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Moduli Spaces of Arrangements of 12 Projective Lines with a Sextic Point

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Abstract

In this paper, we classify moduli spaces of arrangements of 12 lines with a sextic point. We show that these moduli spaces can consist of more than two connected components. We also present defining equations of arrangements whose moduli spaces are not irreducible, and after taking quotients by complex conjugation, we obtain potential Zariski pairs.

Keywords: Zariski pairs; line arrangements; classification of algebraic curves; moduli spaces

MSC: 14H10; 14H15; 14H20; 14H30

1. Introduction

A line arrangement $\mathcal{A}$ in $\mathbb{CP}^2$ is a finite collection of projective lines. The complement of the union of lines in $\mathcal{A}$ is denoted by $M(\mathcal{A})$.

An essential topic in hyperplane arrangement theory is the relation between the topology of complements and the combinatorics of intersection lattices. For line arrangements, Jiang and Yau [1] showed that homeomorphism of the complement always implies lattice isomorphism. However, the converse is not true in general. In [2,3] the authors found a large class of line arrangements, called *nice arrangements*, whose intersection lattices determine the topology of the complements (see also [4–6]).

A *Zariski pair* is a pair of line arrangements that are lattice-isomorphic but have non-isomorphic fundamental groups of their complements. The first Zariski pair was constructed by Rybnikov [7]. On the other hand, Fan [8] and Garber et al. [9] proved that there is no Zariski pair for arrangements of up to eight real lines. This was extended to eight complex lines by Nazir and Yoshinaga [10], and to nine complex lines by Ye [11]. Recently, Amram et al. classified arrangements of 10 and 11 lines with a quintuple point [12–14] and found some “potential Zariski pairs”. (By a *potential Zariski pair* we mean a pair of lattice-isomorphic arrangements whose moduli space has more than one point; thus, by Randell’s theorem, they are not ruled out as Zariski pairs, but the actual computation of their fundamental groups is left for future work.)

For a complex line arrangement $\mathcal{A}$, Randell’s lattice-isotopy theorem [15] and Cohen–Suciu’s theorem [16] imply that arrangements belonging to the same connected component of the moduli space, or to two complex conjugate components, cannot form a Zariski pair.



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The classification of moduli spaces is carried out in three steps. First, we roughly classify intersection lattices according to the number of multiple points. Second, we distinguish cases by the relative position of the sextic point and other multiple points. Third, we write down defining equations involving parameters for each intersection lattice.

The study of moduli spaces of line arrangements with high-multiplicity points not only extends the classification of Zariski pairs but also provides concrete models for moduli spaces of curves with prescribed singularities. These spaces are closely related to the periods of K3 surfaces and may admit interpretations in the context of decorated moduli spaces in theoretical physics, such as those arising in string compactifications or two-dimensional gravity (cf. [17,18]). In this paper we focus on arrangements of 12 lines with a sextic point and show that their moduli spaces can have multiple connected components, yielding new examples of potential Zariski pairs.

The paper is organized as follows. Section 2 collects preliminaries. Section 3 treats arrangements with a point of multiplicity ≥ 7 . Section 4 deals with arrangements containing a sextic point and at least one quadruple point. Section 5 considers sextic point arrangements without quadruple points. Section 6 summarizes the results and discusses broader connections.

2. Preliminaries

Let $\mathcal{A} = \{L_1, \dots, L_n\}$ be a line arrangement in $\mathbb{CP}^2$. A singularity of $\bigcup L_i$ is called a *multiple point* of $\mathcal{A}$ if its multiplicity is at least 3. The set

$$L(\mathcal{A}) = \left\{ \bigcap_{i \in S} L_i \mid S \subseteq \{1, \dots, n\} \right\},$$

partially ordered by reverse inclusion, is the *intersection lattice* of $\mathcal{A}$.

Definition 1. Two line arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ are *lattice-isomorphic*, denoted $\mathcal{A}_1 \sim \mathcal{A}_2$, if their intersection lattices are isomorphic, i.e., there exists a permutation ϕ of $\{1, \dots, n\}$ such that for every non-empty subset $S \subseteq \{1, \dots, n\}$,

$$\dim \left(\bigcap_{i \in S} L_i \right) = \dim \left(\bigcap_{j \in \phi(S)} H_j \right),$$

where $L_i \in \mathcal{A}_1$, $H_j \in \mathcal{A}_2$.

Definition 2. The moduli space of line arrangements with fixed lattice $L(\mathcal{A})$ (or simply of $\mathcal{A}$) is

$$\mathcal{M}_{\mathcal{A}} = \left\{ \mathcal{B} \in ((\mathbb{CP}^2)^*)^n \mid \mathcal{B} \sim \mathcal{A} \right\} / \mathrm{PGL}_{\mathbb{C}}(2).$$

We denote by $\mathcal{M}_{\mathcal{A}}^c$ the quotient of $\mathcal{M}_{\mathcal{A}}$ under complex conjugation.

Definition 3 ([7]). A pair of reduced plane curves $\mathcal{C}_1, \mathcal{C}_2 \subsetneq \mathbb{CP}^2$ is called a *Zariski pair* if there exist tubular neighbourhoods $T(\mathcal{C}_1), T(\mathcal{C}_2)$ and a homeomorphism $h : T(\mathcal{C}_1) \rightarrow T(\mathcal{C}_2)$ with $h(\mathcal{C}_1) = \mathcal{C}_2$, but there is no homeomorphism $f : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ with $f(\mathcal{C}_1) = \mathcal{C}_2$.

Definition 4 ([7]). Let $k \in \mathbb{N}$. An arrangement $\mathcal{A}$ is of *type C_k* if k is the minimum number of lines in $\mathcal{A}$ containing all points of multiplicity at least three.

Definition 5 ([7]). Let $\mathcal{A}$ be a line arrangement of type C_3 . It is called a *simple C_3 arrangement* if there exist three lines $L_1, L_2, L_3 \in \mathcal{A}$ containing all multiple points and one of the following holds:

1. $L_1 \cap L_2 \cap L_3 \neq \emptyset$, or

2. $L_1 \cap L_2 \cap L_3 = \emptyset$ and one of L_1, L_2, L_3 contains only one multiple point apart from the intersections with the other two lines.

Theorem 1 ([7]). If $\mathcal{A}$ is a simple C_3 arrangement, then $\mathcal{M}_{\mathcal{A}}$ is irreducible.

Theorem 2 ([7]). Let $\mathcal{A} = \{L_1, \dots, L_n\}$ and assume L_n passes through at most two multiple points. Set $\mathcal{A}' = \{L_1, \dots, L_{n-1}\}$. Then $\mathcal{M}_{\mathcal{A}}$ is irreducible whenever $\mathcal{M}_{\mathcal{A}'}$ is irreducible.

An arrangement is called *non-reductive* if each of its lines passes through at least three multiple points; otherwise it is *reductive*. Denote by n_r the number of intersection points of multiplicity r .

Lemma 1 ([19]). For an arrangement of k lines in $\mathbb{CP}^2$,

$$\frac{k(k-1)}{2} = \sum_{r \geq 2} \frac{r(r-1)n_r}{2}.$$

Theorem 3 ([19]). If $n_k = n_{k-1} = n_{k-2} = 0$, then

$$n_2 + \frac{3}{4}n_3 \geq k + \sum_{r \geq 5} (2r-9)n_r.$$

Theorem 4 ([12]). Let $\mathcal{A}$ be an arrangement of $n \geq 9$ lines. If there exists a multiple point of multiplicity $\geq n-4$, then $\mathcal{M}_{\mathcal{A}}$ is irreducible.

The following lemma is well known and facilitates computations.

Lemma 2. Let $\{L_1, L_2, L_3\}$ and $\{L_4, L_5, L_6, L_7, L_8\}$ be two pencils of lines intersecting at one point and intersecting transversally in 15 points. Then there exists an automorphism of the dual projective plane such that the eight lines are given by

$$Y = 0, Y = Z, Y = t_1Z, X = 0, X = Z, X = t_2Z, X = t_3Z, X = t_4Z.$$

3. Arrangements of 12 Lines with a Multiple Point of Multiplicity ≥ 7

Theorem 5. Let $\mathcal{A}$ be a non-reductive arrangement of 12 lines possessing a multiple point of multiplicity ≥ 7 . Then $\mathcal{M}_{\mathcal{A}}$ is irreducible.

Proof. If the multiplicity is ≥ 8 , irreducibility follows from Theorem 4. Assume the highest multiplicity equals 7. By Lemma 1 and Theorem 3,

$$54 - 26n_7 \geq \frac{9}{4}(n_3 + n_4 + n_5 + n_6).$$

There are at least $15 - n_7$ multiple points of multiplicity < 7 ; hence $15 - n_7 \leq n_3 + n_4 + n_5$. Substituting gives $n_7 \leq \frac{54}{95} < 1$, a contradiction. $\square$

4. Arrangements with a Sextic Point and a Quadruple Point

Theorem 6. Let $\mathcal{A}$ be a non-reductive arrangement of 12 lines in $\mathbb{CP}^2$ with a sextic point and $n_r = 0$ for $r \geq 7$. Then $n_6 = 1$, $n_5 = 0$ and $n_4 \leq 1$.

Proof. From Lemma 1 and Theorem 3,

$$54 - 18n_6 \geq \frac{9}{4}(n_3 + n_4 + n_5).$$

At least $13 - n_6$ multiple points have multiplicity < 6 , so $\frac{9}{4}(13 - n_6) \leq 54 - 18n_6$, yielding $n_6 \leq 1$; hence $n_6 = 1$.

Similarly,

$$36 - 11n_5 \geq \frac{9}{4}(n_3 + n_4), \quad \frac{9}{4}(12 - n_5) \leq 36 - 11n_5,$$

so $n_5 \leq 1$. If $n_5 = 1$ and the quintuple point is not collinear with the sextic point, some line contains at most two multiple points—contradiction. If they are collinear, label L_{12} as the line through both, and let $L_1 \cap \dots \cap L_5 \cap L_{12}$ be the sextic point and $L_6 \cap L_7 \cap L_8 \cap L_9 \cap L_{12}$ the quintuple point. Then each of L_{10}, L_{11} must pass through four intersection points of the form $L_i \cap L_j$ ($i = 1, \dots, 5, j = 6, 7, 8, 9$), forcing some L_i to have at most two multiple points—contradiction. Thus $n_5 = 0$.

Now

$$36 - 6n_4 \geq \frac{9}{4}n_3, \quad \frac{9}{4}(12 - n_4) \leq 36 - 6n_4,$$

giving $n_4 \leq 2$. If the sextic and quadruple points are not collinear, at least 12 triple points exist; hence $n_4 \leq 1$. If $n_4 = 2$ and they are collinear, let $L_1 \cap \dots \cap L_5 \cap L_{12}$ be the sextic point and $L_6 \cap L_7 \cap L_8 \cap L_{12}$ a quadruple point, with L_{12} at infinity. Non-reductivity forces L_{12} to contain at least three multiple points. If another quadruple point lies on L_{12} , say $L_9 \cap L_{10} \cap L_{11} \cap L_{12}$, then each of L_9, L_{10}, L_{11} must pass through at most four intersection points $L_i \cap L_j$ ($i = 1, \dots, 5, j = 6, 7, 8$), forcing some L_i to have at most two multiple points—contradiction. If the third multiple point on L_{12} is a triple point, a similar counting argument leads to the same contradiction. Therefore $n_4 \leq 1$. $\square$

Theorem 7. Let $\mathcal{A}$ be non-reductive with $n_6 = n_4 = 1$ and $n_r = 0$ ($r \geq 7$). If the sextic and the quadruple point are not collinear, then $\mathcal{M}_{\mathcal{A}} = \emptyset$.

Proof. Let $L_1 \cap \dots \cap L_5 \cap L_9$ be the sextic point and $L_6 \cap L_7 \cap L_8 \cap L_{12}$ the quadruple point, with L_{12} at infinity. Then L_{10}, L_{11} must pass through at most two intersection points of the form $L_i \cap L_j$ ($i \in \{1, \dots, 5, 9\}, j \in \{6, 7, 8, 12\}$). This forces some L_i to have at most two multiple points, contradicting non-reductivity. $\square$

Example 1. When the sextic and quadruple points are collinear (Figure 1), after computation we obtain the equation

$$XYZ(X - Z)(Y - Z)(Y - t_1Z)(X - t_2Z)(X - t_3Z)(X - t_4Z) \\ \times [X + (t_3 - t_4)Y - t_3Z][X + (t_4 - t_2)Y - t_4Z][X + (1 - t_3)Y - Z] = 0,$$

where $t_1 = \frac{t-1}{1-t-t^2}$, $t_2 = t^2$, $t_3 = t$, $t_4 = 1 - t$ and t satisfies $t^3 + t^2 + t - 1 = 0$. The moduli space $\mathcal{M}_{\mathcal{A}}$ consists of three points—a potential Zariski pair.

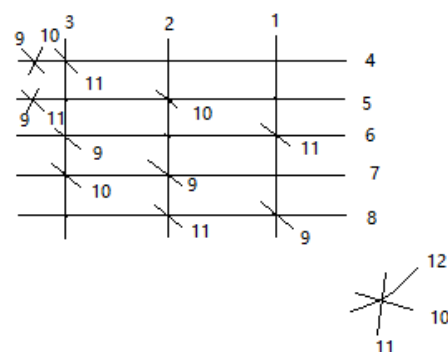


Figure 1. Configuration of Example 1.

5. Arrangements with a Sextic Point and No Quadruple Point

First we bound the number of triple points.

Theorem 8. *Let $\mathcal{A}$ be a non-reductive arrangement of 12 lines with one sextic point and such that all triple points lie on the six lines through the sextic point. Then $12 \leq n_3 \leq 14$.*

Proof. From Lemma 1 and Theorem 3,

$$n_2 + 3n_3 + 6n_4 + 10n_5 + 15n_6 = 66, \quad n_2 + \frac{3}{4}n_3 \geq 11 + n_5 + 3n_6.$$

Eliminating n_2 gives $n_3 \leq 16$. Let $L_1 \cap \dots \cap L_6$ be the sextic point. The remaining six lines each contain at least three triple points. Let a, b, c, d be the numbers among $\{L_7, \dots, L_{12}\}$ passing through 3, 4, 5, 6 triple points respectively. If $d \neq 0$, one line would have six triple points, but it can intersect the other five lines in at most five points, so $d = 0$.

If $n_3 = 16$, the system

$$a + b + c + d = 6, \quad 3a + 4b + 5c + 6d = 32$$

has four solutions, all impossible by a similar counting argument. If $n_3 = 15$, the system

$$a + b + c + d = 6, \quad 3a + 4b + 5c + 6d = 30$$

has six solutions, which is again impossible. Hence $n_3 \leq 14$. Each of $L_1, \dots, L_6$ contains at least two triple points, so $n_3 \geq 12$. $\square$

Example 2. *An arrangement with exactly 13 triple points, where $L_1 \cap L_2 \cap L_3$ is an extra triple point (Figure 2). Its equation is*

$$XY(Y - Z)(Y - t_1Z)(X - Z)(X - t_2Z)(X - t_3Z)(X - t_4Z)(X - t_5Z) \\ \times [X + t_3Y - t_3Z][X + (1 - t_4)Y - Z][X + (t_5 - t_3)Y - t_5Z] = 0,$$

with $t_1 = 1 - \frac{1}{2t - t^2}$, $t_2 = \frac{t - t^2 + 1}{2 - t}$, $t_3 = t$, $t_4 = \frac{1}{2 - t}$, $t_5 = 2t - t^2$ and $t^3 - 4t^2 + 3t + 1 = 0$. $\mathcal{M}_{\mathcal{A}}$ consists of three points—a potential Zariski pair.

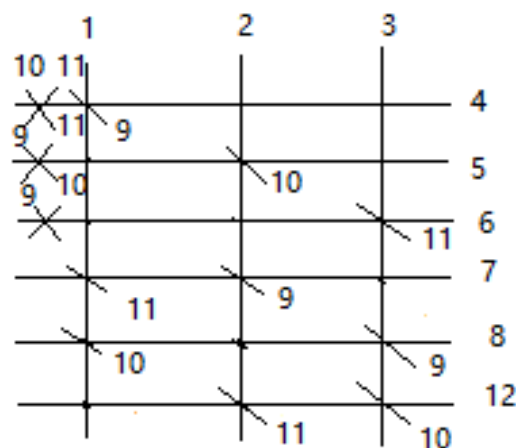


Figure 2. Configuration of Example 2.

Lemma 3. *Let $\mathcal{A} = \{L_1, \dots, L_n\}$ and suppose L_n passes through at least two multiple points. Set $\mathcal{A}' = \{L_1, \dots, L_{n-1}\}$. Then $\mathcal{M}_{\mathcal{A}}$ (resp. $\mathcal{M}_{\mathcal{A}}^c$) is either a point or empty whenever $\mathcal{M}_{\mathcal{A}'}$ (resp. $\mathcal{M}_{\mathcal{A}'}^c$) is a point.*

Proof. If $\mathcal{M}_{\mathcal{A}}$ contained two distinct points $\mathcal{A}, \tilde{\mathcal{A}}$, then after applying an element of $PGL_{\mathbb{C}}(2)$, we would have two distinct arrangements with the same $\mathcal{A}'$. Since L_n is uniquely determined by the two multiple points it passes through, this would force $\mathcal{A} = \tilde{\mathcal{A}}$, a contradiction. $\square$

Theorem 9. Let $\mathcal{A}$ be non-reductive with one sextic point and 14 triple points, all on the six lines through the sextic point. Then $\mathcal{M}_{\mathcal{A}}^c$ is irreducible (in fact a single point).

Proof. Let $L_1 \cap \dots \cap L_6$ be the sextic point. Exactly two of $L_1, \dots, L_6$ contain three triple points; the rest contain two. Let a, b, c be the numbers of lines among $\{L_7, \dots, L_{12}\}$ passing through 3, 4, 5 triple points respectively. Then

$$a + b + c = 6, \quad 3a + 4b + 5c = 28,$$

so $(a, b, c) = (1, 0, 5)$ or $(0, 2, 4)$. By symmetry we may assume that L_6 contains only two triple points and remove it. Then $\mathcal{A}' = \mathcal{A} \setminus \{L_6\}$ is an 11-line arrangement with one quintuple point and 12 triple points, all in the pencil of the quintuple point. By [12] (Theorem 6.6), $\mathcal{M}_{\mathcal{A}'}$ is a single point. Lemma 3 then forces $\mathcal{M}_{\mathcal{A}}^c$ to be a point (hence irreducible). $\square$

Corollary 1. If $\mathcal{A}$ is non-reductive with $n_6 = 1$, $n_4 = 0$, and all triple points lie on the six lines through the sextic point, then $n_3 > 13$ implies that $\mathcal{A}$ cannot be part of a Zariski pair.

When $n_3 \leq 13$, many configurations exist with $\mathcal{M}_{\mathcal{A}}$ having more than one point. We present three examples.

Example 3. Thirteen triple points, all on the six lines (Figure 3). Equation:

$$\begin{aligned} &XYZ(X-Z)(Y-Z)(X-t_1Z)(X-t_2Z)(X-t_3Z) \\ &\times [X+(t_2-1)Y-t_2Z][X+(t_2-t_1)Y-t_2Z](X-t_3Y)[X+(t_1-t_2)Y-t_1Z] = 0, \end{aligned}$$

with $t_1 = \frac{1}{2}$, $t_2 = \pm\sqrt{\frac{1}{2}}$, $t_3 = 1 \mp \sqrt{\frac{1}{2}}$. $\mathcal{M}_{\mathcal{A}}$ consists of two points.

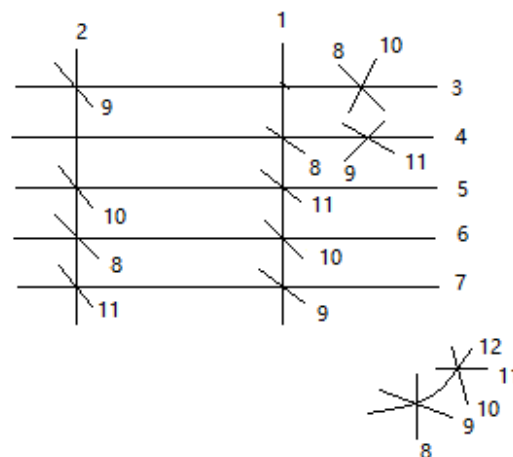


Figure 3. Configuration of Example 3.

Example 4. Twelve triple points, all on the six lines (Figure 4). Equation:

$$\begin{aligned} &XYZ(X-Z)(Y-Z)(X-t_1Z)(X-t_2Z)(X-t_3Z)(X+t_2Y-t_2Z) \\ &\times [X+(1-t_2)Y-Z](X-t_3Y)[X+(t_3-t_1)Y-t_3Z] = 0, \end{aligned}$$

with $t_1 = t^2 - 4t$, $t_2 = t$, $t_3 = -t$ and $t^2 - 5t + 2 = 0$. $\mathcal{M}_{\mathcal{A}}$ consists of two points.

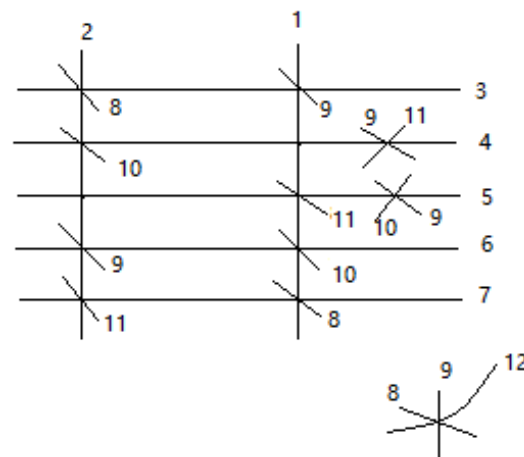


Figure 4. Configuration of Example 4.

Example 5. Twelve triple points, all on the six lines (Figure 5). Equation:

$$XYZ(X-Z)(Y-Z)(X-t_1Z)(X-t_2Z)(X-t_3Z) \\ \times [X+(1-t_2)Y-Z][X+(t_2-t_3)Y-t_2Z](X-t_1Y)[X+t_3Y-t_3Z] = 0,$$

with $t_1 = t - 1$, $t_2 = t$, $t_3 = \frac{t-1}{t-2}$ and $t^3 - 3t^2 + 2t - 2 = 0$. $\mathcal{M}_{\mathcal{A}}$ consists of three points.

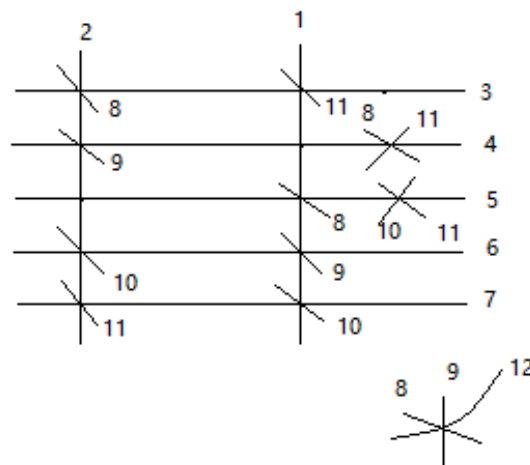


Figure 5. Configuration of Example 5.

6. Conclusions

We have classified the moduli spaces of arrangements of 12 projective lines in $\mathbb{CP}^2$ that contain a sextic point. The main results are:

- For non-reductive arrangements with a sextic point and no points of multiplicity ≥ 7 , necessarily $n_6 = 1$, $n_5 = 0$ and $n_4 \leq 1$ (Theorem 6).
- When a sextic and a quadruple point coexist and are not collinear, the moduli space is empty (Theorem 7); when they are collinear, we constructed examples whose moduli spaces consist of three points, providing potential Zariski pairs (Example 1).
- For arrangements with a sextic point and no quadruple point, $12 \leq n_3 \leq 14$. For $n_3 = 14$ the quotient moduli space $\mathcal{M}_{\mathcal{A}}^c$ is irreducible (a single point, Theorem 9). For $n_3 = 12$ or 13 we constructed examples where $\mathcal{M}_{\mathcal{A}}$ consists of two or three points (Examples 3–5), again yielding potential Zariski pairs.

These examples show that moduli spaces of 12-line arrangements with a sextic point can have multiple connected components, and the number of components depends on the combinatorial configuration.

Broader Connections

The configurations studied here are closely related to moduli spaces of sextic curves on $K3$ surfaces. Recent work by Yu et al. [18] shows that moduli of sextic curves with ADE singularities correspond to period points in the moduli space of $K3$ surfaces. Similarly, line arrangements appear in the study of moduli spaces of del Pezzo surfaces, where they are used to construct compactifications via K-stability [20]. Moreover, Guerville-Ballé [21] recently constructed arrangements of eleven lines whose moduli spaces consist of four connected components using the splitting polygon structure, complementing our results for twelve lines. The relation between line arrangements and Zariski pairs continues to be explored, as demonstrated by Bannai et al. [17] in the context of conic-line arrangements. Our explicit equations provide concrete models for such geometric structures and may admit interpretations in the context of *decorated moduli spaces* in two-dimensional gravity or as configurations of D-branes in string compactifications.

A summary of the main configurations and their properties is given in Table 1.

Table 1. Summary of 12-line arrangements with a sextic point.

Configuration	n_6	n_4	n_3	Moduli Space	Irreducible?	Potential Zariski Pair?
quadruple (collinear)	1	1	–	3 points	No	Yes
13 triple points (Example 2)	1	0	13	3 points	No	Yes
13 triple points (Example 3)	1	0	13	2 points	No	Yes
12 triple points (Example 4)	1	0	12	2 points	No	Yes
12 triple points (Example 5)	1	0	12	3 points	No	Yes
14 triple points (Theorem 9)	1	0	14	1 point (quotient)	Yes (quotient)	No

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