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The Space of Interval-Valued Abel-Convergent Sequences and Their Fundamental Properties

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Abstract

This study introduces and investigates the space of interval-valued Abel-convergent sequences, extending the classical notion of Abel convergence to sequences whose terms are closed and bounded intervals rather than real numbers. A suitable metric structure is defined on this space, and it is shown that the space is complete with respect to the introduced metric. Additionally, it is proven that the space of interval-valued Abel-convergent sequences forms a quasilinear subspace. Moreover, several inclusion relations are examined, and a norm is defined under which the space becomes a normed quasilinear space.

Keywords: Abel convergence; interval-valued sequences; sequence spaces; quasilinear space

MSC: 40G10; 46A45; 46A99

1. Introduction

The investigation of convergence methods has consistently been a central topic in mathematical analysis. Among these methods, Abel convergence is particularly notable due to its deep connections with summability theory and its effectiveness in addressing divergent series [1,2]. Recent developments have focused on generalizing classical convergence concepts to broader mathematical structures. A prominent example is the study of interval-valued sequences, in which each term is represented by a closed and bounded interval rather than a single real number. Numerous studies have been devoted to interval analysis, exploring both its theoretical foundations and practical applications. Some notable contributions include [3–10].

The concept of quasilinear spaces was first introduced by Aseev [11] as a generalization of classical linear spaces. This pioneering work has since influenced a wide range of studies, including [12–17].

In this context, the present work introduces and examines the space of interval-valued Abel-convergent sequences, extending the traditional notion of Abel convergence to sequences with interval elements. Furthermore, we establish that this newly defined space possesses a quasilinear structure, providing a richer framework for the analysis of interval-valued sequences. Before presenting our main results, we briefly review essential background material on Abel convergence and the convergence theory of interval-valued sequences that will encourage the subsequent developments.

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2. Preliminaries

Definition 1 ([1]). *Let (a_n) be a sequence of complex numbers. The sequence (a_n) is said to be Abel-convergent to L if* $\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} a_n x^n = L$.

Definition 2 ([6]). *An interval X is defined as* $X = \left[\underline{X}, \overline{X}\right] = \left\{x \in \mathbb{R} : \underline{X} \le x \le \overline{X}\right\}$ *where* $\underline{X}$ *is the lower bound and* $\overline{X}$ *is the upper bound.*

Definition 3 ([6]). *An interval that coincides with a single real number can be written as* $[x, x]$ *for some* $x \in \mathbb{R}$. *Such an interval is called a degenerate (or derived) interval. This type of interval represents an ordinary real number, i.e.,* $x = [x, x]$.

Given two intervals $X = \left[\underline{X}, \overline{X}\right]$ and $Y = \left[\underline{Y}, \overline{Y}\right]$, it can be said $X < Y \Leftrightarrow \overline{X} < \underline{Y}$. Furthermore $X \subseteq Y \Leftrightarrow \underline{Y} \le \underline{X}$, $\overline{X} \le \overline{Y}$.

Arithmetic operations on intervals are expressed by using endpoint formulas as follows:

$$X + Y = \left[\underline{X} + \underline{Y}, \overline{X} + \overline{Y}\right], \quad X - Y = \left[\underline{X} - \overline{Y}, \overline{X} - \underline{Y}\right]$$

$$S = \left\{\underline{X}\underline{Y}, \underline{X}\overline{Y}, \overline{X}\underline{Y}, \overline{X}\overline{Y}\right\} \quad then \quad X.Y = [\min S, \max S]$$

$$\frac{X}{Y} = X.\frac{1}{Y} = \left[\underline{X}, \overline{X}\right].\left[\frac{1}{\overline{Y}}, \frac{1}{\underline{Y}}\right]; 0 \notin Y$$

$$\alpha X = \alpha\left[\underline{X}, \overline{X}\right] = \begin{cases} [\alpha\underline{X},\ \alpha\overline{X}], & \alpha \ge 0, \\ [\alpha\overline{X},\ \alpha\underline{X}], & \alpha < 0. \end{cases}$$

Definition 4 ([6]). *Let* $I = \left\{X \subset \mathbb{R} : X = \left[\underline{X}, \overline{X}\right], \underline{X} \le x \le \overline{X}, x \in \mathbb{R}\right\}$. *An interval-valued sequence is a function* $f : \mathbb{N} \to I$, *where the domain is the set of natural numbers* $\mathbb{N}$, *and the range is the set of all real intervals I.*

The class of all interval sequence spaces is denoted by $w(I) = \{(X_n) : n \in \mathbb{N}, X_n \in I\}$.

Definition 5 ([6]). *The set of all real valued interval numbers I is metric space with the metric h called the Hausdorff metric, where h is defined as* $h(X, Y) = max\left\{\left|\underline{X} - \underline{Y}\right|, \left|\overline{X} - \overline{Y}\right|\right\}$.

Definition 6 ([18]). *Let (X_n) be an interval-valued sequence and let* $X_0 = \left[\underline{X_0}, \overline{X_0}\right]$ *be an interval number. If there is a* $N \in \mathbb{N}$ *for which the inequality* $h(X_n, X_0) < \varepsilon$ *is provided for all* $\varepsilon > 0$ *and for all* $n > N$, *then the sequence (X_n) is said to converge to* X_0. *This is denoted by* $\lim_{n\to\infty} X_n = X_0$ *or simply* $X_n \to X_0$ $(n \to \infty)$.

Example 1. *Let* $X_n = \left[1 - \frac{1}{n^2}, 2 + \frac{1}{4n^2}\right]$. *It can be observed that* $\lim_{n\to\infty} X_n = [1, 2]$.

Example 2. *Let* $X_n = [n, n + 2]$. *Then the sequence diverges, since the endpoints tend to infinity as* $n \to \infty$.

In [7], some sequence spaces are defined as follows.

$$I_{C_0} = \left\{(U_k) \in w(I) : \lim_{k\to\infty} U_k = [0, 0]\right\}$$

$$I_C = \left\{ (U_k) \in w(I) : \lim_{k\to\infty} U_k = U_0, \quad U_0 \in I \right\}$$

$$I_{\ell_\infty} = \left\{ (U_k) \in w(I) : \sup_k \left\{ |\underline{U}_k|, |\overline{U}_k| \right\} < \infty \right\}.$$

Definition 7 ([11]). *A set X is referred to as a* quasilinear space *(QLS) when it is equipped with a partial ordering "$\preccurlyeq$", an addition operation, and a real scalar multiplication, and these structures satisfy the following axioms. For any elements $p, q, r, s \in X$ and $\alpha, \gamma \in \mathbb{K}$, the requirements below must hold:*

$$p \preccurlyeq p,$$

$$p \preccurlyeq r \quad \text{whenever} \quad p \preccurlyeq q \text{ and } q \preccurlyeq r,$$

$$p = q \quad \text{if} \quad p \preccurlyeq q \text{ and } q \preccurlyeq p,$$

$$p + q = q + p,$$

$$p + (q + r) = (p + q) + r,$$

$$\text{there exists } \mathbf{0} \in X \text{ such that } p + \mathbf{0} = p,$$

$$\alpha(\gamma p) = (\alpha\gamma)p,$$

$$\alpha(p + q) = \alpha p + \alpha q,$$

$$1 \cdot p = p,$$

$$0 \cdot p = \mathbf{0},$$

$$(\alpha + \gamma)p \preccurlyeq \alpha p + \gamma p,$$

$$p + r \preccurlyeq q + s \quad \text{whenever} \quad p \preccurlyeq q \text{ and } r \preccurlyeq s,$$

$$\alpha p \preccurlyeq \alpha q \quad \text{whenever} \quad p \preccurlyeq q.$$

A linear space is a QLS with the relation "$p \preccurlyeq q \iff p = q$".

Let X be a quasilinear space and assume that $Y \subseteq X$. We say that Y is a *subspace* of X provided that Y itself is a quasilinear space equipped with the same partial order as X. Equivalently, Y is a subspace of the quasilinear space X precisely when for all $x, y \in Y$ and all $\alpha, \beta \in \mathbb{K}$, one has $\alpha x + \beta y \in Y$ (see [8]).

Remark 1 ([9]). *The space $w(I)$ forms a quasilinear space under the operations defined below: For any $U = (U_k)$ and $V = (V_k)$ in $w(I)$, where*

$$U_k = [\underline{U}_k, \overline{U}_k], \quad V_k = [\underline{V}_k, \overline{V}_k], \quad k \in \mathbb{N},$$

the algebraic operations are defined component-wise by

$$U + V = \left([\underline{U}_k + \underline{V}_k,\ \overline{U}_k + \overline{V}_k] \right), \quad \text{and} \quad \mu U = \left(\mu [\underline{U}_k, \overline{U}_k] \right),$$

where

$$\mu [\underline{U}_k, \overline{U}_k] = \begin{cases} [\mu \underline{U}_k,\ \mu \overline{U}_k], & \mu \geq 0, \\ [\mu \overline{U}_k,\ \mu \underline{U}_k], & \mu < 0. \end{cases}$$

Moreover, the partial order on $w(I)$ is defined by

$$U \preceq V \quad \text{if and only if} \quad [\underline{U}_k, \overline{U}_k] \subseteq [\underline{V}_k, \overline{V}_k], \quad \text{for all } k \in \mathbb{N}.$$

Definition 8. *[11] Let X be a QLS. A function $\|.\| : X \to \mathbb{R}$ is said to be a norm on X if for all $u, v \in X$ and $\mu \in \mathbb{R}$:*

$$\begin{aligned}
&\|u\| > 0 \quad \text{for } u \neq \theta,\\
&\|u + v\| \leq \|u\| + \|v\|,\\
&\|\mu u\| = |\mu|\|u\|,\\
&u \preceq v \Rightarrow \|u\| \leq \|v\|,\\
&\forall \delta > 0 \; \exists u_\delta \in X \text{ such that } \|u_\delta\| \leq \delta \text{ and } u \preceq v + u_\delta \Rightarrow u \preceq v
\end{aligned}$$

then, the space X equipped with such a norm is called a normed quasilinear space.

3. Main Results

In this part of the study, we outline our primary results related to the Abel method applied to sequences whose terms are intervals. Our work begins by proposing a formal definition of Abel convergence for such interval-valued sequences.

Definition 9. *Let $U = (U_k)$ be an interval-valued sequence. We say that the sequence U is Abel-convergent to the interval $V = [\underline{V}, \overline{V}]$ if $\lim_{x \to 1^-} h\left((1-x)\sum_{k=0}^{\infty} U_k x^k, V\right) = 0$.*

Throughout this study, for 0 < x < 1, we will have

$$F_U(x) = (1-x)\sum_{k=0}^{\infty} U_k x^k = [\underline{F}_U(x), \overline{F}_U(x)] = \left[(1-x)\sum_{k=0}^{\infty} \underline{U}_k x^k, (1-x)\sum_{k=0}^{\infty} \overline{U}_k x^k\right].$$

Once this definition is set, we move on to describe the structure of the space formed by Abel-convergent interval-valued sequences. Based on the previously introduced condition, we construct the class consisting of all interval-valued sequences that are Abel-convergent. Here, A_I denotes the space of interval-valued sequences that converge in the Abel sense, while A_{I_0} refers to the space of those that are Abel-null in the interval-valued setting. For $\varepsilon > 0$, that is

$$A_I = \left\{(U_k) \in w(I) : h\left((1-x)\sum_{k=0}^{\infty} U_k x^k, V\right) < \varepsilon, V = [\underline{V}, \overline{V}] \in I\right\}$$

$$A_{I_0} = \left\{(U_k) \in w(I) : h\left((1-x)\sum_{k=0}^{\infty} U_k x^k, [0,0]\right) < \varepsilon, [0,0] \in I\right\}$$

Now, we show that A_I is well-defined; that is, it contains at least one element.

Example 3. *Let us take the sequence as $(U_n) = \left[1 - \frac{1}{n}, 1 + \frac{1}{n}\right]$. Let $V = [1,1]$ be the number to which the sequence may be Abel-convergent. Now, let us demonstrate this convergence. It is known that $\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$ and $\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x)$ for $0 < x < 1$. Based on these, we have*

$$\begin{aligned}
&\lim_{x \to 1^-} h\left((1-x)\sum_{n=1}^{\infty}\left[1 - \tfrac{1}{n}, 1 + \tfrac{1}{n}\right]x^n, [1,1]\right)\\
&= \lim_{x \to 1^-} h\left(\left[(1-x)\sum_{n=1}^{\infty}\left(1 - \tfrac{1}{n}\right)x^n, (1-x)\sum_{n=1}^{\infty}\left(1 + \tfrac{1}{n}\right)x^n\right], [1,1]\right)
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{x\to 1^-} max\left\{\left|(1-x)\sum_{n=1}^{\infty}\left(1-\tfrac{1}{n}\right)x^n-1\right|,\left|(1-x)\sum_{n=1}^{\infty}\left(1+\tfrac{1}{n}\right)x^n-1\right|\right\}\\
&= \lim_{x\to 1^-} max\left\{\left|(1-x)\sum_{n=1}^{\infty}x^n-(1-x)\sum_{n=1}^{\infty}\tfrac{x^n}{n}-1\right|,\left|(1-x)\sum_{n=1}^{\infty}x^n+(1-x)\sum_{n=1}^{\infty}\tfrac{x^n}{n}-1\right|\right\}\\
&= \lim_{x\to 1^-}(1-x)\ln(1-x)=0
\end{aligned}
$$

Consequently, the sequence (U_n) *is Abel-convergent to* V.

Now, we present a thorough demonstration showing that the function we introduced indeed satisfies the requirements of a metric. This verification guarantees that the collection of Abel-convergent interval-valued sequences forms a metric space when equipped with the metric defined below.

Theorem 1. *The space* A_I *is a metric space with the function* d *defined as*

$$
\begin{gathered}
d: A_I \times A_I \to \mathbb{R}\\
(U,Q)\to d(U,Q)=\sup_{0<x<1}\left\{max\left\{\left|\underline{F}_U(x)-\underline{F}_Q(x)\right|,\left|\overline{F}_U(x)-\overline{F}_Q(x)\right|\right\}\right\}.
\end{gathered}
$$

Proof. M1- Let $U, Q \in A_I$ and $V_1, V_2 \in I$. For every $\varepsilon_1, \varepsilon_2 > 0$, there are $\delta_1, \delta_2 > 0$ such that

$$
h\left((1-x)\sum_{k=0}^{\infty}U_k x^k, V_1\right)=max\left\{\left|\underline{F}_U(x)-\underline{V}_1\right|,\left|\overline{F}_U(x)-\overline{V}_1\right|\right\}<\varepsilon_1
$$

and

$$
h\left((1-x)\sum_{k=0}^{\infty}Q_k x^k, V_2\right)=max\left\{\left|\underline{F}_Q(x)-\underline{V}_2\right|,\left|\overline{F}_Q(x)-\overline{V}_2\right|\right\}<\varepsilon_2.
$$

Then, we have $\left|\underline{F}_U(x)-\underline{V}_1\right|<\varepsilon_1, \left|\overline{F}_U(x)-\overline{V}_1\right|<\varepsilon_1, \left|\underline{F}_Q(x)-\underline{V}_2\right|<\varepsilon_2, \left|\overline{F}_Q(x)-\overline{V}_2\right|<\varepsilon_2$ and thus we obtain:

$$
\begin{aligned}
\left|\underline{F}_U(x)-\underline{F}_Q(x)\right| &= \left|\underline{F}_U(x)-\underline{F}_Q(x)-\underline{V}_1+\underline{V}_1-\underline{V}_2+\underline{V}_2\right|\\
&\leq \left|\underline{F}_U(x)-\underline{V}_1\right|+\left|\underline{F}_Q(x)-\underline{V}_2\right|+\left|\underline{V}_1-\underline{V}_2\right|<\infty
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\overline{F}_U(x)-\overline{F}_Q(x)\right| &= \left|\overline{F}_U(x)-\overline{F}_Q(x)-\overline{V}_1+\overline{V}_1-\overline{V}_2+\overline{V}_2\right|\\
&\leq \left|\overline{F}_U(x)-\overline{V}_1\right|+\left|\overline{F}_Q(x)-\overline{V}_2\right|+\left|\overline{V}_1-\overline{V}_2\right|<\infty.
\end{aligned}
$$

Hence, d is a real-valued, finite, and non-negative function.

M2- Let $d(U,Q)=0$. Then, $F_U(x)=F_Q(x)$ for all $x\in(0,1)$. Hence,

$$
(1-x)\sum_{k=0}^{\infty}\underline{U}_k x^k=(1-x)\sum_{k=0}^{\infty}\underline{Q}_k x^k
$$

and

$$
(1-x)\sum_{k=0}^{\infty}\overline{U}_k x^k=(1-x)\sum_{k=0}^{\infty}\overline{Q}_k x^k
$$

for all $x\in(0,1)$. Since $1-x\neq 0$ on $(0,1)$, it follows that

$$
\sum_{k=0}^{\infty}(\underline{U}_k-\underline{Q}_k)x^k=0 \quad \text{and} \quad \sum_{k=0}^{\infty}(\overline{U}_k-\overline{Q}_k)x^k=0
$$

for all $x \in (0,1)$. By the uniqueness theorem for power series, we obtain

$$\underline{U}_k = \underline{Q}_k \quad \text{and} \quad \overline{U}_k = \overline{Q}_k \quad \text{for all} \quad k \in \mathbb{N}.$$

Therefore $U_k = Q_k$ for all k, and hence $U = Q$.
M3-

$$\begin{aligned} d(U,Q) &= \sup_{0<x<1} \{max\{|\underline{F}_U(x) - \underline{F}_Q(x)|, |\overline{F}_U(x) - \overline{F}_Q(x)|\}\} \\ &= \sup_{0<x<1} \{max\{|\underline{F}_Q(x) - \underline{F}_U(x)|, |\overline{F}_Q(x) - \overline{F}_U(x)|\}\} \\ &= d(Q,U) \end{aligned}$$

M4- Let $U, Q, Z \in A_I$. Let us show that $d(U,Q) \leq d(U,Z) + d(Z,Q)$.
$Z \in A_I \Rightarrow F_Z(x) = (1-x)\sum_{k=0}^{\infty} Z_k x^k$ and there exists $V_3 \in I$ such that $h(F_Z(x), V_3) < \varepsilon$, which means $\sup_{0<x<1} \{max\{|\underline{F_Z}(x) - \underline{V_3}|, |\overline{F_Z}(x) - \overline{V}_3|\}\} < \varepsilon$. As a result,

$$\begin{aligned} d(U,Q) &= \sup_{0<x<1} \{max\{|\underline{F}_U(x) - \underline{F}_Q(x)|, |\overline{F}_U(x) - \overline{F}_Q(x)|\}\} \\ &= \sup_{0<x<1} \{max\{|\underline{F}_U(x) - \underline{F}_Z(x) + \underline{F_Z}(x) - \underline{F}_Q(x)|, |\overline{F}_U(x) - \overline{F}_Z(x) + \overline{F}_Z(x) - \overline{F}_Q(x)|\}\} \\ &\leq \sup_{0<x<1} \{max\{|\underline{F}_U(x) - \underline{F}_Z(x)| + |\underline{F}_Z(x) - \underline{F}_Q(x)|, |\overline{F}_U(x) - \overline{F}_Z(x)| + |\overline{F}_Z(x) - \overline{F}_Q(x)|\}\} \\ &\leq \sup_{0<x<1} \{max\{|\underline{F}_U(x) - \underline{F}_Z(x)|, |\overline{F}_U(x) - \overline{F}_Z(x)|\}\} \\ &+ \sup_{0<x<1} \{max\{|\underline{F}_Z(x) - \underline{F}_Q(x)|, |\overline{F}_Z(x) - \overline{F}_Q(x)|\}\} \\ &= d(U,Z) + d(Z,Q). \end{aligned}$$

□

After confirming the metric structure, we turn our attention to the issue of completeness. The theorem that follows demonstrates that the metric space constructed above is, in fact, complete.

Theorem 2. *(A_I, d) is a complete space.*

Proof. Let $\left(U^{(m)}\right)$ be a Cauchy sequence in (A_I, d), where

$$U^{(m)} = (U_k^{(m)}) = ([\underline{U}_k^{(m)}, \overline{U}_k^{(m)}]).$$

Since $\left(U^{(m)}\right)$ is Cauchy in (A_I, d), for every $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that

$$d(U^{(m)}, U^{(p)}) < \varepsilon$$

for all $m, p \geq m_0$. By the definition of d, this implies

$$\sup_{0<x<1} \left\{ \max \left\{ |\underline{F}_{U^{(m)}}(x) - \underline{F}_{U^{(p)}}(x)|, \, |\overline{F}_{U^{(m)}}(x) - \overline{F}_{U^{(p)}}(x)| \right\} \right\} < \varepsilon$$

where

$$\underline{F}_{U^{(m)}}(x) = (1-x)\sum_{k=0}^{\infty} \underline{U}_k^{(m)} x^k, \qquad \overline{F}_{U^{(m)}}(x) = (1-x)\sum_{k=0}^{\infty} \overline{U}_k^{(m)} x^k.$$

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Hence, $(\underline{F}_{U^{(m)}})$ and $(\overline{F}_{U^{(m)}})$ are uniformly Cauchy on $(0,1)$. Therefore, there exist functions $f, g : (0,1) \to \mathbb{R}$ such that

$$\underline{F}_{U^{(m)}}(x) \to f(x), \qquad \overline{F}_{U^{(m)}}(x) \to g(x)$$

uniformly on $(0,1)$. Now, for each m,

$$\underline{F}_{U^{(m)}}(x) = \underline{U}_0^{(m)} + \sum_{k=1}^{\infty} (\underline{U}_k^{(m)} - \underline{U}_{k-1}^{(m)}) x^k,$$

and similarly,

$$\overline{F}_{U^{(m)}}(x) = \overline{U}_0^{(m)} + \sum_{k=1}^{\infty} (\overline{U}_k^{(m)} - \overline{U}_{k-1}^{(m)}) x^k.$$

Passing to the limit in these representations, for suitable sequences (c_k) and (d_k), we obtain power series expansions

$$f(x) = c_0 + \sum_{k=1}^{\infty} c_k x^k, \qquad g(x) = d_0 + \sum_{k=1}^{\infty} d_k x^k,$$

where

$$c_0 = \lim_{m\to\infty} \underline{U}_0^{(m)}, \qquad c_k = \lim_{m\to\infty} (\underline{U}_k^{(m)} - \underline{U}_{k-1}^{(m)}) \quad (k \geq 1),$$

and

$$d_0 = \lim_{m\to\infty} \overline{U}_0^{(m)}, \qquad d_k = \lim_{m\to\infty} (\overline{U}_k^{(m)} - \overline{U}_{k-1}^{(m)}) \quad (k \geq 1).$$

Define recursively

$$\underline{U}_0 = c_0, \qquad \underline{U}_k = \underline{U}_{k-1} + c_k \quad (k \geq 1),$$

and

$$\overline{U}_0 = d_0, \qquad \overline{U}_k = \overline{U}_{k-1} + d_k \quad (k \geq 1).$$

Set $U_k = [\underline{U}_k, \overline{U}_k], U = (U_k)$. Then, by construction,

$$f(x) = (1-x) \sum_{k=0}^{\infty} \underline{U}_k x^k, \qquad g(x) = (1-x) \sum_{k=0}^{\infty} \overline{U}_k x^k,$$

and hence

$$F_U(x) = [f(x), g(x)].$$

Since each $U^{(m)} \in A_I$, there exists $V^{(m)} \in I$ such that

$$\lim_{x\to 1^-} F_{U^{(m)}}(x) = V^{(m)}.$$

Passing to the limit, we obtain

$$\lim_{x\to 1^-} F_U(x) = V$$

for some $V \in I$. Therefore, $U \in A_I$. Finally, from the uniform convergence of $F_{U^{(m)}}$ to F_U, it follows that

$$d(U^{(m)}, U) \to 0 \qquad (m \to \infty).$$

Therefore, every Cauchy sequence in (A_I, d) converges to an element of A_I, and hence (A_I, d) is complete. □

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Theorem 3. *A_I is a quasilinear subspace.*

Proof. Since $A_I \subset w(I)$ and $w(I)$ is quasilinear space (see [9]), it is sufficient to show that for all $\alpha \in \mathbb{K}$ and for all $U, Q \in A_I$, we have $U + Q \in A_I$ and $\alpha U \in A_I$.

There exist $V_1, V_2 \in I$ such that

$$U \in A_I \Rightarrow \lim_{x \to 1^-} max\left\{\left|\underline{F}_U(x) - \underline{V}_1\right|, \left|\overline{F}_U(x) - \overline{V}_1\right|\right\} = 0 \tag{1}$$

and

$$Q \in A_I \Rightarrow \lim_{x \to 1^-} max\left\{\left|\underline{F}_Q(x) - \underline{V}_2\right|, \left|\overline{F}_Q(x) - \overline{V}_2\right|\right\} = 0. \tag{2}$$

By using (1) and (2), we have $\lim_{x \to 1^-} |\underline{F}_U(x) - \underline{V}_1| = 0$, $\lim_{x \to 1^-} \left|\overline{F}_U(x) - \overline{V}_1\right| = 0$, $\lim_{x \to 1^-} |\underline{F}_Q(x) - \underline{V}_2| = 0$ and $\lim_{x \to 1^-} \left|\overline{F}_Q(x) - \overline{V}_2\right| = 0$. Hence, there exist $V = \left[\underline{V}_1 + \underline{V}_2, \overline{V}_1 + \overline{V}_2\right]$ such that

$$\lim_{x \to 1^-} \left(\underline{F}_U(x) + \underline{F}_Q(x)\right) = \underline{V}, \quad \lim_{x \to 1^-} \left(\overline{F}_U(x) + \overline{F}_Q(x)\right) = \overline{V}.$$

It follows that

$$\lim_{x \to 1^-} \left|\underline{F}_U(x) + \underline{F}_Q(x) - \underline{V}\right| = 0, \lim_{x \to 1^-} \left|\overline{F}_U(x) + \overline{F}_Q(x) - \overline{V}\right| = 0.$$

Thus, we get

$$\lim_{x \to 1^-} max\left\{\left|\underline{F}_U(x) + \underline{F}_Q(x) - \underline{V}\right|, \left|\overline{F}_U(x) + \overline{F}_Q(x) - \overline{V}\right|\right\} = 0$$

which means

$$\lim_{x \to 1^-} h\left((1-x)\sum_{k=0}^{\infty}(U_k + Q_k)x^k, V\right) = 0 \Rightarrow U + Q \in A_I.$$

To show that $\alpha U \in A_I$:

Let $\alpha \geq 0$. Since

$$\lim_{x \to 1^-} \underline{F}_U(x) = \underline{V}_1, \quad \lim_{x \to 1^-} \overline{F}_U(x) = \overline{V}_1,$$

we obtain

$$\lim_{x \to 1^-} \alpha\underline{F}_U(x) = \alpha\underline{V}_1, \quad \lim_{x \to 1^-} \alpha\overline{F}_U(x) = \alpha\overline{V}_1.$$

Then, we get

$$\lim_{x \to 1^-} h\left((1-x)\sum_{k=0}^{\infty}\alpha U_k x^k, V\right) = 0 \Rightarrow \alpha U \in A_I$$

where $V = \left[\alpha\underline{V}_1, \alpha\overline{V}_1\right]$.

If $\alpha < 0$, then set $V = \left[\alpha\overline{V}_1, \alpha\underline{V}_1\right]$ and similarly one obtains $\alpha U \in A_I$. Therefore, A_I is a quasilinear subspace. □

Theorem 4. *An interval-valued convergent sequence is Abel-convergent.* $(I_C \subset A_I)$

Proof. Let $U_k = [\underline{U}_k, \overline{U}_k]$ be an interval sequence. If (U_k) converges to U, then we write

$$\lim_{k\to\infty} h(U_k, U) = 0 \Rightarrow \underline{U}_k \to \underline{U} \quad \text{and} \quad \overline{U}_k \to \overline{U}.$$

Also, using the fact that a convergent sequence is Abel-convergent to the same limit, we get

$$\begin{aligned} \lim_{x\to 1^-} F_U(x) &= \lim_{x\to 1^-} \left[(1-x)\sum_{k=0}^{\infty} \underline{U}_k x^k, (1-x)\sum_{k=0}^{\infty} \overline{U}_k x^k \right] \\ &= \left[\lim_{x\to 1^-} (1-x)\sum_{k=0}^{\infty} \underline{U}_k x^k, \lim_{x\to 1^-} (1-x)\sum_{k=0}^{\infty} \overline{U}_k x^k \right] \\ &= [\underline{U}, \overline{U}]. \end{aligned}$$

This completes the proof. □

To study the relationship between the spaces I_C and A_I, we evaluate the Abel transform at a sequence of points approaching 1.

Theorem 5. *Let (x_n) be a sequence in $(0,1)$ such that $x_n \to 1^-$ as $n \to \infty$. Define the operator*

$$T : I_C \to A_I \quad by \quad T(U) = V = (V_n)$$

where

$$V_n = (1-x_n)\sum_{k=0}^{n} U_k x_n^k \qquad (n \in \mathbb{N}).$$

Then T is well defined and linear.

Proof. Let $U = (U_k) \in I_C$. Then there exists $L \in I$ such that $U_k \to L \quad (k \to \infty)$. Since every convergent interval-valued sequence is Abel-convergent to the same limit, we have

$$\lim_{x\to 1^-} (1-x)\sum_{k=0}^{\infty} U_k x^k = L.$$

Now define

$$V_n = (1-x_n)\sum_{k=0}^{n} U_k x_n^k.$$

Because $x_n \to 1^-$ and (U_k) converges to L, the sequence of partial Abel sums (V_n) converges to L. Hence $V = (V_n)$ is a convergent interval-valued sequence; that is, $V \in I_C$. Since $I_C \subset A_I$, it follows that

$$T(U) = V \in A_I.$$

Therefore, T is well defined.

Next, let $U, Q \in I_C$ and $\alpha \in \mathbb{R}$. For each $n \in \mathbb{N}$,

$$T(U+Q)_n = (1-x_n)\sum_{k=0}^{n}(U_k+Q_k)x_n^k = (1-x_n)\sum_{k=0}^{n} U_k x_n^k + (1-x_n)\sum_{k=0}^{n} Q_k x_n^k = T(U)_n + T(Q)_n.$$

Also,

$$T(\alpha U)_n = (1-x_n)\sum_{k=0}^{n} \alpha U_k x_n^k = \alpha(1-x_n)\sum_{k=0}^{n} U_k x_n^k = \alpha T(U)_n.$$

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Thus,

$$T(U+Q) = T(U) + T(Q) \quad \text{and} \quad T(\alpha U) = \alpha T(U),$$

so T is linear. □

Theorem 6. *The space A_I is a subspace of I_{ℓ_∞}.*

Proof. It is known that

$$I_{\ell_\infty} = \left\{ (U_k) \in w(I) : \sup_k \left\{ |\underline{U}_k|, |\overline{U}_k| \right\} < \infty \right\}.$$

Now

$$\begin{aligned} U \in A_I \quad &\Rightarrow \quad \lim_{x\to 1^-} h\left((1-x)\sum_{k=0}^{\infty} U_k x^k, V \right) = 0 \\ &\Rightarrow \quad \lim_{x\to 1^-} \max\left\{ \left| (1-x)\sum_{k=0}^{\infty} \underline{U}_k x^k - \underline{V} \right|, \left| (1-x)\sum_{k=0}^{\infty} \overline{U}_k x^k - \overline{V} \right| \right\} = 0 \\ &\Rightarrow \quad \lim_{x\to 1^-} (1-x)\sum_{k=0}^{\infty} \underline{U}_k x^k = \underline{V} \qquad \textit{and} \qquad \lim_{x\to 1^-} (1-x)\sum_{k=0}^{\infty} \overline{U}_k x^k = \overline{V}. \end{aligned}$$

Hence, the sequences $(\underline{U}_k), (\overline{U}_k)$ are Abel-convergent, and therefore they are bounded real sequences. That is,

$$\sup_k \left\{ |\underline{U}_k|, |\overline{U}_k| \right\} < \infty \Rightarrow U \in I_{l_\infty}$$

which implies $A_I \subset I_{l_\infty}$. To show that it is a subspace, it remains to be proven that $U+Q \in A_I, \alpha U \in A_I$ for all $\alpha \in \mathbb{K}$ and all $U, Q \in A_I$. This can be shown, as in Theorem 3 and A_I is a subspace of I_{l_∞}. □

Theorem 7. *The space A_I is a normed quasilinear space with the function $\|.\|_A$ defined by*

$$\|.\|_A : A_I \to \mathbb{R}$$
$$\|U\|_A = \sup_{0<x<1} \left\{ max\left\{ |\underline{F}_U(x)|, |\overline{F}_U(x)| \right\} \right\}, \quad U = (U_k) \in A_I.$$

Proof. N1- Let $U \in A_I$. By the definition of the space A_I, the functions $\underline{F}_U(x)$ and $\overline{F}_U(x)$ are real-valued and bounded on the interval $(0,1)$. Hence, the function

$$x \mapsto \max\left\{ |\underline{F}_U(x)|, |\overline{F}_U(x)| \right\}$$

is bounded on $(0,1)$. Therefore, the supremum over $0 < x < 1$ exists and is finite, which shows that $\|U\|_A$ is well defined, and it follows directly from the definition that $\|U\|_A > 0$ for $U = (U_k) \neq [0,0]$.

N2- Let $U = (U_k)$ and $V = (V_k)$ be arbitrary elements of A_I, where $U_k = [\underline{U}_k, \overline{U}_k]$ and $V_k = [\underline{V}_k, \overline{V}_k]$. Then, for each $x \in (0,1)$,

$$\underline{F}_{U+V}(x) = \underline{F}_U(x) + \underline{F}_V(x), \qquad \overline{F}_{U+V(x)} = \overline{F}_U(x) + \overline{F}_V(x).$$

Using the triangle inequality for real numbers, we obtain

$$|\underline{F}_{U+V}(x)| \leq |\underline{F}_U(x)| + |\underline{F}_V(x)|,$$

$$|\overline{F}_{U+V(x)}| \leq |\overline{F}_U(x)| + |\overline{F}_V(x)|.$$

Hence,

$$\max\left(|\underline{F}_{U+V}(x)|, |\overline{F}_{U+V}(x)|\right) \leq \max\left(|\underline{F}_U(x)|, |\overline{F}_U(x)|\right) + \max\left(|\underline{F}_V(x)|, |\overline{F}_V(x)|\right).$$

Taking supremum over $0 < x < 1$, we conclude that

$$\|U + V\|_A \leq \|U\|_A + \|V\|_A.$$

N3- Let $\mu \in \mathbb{R}$ and $U = (U_k) \in A_I$. Then,

$$\underline{F}_{\mu U}(x) = \mu \underline{F}_U(x), \qquad \overline{F}_{\mu U}(x) = \mu \overline{F}_U(x),$$

for all $x \in (0, 1)$. Therefore,

$$\|\mu U\|_A = \sup_{0<x<1} \max\left(|\mu \underline{F}_U(x)|, \, |\mu \overline{F}_U(x)|\right) = |\mu| \, \|U\|_A.$$

N4- Assume that $U = (U_k), V = (V_k) \in A_I$ satisfy $U \preceq V$. Then, $\underline{V}_k \leq \underline{U}_k$ and $\overline{U}_k \leq \overline{V}_k$ for all k, which implies

$$\max\left(|\underline{F}_U(x)|, \, |\overline{F}_U(x)|\right) \leq \max\left(|\underline{F}_V(x)|, \, |\overline{F}_V(x)|\right),$$

and hence taking supremum over $0 < x < 1$, we obtain

$$\|U\|_A \leq \|V\|_A.$$

N5- Let $\delta > 0$ be given and let $U, V \in A_I$. Assume that there exists an element $U_\delta \in A_I$ such that

$$U \preceq V + U_\delta \quad \text{and} \quad \|U_\delta\|_A \leq \delta.$$

Then, by the definition of interval inclusion, we have

$$\underline{V}_k + \underline{U}_{\delta,k} \leq \underline{U}_k \qquad \textit{and} \qquad \overline{U}_k \leq \overline{V}_k + \overline{U}_{\delta,k} \quad \text{for all } k \in \mathbb{N}.$$

Since $\|U_\delta\|_A \leq \delta$, letting $\delta \to 0$ yields

$$\underline{V}_k \leq \underline{U}_k \qquad \textit{and} \qquad \overline{U}_k \leq \overline{V}_k \quad \text{for all } k \in \mathbb{N}.$$

Hence, $U \preceq V$, which completes the proof.

Therefore, all the axioms of a normed quasilinear space are satisfied. Hence, A_I equipped with the norm $\|\cdot\|$ is a normed quasilinear space. □

Corollary 1.

$$d(U, Q) = \|U - Q\|_A = \sup_{0<x<1} \left\{ max\left\{ |\underline{F}_U(x) - \underline{F}_Q(x)|, |\overline{F}_U(x) - \overline{F}_Q(x)| \right\} \right\}$$

defines a metric on A_I known as the norm metric. As proved earlier, the metric space (A_I, d) is complete. This shows that A_I is complete as a normed quasilinear space. Consequently, A_I is a Banach quasilinear space.

4. Conclusions

This work introduced Abel convergence for interval-valued sequences and defined the corresponding space together with a suitable metric. It was shown that this metric space

is complete and forms a quasilinear subspace. In addition, its relationships with several existing interval-valued sequence spaces were examined. Further, a norm was defined on this space and it was shown that the space forms a normed quasilinear space. These results clarify the position of Abel convergence within interval sequence theory and provide a basis for further study.

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