable derivatives and impulsive perturbations at the moments* $\{\eta_k\}$, $k = 1, 2, \ldots$. *The conformable framework offers modeling flexibility compared with the modeling by the integer-order derivatives, and at the same time avoiding the complications inherent to classical fractional derivatives. The consideration of diffusion terms is motivated by the fact that EVD evolves not only in time but also in the space that was considered in [3]. Impulsive perturbations in the susceptible subpopulation* $\zeta_1(t, z)$ *can be in the form of impulsive vaccinations or other impulsive treatments such as quarantining. Impulsive effects on* $\zeta_4(t, z)$ *may be due to environmental sanitation as in [40]. Taking such impulsive control allows for a rapid reduction in the number of susceptible individuals, as well as the introduction of disinfectant into the environment.*

Remark 4. *Although some previously proposed EVD models have incorporated aspects such as reaction–diffusion terms, conformable derivatives, and impulsive effects to some extent, substantial gaps still remain. Some reaction–diffusion models do not involve impulsive perturbations and conformable derivatives [11,12]. The existing very limited number of studies on conformable EVD models do not consider the synergistic impact of impulsive treatment strategies and diffusion terms [29–31]. Impulsive treatment strategies for virus diseases' mathematical models have been proposed by numerous authors [39,40], including EVD models [41], without considering conformable derivatives. Thus, the introduced model (4) generalizes some existing impulsive, reaction–diffusion, and conformable EVD models, as well as numerous models introduced in virology and medicine.*

Remark 5. *Impulsive perturbations can also be considered for the population of infected individuals $\zeta_2(t,z)$ as in [41], which is a subject of our future research. In this case, we plan to analyze the impact of their migration.*

We will study model (4) under the assumption that the four subpopulations do not move across $\partial\Omega$. Hence, the following boundary and initial conditions hold:

$$\frac{\partial \zeta_j(t,z)}{\partial \nu} = 0,\ t \geq \eta_0,\ z \in \partial\Omega,\ j = 1,2,3,4, \tag{5}$$

$$\zeta_j(\eta_0^+, z) = \Phi_{0j}(z) \geq 0,\ z \in \Omega,\ j = 1,2,3,4, \tag{6}$$

where η_0 is the initial time, $0 \leq \eta_0 < \eta_1 < \cdots < \eta_k < \ldots$, $\Phi_{0j}(z)$ are continuous real-valued, defined on Ω functions, $\Phi_0 = (\Phi_{01}, \Phi_{02}, \Phi_{03}, \Phi_{04})^T$, and $\frac{\partial}{\partial \nu}$ is the outward normal derivative on $\partial\Omega$.

According to the theory of impulsive conformable systems [44,46,51], the solution $\zeta(t,z)$ of the initial boundary value problem (IBVP) (4)–(6),

$$\zeta(t,z) = \zeta(t,z;\eta_0,\Phi_0),$$

is a piecewise continuous function with points of discontinuity of the first kind η_k at which it is left continuous and has time α-conformable derivatives on $\mathbb{R}_+$ for $z \in \Omega$.

For the purposes of our analysis, we will assume that the solution $\zeta(t,z) = \zeta(t,z;\eta_0,\Phi_0)$ exists and is uniquely determined by the initial data $\eta_0 \in \mathbb{R}_+$ and $\Phi_0 \in \mathbb{R}_+^4$. For some existence and uniqueness criteria of impulsive conformable models, we refer to [44,46,51] and the references cited therein.

Finally, some results for diffusion systems will be given.

We introduce the following notation

$$||\zeta(t,\cdot)||_2 = \left[\int_\Omega \sum_{j=1}^{4} \zeta_j^2(t,z)dz\right]^{1/2}$$

for a function $\zeta(t,z) \in \mathbb{R}^4$, $\zeta(t,z) = (\zeta_1(t,z), \zeta_2(t,z), \zeta_3(t,z), \zeta_4(t,z))^T$, $t \in \mathbb{R}_+$, $z \in \Omega$, and recall the next Poincaré-type integral lemma [44,51].

Lemma 2. *For a bounded domain $\Omega \subseteq \mathbb{R}^n$ with a smooth boundary $\partial\Omega$, and a real-valued function $\mathcal{U} : \Omega \to \mathbb{R}_+$ belonging to $H^1(\Omega)$, such that $\left.\frac{\partial \mathcal{U}(z)}{\partial \nu}\right|_{\partial\Omega} = 0$, we have*

$$\theta_1 \int_\Omega |\mathcal{U}(z)|^2\, dz \leq \int_\Omega |\nabla \mathcal{U}(z)|^2\, dz,$$

where ∇ is the gradient operator and θ_1 is the smallest positive eigenvalue of the boundary problem

$$\begin{cases} -\Delta\chi(z) = \theta\chi(z),\ z \in \Omega, \\ \dfrac{\partial \chi(z)}{\partial \nu} = 0,\ z \in \partial\Omega. \end{cases}$$

2.3. Stability of Sets Concepts

Since our aim is to adopt the concept of stability of sets to the proposed model (4), in this subsection, definitions related to this concept introduced in [47] and applied in [36,49–51] will be stated.

Let $\mathbb{R}^4_+ = \{(\zeta_1, \zeta_2, \zeta_3, \zeta_4)^T : \zeta_j \geq 0,\ j = 1,2,3,4\}$. We will consider a set $\Theta \subset \mathbb{R}_+ \times \Omega \times \mathbb{R}^4_+$, and related sets, defined by:

- $\Theta(t,z) = \{\zeta \in \mathbb{R}^4_+ : (t,z,\zeta) \in \Theta,\ t \in \mathbb{R}_+,\ z \in \Omega\}$;
- $\Theta(t,z)(\rho) = \{\zeta \in \mathbb{R}^4_+ :\ d(\zeta, \Theta(t,z)) < \rho\}$ $(\rho > 0)$;
- $\overline{\Theta}(t,z)(\rho) = \{\zeta \in \mathbb{R}^4_+ :\ d(\zeta, \Theta(t,z)) \leq \rho\}$;
- $S_\rho = \{\zeta \in \mathbb{R}^4 :\ ||\zeta||_2 < \rho\}$,

where

$$d(\zeta, \Theta(t,z)) = \inf_{\tilde{\zeta} \in \Theta(t,z)} ||\zeta - \tilde{\zeta}||_2$$

denotes the distance between $\zeta \in \mathbb{R}^4_+$ and $\Theta(t,z)$.

Assuming that for any $t \in \mathbb{R}_+$ and $z \in \Omega$, the set $\Theta(t,z)$ is not empty and $\Theta(t,z) \subset B$ for a compact set $B \subset \mathbb{R}^4_+$, we first state the definition of boundedness [51].

Definition 5. *The solutions of the impulsive conformable EVD model (4) are:*

(a) Equi-Θ-bounded, if for any $\eta_0 \in \mathbb{R}_+$ and any positive constants $\rho,\ \phi > 0$ there exists a constant $\beta = \beta(\eta_0, \rho, \phi) > 0$ such that $z \in \Omega$ and $\Phi_0 \in S_\rho \cap \overline{\Theta}(\eta_0^+, z)(\phi)$ imply $\zeta(t,z;\eta_0,\Phi_0) \in \Theta(t,z)(\beta),\ t \geq \eta_0$;

(b) Uniformly Θ-bounded, if the number β in (a) depends only on ϕ.

The concepts of set stability are defined by the next definition [51].

Definition 6. *The set Θ is said to be:*

(a) Stable with respect to system (4), if for any $\eta_0 \in \mathbb{R}_+$ and for any positive constants $\rho > 0$ and $v > 0$, there exists a constant $\delta = \delta(\eta_0, \rho, v) > 0$ such that $z \in \Omega$ and $\Phi_0 \in S_\rho \cap \Theta(\eta_0^+, z)(\delta)$ imply $\zeta(t,z;\eta_0,\Phi_0) \in \Theta(t,z)(v),\ t \geq \eta_0$;

(b) Uniformly stable with respect to system (4), if the number δ in (a) depends only on v;

(c) Uniformly globally attractive with respect to system (4), if for any positive constants $\lambda > 0$ and $v > 0$, there exists a constant $\iota = \iota(\lambda, v) > 0$ such that $\eta_0 \in \mathbb{R}_+$, $\rho > 0$, $z \in \Omega$ and $\Phi_0 \in S_\rho \cap \overline{\Theta}(\eta_0^+, z)(\lambda)$ imply $\zeta(t,z;\eta_0,\Phi_0) \in \Theta(t,z)(v),\ t \geq \eta_0 + \iota$;

(d) Uniformly globally asymptotically stable with respect to system (4), if Θ is uniformly stable and uniformly globally attractive set of system (4), and if the solutions of system (4) are uniformly Θ-bounded;

(e) Uniformly globally exponentially stable with respect to system (4), if there exist a strictly positive constant $\mathbf{N}$ and a nonnegative constant ψ such that

$$d(\zeta(t,z;\eta_0,\Phi_0), \Theta(t,z)) \leq \mathbf{N} d(\Phi_0, \Theta(\eta_0^+, z))(E_\alpha(-\psi, t-\eta_0))^\omega,\ \ t \geq \eta_0,\ z \in \Omega,\ \omega > 0.$$

Remark 6. *Existing studies on EVD models investigated the stability behavior only of single states of the models [3,6–8,29–31,41]. For example, the paper [3] studied the global asymptotic stability of two individual states of (1), namely, the disease-free equilibrium $E_f(\frac{A}{\mu}, 0, 0, 0)$ and the endemic equilibrium $E^*(\zeta_1^*, \zeta_2^*, \zeta_3^*, \zeta_4^*)$, where $\zeta_1^* \in (0, \frac{A}{\mu})$, $\zeta_j^* > 0$, $j = 2,3,4$. The proposed Definition 6 generalizes the stability of individual states to the stability of a more general set Θ. In special cases, where $\Theta(t,z) = \{E_f\}$ or $\Theta(t,z) = \{E^*\}$, the definition can be reduced to stability definitions for such single states. Note that the set Θ is of a very general nature. Thus, Definition 6 extends and generalizes numerous single-state stability notions. In addition, it can be specified to stability of moving invariant sets of integral manifolds [32,35] for particular choices of the set Θ.*

2.4. Impulsive Conformable Lyapunov Method

We first define the following sets:

$$\Psi_k = \{(t,\zeta) : t \in (\eta_{k-1}, \eta_k),\ \zeta \in \mathbb{R}^4_+\},\ k = 1,2,\ldots,\ \Psi = \bigcup_{k=1}^{\infty} \Psi_k.$$

The main method that will be used in the proofs of our main results is the Lyapunov function method extended to the impulsive conformable approach. To this end, we will apply Lyapunov functions, defined by the next definition.

Definition 7 ([44,51]). *A nonnegative Lyapunov-type function* Υ *belongs to class* ${}^{\Theta}\Upsilon^{\alpha}_{\eta_k}$*, if:*

(i) $\Upsilon(t,\zeta) = 0$ *for* $\zeta \in \Theta(t,z)$, $(t,z) \in \mathbb{R}_+ \times \Omega$, $t \geq \eta_k$, $k = 1,2,\ldots$;

(ii) Υ *is continuous in* Ψ, α*-conformable differentiable in* t *and locally Lipschitz continuous with respect to* ζ *on each of the sets* Ψ_k;

(iii) For each $k = 1,2,\ldots$ *and* $\zeta \in \mathbb{R}^4_+$*, there exist the finite limits*

$$\Upsilon(\eta_k^-,\zeta) = \Upsilon(\eta_k,\zeta) = \lim_{\substack{t\to\eta_k \\ t<\eta_k}} \Upsilon(t,\zeta), \quad \Upsilon(\eta_k^+,\zeta) = \lim_{\substack{t\to\eta_k \\ t>\eta_k}} \Upsilon(t,\zeta).$$

In order to present some main definitions and lemmas related to the impulsive conformable Lyapunov function technique, we will introduce the following notations:

$$\kappa\zeta = (d_1\Delta\zeta_1, d_2\Delta\zeta_2, d_3\Delta\zeta_3, d_4\Delta\zeta_4)^T,$$

$$F(\zeta(t,z)) = (F_1(\zeta(t,z)), F_2(\zeta(t,z)), F_3(\zeta(t,z)), F_4(\zeta(t,z))),$$

where

$$F_1(\zeta(t,z)) = A - \mu\zeta_1(t,z) - f(\zeta_1(t,z),\zeta_2(t,z))\zeta_2(t,z)) - g(\zeta_1(t,z),\zeta_3(t,z))\zeta_3(t,z)$$
$$-h(\zeta_1(t,z),\zeta_4(t,z))\zeta_4(t,z),$$
$$F_2(\zeta(t,z)) = f(\zeta_1(t,z),\zeta_2(t,z))\zeta_2(t,z) + g(\zeta_1(t,z),\zeta_3(t,z))\zeta_3(t,z) + h(\zeta_1(t,z),\zeta_4(t,z))\zeta_4$$
$$-(\mu + d + r)\zeta_2(t,z),$$
$$F_3(\zeta(t,z)) = (\mu + d)\zeta_2(t,z) - b\zeta_3(t,z),$$
$$F_4(\zeta(t,z)) = \sigma\zeta_2(t,z) + \gamma\zeta_3(t,z) - \eta\zeta_4(t,z),$$

and

$$J_k(\zeta(\eta_k,z)) = (J_{1k}(\zeta_1(\eta_k,z)), 0, 0, J_{4k}(\zeta_4(\eta_k,z)))^T,\ k = 1,2,\ldots.$$

Using the above notations, we can rewrite model (4) in the following form

$$\begin{cases} \mathcal{T}^{\alpha}_{\eta_k}\zeta(t,z) = \kappa\zeta(t,z) + F(\zeta(t,z)) = \bar{F}(\zeta(t,z)),\ t \neq \eta_k, \\ \zeta(\eta_k^+,z) = \zeta(\eta_k,z) + J_k(\zeta(\eta_k,z)),\ k = 1,2,\ldots. \end{cases} \tag{7}$$

We will define the α-time-conformable derivative of a function $\Upsilon \in^{\Theta} \Upsilon^{\alpha}_{\eta_k}$ with respect to the solution $\zeta = \zeta(t,z;\eta_0,\Phi_0)$ of the problem (4) (or (7)) as [44,51]

$${}^{+}\mathcal{T}^{\alpha}_{\eta_k}\Upsilon(t,\zeta) = \lim_{\varepsilon\to 0^+} \sup \frac{\Upsilon(t + \varepsilon(t-\eta_k)^{1-\alpha},\ \zeta + \varepsilon(t-\eta_k)^{1-\alpha}\bar{F}(\zeta)) - \Upsilon(t,\zeta)}{\varepsilon}. \tag{8}$$

If $\Upsilon(t,\zeta(t,\cdot)) = \Upsilon(\zeta(t,\cdot))$, $0 < \alpha \leq 1$, Υ is differentiable on ζ, and $\zeta(t,\cdot)$ is α-conformable differentiable with respect to t, we have

$$^{+}\mathcal{T}^{\alpha}_{\eta_k}\,\Upsilon(t,\zeta) = \Upsilon'(\zeta(t,\cdot))\,\mathcal{T}^{\alpha}_{\eta_k}\zeta(t,\cdot). \tag{9}$$

When using the impulsive conformable Lyapunov technique, the following comparison lemma will be also useful [44,51].

Lemma 3. *If for $t \in \mathbb{R}_+$ and $\zeta \in \mathbb{R}^4_+$ the function $\Upsilon \in^{\Theta} \Upsilon^{\alpha}_{\eta_k}$ satisfies*

(i)

$$\Upsilon(\eta_k^+,\zeta(\eta_k,\cdot) + J_k(\zeta(\eta_k,\cdot))) \leq \Upsilon(\eta_k,\zeta(\eta_k,\cdot)),\ k = 1,2,\ldots,$$

(ii)

$$^{+}\mathcal{T}^{\alpha}_{\eta_k}\,\Upsilon(t,\zeta(t,\cdot)) \leq -\varpi\Upsilon(t,\zeta(t,\cdot)),\ \varpi \geq 0,\ t \neq \eta_k,\ k = 1,2,\ldots,$$

then, the following estimate

$$\Upsilon(t,\zeta(t,\cdot)) \leq \Upsilon(\eta_0^+,\Phi_0)E_{\alpha}(-\varpi,t-\eta_0),\ t \geq \eta_0$$

holds.

3. Main Stability of Sets Criteria

Introduce the following hypotheses:

Hypothesis 1. *The functions $f(\zeta_1,\zeta_2)$, $g(\zeta_1,\zeta_3)$, and $h(\zeta_1,\zeta_4)$, are continuously differentiable and*

$$f(0,\zeta_2) = 0,\ \frac{\partial f}{\partial \zeta_1}(\zeta_1,\zeta_2) > 0,\ \frac{\partial f}{\partial \zeta_2}(\zeta_1,\zeta_2) \leq 0, \text{ for all } \zeta_1,\zeta_2 \geq 0,$$

$$g(0,\zeta_3) = 0,\ \frac{\partial g}{\partial \zeta_1}(\zeta_1,\zeta_3) > 0,\ \frac{\partial g}{\partial \zeta_3}(\zeta_1,\zeta_3) \leq 0, \text{ for all } \zeta_1,\zeta_3 \geq 0,$$

$$h(0,\zeta_4) = 0,\ \frac{\partial h}{\partial \zeta_1}(\zeta_1,\zeta_4) > 0,\ \frac{\partial h}{\partial \zeta_4}(\zeta_1,\zeta_4) \leq 0, \text{ for all } \zeta_1,\zeta_4 \geq 0$$

in the interior of $\mathbb{R}^2_+$.

Hypothesis 2. *The functions $f(\zeta_1,\zeta_2)$, $g(\zeta_1,\zeta_3)$, and $h(\zeta_1,\zeta_4)$, are bounded and Lipschitz, i.e., there exist positive constants constants B_i, $i = 1,2,3$ and constants L_i, $i = 1,2,3$, with*

$$f(\zeta_1,\zeta_2) \leq B_1,\ \ g(\zeta_1,\zeta_3) \leq B_2,\ \ h(\zeta_1,\zeta_4) \leq B_3,$$

$$|f(\zeta_1,\zeta_2) - f(\overline{\zeta}_1,\overline{\zeta}_2)| \leq L_1\big(|\zeta_1 - \overline{\zeta}_1| + |\zeta_2 - \overline{\zeta}_2|\big),$$
$$|g(\zeta_1,\zeta_3) - g(\overline{\zeta}_1,\overline{\zeta}_3)| \leq L_2\big(|\zeta_1 - \overline{\zeta}_1| + |\zeta_3 - \overline{\zeta}_3|\big),$$
$$|h(\zeta_1,\zeta_4) - g(\overline{\zeta}_1,\overline{\zeta}_4)| \leq L_3\big(|\zeta_1 - \overline{\zeta}_1| + |\zeta_4 - \overline{\zeta}_4|\big),$$

for all $\zeta_j,\overline{\zeta}_j \in \mathbb{R}_+$, $\zeta_j \neq \overline{\zeta}_j$, $j = 1,2,3,4$.

Hypothesis 3. *The functions $\zeta_j(t,z)$, $j = 1,2,3,4$ are such that for $t \neq \eta_k$, $k = 1,2,\ldots,$*

$$\zeta_j(t,z) \leq \hat{\zeta}_j,\ \hat{\zeta}_j = const > 0,\ (t,z) \in \mathbb{R}_+ \times \Omega.$$

Hypothesis 4. *The impulsive control functions J_{jk}, $j = 1, 4$, $k = 1, 2, \ldots$ are such that*

$$J_{jk}(\zeta_j(\eta_k, z)) = -\delta_{jk}\zeta_j(\eta_k, z), \quad 0 < \delta_{jk} < 1,$$

where δ_{jk} are constants, $j = 1, 4$, $k = 1, 2, \ldots$.

Remark 7. *The biological meaning of the restrictions in H1–H3 is the same as in [3]. These hypotheses provide the existence and uniqueness of the solutions of the impulse-free model (1) and the references cited therein. The bounds in H3 are justified by the habitat's carrying capacity. Adding the Hypothesis H4, we guarantee the existence and uniqueness of the solutions of system (4).*

Consider a nonempty set $\Theta \subset \mathbb{R}_+ \times \Omega \times \mathbb{R}_+^4$. Since some concepts of stability of sets require the Θ-boundedness of the solution, we first propose the next boundedness result.

Theorem 1. *Under Hypotheses H1–H4, if the model's parameters satisfy*

$$\varpi_1 = \Theta_1 - \Big(\frac{1}{2}(B_1 + B_2 + B_3) + 2(\hat{\zeta}_2 L_1 + \hat{\zeta}_3 L_2 + \hat{\zeta}_4 L_3)\Big) \geq 0,$$

$$\varpi_2 = \Theta_2 + \frac{1}{2}(\mu + d + 2r - \sigma) - \Big(\frac{1}{2}(3B_1 + B_2 + B_3) + 2\hat{\zeta}_2 L_1 + \hat{\zeta}_3 L_2 + \hat{\zeta}_4 L_3\Big) \geq 0,$$

$$\varpi_3 = \Theta_3 + b - \Big(\frac{1}{2}(\mu + d + \gamma) + B_2 + \hat{\zeta}_3 L_2\Big) \geq 0,$$

$$\varpi_4 = \Theta_4 + \eta - \Big(\frac{1}{2}(\sigma + \gamma) + B_3 + \hat{\zeta}_4 L_3\Big) \geq 0,$$

where $\Theta_j = n\theta_1 d_j$, $j = 1, \ldots, 4$, then the solutions of system (4) are uniformly Θ-bounded.

Proof. Assume that $\eta_0 \in \mathbb{R}_+$, $z \in \Omega$ and Φ_0 is the continuous initial function in (6). Let $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)^T$ $\zeta(t, z) = \zeta(t, z; \eta_0, \Phi_0)$ be the solution of model (4) that satisfies the boundary and initial conditions (5) and (6), i.e.,

$$\zeta(t, z) = \zeta(t, z; \eta_0, \Phi_0).$$

The fact that the set $\Theta(t, z)$ is not empty implies the existence of at least one solution of (4) $\overline{\zeta} = \overline{\zeta}(t, z) = (\overline{\zeta}_1(t, z), \overline{\zeta}_2(t, z), \overline{\zeta}_3(t, z), \overline{\zeta}_4(t, z))^T \in \Theta(t, z)$ that satisfies the boundary condition (5).

Then, for any $t \neq \eta_k$, $k = 1, 2, \ldots$, for the time conformable derivative of order α of the function

$$\overline{\Upsilon}(t, \zeta(t, \cdot)) = \frac{1}{2}\int_\Omega \sum_{j=1}^{4} \big((\zeta_j(t, z) - \overline{\zeta}_j(t, z)\big)^2 dz,$$

we have

$$\begin{aligned}
\mathcal{T}_{\eta_k}^{\alpha}\overline{\Upsilon}(t, \zeta(t, \cdot)) = \int_\Omega \big(\zeta_1(t, z) - \overline{\zeta}_1(t, z)\big)\Big[&d_1\Delta\big(\zeta_1(t, z) - \overline{\zeta}_1(t, z)\big) - \mu\big(\zeta_1(t, z) - \overline{\zeta}_1(t, z)\big) \\
&- f(\zeta_1(t, z), \zeta_2(t, z))\zeta_2(t, z) - g(\zeta_1(t, z), \zeta_3(t, z))\zeta_3(t, z) - h(\zeta_1(t, z), \zeta_4(t, z))\zeta_4(t, z) \\
&+ f\overline{\zeta}_1(t, z), \overline{\zeta}_2(t, z))\overline{\zeta}_2(t, z) + g(\overline{\zeta}_1(t, z), \overline{\zeta}_3(t, z))\overline{\zeta}_3(t, z) + h(\overline{\zeta}_1(t, z), \overline{\zeta}_4(t, z))\overline{\zeta}_4(t, z)\Big] dz \\
+ \int_\Omega \big(\zeta_2(t, z) - \overline{\zeta}_2(t, z)\big)\Big[&d_2\Delta\big(\zeta_2(t, z) - \overline{\zeta}_2(t, z)\big) + f(\zeta_1(t, z), \zeta_2(t, z))\zeta_2(t, z) \\
&+ g(\zeta_1(t, z), \zeta_3(t, z))\zeta_3(t, z) + h(\zeta_1(t, z), \zeta_4(t, z))\zeta_4(t, z) - f(\overline{\zeta}_1(t, z), \overline{\zeta}_2(t, z))\overline{\zeta}_2(t, z) \\
&- g(\overline{\zeta}_1(t, z), \overline{\zeta}_3(t, z))\overline{\zeta}_3(t, z) - h(\overline{\zeta}_1(t, z), \overline{\zeta}_4(t, z))\overline{\zeta}_4(t, z) - (\mu + d + r)\big(\zeta_2(t, z) - \overline{\zeta}_2(t, z)\big)\Big] dz
\end{aligned}$$

$$+\int_\Omega \left(\zeta_3(t,z)-\overline{\zeta}_3(t,z)\right)\Big[d_3\Delta\left(\zeta_3(t,z)-\overline{\zeta}_3(t,z)\right)$$

$$+(\mu+d)\left(\zeta_2(t,z)-\overline{\zeta}_2(t,z)\right)-b\left(\zeta_3(t,z)-\overline{\zeta}_3(t,z)\right)\Big]dz$$

$$+\int_\Omega \left(\zeta_4(t,z)-\overline{\zeta}_4(t,z)\right)\Big[d_4\Delta\left(\zeta_4(t,z)-\overline{\zeta}_4(t,z)\right)$$

$$+\sigma\left(\zeta_2(t,z)-\overline{\zeta}_2(t,z)\right)+\gamma\left(\zeta_3(t,z)-\overline{\zeta}_3(t,z)\right)-\eta\left(\zeta_4(t,z)-\overline{\zeta}_4(t,z)\right)\Big]dz.$$

We apply the boundary conditions (5) to the difference $\zeta(t,z)-\overline{\zeta}(t,z)$, and using the Gauss formula and Lemma 2, we obtain

$$\int_\Omega \left(\zeta_j(t,z)-\overline{\zeta}_j(t,z)\right)\Big[d_j\Delta\left(\zeta_j(t,z)-\overline{\zeta}_j(t,z)\right)\Big]dz$$

$$\leq -d_j\sum_{i=1}^{n}\int_\Omega\left(\frac{\zeta_j(t,z)-\overline{\zeta}_j(t,z)}{\partial z_i}\right)^2dz=-nd_j\int_\Omega\left(\frac{\zeta_j(t,z)-\overline{\zeta}_j(t,z)}{\partial z_i}\right)^2dz$$

$$\leq -\Theta_j\int_\Omega\left(\zeta_j(t,z)-\overline{\zeta}_j(t,z)\right)^2dz,\ j=1,\ldots,4. \tag{10}$$

From (10) and the Hypotheses H1–H3, we also have

$$\int_\Omega \left(\zeta_1(t,z)-\overline{\zeta}_1(t,z)\right)\Big[d_1\Delta\left(\zeta_1(t,z)-\overline{\zeta}_1(t,z)\right)-\mu\left(\zeta_1(t,z)\overline{\zeta}_1(t,z)\right)$$

$$-f(\zeta_1(t,z),\zeta_2(t,z))\zeta_2(t,z)-g(\zeta_1(t,z),\zeta_3(t,z))\zeta_3(t,z)-h(\zeta_1(t,z),\zeta_4(t,z))\zeta_4(t,z)$$

$$+f(\overline{\zeta}_1(t,z),\overline{\zeta}_2(t,z))\overline{\zeta}_2(t,z)+g(\overline{\zeta}_1(t,z),\overline{\zeta}_3(t,z))\overline{\zeta}_3(t,z)+h(\overline{\zeta}_1(t,z),\overline{\zeta}_4(t,z))\overline{\zeta}_4(t,z)\Big]dz$$

$$\leq(-\Theta_1+\hat{\zeta}_2L_1+\hat{\zeta}_3L_2+\hat{\zeta}_4L_3)\int_\Omega\left(\zeta_1(t,z)-\overline{\zeta}_1(t,z)\right)^2dz$$

$$+(B_1+\hat{\zeta}_2L_1)\int_\Omega\left|\zeta_1(t,z)-\tilde{\zeta}_1(t,z)\right|\left|\zeta_2(t,z)-\overline{\zeta}_2(t,z)\right|dz$$

$$+(B_2+\hat{\zeta}_3L_2)\int_\Omega\left|\zeta_1(t,z)-\tilde{\zeta}_1(t,z)\right|\left|\zeta_3(t,z)-\overline{\zeta}_3(t,z)\right|dz$$

$$+(B_3+\hat{\zeta}_4L_3)\int_\Omega\left|\zeta_1(t,z)-\tilde{\zeta}_1(t,z)\right|\left|\zeta_4(t,z)-\overline{\zeta}_4(t,z)\right|dz$$

$$\leq\Big(-\Theta_1+\frac{1}{2}\left(B_1+B_2+B_3\right)+\frac{3}{2}\left(\hat{\zeta}_2L_1+\hat{\zeta}_3L_2+\hat{\zeta}_4L_3\right)\Big)\int_\Omega\left(\zeta_1(t,z)-\overline{\zeta}_1(t,z)\right)^2dz$$

$$+\frac{1}{2}(B_1+\hat{\zeta}_2L_1)\int_\Omega\left(\zeta_2(t,z)-\overline{\zeta}_2(t,z)\right)^2dz+\frac{1}{2}(B_2+\hat{\zeta}_3L_2)\int_\Omega\left(\zeta_3(t,z)-\overline{\zeta}_3(t,z)\right)^2dz$$

$$+\frac{1}{2}(B_3+\hat{\zeta}_4L_3)\int_\Omega\left(\zeta_4(t,z)-\overline{\zeta}_4(t,z)\right)^2dz.$$

Using the same hypotheses and argumentation, we obtain

$$\int_\Omega \left(\zeta_2(t,z)-\overline{\zeta}_2(t,z)\right)\Big[d_2\Delta\left(\zeta_2(t,z)-\overline{\zeta}_2(t,z)\right)+f(\zeta_1(t,z),\zeta_2(t,z))\zeta_2(t,z)$$

$$+g(\zeta_1(t,z),\zeta_3(t,z))\zeta_3(t,z)+h(\zeta_1(t,z),\zeta_4(t,z))\zeta_4(t,z)-f(\overline{\zeta}_1(t,z),\overline{\zeta}_2(t,z))\overline{\zeta}_2(t,z)$$

$$-g(\overline{\zeta}_1(t,z),\overline{\zeta}_3(t,z))\overline{\zeta}_3(t,z)-h(\overline{\zeta}_1(t,z),\overline{\zeta}_4(t,z))\overline{\zeta}_4(t,z)\Big]dz$$

$$\leq\Big(-(\Theta_2+\mu+d+r)+\hat{\zeta}_2L_1+B_1\Big)\int_\Omega\left(\zeta_2(t,z)-\overline{\zeta}_2(t,z)\right)^2dz$$

$$+(\hat{\zeta}_2L_1+\hat{\zeta}_3L_2+\hat{\zeta}_4L_3)\int_\Omega|\zeta_1(t,z)-\overline{\zeta}_1(t,z)||\zeta_2(t,z)-\overline{\zeta}_2(t,z)|dz$$
$$+(\hat{\zeta}_3L_2+B_2)\int_\Omega|\zeta_3(t,z)-\overline{\zeta}_3(t,z)||\zeta_2(t,z)-\overline{\zeta}_2(t,z)|dz$$
$$+(\hat{\zeta}_4L_3+B_3)\int_\Omega|\zeta_4(t,z)-\overline{\zeta}_4(t,z)||\zeta_2(t,z)-\overline{\zeta}_2(t,z)|dz$$

$$\leq\Big(-(\Theta_2+\mu+d+r)+\frac{1}{2}(2B_1+B_2+B_3)+\frac{3}{2}\hat{\zeta}_2L_1+\hat{\zeta}_3L_2+\hat{\zeta}_4L_3\Big)\int_\Omega\big(\zeta_2(t,z)-\overline{\zeta}_2(t,z)\big)^2dz$$
$$+\frac{1}{2}(\hat{\zeta}_2L_1+\hat{\zeta}_3L_2+\hat{\zeta}_4L_3)\int_\Omega\big(\zeta_1(t,z)-\overline{\zeta}_1(t,z)\big)^2dz$$
$$+\frac{1}{2}(B_2+\hat{\zeta}_3L_2)\int_\Omega\big(\zeta_3(t,z)-\overline{\zeta}_3(t,z)\big)^2dz+\frac{1}{2}(B_3+\hat{\zeta}_4L_3)\int_\Omega\big(\zeta_4(t,z)-\overline{\zeta}_4(t,z)\big)^2dz,$$

and

$$\int_\Omega\big(\zeta_3(t,z)-\overline{\zeta}_3(t,z)\big)\Big[d_3\Delta\big(\zeta_3(t,z)-\overline{\zeta}_3(t,z)\big)$$
$$+(\mu+d)\big(\zeta_2(t,z)-\overline{\zeta}_2(t,z)\big)-b\big(\zeta_3(t,z)-\overline{\zeta}_3(t,z)\big)\Big]dz$$
$$\leq\Big(-(\Theta_3+b)+\frac{1}{2}(\mu+d)\Big)\int_\Omega\big(\zeta_3(t,z)-\overline{\zeta}_3(t,z)\big)^2dz$$
$$+\frac{1}{2}(\mu+d)\int_\Omega\big(\zeta_2(t,z)-\overline{\zeta}_2(t,z)^2\big)dz,$$

and

$$\int_\Omega\big(\zeta_4(t,z)-\overline{\zeta}_4(t,z)\big)\Big[d_4\Delta\big(\zeta_4(t,z)-\overline{\zeta}_4(t,z)\big)$$
$$+\sigma\big(\zeta_2(t,z)-\tilde{\zeta}_2(t,z)\big)+\gamma\big(\zeta_3(t,z)-\overline{\zeta}_3(t,z)\big)-\eta\big(\zeta_4(t,z)-\overline{\zeta}_4(t,z)\big)\Big]dz$$
$$\leq\Big(-(\Theta_4+\eta)+\frac{1}{2}\sigma+\frac{1}{2}\gamma\Big)\int_\Omega\big(\zeta_4(t,z)-\overline{\zeta}_4(t,z)\big)^2dz$$
$$+\frac{1}{2}\sigma\int_\Omega\big(\zeta_2(t,z)-\overline{\zeta}_2(t,z)\big)^2dz+\frac{1}{2}\gamma\int_\Omega\big(\zeta_3(t,z)-\overline{\zeta}_3(t,z)\big)^2dz.$$

Then, from all the above estimates and the condition of Theorem 1, we get for $t\neq\eta_k$, $k=1,2,\ldots,$

$$\mathcal{T}^{\alpha}_{\eta_k}\overline{\Upsilon}(t,\zeta(t,\cdot))\leq-\int_\Omega\sum_{j=1}^{4}\varpi_j\big((\zeta_j(t,z)-\overline{\zeta}_j(t,z)\big)^2dz\leq0. \tag{11}$$

For $t=\eta_k$, $k=1,2,\ldots,$ using H4, we get $0<1-\delta_{jk}<1$, $j=1,4$, and hence

$$\int_\Omega\big(\zeta_j(t^+,z)-\overline{\zeta}_j(t^+,z)\big)^2dz=\int_\Omega\big(\zeta_j(t,z)-\delta_{jk}\zeta_j(t,z)-\overline{\zeta}_j(t,z)+\delta_{jk}\overline{\zeta}_j(t,z)\big)^2dz$$
$$=\int_\Omega(1-\delta_{jk})^2\big(\zeta_j(t,z)-\overline{\zeta}_j(t,z)\big)^2dz$$
$$<\int_\Omega\big(\zeta_j(t,z)-\overline{\zeta}_j(t,z)\big)^2dz,\ j=1,4, \tag{12}$$

and

$$\int_\Omega\big(\zeta_j(t^+,z)-\overline{\zeta}_j(t^+,z)\big)^2dz=\int_\Omega\big(\zeta_j(t,z)-\overline{\zeta}_j(t,z)\big)^2dz,\ j=2,3. \tag{13}$$

Define a Lyapunov-type function

$$\Upsilon(t,\zeta) = \frac{1}{2}d^2(\zeta,\Theta(t,z)) = \frac{1}{2}\left(\inf_{\overline{\zeta}\in\Theta(t,z)} ||\zeta-\overline{\zeta}||_2\right)^2. \tag{14}$$

It is clear that $\Upsilon \in {}^{\Theta}\Upsilon^{\alpha}_{\eta_k}$. Also, from the fact that $(\inf x)^2 = \inf(x^2)$ for $x \geq 0$, for function (14), we have

$$\Upsilon(t,\zeta) = \inf_{\overline{\zeta}\in\Theta(t,z)} \frac{1}{2}\int_{\Omega}\sum_{j=1}^{4}\left((\zeta_j(t,z)-\overline{\zeta}_j(t,z)\right)^2 dz. \tag{15}$$

Then, from (12), (13) and (11), for $t \geq \eta_0$ and $\zeta \in \mathbb{R}^4_+$, we obtain

$$\Upsilon(\eta_k^+,\zeta(\eta_k,\cdot)+J_k(\zeta(\eta_k,\cdot))) \leq \Upsilon(\eta_k,\zeta(\eta_k,\cdot)),\ k=1,2,\ldots, \tag{16}$$

and

$${}^{+}\mathcal{T}^{\alpha}_{\eta_k}\,\Upsilon(t,\zeta(t,\cdot)) \leq 0,\ t\neq\eta_k,\ k=1,2,\ldots. \tag{17}$$

Hence, (16), (17) and Lemma 3 imply that

$$\Upsilon(t,\zeta(t,\cdot)) \leq \Upsilon(\eta_0^+,\Phi_0),\ t\geq\eta_0. \tag{18}$$

Finally, let $\phi > 0$, and the constant $\beta = \beta(\phi) > 0$ be chosen so that $\phi < \beta$. For a $\rho > 0$, let $\Phi_0 \in S_\rho \cap \overline{\Theta}(\eta_0^+,z)(\phi)$.

We can conclude from (14) and (18) that for $t \geq \eta_0$,

$$\frac{1}{2}d^2(\zeta,\Theta(t,z)) = \Upsilon(t,\zeta(t,z)) \leq \Upsilon(\eta_0^+,\Phi_0) = \frac{1}{2}d^2(\Phi_0,\Theta(\eta_0^+,z)) \leq \frac{1}{2}\phi^2 < \frac{1}{2}\beta^2.$$

Therefore, $\zeta(t,z;\eta_0,\Phi_0) \in \Theta(t,z)(\beta)$ for $t \geq \eta_0$, $z \in \Omega$, which according to Definition 5(b), means that the solutions of model (4) are uniformly Θ-bounded. □

Next, we will propose uniform global asymptotic stability of sets criteria.

Theorem 2. *If conditions of Theorem 1 hold and, in addition, there exists a function* $\psi \in C[\mathbb{R}_+,\mathbb{R}_+]:$ *ψ is strictly increasing,* $\psi(0)=0$, *such that*

$$\min\{\omega_1,\omega_2,\omega_2,\omega_4\} \geq \frac{1}{2}\psi(d(\zeta,\Theta(t,z))),\ \zeta\in\mathbb{R}^4_+,\ z\in\Omega,\ t\neq\eta_k,\ k=1,2,\ldots, \tag{19}$$

then the set Θ is uniformly globally asymptotically stable with respect to model (4).

Proof. Assume again that $\eta_0 \in \mathbb{R}_+$, $z \in \Omega$ and Φ_0 is the continuous initial function in (6). Let $\zeta = (\zeta_1,\zeta_2,\zeta_3,\zeta_4)^T$, $\zeta(t,z) = \zeta(t,z;\eta_0,\Phi_0)$ be the solution of model (4) which satisfies the boundary and initial conditions (5) and (6), i.e.,

$$\zeta(t,z) = \zeta(t,z;\eta_0,\Phi_0).$$

Since all the conditions of Theorem 1 are satisfied, the solutions of model (4) are uniformly Θ-bounded.

Next, we have to prove that Θ is uniformly stable and uniformly globally attractive set with respect to system (4).

Let $v > 0$, and choose $\delta = \delta(v) > 0$ so that $\delta < v$. For given $\rho > 0$ and $z \in \Omega$, we have $\Phi_0 \in S_\rho \cap \Theta(\eta_0^+,z)(\delta)$.

For the Lyapunov-type function defined by (14), using the same steps as in the proof of Theorem 1, we get (18) and

$$\frac{1}{2}d^2(\zeta,\Theta(t,z)) = \Upsilon(t,\zeta(t,z)) \leq \Upsilon(\eta_0^+,\Phi_0) = \frac{1}{2}d^2(\Phi_0,\Theta(\eta_0^+,z)) \leq \frac{1}{2}\delta^2 < \frac{1}{2}v^2,\ t \geq \eta_0.$$

Hence,

$$\zeta(t,z;\eta_0,\Phi_0) \in \Theta(t,z)(v),\ t \geq \eta_0,$$

which proves the uniform stability of the set Θ with respect to (4).

The last part is the proof of the uniform global attractivity of the set Θ with respect to (4).

From (11) and (19), we obtain

$$^{+}\mathcal{T}^{\alpha}_{\eta_k}\,\Upsilon(t,\zeta(t,\cdot)) \leq -\psi(d(\zeta,\Theta(t,\cdot))),\ t \neq \eta_k,\ k = 1,2,\ldots. \tag{20}$$

For given $\lambda > 0$ and $v > 0$, let $\Phi_0 \in S_\rho \cap \overline{\Theta}(\eta_0^+,z)(\lambda)$ for $\rho > 0$. Let the number $\delta = \delta(v) > 0$ be such that $\delta < v$.

We will show that there exists an $\iota = \iota(\lambda,v) > 0$ such that for some $\bar{t} \in [\eta_0,\eta_0+\iota]$ and for $z \in \Omega$, the following inequality holds

$$d(\zeta(\bar{t},z),\Theta(\bar{t},z)) < \delta. \tag{21}$$

If we assume that inequality (21) does not hold, then for any $\iota > 0$, there exists a solution $\zeta(t,z;\eta_0,\Phi_0)$ of model (4) such that for $\eta_0 \in \mathbb{R}_+$, $\rho > 0$, $z \in \Omega$, $\Phi_0 \in S_\rho \cap \overline{\Theta}(\eta_0^+,z)(\lambda)$,

$$d(\eta(t,z),\Theta(t,z)) \geq \delta,\ t \in [\eta_0,\eta_0+\iota],\ z \in \Omega.$$

Using (11) and (20) and applying Lemma 1, we have

$$\Upsilon(t,\zeta(t,\cdot)) - \Upsilon(\eta_0^+,\Phi_0) = I^{\alpha}_{\eta_0}\,{}^{+}\mathcal{T}^{\alpha}_{\eta_0}\,\Upsilon(t,\zeta(t,\cdot)) \leq -\int_{\eta_0}^{t}(s-\eta_0)^{\alpha-1}\psi(d(\zeta(s,\cdot),\Theta(s,\cdot)))ds.$$

Since the Lyapunov-type function $\Upsilon(t,\zeta(t,\cdot))$ is such that

$$\lim_{t\to\infty}\Upsilon(t,\zeta(t,\cdot)) = \Upsilon^* \geq 0,$$

then

$$\int_{\eta_0}^{\infty}(t-\eta_0)^{\alpha-1}\psi(d(\zeta(t,\cdot),\Theta(t,\cdot)))dt \leq \frac{1}{2}\lambda^2 - \Upsilon^*.$$

From the fact that $\Upsilon^* \leq \frac{1}{2}\lambda^2$, it follows that the number $\iota = \iota(\lambda,v) > 0$ can be chosen so that

$$\frac{\iota^{\alpha}}{\alpha} = \int_{\eta_0}^{\eta_0+\iota}(t-\eta_0)^{\alpha-1}dt > \frac{\frac{1}{2}\lambda^2 - \Upsilon^* + 1}{\psi(\delta)}.$$

Then, we obtain the next contraction

$$\frac{1}{2}\lambda^2 - \Upsilon^* \geq \int_{\eta_0}^{\infty}(t-\eta_0)^{\alpha-1}\psi(d(\zeta(t,\cdot),\Theta(t,\cdot)))dt$$

$$\geq \int_{\eta_0}^{\eta_0+\iota} (t-\eta_0)^{\alpha-1}\psi(d(\zeta(t,\cdot),\Theta(t,\cdot)))dt \geq \psi(\delta)\int_{\eta_0}^{\eta_0+\iota}(t-\eta_0)^{\alpha-1}dt > \frac{1}{2}\lambda^2 - \Upsilon^* + 1.$$

Hence, there exists an $\iota = \iota(\lambda, v) > 0$ such that for some $\bar{t} \in [\eta_0, \eta_0 + \iota]$ and for $z \in \Omega$, inequality (21) is satisfied.

Then, for $t \geq \bar{t}$ (hence, for any $t \geq \eta_0 + \iota$ as well), we have

$$\frac{1}{2}d^2(\zeta, \Theta(t,z)) = \Upsilon(t, \zeta(t,z)) \leq \Upsilon(\bar{t}, \zeta(\bar{t}, z))$$

$$= \frac{1}{2}d^2(\zeta(\bar{t},z), \Theta(\bar{t},z)) \leq \frac{1}{2}\delta^2 < \frac{1}{2}v^2.$$

Therefore, $\eta_0 \in \mathbb{R}_+$, $\rho > 0$, $z \in \Omega$ and $\Phi_0 \in S_\rho \cap \overline{\Theta}(\eta_0^+, z)(\lambda)$ imply $\zeta(t, z; \eta_0, \Phi_0) \in \Theta(t,z)(v)$, $t \geq \eta_0 + \iota$, which, according to Definition 6(c) means that the set Θ is uniformly globally attractive with respect to system (4).

Since all the requirements of Definition 6(d) are met, the set Θ is uniformly globally asymptotically stable with respect to model (4). □

Remark 8. *Given that establishing rigorous stability results is a problem of both theoretical and practical importance, the stability properties of mathematical models studied in biology and medicine are the most investigated qualitative properties. Particularly, numerous researchers offered stability results for integer-order, fractional-order or conformable EVD models [3,6–8,17,19,29–31,40]. However, all these existing stability results consider only individual states. The stability criteria offered by Theorem 2 extend and generalize the existing stability results to the stability of sets case. For the specific cases, when the set Θt contains only single states studied in [3,29–31], the results there can be obtained as corollaries of Theorem 2.*

Remark 9. *Stability is also a fundamental issue in control systems, such as impulsive control systems [37,38]. Impulsive control strategies for virus diseases' mathematical models have been proposed by some researchers in [39–41]. However, in all existing impulsive control results, the restrictions on the impulsive control instances and impulsive control functions are very rigorous. For example, the authors in [41] considered constants restricted on the open interval $(0,1)$, while the proposed condition H4 is less restrictive. Also, the stability criteria offered by Theorem 2 do not require any conditions on the distance between impulsive control instants as in [39–41]. Hence, the proposed technique improves several recently developed impulsive control strategies.*

Finally, we will propose criteria for uniform global exponential stability of the set Θ. It is well known that the concept of exponential stability is a specific case of the asymptotic stability notion, which is also demonstrated by the next result.

Theorem 3. *If conditions of Theorem 2 hold, and in addition, ψ is a nonnegative constant, then the set Θ is uniformly globally exponentially stable with respect to system (4).*

Proof. Let $t \geq \eta_0$, $z \in \Omega$ and $\zeta = \zeta(t, z; \eta_0, \Phi_0)$ be the solution of the IBVP (4)–(6) corresponding to a continuous initial function Φ_0.

For the Lyapunov-type function Υ, defined by (14), we again obtain (16).

In addition, for a constant function ψ, inequality (20) takes the form

$$^{+}\mathcal{T}^{\alpha}_{\eta_k}\,\Upsilon(t, \zeta(t,\cdot)) \leq -\psi d(\zeta, \Theta(t,\cdot)) = -2\psi\,\Upsilon(t, \zeta(t,\cdot)),\ t \neq \eta_k,\ k = 1, 2, \ldots. \tag{22}$$

Then, (16), (22) and the comparison Lemma 3 imply

$$\Upsilon(t,\zeta(t,\cdot)) \leq \Upsilon(\eta_0^+,\Phi_0)E_\alpha(-2\psi,t-\eta_0),\ t\geq\eta_0,$$

or

$$d(\zeta(t,z;\eta_0,\Phi_0),\Theta(t,z)) \leq d(\Phi_0,\Theta(\eta_0^+,z))E_\alpha^{1/2}(-2\psi,t-\eta_0),\ \ t\geq\eta_0,\ z\in\Omega. \tag{23}$$

Therefore, the set Θ is uniformly globally exponentially stable with respect to system (4). □

Remark 10. *For $\alpha = 1$, estimate (23) becomes*

$$d(\zeta(t,z;\eta_0,\Phi_0),\Theta(t,z)) \leq d(\Phi_0,\Theta(\eta_0^+,z))\exp^{1/2}(-2\psi,t-\eta_0),\ \ t\geq\eta_0,\ z\in\Omega,$$

which guarantees the global exponential stability of the set Θ with respect to the integer-order model (1). Hence, with this research, we generalize and improve existing stability results for EVD models [3,4,6–8,41] to the impulsive conformable setting considering diffusion effects and the extended stability of sets concept.

4. Discussion and an Illustrative Example

The reliance on integer-order ODE [3,6–8] and PDE models [3,11,12] can be restrictive, as it limits the ability to effectively capture the complex dynamics associated with infectious diseases such as EVD. Unlike classical integer-order differential models, fractional models incorporate memory and hereditary effects, which are important in describing disease transmission and progression [16,17]. However, most of the classical fractional derivatives are inconvenient in applications. The conformable derivative provides a relatively simple and mathematically consistent definition of fractional differentiation. Furthermore, the conformable operator simplifies the stability analysis and qualitative study of epidemic models compared with classical fractional derivatives, making it particularly attractive for researchers studying equilibrium points, reproduction numbers, and long-term disease behavior. As the conformable calculus approach provides a more realistic and flexible framework for describing the transmission dynamics and control of Ebola outbreaks, it has been recently applied in a few research papers [29–31].

In addition, the modeling of infectious diseases often requires incorporating realistic intervention strategies that occur suddenly or at discrete time intervals rather than continuously. Impulsive control frameworks facilitate the analysis of long-term disease eradication strategies. By studying the stability of disease-free equilibria under impulsive interventions, researchers can determine optimal intervention schedules and thresholds required to suppress or eliminate outbreaks of EVD [41].

In this study, we propose a hybrid impulsive conformable framework for modeling the spread of the EVD is provided. Different form all existing EVD studies, which investigate the qualitative behavior of single equilibrium states, the stability of sets concept is introduced and criteria are established. Table 2 highlights the comparison between the proposed approach and existing EVD models.

Table 2. Comparison between the proposed approach and existing EVD models.

EVD Models	DTs	CDs	IC	SS
[6–8]	×	×	×	×
[3,11,12]	√	×	×	×
[29–31]	×	√	×	×
[41]	×	×	√	×
The proposed EVD model (4)	√	√	√	√

The abbreviations used in Table 2 are as follows:

- DTs = Diffusion Terms
- CDs = Conformable Derivatives
- IC = Impulsive Control
- SS = Stability of Sets

Since the criteria are formulated as inequalities involving the model parameters, their implementation is relatively straightforward. The following simple algorithm can make the modeling and stability analysis approaches easier to follow and potentially useful for researchers who wish to apply the framework to other epidemiological models.

Algorithmic Procedure for Modeling and Stability Analysis

- Phase 1: Mathematical Modeling
  - Step 1: Identify system variables.
  - Step 2: Introduce diffusion terms.
  - Step 3: Introduce conformable derivatives.
  - Step 4: Introduce impulsive controllers.
  - Step 5: Obtain impulsive conformable equations with diffusion terms describing system dynamics.
- Phase 2: Stability Concept
  - Step 6: Define the stability of sets concept.
  - Step 7: Determine the Set Θ.
- Phase 2: Stability Analysis
  - Step 8: Assess the stability conditions H1–H3 and conditions of Theorems 1–3 for the system's parameters in the continuous part.
  - Step 9: Introduce suitable impulsive controllers which satisfy H4.

The accurate stability analysis and behavioral prediction of the proposed model are expected to accelerate medical and biotechnological studies, as computational predictions are significantly faster and more cost-effective than experimental laboratory procedures.

Example 1. *We consider the impulsive conformable EVD model (4) for* $t \in \mathbb{R}_+$, $z \in \Omega \subset \mathbb{R}^3_+$, Ω*-bounded, with the following particular values of the model's parameters* $A = 50$, $\mu = 0.5$, $f(\zeta_1, \zeta_2) = \beta_1\zeta_1$, $g(\zeta_1, \zeta_3) = \beta_2\zeta_1$, $h(\zeta_1, \zeta_4) = \beta_3\zeta_1$, $\beta_1 = 0.005$, $\beta_2 = 0.001$, $\beta_3 = 0.001$, $d = 0.05$, $r = 0.06$, $b = 0.8$, $\sigma = 0.02$, $\gamma = 0.01$, $\eta = 0.8$, *diffusion coefficients* $d_1 = 0.5$, $d_2 = 0.56$, $d_3 = 0.0034$, $d_4 = 0.00034$, *and impulsive control functions determined by* $\delta_{jk} = \frac{1}{k}$, $j = 1, 4$, $k = 1, 2, \ldots$.

Define the set $\Theta \subset \mathbb{R}_+ \times \Omega \times \mathbb{R}^4_+$ such that

$$\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)^T \in \mathbb{R}^4_+ : \zeta_1 \leq 100,\ \zeta_2 \leq 100,\ \zeta_3 \leq 68.75,\ \zeta_4 \leq 3.359375,$$

and

$$\Theta = \{\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)^T \in \mathbb{R}^4_+ : \ \zeta_1 + \zeta_2 < 100, \ \zeta_3 \leq 68.75, \ \zeta_4 \leq 3.359375\}.$$

Hence, we have that condition H3 is satisfied for

$$\hat{\zeta}_1 = \hat{\zeta}_2 = 100, \ \hat{\zeta}_3 = 68.75, \ \hat{\zeta}_4 = 3.359375.$$

In addition, condition H4 is satisfied for the impulsive functions, and Hypotheses H1 and H2 are satisfied for the functions f, g and h for

$$B_1 = 0.5, \ B_2 = 0.1, \ B_3 = 0.1,$$

$$L_1 = \beta_1 = 0.005, \ L_2 = \beta_2 = 0.001, \ L_3 = \beta_3 = 0.001.$$

Now, we have that

$$\varpi_1 = \Theta_1 - \left(\frac{1}{2}(B_1 + B_2 + B_3) + 2(\hat{\zeta}_2 L_1 + \hat{\zeta}_3 L_2 + \hat{\zeta}_4 L_3)\right) = 0.00578125,$$

$$\varpi_2 = \Theta_2 + \frac{1}{2}(\mu + d + 2r - \sigma) - \left(\frac{1}{2}(3B_1 + B_2 + B_3) + 2\hat{\zeta}_2 L_1 + \hat{\zeta}_3 L_2 + \hat{\zeta}_4 L_3\right) = 0.082890625,$$

$$\varpi_3 = \Theta_3 + b - \left(\frac{1}{2}(\mu + d + \gamma) + B_2 + \hat{\zeta}_3 L_2\right) = 0.36145,$$

$$\varpi_4 = \Theta_4 + \eta - \left(\frac{1}{2}(\sigma + \gamma) + B_3 + \hat{\zeta}_4 L_3\right) = 0.682660625.$$

According to Theorem 1, the solutions of system (4) are uniformly Θ-bounded. Moreover, a nonnegative constant ψ exists such that

$$0 \leq \psi \leq \min\{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$$

for which inequality (22) holds.

Therefore, according to Theorem 3, the set Θ is uniformly globally exponentially stable with respect to system (4).

Remark 11. *The proposed Example illustrates our theoretical boundedness and stability criteria. As the notions of stability of individual states such as equilibria are particular cases of the stability of sets concept, the applicability of the proposed results is extended. For example, our results generalize the boundedness and stability results in [3] to the stability of sets case considering impulsive perturbations and conformable derivatives. In fact, the model's parameters used in our Example are similar to those in [3] and the set $\Theta(t, z)$ includes the equilibrium states studied in [3]. In addition, the proposed new model and the corresponding extended stability results generalize some earlier results on the topic [5,41]. Furthermore, our results have universal applicability and can be easily expanded to the study of many other conformable diffusion processes studied in virology and medicine.*

5. Conclusions

In this paper, we extend existing EVD models by formulating a model using the impulsive conformal approach. Diffusion effects are also taken into account to consider the evolution of the disease not only in time but also in space. To better study the stability behavior of multi-stable systems and answer the question to what extent the initial conditions can be changed without compromising the stability properties established near the steady states, we adapted the concept of set stability to the introduced model. By

utilizing the impulsive conformable Lyapunov function method along with a comparison lemma, sufficient condition for achieving uniform boundedness, uniform global asymptotic stability and uniform global exponential stability of a set of a general nature with respect to the proposed model are established. The criteria are expressed as inequalities between the model's parameters and are convenient for verification. An example is also presented to illustrate the theoretical results. Since the concept of stability of sets is far more general than that for equilibrium points, our results extend and generalize many existing boundedness and stability criteria for EVD models. In addition, the generalized framework and the results obtained can be applied to investigate other classes of impulsive control conformable models. Our future research will be focused on the consideration of models with distributed delays and non-instantaneous impulses based on this study. Discretization of the proposed model, numerical simulations, optimal impulsive vaccination strategies, parameter estimation from epidemiological data, and applications to other infectious diseases are also important and interesting topics for future investigations.

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References

1. Area, I.; Ndairou, F.; Nieto, J.J.; Silva, C.J. Ebola model and optimal control with vaccination constraints. *J. Ind. Manag. Optim.* **2018**, *14*, 427–446. [CrossRef]
2. Rachah, A.; Torres, D.F.M. Mathematical modelling, simulation, and optimal control of the 2014 Ebola outbreak in West Africa. *Discret. Dyn. Nat. Soc.* **2015**, *2015*, 842792. [CrossRef]
3. El Rhoubari, Z.; Hattaf, K.; Yousf, N. A class of Ebola virus disease models with post-death transmission and environmental contamination. In *Mathematical Modelling and Analysis of Infectious Diseases*; Hattaf, K., Dutta, H., Eds.; Springer: Cham, Switzerland, 2020; Volume 302, pp. 295–321.
4. Hattaf, K.; Yousfi, N. Mathematical modeling in virology. In *Emerging and Reemerging Viral Pathogens, Applied Virology Approaches Related to Human, Animal and Environmental Pathogens*; Ennaji, M.M., Ed.; Academic Press: London, UK, 2020; pp. 325–339, ISBN 978-0-12-814966-9.
5. Berge, T.; Lubuma, J.; Tassè, A.J.O.; Tenkam, H.M. Dynamics of host-reservoir transmission of Ebola with spillover potential to humans. *Electr. J. Qual. Theory Differ. Equ.* **2018**, *14*, 1–32.
6. Rhoubari, Z.E.; Besbassi, H.; Hattaf, K.; Yousfi, N. Dynamics of a generalized model for Ebola virus disease. In *Trends in Biomathematics: Mathematical Modeling for Health, Harvesting, and Population Dynamics*; Rubem, P., Mondaini, R.P., Eds.; Springer: Cham, Switzerland, 2019; pp. 35–46.
7. De la Sen, M.; Ibeas, A.; Alonso-Quesada, S.; Nistal, R. On a new epidemic model with asymptomatic and dead-infective subpopulations with feedback controls useful for Ebola disease. *Discret. Dyn. Nat. Soc.* **2017**, *2017*, 4232971. [CrossRef]
8. El Rhoubari, Z.E.; Besbassi, H.; Hattaf, K.; Yousf, N. Mathematical modeling of Ebola virus disease in bat population. *Discret. Dyn. Nat. Soc.* **2018**, *2018*, 5104524. [CrossRef]
9. Elaiw, A.M.; Hobiny, A.D.; Agha, A.D.A. Global dynamics of reaction-diffusion oncolytic M1 virotherapy with immune response. *Appl. Math. Comput.* **2020**, *367*, 124758. [CrossRef]
10. Guo, Z.; Liu, J.; Wang, Y.; Chen, M.; Wang, D.; Xu, D.; Cheng, J. Diffusion models in bioinformatics and computational biology. *Nat. Rev. Bioeng.* **2024**, *2*, 136–154. [CrossRef]
11. Covington, R.; Patton, S.; Walker, E.; Yamazaki, K. Improved uniform persistence for partially diffusive models of infectious diseases: Cases of avian influenza and Ebola virus disease. *Math. Biosci. Eng.* **2023**, *20*, 19686–19709. [CrossRef]
12. Yamazaki, K. Threshold dynamics of reaction–diffusion partial differential equations model of Ebola virus disease. *Int. J. Biomath.* **2018**, *11*, 1850108. [CrossRef]

13. Khan, H.; Alzabut, J.; Tunç, O.; Kaabar, M.K.A. A fractal–fractional COVID-19 model with a negative impact of quarantine on the diabetic patients. *Results Control Optim.* **2023**, *10*, 100199. [CrossRef]
14. Naik, P.A.; Yavuz, M.; Qureshi, S.; Naik, M.U.D.; Owolabi, K.M.; Soomro, A.; Ganie, A.H. Memory impacts in hepatitis C: A global analysis of a fractional-order model with an effective treatment. *Comput. Method Program Biomed.* **2024**, *254*, 108306. [CrossRef] [PubMed]
15. Singh, H.; Srivastava, H.M.; Nieto, J.J. (Eds.) *Handbook of Fractional Calculus for Engineering and Science*, 1st ed.; CRC Press: Boca Raton, FL, USA, 2022.
16. Pan, W.; Li, T.; Ali, S. A fractional order epidemic model for the simulation of outbreaks of Ebola. *Adv. Differ. Equ.* **2021**, *2021*, 161. [CrossRef] [PubMed]
17. Yousef, A. Fractional-order modeling of Ebola Virus Disease (EVD) transmission: Insights from infected animals to humans and healthcare deficiencies. *Alex. Eng. J.* **2025**, *120*, 391–408. [CrossRef]
18. Younoussi, M.E.; Hajhouji, Z.; Hattaf, K.; Yousfi, N. Dynamics of a reaction-diffusion fractional-order model for *M*1 oncolytic virotherapy with CTL immune response. *Chaos Solitons Fract.* **2022**, *157*, 111957. [CrossRef]
19. Naik, P.A.; Farman, M.; Jamil, K.; Nisar, K.S.; Hashmi, M.A.; Huang, Z. Modeling and analysis using piecewise hybrid fractional operator in time scale measure for ebola virus epidemics under Mittag–Leffler kernel. *Sci. Rep.* **2024**, *14*, 24963. [CrossRef]
20. Rosa, S.; Ndaïrou, F. Optimal control applied to piecewise-fractional Ebola model. *Mathematics* **2024**, *12*, 985. [CrossRef]
21. Abdeljawad, T. On conformable fractional calculus. *J. Comput. Appl. Math.* **2015**, *279*, 57–66. [CrossRef]
22. Khalil, R.; Al Horani, M.; Yousef, A.; Sababheh, M. A new definition of fractional derivative. *J. Comput. Appl. Math.* **2014**, *264*, 65–70. [CrossRef]
23. Anderson, D.R.; Ulness, D.J. Newly defined conformable derivatives. *Adv. Dyn. Syst. Appl.* **2015**, *10*, 109–137.
24. Kiskinov, H.; Petkova, M.; Zahariev, A. Why and how the definition of the conformable derivative in the lower terminal should be changed. *Sci. Works Plovdiv Univ.* **2025**, *40*, 17–29.
25. Kiskinov, H.; Petkova, M.; Zahariev, A.; Veselinova, M. Some results about conformable derivatives in Banach spaces and an application to the partial differential equations. *AIP Conf. Proc.* **2021**, *2333*, 120002. [CrossRef]
26. Martynyuk, A.A.; Stamov, G.; Stamova, I. Integral estimates of the solutions of fractional-like equations of perturbed motion. *Nonlinear Anal. Model. Control* **2019**, *24*, 138–149. [CrossRef]
27. Thabet, H.; Kendre, S. Conformable mathematical modeling of the COVID-19 transmission dynamics: A more general study. *Math. Methods Appl. Sci.* **2023**, *46*, 18126–18149. [CrossRef]
28. Soni, K.; Sinha, A.K. Dynamics of epidemic model with conformable fractional derivative. *Nonlinear Sci.* **2025**, *4*, 100040. [CrossRef]
29. Abbas, N.; Zanib, S.A.; Ramzan, S.; Nazir, A.; Shatanawi, W. A conformable mathematical model of Ebola Virus Disease and its stability analysis. *Heliyon* **2024**, *10*, e35818. [CrossRef]
30. Hammouch, Z.; Rasul, R.R.Q.; Ouakka, A.; Elazzouzi, A. Mathematical analysis and numerical simulation of the Ebola epidemic disease in the sense of conformable derivative. *Chaos Soliton Fract.* **2022**, *158*, 112006. [CrossRef]
31. Nazir, A.; Ahmed, N.; Khan, U.; Mohyud-Din, S.T.; Nisar, K.S.; Khan, I. An advanced version of a conformable mathematical model of Ebola virus disease in Africa. *Alex. Eng. J.* **2020**, *59*, 3261–3268. [CrossRef]
32. Bohner, M.; Stamov, G.; Stamova, I.; Spirova, C. On *h*-manifolds stability for impulsive delayed SIR epidemic models. *Appl. Math. Model.* **2023**, *118*, 853–862. [CrossRef]
33. Wang, L.; Chen, L.; Nieto, J.J. The dynamics of an epidemic model for pest control with impulsive effect. *Nonlinear Anal. Real World Appl.* **2010**, *11*, 1374–1386. [CrossRef]
34. Yan, P. Impulsive SUI epidemic model for HIV/AIDS with chronological age and infection age. *J. Theor. Biol.* **2010**, *265*, 177–184. [CrossRef]
35. Stamov, G.; Stamova, I.; Spirova, C. Impulsive reaction-diffusion delayed models in biology: Integral manifolds approach. *Entropy* **2021**, *23*, 1631. [CrossRef] [PubMed]
36. Li, X.; Song, S. *Impulsive Systems with Delays: Stability and Control*, 1st ed.; Science Press & Springer: Singapore, 2022; ISBN 978-981-16-4686-7.
37. Yang, T. *Impulsive Control Theory*, 1st ed.; Springer: Berlin, Germany, 2001.
38. Yang, X.; Peng, D.; Lv, X.; Li, X. Recent progress in impulsive control systems. *Math. Comput. Simul.* **2019**, *155*, 244–268. [CrossRef]
39. Belfo, J.P.; Lemos, J.M. *Optimal Impulsive Control for Cancer Therapy*, 1st ed.; Springer: Cham, Switzerland, 2021; ISBN 978-3-030-50487-8.
40. Wang, L.; She, A.; Xie, Y. The dynamics analysis of Gompertz virus disease model under impulsive control. *Sci. Rep.* **2023**, *13*, 10180. [CrossRef] [PubMed]
41. Njankou, S.D.D.; Nyabadza, F. Modelling the potential influence of human migration and two strains on Ebola virus disease dynamics. *Infect. Dis. Model.* **2020**, *7*, 645e659.
42. Stamov, T.; Stamova, I. Design of impulsive controllers and impulsive control strategy for the Mittag–Leffler stability behavior of fractional gene regulatory networks. *Neurocomputing* **2021**, *424*, 54–62. [CrossRef]

43. Tuz, M. Stability analysis of fractional-order impulsive Cohen–Grossberg neural networks with interdisciplinary applications in neurobiology and chemical kinetics. *Comp. Biol. Chem.* **2026**, *121*, 108785. [CrossRef]
44. Bohner, M.; Stamova, I.; Stamov, G.; Spirova, C. Integral manifolds for impulsive HCV conformable neural network models. *Appl. Math. Sci. Eng.* **2024**, *32*, 2345896. [CrossRef]
45. Kaviya, R.; Priyanka, M.; Muthukumar, P. Mean-square exponential stability of impulsive conformable fractional stochastic differential system with application on epidemic model. *Chaos Solitons Fract.* **2022**, *160*, 112070. [CrossRef]
46. Martynyuk, A.; Stamov, G.; Stamova, I.; Gospodinova, E. Formulation of impulsive ecological systems using the conformable calculus approach: Qualitative analysis. *Mathematics* **2023**, *11*, 2221. [CrossRef]
47. Hale, J.K.; Verduyn Lunel, S.M. *Introduction to Functional Differential Equations*, 1st ed.; Springer: New York, NY, USA, 1993; ISBN 978-0-387-94076-2, 978-1-4612-8741-4, 978-1-4612-4342-7.
48. Li, Y.; Sanfelice, R.G. Finite time stability of sets for hybrid dynamical systems. *Automatica* **2019**, *100*, 200–211. [CrossRef]
49. Li, Z.; Yan, L.; Zhou, X. Global attracting sets and stability of neutral stochastic functional differential equations driven by Rosenblatt process. *Front. Math. China* **2018**, *13*, 87–105. [CrossRef]
50. Xie, S. Stability of sets of functional differential equations with impulse effect. *Appl. Math. Comput.* **2011**, *218*, 592–597. [CrossRef]
51. Stamov, G.; Stamova, I.; Spirova, C. On an impulsive conformable M1 oncolytic virotherapy neural network model: Stability of sets analysis. *Mathematics* **2025**, *13*, 141. [CrossRef]

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