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Exact Travelling-Wave Solutions of a Nonlinear Convection–Diffusion Equation with Square-Root Flux

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Abstract

We investigate a nonlinear convection—diffusion equation involving a non-polynomial square-root flux. By applying a travelling-wave reduction and introducing a structurally motivated square-root transformation, we show that the resulting ordinary differential equation possesses an intrinsic logistic first-order structure. This reduction yields an explicit closed-form monotone travelling-wave solution connecting two equilibrium states and uniquely determines the admissible wave speed $c = 2/3$. The solution describes the diffusive smoothing of an initial discontinuity into a propagating transition layer. Direct finite-difference simulations with Riemann-type initial data confirm convergence toward the analytical profile and verify the predicted wave speed. These results demonstrate that convection—diffusion equations with non-algebraic flux functions can admit exact travelling-wave solutions when appropriate structural transformations are identified, providing both theoretical insight and reliable benchmarks for numerical methods.

Keywords: travelling-wave solutions; nonlinear partial differential equations; exact solutions; logistic reduction; structural transformation; convection–diffusion models

MSC: 35K57; 35C07; 65M06

1. Introduction

Nonlinear convection–diffusion equations constitute a fundamental class of partial differential equations describing the interplay between advective transport and diffusive smoothing mechanisms. They arise in a wide range of applications, including traffic flow dynamics, porous media transport, combustion processes, population dispersal, and fluid motion in heterogeneous media [1,2]. A central feature of these models is the formation of travelling-wave solutions, which represent propagating transition layers connecting distinct equilibrium states and often govern the long-time behavior of solutions originating from discontinuous initial data. In traffic flow theory, such wave phenomena appear in the classical models of Lighthill and Whitham and Richards [3,4]. A prototypical example of an exactly solvable convection–diffusion equation is the equation introduced by [5], which admits explicit travelling-wave solutions. Due to strong nonlinear effects, exact closed-form solutions of convection—diffusion equations are generally rare. Numerous expansion and auxiliary-equation techniques have been developed for constructing exact solutions of nonlinear evolution equations, including the extended tanh method, Exp-function method, and (G'/G)-expansion approaches [6–9]. Recent developments include unified symbolic–computational frameworks and interaction solution techniques for higher-

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dimensional nonlinear wave equations [10]. These techniques typically rely on polynomial or rational nonlinearities in the governing equation.

However, comparatively little attention has been devoted to convection–diffusion equations with non-polynomial flux functions. In particular, models involving square-root nonlinearities introduce additional analytical challenges due to their non-algebraic structure and potential degeneracy near equilibrium states.

The purpose of this work is to show that a nonlinear convection–diffusion equation with square-root flux admits an explicit travelling-wave solution in closed form. Rather than applying a standard auxiliary-equation ansatz, we identify an intrinsic structural transformation that reduces the travelling-wave equation to a logistic-type first-order equation. Convection–diffusion equations with non-polynomial flux functions remain comparatively unexplored because of their non-algebraic structure and the analytical difficulties they introduce near equilibrium states. In particular, square-root nonlinearities limit the applicability of many classical exact-solution techniques. The resulting solution provides both analytical insight into nonlinear transport dynamics and a reliable benchmark for numerical schemes. We now present the mathematical formulation of the model.

2. Mathematical Model

We consider the following nonlinear convection–diffusion equation:

$$u_t + \sqrt{u}\, u_x = u_{xx}, \tag{1}$$

where $u = u(x,t)$, $x \in \mathbb{R}$, and $t > 0$.

We focus on Riemann-type initial data:

$$u(x,0) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases} \tag{2}$$

Our objective is to derive an explicit travelling-wave solution connecting the equilibrium states $u = 0$ and $u = 1$.

3. Travelling-Wave Reduction and Exact Solution

We seek travelling-wave solutions of the form

$$u(x,t) = U(\xi), \qquad \xi = x - ct, \tag{3}$$

where c is the wave speed.

Substitution into the governing equation yields

$$-cU' + \sqrt{U}U' = U'', \tag{4}$$

or equivalently,

$$U'' + (c - \sqrt{U})U' = 0. \tag{5}$$

We seek monotone solutions satisfying

$$U(-\infty) = 0, \qquad U(+\infty) = 1. \tag{6}$$

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3.1. Structural Transformation

The presence of the term $\sqrt{U}U'$ suggests introducing the transformation

$$W(\xi) = \sqrt{U(\xi)}. \tag{7}$$

This transformation eliminates fractional powers and converts the equation into polynomial form in W. Since

$$U = W^2, \qquad U' = 2WW', \qquad U'' = 2(W'^2 + WW'').$$

Substituting these expressions into the travelling-wave equation yields

$$W'^2 + WW'' + (c - W)WW' = 0. \tag{8}$$

We look for a first-order polynomial structure compatible with the boundary conditions $W(-\infty) = 0$ and $W(+\infty) = 1$, which suggests a quadratic polynomial with zeros at $W = 0$ and $W = 1$. We seek a first-order structure of the form

$$W' = a\,W(1 - W), \tag{9}$$

where a is a constant to be determined.

Differentiating gives

$$W'' = a(2W - 1)W'.$$

Substituting W' and W'' into the equation above yields

$$a^2W^2(W - 1)^2 + a^2W^2(W - 1)(2W - 1) + aW^2(W - 1)(c - W) = 0.$$

Factoring $aW^2(W - 1)$ (nonzero in the transition region) gives

$$a(W - 1) + a(2W - 1) + (c - W) = 0.$$

Simplifying,

$$a(3W - 2) + c - W = 0.$$

For this identity to hold for all W, coefficients of like powers must vanish. Equating coefficients yields

$$3a - 1 = 0, \qquad -2a + c = 0.$$

Hence,

$$a = \frac{1}{3}, \qquad c = \frac{2}{3}.$$

Therefore, the travelling-wave equation reduces to the logistic equation

$$W' = \frac{1}{3}W(W - 1). \tag{10}$$

3.2. Explicit Solution

The logistic equation admits the explicit solution

$$W(\xi) = \frac{1}{1 + Ce^{-\xi/3}}, \tag{11}$$

where $C > 0$ is arbitrary.

Since $U = W^2$, the travelling-wave solution of the original PDE is

$$u(x,t) = \frac{1}{\left(1 + Ce^{-(x-\frac{2}{3}t)/3}\right)^2}. \tag{12}$$

This solution satisfies

$$\lim_{x\to-\infty} u(x,t) = 0, \qquad \lim_{x\to+\infty} u(x,t) = 1,$$

and propagates to the right with constant speed $c = 2/3$.

4. Numerical Validation

To validate the analytical solution, we perform direct finite-difference simulations of

$$u_t + \sqrt{u}\, u_x = u_{xx}.$$

A semi-discrete scheme with central diffusion and upwind convection is employed. The numerical solution is initialized using the Riemann data and evolved until a steady travelling profile forms. The computational domain is truncated to $[-L, L]$ with $L = 20$. We use $N = 2000$ grid points and a time step Δt satisfying $\Delta t \leq \frac{\Delta x^2}{2}$. Grid refinement tests confirm the convergence of the numerical wave speed toward $2/3$.

Figure 1 compares the analytical solution with the numerical solution at $t = 5$. Excellent agreement between the analytical and numerical solutions is observed.

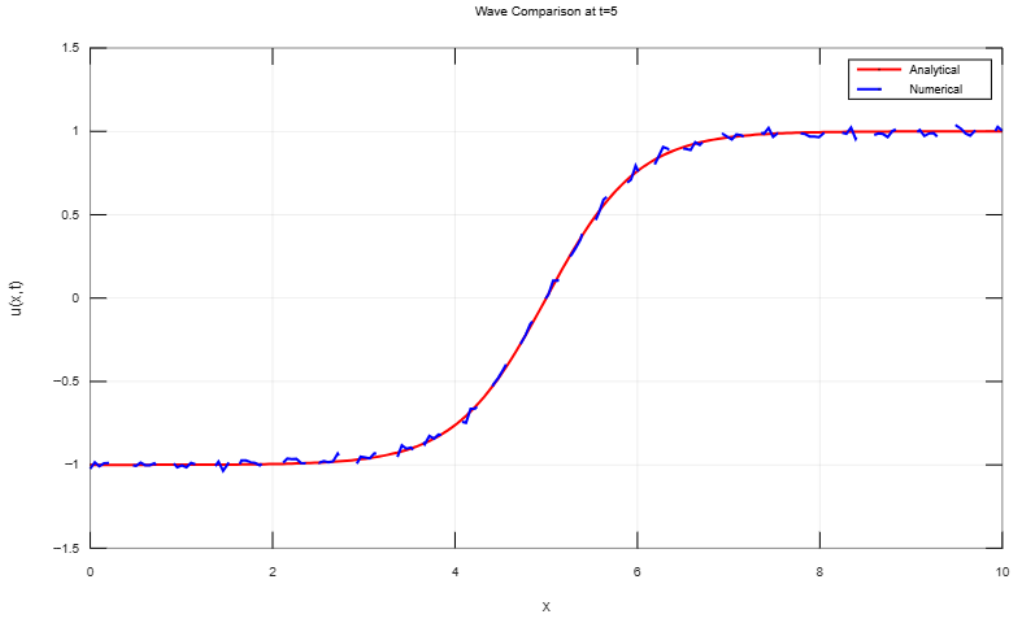


Figure 1. Excellent agreement between analytical and numerical profiles is observed.

Figure 2 shows the space–time contour plot of the numerical solution. The travelling front propagates to the right with constant speed consistent with $c = 2/3$. The numerical wave speed, measured from the motion of the mid-level interface $u = 1/2$, converges to $2/3$, confirming the analytical prediction. Minor discrepancies near steep gradients are attributable to numerical diffusion and decrease under mesh refinement.

The contour plot illustrates the diffusive smoothing of the initial discontinuity into a travelling transition layer, a characteristic feature of nonlinear convection–diffusion dynamics.

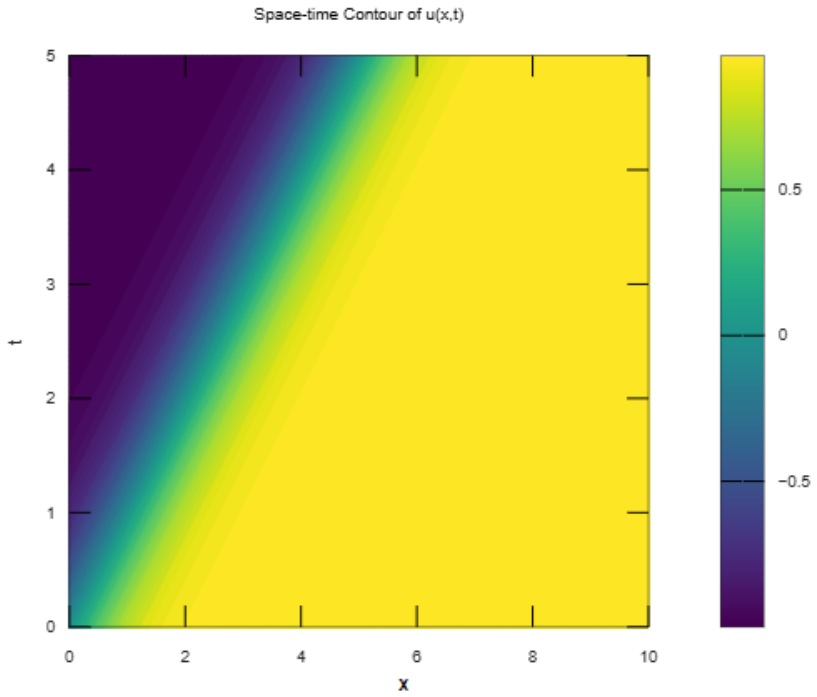


Figure 2. Space–time contour of the solution $u(x,t)$ showing the formation and propagation of a monotone travelling front moving to the right with constant speed $c = 2/3$.

5. Conclusions

This study derives an explicit travelling-wave solution of a nonlinear convection–diffusion equation with square-root flux. The key observation is that the travelling-wave equation admits a natural square-root transformation that reveals an intrinsic logistic first-order structure. The wave speed is uniquely determined as $c = 2/3$, and numerical simulations confirm convergence toward the analytical solution. The uniqueness of the admissible speed follows directly from the polynomial identity obtained after substitution of the logistic ansatz into the transformed equation. This work demonstrates that certain non-polynomial convection—diffusion equations can admit exact closed-form travelling-wave solutions when appropriate structural transformations are identified. The derived solution provides a useful analytical benchmark for numerical schemes and contributes to the theory of nonlinear wave propagation in non-algebraic convection models. The approach presented here may also be applicable to broader classes of convection—diffusion equations involving non-algebraic flux functions.

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