

Article

On Non-Commensurate Fractional-Order System Control

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Abstract

The control systems for models described by non-commensurate fractional-order differential equations are based on their transformation into large commensurate-order systems, which impose difficulties in determining control laws. In this context, in this paper, a new control method for this class of systems is proposed. The results obtained are based on Lyapunov methods for differential equations with fractional exponents and on the application of the Yakubovich–Kalman–Popov lemma adapted for this class of systems. The stability criterion is presented as a frequency criterion and represented graphically by familiar frequency plots similar to those of the Nyquist or Popov type. If the parameters that define the model can be defined in a closed domain, the frequency criterion can be interpreted as “Popov’s circle criterion”. The two numerical applications present two important cases. The first studies the stability criterion in the case where the viscosity coefficients determine non-commensurate fractional-order exponents in the dynamic model of the system. The second example studies the complex problem of the human–machine system in which the human model imposes dynamics determined by non-commensurate fractional-order systems. The proposed investigation methods allow for a reduction in computational effort by several orders of magnitude for non-commensurate fractional-order systems, eliminate stability conditions that use matrix-based criteria for large-scale systems, and introduce standard frequency-domain criteria.

Keywords: fractional-order system; control; stability; frequency criterion; circle criterion**MSC:** 93D10

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1. Introduction

The literature is quite rich in studies on commensurate fractional-order systems (CFOS) that involve transforming into differential equations with integer exponents and using well-known techniques and methods for these systems. Enumerating these methods would be an extremely difficult operation. A summary of these efforts can be found in [1–3]. We will mention a few examples. Conventional methods deriving from the study of transfer functions with fractional exponents are discussed in [4]. In [5], an extension of the Lyapunov direct method to demonstrate the stability of many fractional-order systems is proposed. The solution of fractional-order systems has been studied in [6] in terms of the Mittag–Leffler function. New definitions for the exponential stability of fractional-order systems are given in [7] and sufficient conditions are obtained for exponential stability using the Lyapunov stability. Mittag–Leffler stability and generalized Mittag–Leffler stability has been proposed in [8]. Based on Lyapunov’s second method, a stability criterion for a class

of nonlinear fractional differential system is derived in [9]. The stability of fractional-order systems with the Riemann–Liouville derivative is discussed in [10]. By applying the Mittag–Leffler function, generalized Gronwall inequality, and comparison principle to fractional differential systems, some sufficient conditions ensuring stability and asymptotic stability are given in [11]. CFOS models defined by Caputo derivatives are analyzed in [12,13]. CFOS with nonlinear or linearized components are treated in [14]. By using the Mittag–Leffler function, Laplace transform, and Gronwall–Bellman lemma, sufficient conditions are derived for the asymptotical stability of a class of nonlinear fractional differential systems in [15]. In [16], the Mittag–Leffler stability and the fractional Lyapunov of the second method were proposed. Many results from the stability analysis of fractional nonlinear systems are analyzed in [17,18]. The stability of many fractional-order systems, which can be nonlinear and time-varying, are analyzed in [19]. Stability criteria are formulated in the form of algebraic criteria in [20]. Lyapunov-like functions of the quadratic type are constructed and the fractional derivatives (in the Caputo sense) are used for CFOS stability in [21]. Fractional-order models for a human–machine system have been developed in [22,23]. Modern control techniques for flexible industrial robots are presented in [24].

In the non-commensurate fractional-order systems (NCFOS) domain, we find far fewer papers. Authors prefer, most of the time, to approximate the non-commensurate-order exponent with a commensurate one and to use specific CFOS techniques. Study methods are based on frequency approximation techniques of Oustaloup [25] and the use of classical frequency or Laplace procedures [1]. Motivated by the above discussion, our objective in this paper is to develop new stability criteria for NCFOS models via an approach in which the control laws are defined by frequency criteria. The main contributions of this paper, compared with existing results, are listed as follows:

- The investigation is based on Lyapunov methods for differential equations with fractional exponents and on the application of the Yakubovich–Kalman–Popov lemma adapted for this class of systems.
- In this paper, the stability criterion for NCFOS is presented in the form of a frequency criterion and represented graphically by familiar frequency plots similar to those of the Nyquist or Popov type.
- If the parameters that define the model can be defined in a closed domain, the frequency criterion can be interpreted as “Popov’s circle criterion”.
- The stability criterion for the case where the viscosity coefficients determine non-commensurate fractional-order exponents in the dynamic model of the robotic component is proposed.
- The complex problem of the human–machine system in which the human model imposes dynamics determined by non-commensurate fractional-order exponents is studied. Popov’s circle criterion determines the asymptotic stability conditions depending on the following human parameters: driver reaction time constant and driver inability factor.

This paper is structured as follows: Section 2 presents mathematical preliminaries, Section 3 treats the control techniques proposed in this paper, Section 4 presents two applications: the dynamic model of the robotic component with NCFOS model and the human–machine system control. Then, Sections 4 and 5 present our Discussion and Conclusions.

Notations:

- $D^\nu \zeta$ denotes the ν -order **Caputo** fractional derivative of ζ ; $I^\nu \zeta$ denotes the ν -order fractional integral.
- Bold letters denote the vector or matrix and non-bold letters denote scalar variables.

2. Materials and Methods

2.1. Preliminaries

Theorem 1 [10–12]. The system $D^\nu \zeta(t) = f(\zeta(t)), \zeta(t_0)$, where $D^\nu \zeta(t)$ *Caputo* is the derivative of $\zeta(t)$ for $0 < \nu < 1$, is Mittag–Leffler-stable if there exists a continuously differentiable function $V(t, \zeta)$ that satisfies

$$\xi_1 \|\zeta\| \leq V(t, \zeta(t)) \leq \xi_2 \|\zeta\|^2 \quad (1)$$

$$D^\nu V(t, \zeta(t)) \leq -\xi_3 \|\zeta\|^2 \quad (2)$$

where ξ_1, ξ_2, ξ_3 are positive constants.

Lemma 1 [7,10–14]. Let $\zeta(\cdot) \in R^n$ be a differentiable vector function. Then, the following inequality holds:

$$\frac{1}{2} D^\nu (\zeta^T \zeta) \leq \zeta^T D^\nu (\zeta), \quad \nu \in (0, 1) \quad (3)$$

Definition 1 [1]. The system $D^\nu \zeta(t) = f(D^\mu \zeta(t), \zeta(t)), \zeta \in R^n$ where $0 < \mu < \nu < 1$ and the order of derivation are not integer multiples of a base order α , that is, $\mu, \nu \neq k\alpha, \alpha \in R^+$, is a non-commensurate-order system.

2.2. Main Result

The following example is presented to define the objective of the paper,

$$D^{1.4}x + 0.3x + 2.5D^{0.7}x + 1.4D^{0.3}x + 0.8D^{0.6}x = 0 \quad (4)$$

This equation can be reformulated as

$$x = x_1; D^{0.7}x_1 = x_2; D^{0.7}x_2 = D^{1.4}x_1; D^{0.3}x = D^{0.3}x_1;$$

$$D^{0.7} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.3 & -2.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1.4 \end{bmatrix} D^{0.3}x_1 + \begin{bmatrix} 0 \\ -0.8 \end{bmatrix} D^{2 \times 0.3}x_1 \quad (5)$$

Equation (5) represents an example of a non-commensurate-order system with the orders of derivation $\nu = 0.7$ and $\mu = 0.3$.

Consider the non-commensurate-order system

$$D^\nu x(t) = Ax(t) - \sum_{j=1}^r a_j D^{j\mu} x_1(t) + bu(t) \quad (6)$$

$$y(t) = c^T x(t) \quad (7)$$

where $0 < \nu \leq 1; 0 < \mu < 1$; ν, μ are not integer multiples of a base order α ; $x = [x_1, x_2, \dots, x_n]^T$ is the state vector; $D^\nu x_1 = x_2, D^\nu x_2 = x_3, \dots, D^\nu x_{n-2} = x_{n-1}$; $u, y \in R^1$ are the input and output variables, respectively; and $A \in R^{n \times n}, a_j, b, c \in R^{n \times 1}$. The vectors a_j verify the relation

$$a_j = p_0 \alpha_j a; 0 < \alpha_j < 1, j = 1, 2, \dots, r; 0 < p_0 < 1 \quad (8)$$

The dynamic model (6) can be rewritten as

$$D^\nu x(t) = Ax(t) - \sum_{j=1}^r a_j z_j(t) + bu(t) \quad (9)$$

$$D^{j\mu}x_1(t) = z_j(t), \quad j = 1, 2, \dots, r \quad (10)$$

Consider the control law

$$u(t) = -ky(t) = -\sum_{i=0}^{n-1} k_i D^i x_1(t) \quad (11)$$

where $k_i, i = 0, 1, \dots, n-1$ are the control gains, ($k_i > 0$),

$$A^* = A - kbc^T \quad (12)$$

and A^* becomes Hurwitz [1,4].

Theorem 2. The system (6), (7) with the control law (11), is Mittag-Leffler-stable if the following conditions are satisfied:

$$\operatorname{Re}(d^T(j\omega I - A^*)^{-1}a) \geq -\frac{\alpha^*}{2r^2}, \quad \omega \in (-\infty, +\infty) \quad (13)$$

$$\left(1 - \frac{1}{r}\right) \lambda_{\min}(Q) > rp_0^2 \quad (14)$$

where j is the complex variable and I is the unit matrix and $Q = qq^T$ is the solution of Lyapunov equation $A^{*T}P + PA^* = -qq^T$.

Proof. Consider the following generalized Lyapunov function [26–29]:

$$V(x) = x^T Px + \frac{1}{2}rp_0x_1^2 \quad (15)$$

that satisfies the condition

$$\xi_1 \|x\| \leq V(x) \leq \xi_2 \|x\|^2 \quad (16)$$

where $P > 0$, $P = P^T$, $0 < p_0 < 1$ and $\xi_1 = \lambda_{\min}(P)$; $\xi_2 = \lambda_{\max}(P)$. The following fractional-order derivative of (15) will be defined. This derivative is denoted by $(D^{\nu;\mu}(V))$,

$$D^{\nu;\mu}(V) = D^\nu(x^T Px) + \sum_{j=1}^r D^{j\mu} \left(\frac{1}{2}p_0x_1^2 \right) \quad (17)$$

According to Theorem 2 and because $x_1 \in x$, the stability condition (2) becomes

$$D^{\nu;\mu}(V) \leq -\xi_3 \|x\| \quad (18)$$

From Lemma 1, it follows that

$$D^\nu(x^T Px) + \sum_{j=1}^r D^{j\mu} \left(\frac{1}{2}p_0x_1^2 \right) \leq (D^\nu x^T)Px + x^T P(D^\nu x) + \sum_{j=1}^r p_0x_1 D^{j\mu}x_1 \quad (19)$$

The derivative of (15) along the solutions of (9), (10) is

$$D^{\nu;\mu}(V) \leq x^T(A^{*T}P + PA^*)x - 2x^T \sum_{j=1}^r (P\alpha_j a z_j - x_1 z_j) p_0 \quad (20)$$

In the following, we will use a completion-of-squares technique. The first term of (20) can be rewritten as

$$\begin{aligned} x^T(A^{*T}P + PA^*)x &= (r^2) \frac{x}{r^T} (A^{*T}P + PA^*) \frac{x}{r} - r \frac{x}{r^T} (A^{*T}P + PA^*) \frac{x}{r} + r \frac{x}{r^T} (A^{*T}P + PA^*) \frac{x}{r} = \\ &= (r^2 - r) \frac{x}{r^T} (A^{*T}P + PA^*) \frac{x}{r} + \sum_{i=1}^r ((\frac{x}{r^T} (A^{*T}P + PA^*) \frac{x}{r} \end{aligned} \quad (21)$$

Using $x_1 = \mathbf{x}^T \mathbf{d}; \mathbf{d} = [1 \ 0 \ 0..0]^T$, the second term of (20) becomes

$$2\mathbf{x}^T(P\alpha_j \mathbf{a} z_j p_0 - x_1 z_j p_0) = 2\frac{\mathbf{x}^T}{r}(P\alpha_j \mathbf{a} - \frac{r\mathbf{d}}{2})p_0 z_j \quad (22)$$

Substituting these results into (20) yields

$$D^{v;\mu}(V) \leq (r^2 - r)\frac{\mathbf{x}^T}{r}(A^{*T}P + PA^*)\frac{\mathbf{x}}{r} + \sum_{i=1}^r ((\frac{\mathbf{x}^T}{r}(A^{*T}P + PA^*)\frac{\mathbf{x}}{r} - 2\mathbf{x}^T(P\alpha_j \mathbf{a} - \frac{\mathbf{d}}{2})p_0 z_j) \quad (23)$$

Adding and subtracting the term $(p_0 z_j)^2$ results in

$$D^{v;\mu}(V) \leq (r^2 - r)\frac{\mathbf{x}^T}{r}(A^{*T}P + PA^*)\frac{\mathbf{x}}{r} + \sum_{i=1}^r ((\frac{\mathbf{x}^T}{r}(A^{*T}P + PA^*)\frac{\mathbf{x}}{r} - 2\frac{\mathbf{x}^T}{r}(P\alpha_j \mathbf{a} - \frac{r\mathbf{d}}{2})p_0 z_j) - (p_0 z_j)^2 + (p_0 z_j)^2) \quad (24)$$

Providing that A^* is Hurwitz and α_j satisfies (8) results in

$$A^{*T}P + PA^* = -\mathbf{q}\mathbf{q}^T \leq A^{*T}P\alpha_j + P\alpha_j A^* = -\sqrt{\alpha_j}\mathbf{q}\sqrt{\alpha_j}\mathbf{q}^T \quad (25)$$

Substituting this inequality into (24) yields

$$D^{v;\mu}(V) \leq (r^2 - r)\frac{\mathbf{x}^T}{r}(A^{*T}P + PA^*)\frac{\mathbf{x}}{r} + \sum_{i=1}^r ((\frac{\mathbf{x}^T}{r}(A^{*T}P\alpha_j + P\alpha_j A^*)\frac{\mathbf{x}}{r} - 2\frac{\mathbf{x}^T}{r}(P\alpha_j \mathbf{a} - \frac{r\mathbf{d}}{2})p_0 z_j) - (p_0 z_j)^2 + (p_0 z_j)^2) \quad (26)$$

or

$$D^{v;\mu}(V) \leq (r^2 - r)\frac{\mathbf{x}^T}{r}(A^{*T}P + PA^*)\frac{\mathbf{x}}{r} + \sum_{j=1}^r ((\frac{\mathbf{x}^T}{r}(A^{*T}P\alpha_j + P\alpha_j A^*)\frac{\mathbf{x}}{r} - 2\frac{\mathbf{x}^T}{r}(P\alpha_j \mathbf{a} - \frac{r\mathbf{d}}{2})p_0 z_j) - (p_0 z_j)^2) + \sum_{j=1}^r (p_0 z_j)^2 \quad (27)$$

Provided that (11) and (25) are verified, the pairs $(A^*, \mathbf{a})$, $(A^*, \mathbf{d})$ are controllable and observable, respectively, and if the frequency conditions

$$\operatorname{Re}(r\mathbf{d}^T(\mathbf{j}\omega\mathbf{I} - A^*)^{-1}r\mathbf{a})) + \frac{\alpha_j}{2} \geq 0, \quad j = 1, 2, \dots, r; \omega \in (-\infty, +\infty) \quad (28)$$

are satisfied, considering $\mathbf{q}$ as a solution of Lyapunov Equation (25) and applying the substitutions of the Yakubovich–Kalman–Popov lemma [30],

$$P\alpha_j r\mathbf{a} - \frac{r\mathbf{d}}{2} = \sqrt{\alpha_j}\mathbf{q}, \quad j = 1, 2, \dots, r \quad (29)$$

the inequality (27) becomes

$$D^{v;\mu}(V) \leq -(r^2 - r)\frac{\mathbf{x}^T}{r}\mathbf{q}\mathbf{q}^T\frac{\mathbf{x}}{r} + \sum_{j=1}^r (-\frac{\mathbf{x}^T}{r}(\sqrt{\alpha_j}\mathbf{q}\sqrt{\alpha_j}\mathbf{q}^T)\frac{\mathbf{x}}{r} - 2\frac{\mathbf{x}^T}{r}\sqrt{\alpha_j}\mathbf{q}p_0 z_j) - (p_0 z_j)^2 + \sum_{j=1}^r (p_0 z_j)^2 \quad (30)$$

$$D^{v;\mu}(V) \leq -(r^2 - r)\frac{\mathbf{x}^T}{r}\mathbf{q}\mathbf{q}^T\frac{\mathbf{x}}{r} - \sum_{j=1}^r (\frac{\mathbf{x}^T}{r}\sqrt{\alpha_j}\mathbf{q} + p_0 z_j)^2 + \sum_{j=1}^r (p_0 z_j)^2 \quad (31)$$

Provided that $\|D^\rho \mathbf{x}\|_2 \leq c \|\mathbf{x}\|_2$, ($c > 0$) results in

$$(D^{j\mu} x_1)^2 < \mathbf{x}^T \mathbf{x}, \quad j = 1, 2, \dots, r \quad (32)$$

and considering that the second term is negative yields

$$D^{v;\mu}(V) \leq -\left((1 - \frac{1}{r})\lambda_{\min}(\mathbf{Q}) - rp_0^2\right)\mathbf{x}^T \mathbf{x} \quad (33)$$

where $\lambda_{\min(Q)}$ is the minimum eigenvalue of $Q = qq^T$. This inequality leads to the asymptotic stability of the system (6), (7) with the control law (11) if the relation

$$\left(1 - \frac{1}{r}\right) \lambda_{\min(Q)} - rp_0^2 > 0 \quad (34)$$

is verified and the frequency conditions (28) are satisfied. This last relation can be simplified if we define

$$\alpha^* = \min_j \alpha_j \quad (35)$$

$$\operatorname{Re}(H(j\omega)) > -\frac{\alpha^*}{r^2}, \quad \omega \in (-\infty, +\infty) \quad (36)$$

where

$$H(j\omega) = d^T(j\omega I - A^*)^{-1}a \quad (37)$$

□

Remark 1. If the condition

$$\alpha_{\min} \leq \alpha_j \leq \alpha_{\max} \quad (38)$$

is verified, the frequency condition (36) can be rewritten as ([30], Theorem 7.1, pp. 269).

$$\operatorname{Re} \left[\frac{\frac{2r^2}{\alpha_{\max}} + H(j\omega)}{\frac{2r^2}{\alpha_{\min}} + H(j\omega)} \right] > 0 \quad (39)$$

This condition can be interpreted as “Popov’s circle criterion” for this class of systems. Therefore, the conditions of Theorem 2 are satisfied if the plot of $H(j\omega)$ does not enter the circle $C(-\frac{2r^2}{\alpha_{\min}}, -\frac{2r^2}{\alpha_{\max}})$.

Remark 2. Theorem 2 determines the sufficient conditions for the Mittag–Leffler stability of the system (6), (7) characterized by fractional derivatives with fractional non-commensurate orders of derivation, ν and μ . This result is extendable to fractional-order systems characterized by the following derivation orders $\nu, \mu_1, \mu_2, \dots, \mu_s$,

$$D^\nu x(t) = Ax(t) - \sum_{i=1}^s \sum_{j=1}^{r_i} a_{j,i} D^{j\mu_i} x_1(t) + bu(t) \quad (40)$$

Extension of Theorem 2. The system (40) with the control law (11) is Mittag–Leffler-stable if the following conditions are satisfied:

$$\operatorname{Re}(d^T(j\omega I - A^*)^{-1}a) \geq -\frac{\alpha^*}{r^2}, \quad \omega \in (-\infty, +\infty) \quad (41)$$

$$\left(1 - \frac{1}{r}\right) \lambda_{\min(Q)} > rp_0^2 \quad (42)$$

where

$$\alpha^* = \min_{j,i} \alpha_{j,i}; \quad r = \sum_{i=1}^s r_i \quad (43)$$

Example 1. We consider the model

$$D^{0.7} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.3 & -2.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -16 \end{bmatrix} D^{0.3} x_1 + \begin{bmatrix} 0 \\ -8 \end{bmatrix} D^{0.5} x_1 \quad (44)$$

where $\nu = 0.7$ and $\mu_1 = 0.3, \mu_2 = 0.5, r_1 = 1, r_2 = 1, r = r_1 + r_2 = 2$. The matrix A is Hurwitz-stable with $\lambda_1 = -0.1264, \lambda_2 = -2.3736$. The vector a is selected as $a = \begin{bmatrix} 0 & 32 \end{bmatrix}^T$ and from (11) we obtain $\alpha_1 = 0.2, \alpha_2 = 0.4$ and $\alpha^* = \min(\alpha_1, \alpha_2) = 0.2$. The vector d is obtained from (18) as $d = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$. Consider $p_0 = 0.1$. From (25), (29) yields $\lambda_{1(Q)} = 1.7693, \lambda_{2(Q)} = 0.2437$.

Stability conditions (41) and (42) become

$$\operatorname{Re}(d^T(j\omega I - A)^{-1}a) \geq -0.025; \quad \omega \in (-\infty, +\infty) \quad (45)$$

$$\lambda_{\min(Q)} > 0.04 \quad (46)$$

Remark (Figure 1) that both conditions are verified.

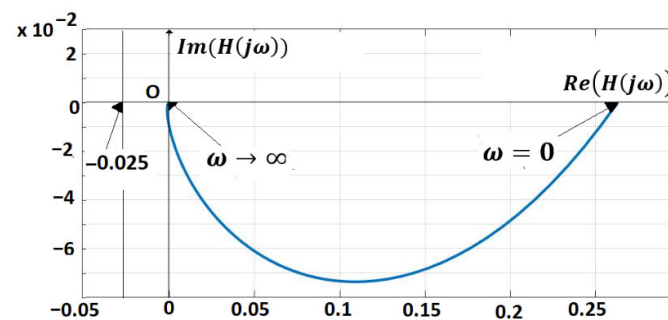


Figure 1. Frequency plot of the model (4), (5).

Remark 3. The criterion formulated by Theorem 2 for non-commensurate fractional-order systems can also be used for commensurate fractional-order systems.

3. Results

3.1. Robot Link Control

The control of flexible-joint robots is an extremely attractive research topic. Fuzzy techniques for model approximation and fuzzy controllers are proposed in [31]. Other papers including [1,24,32] use integer-order models for viscous components and implement fractional-order controllers. In [33], a prescribed performance neuro-adaptive control scheme is proposed for a single-link flexible-joint robotic manipulator with unknown dead-zone and actuator faults. The control law is developed by a backstepping technique to ensure transient/steady-state performance and adaptive neural networks are employed for uncertainty approximation. An adaptive control algorithm in the workspace for a robot manipulator is discussed in [34].

In [35], the output feedback control of single-link flexible-joint robot manipulators with matched disturbances and parametric uncertainties is shown. Two different approaches based on sliding mode control (SMC) and integral SMC have been considered in [36]. A Proportional–Integral–Derivative-type fuzzy logic controller that is based on direct fuzzy relations for a flexible robot component is proposed in [37]. The self-tuning controller is designed by the control of error and change in error to adapt according to the system variations.

Consider a flexible joint of classical robot architecture, shown in Figure 2.

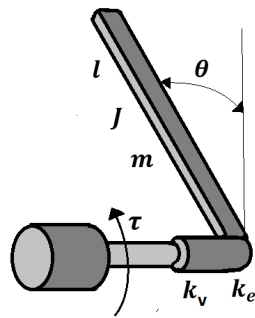


Figure 2. Flexible robot link.

The dynamic model is determined by the following equation [25,32]:

$$\ddot{\theta}(t) + k_v D^{0.7} \theta(t) + k_e \theta(t) + mgl \sin \theta = \tau(t) \quad (47)$$

where θ is the angular position; $D^{0.7} \theta(t)$ is the Caputo 0.7-fractional-order derivative; and J, k_v, k_e, l, m are technological and geometrical parameters, where equivalent inertial moment $J = 0.028 \text{ kg m}^2$, elastic coefficient $k_e = 2.8 \text{ Nm/rad}$, length $l = 0.7 \text{ m}$, mass $m = 0.18 \text{ kg}$, and τ is the actuator torque. The viscosity coefficient is considered to be $k_v = \frac{0.22 \text{ Nm}}{\text{rad}} \in [0.1; 0.3] \frac{\text{Nm}}{\text{rad}}$. Using a classical linear approximation of $\sin(\theta) \approx \rho\theta$, $\theta \in [\frac{\pi}{2} - \delta_1, \frac{\pi}{2} - \delta_2]$, the model is linearized as [24]

$$\ddot{\theta}(t) + k_v D^{0.7} \theta(t) + k_e \theta(t) + mgl\rho\theta(t) = \tau(t), \theta \in [\frac{\pi}{2} - \frac{\pi}{15}, \frac{\pi}{2} - \frac{3\pi}{15}] \quad (48)$$

or

$$\begin{aligned} \dot{\theta}_1(t) &= \theta_2(t) \\ \dot{\theta}_2(t) &= -(k_e + mgl\rho)\theta(t) + \tau(t) - k_v D^{0.7} \theta(t) \end{aligned} \quad (49)$$

The model (48) offers an example of a non-commensurable system. Classical methods [1] study the control and stability of the system by transforming it into a 70th-order differential model, which involves major difficulties either through a matrix method or through Laplace transform.

Based on the method presented in the previous sections, we define the following state vectors:

$$\mathbf{x} = [\theta_1 \ \theta_2]^T, \theta = \theta_1 \quad (50)$$

We consider that the output is defined by

$$y_1(t) = \mathbf{c}^T \mathbf{x}(t); \mathbf{c} = [c_1 \ c_2]^T \quad (51)$$

Assume that the control law is determined by the components with physical significance:

$$\tau(t) = -ky_1(t) = -k\mathbf{c}^T \mathbf{x}(t); \quad (52)$$

$$\tau(t) = -k_1\theta(t) - k_2\dot{\theta}(t) \quad (53)$$

The model (48) becomes

$$\dot{\mathbf{x}}(t) = \mathbf{A}^* \mathbf{x}(t) - \mathbf{a} D^{0.7} \theta(t) \quad (54)$$

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 \\ -(k_1 + k_e + mgl\rho) & -k_2 \end{bmatrix} \quad (55)$$

$$a = \begin{bmatrix} 0 \\ k_v \end{bmatrix} \quad (56)$$

and A^* by the law of control (53) with $k_1 = 24$, $k_2 = 0.85$ becomes a Hurwitz matrix ($\lambda_1 = -0.7423$, $\lambda_2 = -1.5734$). The controllability and observability of (A^*, a) , (A^*, d) , respectively, are obtained from

$$\text{rank}([a \ A^* \ a]) = \text{rank} \left(\begin{bmatrix} 0 & k_v \\ k_v & -k_v k_2 \end{bmatrix} \right) = 2 \quad (57)$$

$$\text{rank} \left(\begin{bmatrix} d \\ d \ A^* \end{bmatrix} \right) = \text{rank} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 2 \quad (58)$$

We use Theorem 2 and the Lyapunov function (13) proposed with $p_0 = 0.1$, and the matrix P is determined as a solution of Yakubovich–Kalman–Popov lemma (25), (29) with $\lambda_{\min}(Q) = 0.8245$, $r = 2$, $d = [1 \ 0]^T$. The k_v viscosity coefficient domain allows us to define $0.1 < \alpha < 0.3$. The asymptotic stability is verified by Popov's circle Theorem. The plot of $H(j\omega)$ does not enter the circle $C(-80, -26.6)$, as shown in Figure 3, and the second condition (37) is easily verified. MATLAB/SIMULINK R2020 and techniques based on the Mittag-Leffler functions are used for the simulation of the model [4,10].

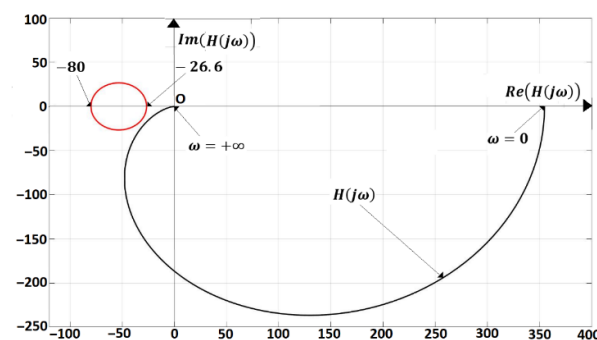


Figure 3. The plot of $H(j\omega)$ for flexible robot link control system.

The characteristics of the control system are given by the trajectories of the system for a desired position $\theta_d = \frac{\pi}{2} - \frac{2\pi}{15}$, as shown in Figure 4. (The rise time is 0.4 s, the settling time is 0.9 s, and the overshoot is 2% with the steady-state error $e_s \cong 0.03$ rad).

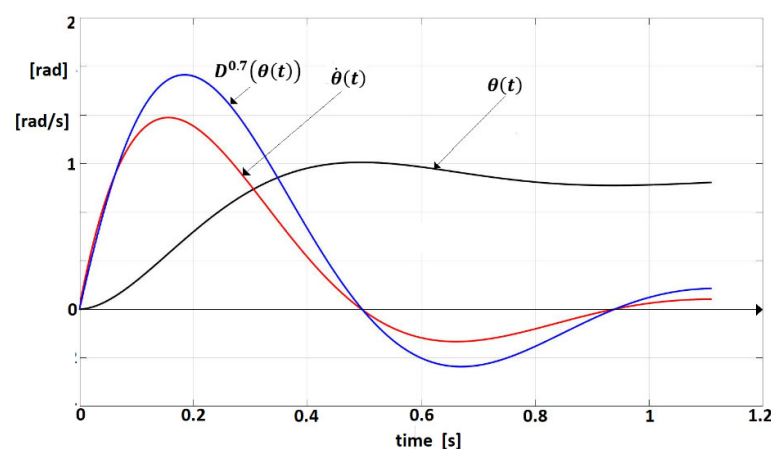


Figure 4. Trajectories of flexible robot link control system.

3.2. Human–Machine Control System

Traditionally, in human–machine systems, humans are assumed to respond to objective machine output variables, which might be translational or angular displacements, velocities, or visual field information derived from the vehicle motion. These variables are processed by the visual system and the central nervous system prior to the generation of a response action. The human model and its main parameters define key factors that ensure the global performances of the motion. There is rich classical literature that investigates the normal human model. One of the most representative models is described in [22]. In the area of human modeling, many techniques and algorithms are derived from control theory, where the human is assumed to act as a controller with the feedback determined by the error from previewed target points [23,38,39]. The optimal control models that ensure a good quality of motion are studied in [40,41] but they are not connected to human performance. Similar studies have been developed in [42,43]. Also, references [39,44–46] introduce a data-driven Brain Emotional Learning-Based Intelligent Controller integrated with a Proportional–Integral–Derivative structure for a flexible joint manipulator system. The controller is designed to enhance trajectory accuracy and dynamic performance in nonlinear plants. The self-tuning controller is designed by the control of error and change in error to adapt according to system variations.

The driving system of an electric vehicle by a human operator is one of the most representative examples of human–machine control (Figure 5). We will consider the fractional-order model with fractional exponent $\mu = 0.7$ [38].

$$D^{0.7}x_h(t) = -\tau_h x_h(t) + w(t) \quad (59)$$

where x_h represents the state of human operator, τ_h is its reaction time constant of the operator, and w is the visual information through which the vehicle movement is controlled:

$$w(t) = \theta(t) \quad (60)$$

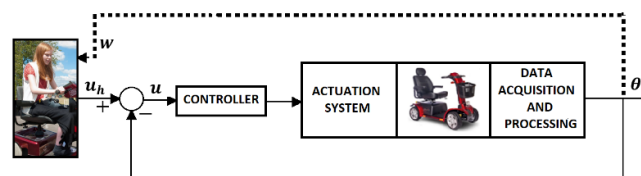


Figure 5. Human–machine control system.

The human operator exercises the control function through the command u_h , which determines the desired, prescribed variable in the vehicle control system:

$$u_h(t) = \chi x_h(t) \quad (61)$$

where χ is a coefficient determined by the operator's driving inability:

$$1 < \chi \leq \rho. \quad (62)$$

Reduced driving skills, the presence of physical disabilities, the presence of Parkinson's disease, or intoxication are interpreted as $\chi \rightarrow \rho \gg 1$, and high performance determines that $\chi \rightarrow 1$.

An electric vehicle with front-wheel-steering control is proposed. The steering system transmits the steering wheel of the operator to the wheels by universal joints. The dynamic model of the steering system is represented by a second-order ODE, taking into account

inertial and stiffness components, and damping and driving torque. In this model, some components were neglected.

$$J_s \ddot{\theta}(t) = -c_d \dot{\theta}(t) - c_s \theta(t) + b_s \tau(t) \quad (63)$$

where θ is the average angle of the two front wheel angles, inertial component $J_s = 0.008 \text{ kg m}^2$, c_s is the stiffness of the steering system ($c_s = 2100 \text{ Nm/rad}$), damping coefficient is c_d ($c_d = 0.136 \frac{\text{Nm s}}{\text{rad}}$), b_s is a coefficient relative to the steering ratio [41], and τ is the driving torque. The model of (63) can be rewritten as $(\theta(t) = \theta_1(t))$:

$$\dot{\theta}_1(t) = \theta_2(t) \quad (64)$$

$$\dot{\theta}_2(t) = -c_e/J_s \theta_1(t) - c_d/J_s \theta_2(t) + b_s/J_s \tau(t) \quad (65)$$

or

$$\dot{\mathbf{x}}_{\theta}(t) = \mathbf{A} \mathbf{x}_{\theta}(t) + \mathbf{b} \tau(t) \quad (66)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -c_e/J_s & -c_d/J_s \end{bmatrix}; \mathbf{b} = \begin{bmatrix} 0 \\ b_s/J_s \end{bmatrix} \quad (67)$$

where $\mathbf{x}_{\theta}$ is the state vector, $\mathbf{x}_{\theta} = [\theta_1 \ \theta_2]^T$, and τ is determined by two components:

$$\tau(t) = u_h(t) + u_c(t) \quad (68)$$

The first component is imposed by the human operator and the second represents the internal control that ensures the desired performance:

$$u_c(t) = -k_1 \theta(t) - k_2 \dot{\theta}(t) \quad (69)$$

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 \\ -c_e/J_s - k_1 b_s/J_s & -c_d/J_s - k_2 b_s/J_s \end{bmatrix} \quad (70)$$

and $\mathbf{A}^*$ is Hurwitz ($k_1 = 14$, $k_2 = 0.35$; $\lambda_1 = -1.8455$, $\lambda_2 = -0.3268$).

The controllability and observability of $(\mathbf{A}^*, \mathbf{a})$, $(\mathbf{A}^*, \mathbf{d})$, respectively, are obtained from

$$\text{rank}([\mathbf{a} \ \mathbf{A}^* * \mathbf{a}]) = \text{rank} \left(\begin{bmatrix} 0 & b_s/J_s \\ b_s/J_s & (-c_d/J_s - k_2 b_s/J_s) b_s/J_s \end{bmatrix} \right) = 2 \quad (71)$$

$$\text{rank} \left(\begin{bmatrix} \mathbf{d} \\ \mathbf{d} * \mathbf{A}^* \end{bmatrix} \right) = \text{rank} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 2 \quad (72)$$

The quality as a driver of the human operator is defined by the inability factor, $\chi \in [1; 2]$, and the reaction time constant, $\tau_h \in [0.1; 0.2] \text{ s}$.

For the global stability analysis of the human-machine system, the following Lyapunov function is proposed:

$$V(\mathbf{x}_{\theta}, x_h) = \mathbf{x}_{\theta}^T \mathbf{P} \mathbf{x}_{\theta} + \frac{1}{2} \chi x_h^2 \quad (73)$$

with the derivative

$$D^{1.0.7}(V) \leq D^1(\mathbf{x}_{\theta}^T \mathbf{P} \mathbf{x}_{\theta}) + D^{0.7} \left(\frac{1}{2} \chi x_h^2 \right) \quad (74)$$

According to Theorem 2, the Mittag–Leffler stability of the human–machine system (59), (63) can be verified by Popov’s circle criterium (39):

$$\operatorname{Re} \left[\frac{\frac{1}{\sigma_a} + H(j\omega)}{\frac{1}{\sigma_b} + H(j\omega)} \right] > 0 \quad (75)$$

where

$$H(j\omega) = d^T(j\omega I - A^*)^{-1} \chi b \quad (76)$$

$$\sigma = \chi \tau_h \quad (77)$$

The plot of $H(j\omega)$ does not enter the circle $C(-\frac{1}{\sigma_a}, -\frac{1}{\sigma_b})$, where

$$\sigma_b = \chi_{\max} \tau_{h,\max}; \sigma_a = \chi_{\min} \tau_{h,\min} \quad (78)$$

Figure 6 presents the frequency plots for three cases and the restriction domain defined by the circle $C : (-\frac{1}{\sigma_a}, -\frac{1}{\sigma_b})$. Curve *a* represents the evolution of the system for an operator with normal abilities, $\chi = 1$, and $\tau_h = 0.15$ s. Curve *c* defines a critical behavior in which the operator, either due to physical or mental disabilities, or due to external causes, has $\tau_h = 0.2$ s and $\chi = 2$, determining a global unstable evolution of the system, although the internal control parameters ensure stability (through the controller) ($H_c \cap C = (M_1 : x_c = -4.3254, y_c = -2.7543; H_c \cap C = (M_2 : x_c = 0.6523, y_c = -2.2486)$). Plot *b* defines a behavior at the limit $H_b \cap C = (M_0 : x_b = -2.4687, y_b = -1.0365)$.

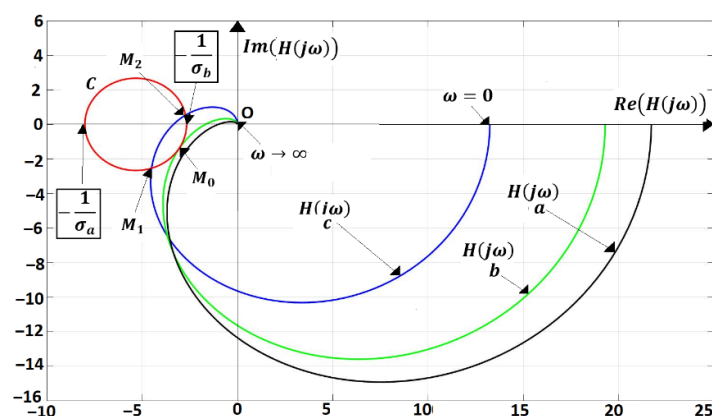


Figure 6. The frequency plots of human–machine control system.

In Figure 7 are shown the trajectories $\theta(t)$ for two human operators $x_h(t)$ driving with the same time constant $\tau_h = 0.15$ s but with different inability factors, $\chi = 1$ (trajectory a—the rise time is 3.8 s, the settling time is 4.5 s, and the overshoot is 1% with the steady-state error $e_s \cong 0.05$ rad) and $\chi = 1.4$ (trajectory b—the rise time is 1.6 s, the settling time is 6 s, and the overshoot is 24% with the steady-state error $e_s \cong 0.4$ rad). It is noted that in both cases, the system is stable, but the driving performance is low in case b.

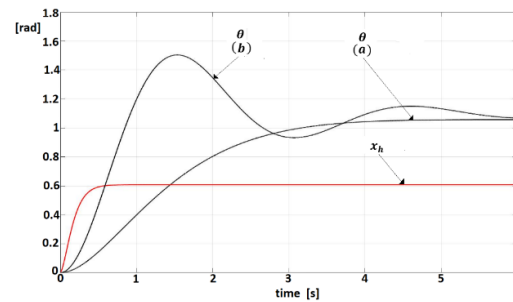


Figure 7. The trajectories of human–machine control system.

4. Discussion

Various approximation methods are presented in the literature for this class of system. References [47,48] present a novel softsign-function-based fractional-order Proportional–Integral–Derivative controller optimized using the fungal growth optimizer for the stabilization and precise position control of an unstable magnetic ball suspension system. The proposed controller introduces a smooth nonlinear softsign function into the conventional structure to limit abrupt control actions and improve transient smoothness while preserving the flexibility of fractional dynamics. Also, this system improves the damping properties of the closed-loop dynamics by moderating the proportional and derivative responses. As a result, more stable and energy-efficient behavior is achieved. Unlike the frequency-domain criterion used in our paper, the softsign-function-based technique employs analytical stability evaluation methods via the Laplace transform.

In references [25,49,50], the Oustaloup approximation method of s^α is analyzed to provide a tool that satisfies the requirements of approximation of the error in magnitude and/or phase in a specific frequency range for the minimal possible order of the approximation. Also, this approximation technique transforms the fractional system into a higher-order system in frequency or into Laplace [1,2]. For example, the non-commensurate system

$$G(s) = \frac{5s^{0.6} + 2}{s^{3.3} + 3.1s^{2.6} + 2.89s^{1.9} + 2.5s^{1.2} + 1.2} \quad (79)$$

is approximated as

$$G(s) = \frac{39.97s^{20} + 7.68 \times 10^4 + \dots 7.703 \times 10^8 s + 1.577 \times 10^6}{s^{23} + 2211s^{22} + \dots 4.275 \times 10^8 s + 9.103 \times 10^5} \quad (80)$$

and analyzed by known techniques, but interpretation is difficult due to the order size of the system. The method proposed in our paper allows for the transformation of the system (63) into the following model:

$$D^{0.7}x(t) = Ax(t) + b_1 D^{0.6}x(t) + b_2 x(t) \quad (81)$$

where

$$\begin{aligned} D^{0.5}x &= x_1; D^{0.7}x_1 = x_2; D^{0.7}x_2 = x_3; D^{0.7}x_3 = x_4; \\ D^{0.7}x_4 &= -3.1x_4 - 2.89x_3 - 2.5x_2 + 5D^{0.6}x + 0.8x \\ x &= [x_1 \ x_2 \ x_3 \ x_4]^T \end{aligned} \quad (82)$$

The stability conditions can be inferred from Theorem 2.

Fractional-order commensurate systems [3] can also be studied by the method proposed in this paper. Let us consider the following example:

$$G(s) = \frac{1}{s^{0.8} + 2.3s^{0.6} + 1.5s^{0.4} + 0.6s^{0.2} + 3.6s^{0.1} + 5.7} \quad (83)$$

The usual methods suggested in the literature [1–3,10] are based on the substitution $\rho = s^{0.1}$ and the determination of the roots of an 8th-order characteristic equation followed by the selection of the roots present on the first Riemann sheet. Another procedure [5,6] consists of the selection of a state vector:

$$x = [x \ D^{0.1}x \ D^{0.2}x \ D^{0.3}x \ D^{0.4}x \ D^{0.5}x \ D^{0.6}x \ D^{0.7}x]^T \quad (84)$$

and the transformation of the model into a fractional differential equation of order 0.1 with appropriate dimensions of the matrix components:

$$D^{0.1}x(t) = Ax(t) \quad (85)$$

where $A : [8 \times 8]$.

Our method uses a state vector,

$$x = [x \ D^{0.2}x \ D^{0.4}x \ D^{0.6}x]^T, \quad (86)$$

and transforms (67) into the system

$$D^{0.2}x(t) = Ax(t) + bD^{0.1}x \quad (87)$$

where $A : [4 \times 4]$, $b : [4 \times 1]$ and the frequency criterion (33) can be applied.

Non-commensurate fractional-order systems are also known in the literature [26,27,51] as multi-order fractional systems. In contrast to the frequency-domain techniques used in our paper for stability evaluation, these papers utilize various analytical methods. In [51], multi-order systems are transformed into a higher-dimensional system with the same Caputo derivative order lying in $(0, 1)$. The stability of the solution to the original system is studied through the analysis of its equivalent system. In [27], fractional differential equations are changed into equations with the same fractional orders, providing that the multi-orders are rational numbers, so the known theorems of existence, uniqueness, and asymptotic stability are applied. A vector Lyapunov-like function method is proposed in [26]. This method reduces the asymptotic stability problem to verify a Hurwitz property on a suitable matrix. Matrix criteria are used to infer the system stability. Then, the results for integer-order systems are generalized to multi-order systems to obtain robust results for multi-order multiple-time-varying delay linear positive systems.

5. Conclusions

Non-commensurate fractional-order systems frequently appear in various physical models, the fractional exponent being determined by certain material properties or by certain evolution laws. In this sense, this paper aims to determine methods for analyzing the stability of control systems for non-commensurate fractional-order models. The results obtained are based on Lyapunov methods for differential equations with fractional exponents and on the application of the Yakubovich–Kalman–Popov lemma adapted for this class of systems. The stability criterion is presented for NCFOS in the form of a frequency criterion and represented graphically by familiar frequency plots similar to those of the Nyquist or Popov type. If the parameters that define the model can be defined in a closed

domain, the frequency criterion can be interpreted as “Popov’s circle criterion”. The two numerical applications present important cases. The first studies the stability criterion in the case where the viscosity coefficients determine non-commensurate fractional-order exponents in the dynamic model of the system. The second example studies the complex problem of the human–machine system in which the human model imposes dynamics determined by non-commensurate fractional-order exponents. In both cases, Popov’s circle criterion determines the asymptotic stability conditions.

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