

Article

Existence, Uniqueness and Solutions for Diffusion and Advection Effects for Predator–Prey Model with Holling Type II Interaction Function

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Abstract

The present work is focused on a predator–prey model with the Holling type II interaction function, which is influenced by diffusion, advection and nonlinear reaction effects. Firstly, we show that the solutions of this dynamical model are bounded and unique. Secondly we use the Lyapunov function and then show that the equilibrium points are globally stable. Thirdly, we obtain the solution profile when the diffusion coefficient is small. For this purpose we introduce self-similar structures to convert the nonlinear partial differential equations into nonlinear ordinary differential equations and then use the singular perturbation technique to solve these equations. Fourthly, we use the Hamiltonian and Lighthill’s technique to obtain upper stationary solutions for a small coefficient of the advection term. Lastly, we consider a large diffusion coefficient and obtain the asymptotic profiles of nonstationary solutions with the help of nonlinear point scaling.

Keywords: Holling-type interaction function; advection; Lighthill’s technique; existence and uniqueness; predator–prey model

MSC: 35A01; 35A35; 35A25

1. Introduction

The study of diffusion and advection effects in predator–prey models with Holling type II interaction functions has received significant attention from both theoretical and practical perspectives. The foundational work of Lotka [1] and Volterra [2] introduced key mathematical models to inspect predator–prey dynamics. Many other researchers have worked on these models based on trophic interactions, such as explaining the construction of predator–prey reaction–diffusion systems in [3], supplementary food sources in [4], prey refuge in [5], hunting cooperation in [6], the fear effect in [7], the Allee effect in [8], refuge and harvesting in [9,10], and dissolved oxygen–plankton in [11]. Although such models often reply on simplifying assumptions, considerable efforts have been devoted to validating their relevance in real-world contexts, particularly in assessing whether predators can regulate prey populations. Prominent frameworks like Holling’s type II functional response [12] and the Lotka–Volterra model [13] have been central in these investigations. Berryman expanded upon predation models using Holling’s type II response but did not



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consider the prey refuge—an ecologically important factor. This issue was later resolved by Kar [14].

The interaction between predator and prey populations in ecological systems can be principally designed in two ways: (1) one predator and multiple prey species or (2) one predator and one prey species [15]. Erbach et al. [16] examined a generalized predator–prey model, while Magal et al. [17] extended it by infusing it with historical prey population effects. Although the authors in the above three references deeply investigated trophic dynamics involving specialized or generalized predator–prey models, they did not consider prey refuges. Sen et al. [18] and Mondal et al. [19] are known as pioneers in investigating the combined influence of prey refuge and Beverton–Holt-like supplementary food on the prey and predator. The proposed model precisely analyzes stability transitions and other dynamical behaviors, incorporating an Allee effect in prey growth and cooperative hunting among predators. In addition to the Holling type II framework, providing alternative food sources for predators when prey seek refuge emerges as a biologically rational strategy. The model reveals that predators consume fewer prey relative to their population, granting prey greater potential to find protection. One possible approach is a predator–prey model where the prey manifests logistic growth. Furthermore, the predator’s food supply can be modeled using a Beverton–Holt-like functional form to account for alternative resources. As a result, one gets

$$\begin{aligned}\frac{\partial u}{\partial t} &= a_1 u \left(1 - \frac{u}{K}\right) - \psi(u, v)v \\ \frac{\partial v}{\partial t} &= e\psi(u, v)v - a_4 v + \frac{a_5 v}{1 + a_6 v},\end{aligned}\quad (1)$$

where $u(0) \geq 0$, $v(0) \geq 0$. Here u and v denote densities of prey and predator species respectively at any time t , e ($0 < e < 1$) signifies the conversion factor from prey into predator density, a_4 implies the natural mortality rate of the predator, a_1 is the logistic growth coefficient of the prey species and K is the prey species holding capacity. The predator species propagates with a Beverton–Holt-like function (adding extra food sources to the predator) of the form $\frac{a_5 v}{1 + a_6 v}$ when the primary prey is absent. The predator’s maximum rate of reproduction is a_5 , which drops with density, and a_6 represents the resilience of biomass dependence. We steadily presume $a_5 > 4$ in order to permit the predator species to persist even in the absence of the primary prey. Ecologically, this imposed condition $a_5 > 4$ represents the minimum threshold for food that is required for the predator to survive and reproduce. If natural prey is insufficient, then additional food is required by the predator for their basic energy needs. When a_5 is too small, then the supplementary food cannot compensate for prey shortage and this causes the predator to decline. Therefore $a_5 > 4$ gives a guarantee that the supply of supplementary food is enough to sustain the predator population. The interaction function, represented as $\psi(u, v)$, specifies how the predator eats the prey species. A generalist predator–prey model is used when $a_5 > a_4$. The limitation $a_5 > a_4$ must be secured throughout the inquiry in order to fully understand the complexity of system dynamics brought about by the Beverton–Holt-like additional food. The response function $\psi(u, v)$ for predation is here concluded to be the Holling type II interaction function, with a_2 representing the maximum number of prey that the predator may eat in a given amount of time and a_3 representing the half-saturation constant. A continuous percentage of the prey refuge, mu , from the prey species is incorporated into the interacting species model (1): $0 \leq mu < u$; m is a positive constant. Across the study, it is assumed that $0 \leq m < 1$; that is, for a fundamental biological system, the acceptable range of prey refuge is $0 < (1 - m) \leq 1$. If $m = 0$ then there is no refuge and all prey are exposed to the predator; for $0 < m < 1$ then some prey are protected and some are

not, which is close to a real ecosystem; and if $m = 1$ then all prey are protected and the predator lives without food, which is unrealistic. Consequently, the model assumes the following form when this refuge phenomenon is included in the response function $\psi(u, v)$, i.e., $\psi(u, v) = \frac{a_2(1-m)u}{a_3 + (1-m)u}$. As a result, model (1) becomes (2) with the addition of the Holling type II interaction function and constant-proportion prey refuge.

$$\begin{aligned}\frac{\partial u}{\partial t} &= a_1 u \left(1 - \frac{u}{K}\right) - \frac{a_2(1-m)uv}{a_3 + (1-m)u} \\ \frac{\partial v}{\partial t} &= e \frac{a_2(1-m)uv}{a_3 + (1-m)u} - a_4 v + \frac{a_5 v}{1 + a_6 v}.\end{aligned}\quad (2)$$

The addition of advection and diffusion terms in the predator–prey model remarkably enhances the consideration of the system’s dynamic. Due to environment factors or other factors, the prey and predator move directly, which is represented by the advection term. The diffusion term represents the random movement of the prey and predator. This addition of diffusion and advection terms is a modification of the previous model and obtains the conditions in which different type of solutions exist. Now we assume that the prey refuge model depends on time as well as space; then after including the diffusion and advection effects in the above-mentioned model, we have

$$\begin{aligned}\frac{\partial u}{\partial t} &= D_1 \frac{\partial^2 u}{\partial x^2} + \bar{\epsilon}_1 \frac{\partial u}{\partial x} + a_1 u \left(1 - \frac{u}{K}\right) - \frac{a_2(1-m)uv}{a_3 + (1-m)u} \\ \frac{\partial v}{\partial t} &= D_2 \frac{\partial^2 v}{\partial x^2} + \bar{\epsilon}_1 \frac{\partial v}{\partial x} + \frac{ea_2(1-m)uv}{a_3 + (1-m)u} - a_4 v + \frac{a_5 v}{1 + a_6 v},\end{aligned}\quad (3)$$

where D_1 and D_2 are the coefficients of diffusion and $\bar{\epsilon}_1$ is the coefficient of advection, with $u(x, 0) = u_0(x)$, $v(x, 0) = v_0(x)$, $u(0, t) = 0$, $u(l\pi, t) = 0$, $v(0, t) = 0$ and $v(l\pi, t) = 0$. In order to reduce the number of system parameters, the model mentioned above must be non-dimensionalized by adding $\tilde{u} = \frac{u}{K}$, $\tilde{v} = \frac{ev}{K}$, $\tilde{x} = \frac{x}{l}$ and $\tilde{t} = a_1 t$. After using this approach, the redesigned model looks like this (tildes dropped):

$$\begin{aligned}\frac{\partial u}{\partial t} &= d_1 \frac{\partial^2 u}{\partial x^2} + \epsilon_1 \frac{\partial u}{\partial x} + u(1 - u) - \frac{\alpha(1-m)uv}{\beta + (1-m)u} \\ \frac{\partial v}{\partial t} &= d_2 \frac{\partial^2 v}{\partial x^2} + \epsilon_1 \frac{\partial v}{\partial x} + \frac{\gamma(1-m)uv}{\beta + (1-m)u} - \delta_1 v + \frac{\epsilon v}{1 + \rho v},\end{aligned}\quad (4)$$

where $u(x, 0) = u_0(x)$, $v(x, 0) = v_0(x)$, $u(0, t) = 0$, $u(\pi, t) = 0$, $v(0, t) = 0$ and $v(\pi, t) = 0$ with $d_1 = \frac{D_1}{2a_1 l^2}$, $d_2 = \frac{D_2}{2a_1 l^2}$, $\epsilon_1 = \frac{\bar{\epsilon}_1}{a_1 l}$, $\alpha = \frac{a_2}{ea_1}$, $\beta = \frac{a_3}{K}$, $\gamma = \frac{ea_2}{a_1}$, $\delta_1 = \frac{a_4}{a_1}$, $\epsilon = \frac{a_5}{a_1}$, $\rho = \frac{a_6 K}{e}$.

In our analysis, we take $\epsilon = 1$ for simplicity, which does not affect the generality. In this paper we consider a predator–prey model with the Holling type II interaction function, under diffusion, advection and reaction effects. The governing equations of this model are nonlinear partial differential equations. In Section Two, we show that the solutions of these partial differential equations are bounded and unique. In Section Three, we consider the Lyapunov function and then show that the equilibrium points are globally stable. In Section Four, we develop the solutions for a large domain with a small coefficient of diffusion. In Section Five, we take a small coefficient of the advection term and apply the Hamiltonian and Lighthill’s technique for upper stationary solutions. In Section Six, we take a large diffusion coefficient and use the nonlinear point scaling technique to obtain the asymptotic nonstationary solutions.

2. Existence and Uniqueness

To prove the existence, we consider the following theorem.

Theorem 1. Given that $(u_0(x), v_0(x)) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then the solutions $u(x, t)$ and $v(x, t)$ are bounded on $B_\gamma \times [0, T]$ with $0 < \gamma < \infty$, where B_γ means the ball of radius γ .

Proof. To start with, consider the test function $\xi_2 \in C^\infty(\mathbb{Q}_T)$, where $\mathbb{Q}_T = \mathbb{R} \times [\tau, T]$ with $0 < \tau < t < T$. Also consider the cut-off function η_{ϕ_2} , with a constant $\phi_2 \rightarrow 0^+$ and $\gamma \gg \gamma_0 > 0$, where γ refers to a spatial dimension, and then a cut = off function defined as follows (refer to [20] for additional insights):

$$\begin{aligned} \eta_{\phi_2} &\in C_0^\infty(x, t), \quad 0 \leq \eta_{\phi_1} \leq 1, \\ \eta_{\phi_2} &= 1 \text{ in } B_{\gamma-\phi_2}, \quad \eta_{\phi_2} = 0 \text{ in } \mathbb{R}^2 - B_{\gamma-\phi_2}, \end{aligned} \quad (5)$$

so that

$$|\nabla \eta_{\phi_2}| \leq \frac{a_1}{\phi_2}, \quad |\Delta \eta_{\phi_1}| \leq \frac{a_1}{\phi_2^2}. \quad (6)$$

Multiplying the first equation of (4) by $\xi_2 \eta_{\phi_2}$, integrating over $B_\gamma \times [\tau, T]$, and assuming u and $\frac{\partial u}{\partial x}$ are zero at the boundary, we get

$$\begin{aligned} \int_{B_\gamma} u(x, t) \xi_2(x, t) \eta_{\phi_2} dx &= \int_{B_\gamma} u(x, \tau) \xi_2(x, \tau) \eta_{\phi_2} dx - \int_\tau^t \int_{B_\gamma} \left[u \frac{\partial \xi_2}{\partial s} \eta_{\phi_2} + u \xi_2 \frac{\partial \eta_{\phi_2}}{\partial s} \right] dx ds \\ &= \int_\tau^t \int_{B_\gamma} \left[d_1 u \frac{\partial^2 \xi_2}{\partial x^2} \eta_{\phi_2} + 2d_1 u \frac{\partial \xi_2}{\partial x} \frac{\partial \eta_{\phi_2}}{\partial x} + d_1 u \xi_2 \frac{\partial^2 \eta_{\phi_2}}{\partial x^2} - \epsilon_1 u \frac{\partial \xi_2}{\partial x} \eta_{\phi_2} \right. \\ &\quad \left. - \epsilon_1 u \xi_2 \frac{\partial \eta_{\phi_2}}{\partial x} + u(1-u) \xi_2 \eta_{\phi_2} - \frac{\alpha(1-m)uv}{\beta + (1-m)u} \xi_2 \eta_{\phi_2} \right] dx ds, \end{aligned}$$

which implies that

$$\begin{aligned} \int_{B_\gamma} u(x, t) \xi_2(x, t) \eta_{\phi_2} dx &= \int_{B_\gamma} u(x, \tau) \xi_2(x, \tau) \eta_{\phi_2} dx + \int_\tau^t \int_{B_\gamma} \left[u \frac{\partial \xi_2}{\partial s} \eta_{\phi_2} dx + u \xi_2 \frac{\partial \eta_{\phi_2}}{\partial s} + d_1 u \frac{\partial^2 \xi_2}{\partial x^2} \eta_{\phi_2} \right. \\ &\quad + 2d_1 u \frac{\partial \xi_2}{\partial x} \frac{\partial \eta_{\phi_2}}{\partial x} + d_1 u \xi_2 \frac{\partial^2 \eta_{\phi_2}}{\partial x^2} - \epsilon_1 u \frac{\partial \xi_2}{\partial x} \eta_{\phi_2} - \epsilon_1 u \xi_2 \frac{\partial \eta_{\phi_2}}{\partial x} + u(1-u) \xi_2 \eta_{\phi_2} \\ &\quad \left. - \frac{\alpha(1-m)uv}{\beta + (1-m)u} \xi_2 \eta_{\phi_2} \right] dx ds. \end{aligned} \quad (7)$$

Similarly multiplying the second equation of (4) by $\xi_2 \eta_{\phi_2}$, integrating over $B_\gamma \times [\tau, T]$, and assuming v and $\frac{\partial v}{\partial x}$ are zero at the boundary, we get

$$\begin{aligned} \int_{B_\gamma} v(x, t) \xi_2(x, t) \eta_{\phi_2} dx &= \int_{B_\gamma} v(x, \tau) \xi_2(x, \tau) \eta_{\phi_2} dx + \int_\tau^t \int_{B_\gamma} \left[v \frac{\partial \xi_2}{\partial s} \eta_{\phi_2} dx + v \xi_2 \frac{\partial \eta_{\phi_2}}{\partial s} + d_2 v \frac{\partial^2 \xi_2}{\partial x^2} \eta_{\phi_2} \right. \\ &\quad + 2d_2 v \frac{\partial \xi_2}{\partial x} \frac{\partial \eta_{\phi_2}}{\partial x} + d_2 v \xi_2 \frac{\partial^2 \eta_{\phi_2}}{\partial x^2} - \epsilon_1 v \frac{\partial \xi_2}{\partial x} \eta_{\phi_2} - \epsilon_1 v \xi_2 \frac{\partial \eta_{\phi_2}}{\partial x} + \frac{\gamma(1-m)uv}{\beta + (1-m)u} \xi_2 \eta_{\phi_2} \\ &\quad \left. - \delta_2 v \xi_2 \eta_{\phi_2} + \frac{v}{1+\rho v} \xi_2 \eta_{\phi_2} \right] dx ds. \end{aligned} \quad (8)$$

For sufficiently large $\gamma \gg \gamma_0 > 1$, from [21] the following results hold:

$$\int_\tau^t u ds \leq c_p(\tau) x^{\frac{2}{m-1}}, \quad \int_\tau^t \frac{\partial u}{\partial x} ds \leq \frac{2}{m-1} c_p(\tau) x^{\frac{3-m}{m-1}}, \quad (9)$$

and

$$\int_\tau^t v ds \leq c_q(\tau) x^{\frac{2}{m-1}}, \quad \int_\tau^t \frac{\partial v}{\partial x} ds \leq \frac{2}{m-1} c_q(\tau) x^{\frac{3-m}{m-1}}, \quad (10)$$

where $c_p(\tau)$ and $c_q(\tau)$ are constants depending on fixed time τ . Admitting $\tau \leftarrow t$, the defined functions ξ_2 and η_{ϕ_1} exhibit local stationarity. Consequently, we have

$$\begin{aligned}
 \int_{\tau}^t \int_{B_{\gamma}} u \frac{\partial \xi_2}{\partial s} \eta_{\phi_2} dx ds &\leq \int_{B_{\gamma}} c_p(\tau) x^{\frac{2}{m-1}} \frac{\partial \xi_2}{\partial s} \eta_{\phi_2} dx \\
 \int_{\tau}^t \int_{B_{\gamma}} u \xi_2 \frac{\partial \eta_{\phi_2}}{\partial s} dx ds &\leq \int_{B_{\gamma}} c_p(\tau) \xi_2 x^{\frac{2}{m-1}} \frac{\partial \eta_{\phi_2}}{\partial s} dx \\
 d_1 \int_{\tau}^t \int_{B_{\gamma}} u \frac{\partial^2 \xi_2}{\partial x^2} \eta_{\phi_1} dx ds &\leq d_1 \int_{\tau}^t \int_{B_{\gamma}} u \frac{\partial^2 \xi_2}{\partial x^2} \eta_{\phi_1} dx ds \\
 &\leq d_1 \int_{B_{\gamma}} c_p(\tau) x^{\frac{2}{m-1}} \frac{\partial^2 \xi_2}{\partial x^2} \eta_{\phi_1} dx \\
 2d_1 \int_{\tau}^t \int_{B_{\gamma}} u \frac{\partial \xi_2}{\partial x} \frac{\partial \eta_{\phi_2}}{\partial x} dx ds &\leq 2d_1 \int_{B_{\gamma}} c_p(\tau) x^{\frac{2m}{m-1}} \frac{\partial \xi_2}{\partial x} \frac{\partial \eta_{\phi_2}}{\partial x} dx \\
 d_1 \int_{\tau}^t \int_{B_{\gamma}} u \xi_2 \frac{\partial^2 \eta_{\phi_2}}{\partial x^2} dx ds &\leq d_1 \int_{B_{\gamma}} c_p(\tau) x^{\frac{2m}{m-1}} \xi_2 \frac{\partial^2 \eta_{\phi_2}}{\partial x^2} dx \\
 -\epsilon_1 \int_{\tau}^t \int_{B_{\gamma}} u \frac{\partial \xi_2}{\partial x} \eta_{\phi_2} dx ds &\leq \epsilon_1 \int_{B_{\gamma}} c_p(\tau) x^{\frac{2}{m-1}} \frac{\partial \xi_2}{\partial x} \eta_{\phi_2} dx \\
 -\epsilon_1 \int_{\tau}^t \int_{B_{\gamma}} u \xi_2 \frac{\partial \eta_{\phi_2}}{\partial x} dx ds &\leq \epsilon_1 \int_{B_{\gamma}} c_p(\tau) x^{\frac{2}{m-1}} \xi_2 \frac{\partial \eta_{\phi_2}}{\partial x} dx, \tag{11}
 \end{aligned}$$

and

$$\begin{aligned}
 \int_{\tau}^t \int_{B_{\gamma}} v \frac{\partial \xi_1}{\partial s} \eta_{\phi_2} dx ds &\leq \int_{B_{\gamma}} c_q(\tau) x^{\frac{2}{m-1}} \frac{\partial \xi_2}{\partial s} \eta_{\phi_2} dx \\
 \int_{\tau}^t \int_{B_{\gamma}} v \xi_2 \frac{\partial \eta_{\phi_2}}{\partial s} dx ds &\leq \int_{B_{\gamma}} c_q(\tau) \xi_2 x^{\frac{2}{m-1}} \frac{\partial \eta_{\phi_2}}{\partial s} dx \\
 d_2 \int_{\tau}^t \int_{B_{\gamma}} v \frac{\partial^2 \xi_2}{\partial x^2} \eta_{\phi_2} dx ds &\leq d_2 \int_{\tau}^t \int_{B_{\gamma}} v \frac{\partial^2 \xi_2}{\partial x^2} \eta_{\phi_2} dx ds \\
 &\leq d_2 \int_{B_{\gamma}} c_q(\tau) x^{\frac{2}{m-1}} \frac{\partial^2 \xi_2}{\partial x^2} \eta_{\phi_2} dx \\
 2d_2 \int_{\tau}^t \int_{B_{\gamma}} v \frac{\partial \xi_2}{\partial x} \frac{\partial \eta_{\phi_2}}{\partial x} dx ds &\leq 2d_2 \int_{B_{\gamma}} c_q(\tau) x^{\frac{2m}{m-1}} \frac{\partial \xi_2}{\partial x} \frac{\partial \eta_{\phi_2}}{\partial x} dx \\
 d_2 \int_{\tau}^t \int_{B_{\gamma}} v \xi_2 \frac{\partial^2 \eta_{\phi_2}}{\partial x^2} dx ds &\leq d_2 \int_{B_{\gamma}} c_q(\tau) x^{\frac{2m}{m-1}} \xi_2 \frac{\partial^2 \eta_{\phi_2}}{\partial x^2} dx \\
 -\epsilon_1 \int_{\tau}^t \int_{B_{\gamma}} v \frac{\partial \xi_2}{\partial x} \eta_{\phi_1} dx ds &\leq \epsilon_1 \int_{B_{\gamma}} c_q(\tau) x^{\frac{2}{m-1}} \frac{\partial \xi_2}{\partial x} \eta_{\phi_1} dx \\
 -\epsilon_1 \int_{\tau}^t \int_{B_{\gamma}} v \xi_2 \frac{\partial \eta_{\phi_2}}{\partial x} dx ds &\leq \epsilon_1 \int_{B_{\gamma}} c_q(\tau) x^{\frac{2}{m-1}} \xi_2 \frac{\partial \eta_{\phi_2}}{\partial x} dx. \tag{12}
 \end{aligned}$$

For a fixed γ , the following holds:

$$\begin{aligned}
 \int_{B_{\gamma}} d_1 c_p(\tau) x^{\frac{2}{m-1}} \frac{\partial^2 \xi_2}{\partial x^2} \eta_{\phi_1} dx &= d_1 c_p(\tau) x^{\frac{2}{m-1}} \left(\frac{\partial \xi_2}{\partial x} \eta_{\phi_1} \right)_{\partial B_{\gamma}} - d_1 c_p(\tau) \int_{B_{\gamma}} \frac{2}{m-1} x^{\frac{2}{m-1}-1} \frac{\partial \xi_2}{\partial x} \eta_{\phi_1} dx \\
 &\quad - d_1 c_p(\tau) \int_{B_{\gamma}} x^{\frac{2}{m-1}} \frac{\partial \xi_2}{\partial x} \frac{\partial \eta_{\phi_1}}{\partial x} dx. \tag{13}
 \end{aligned}$$

Note that, when $\gamma \gg 1$, then $(\frac{\partial \xi_2}{\partial x} \eta_{\phi_1})_{\partial B_{\gamma}} \ll 1$. Now, considering (5) and (6), from (13) we have

$$\begin{aligned} \int_{B_\gamma} d_1 c_p(\tau) x^{\frac{2}{m-1}} \frac{\partial^2 \xi_2}{\partial x^2} \eta_{\phi_1} dx &\leq \frac{2}{m-1} \int_{B_\gamma} d_1 c_p(\tau) x^{\frac{3-m}{m-1}} \left| \frac{\partial \xi_2}{\partial x} \right| dx + \int_{B_\gamma} d_1 c_p(\tau) x^{\frac{2}{m-1}} \left| \frac{\partial \xi_2}{\partial x} \right| \frac{a_1}{\phi_1} dx \\ &\leq \frac{2}{m-1} d_1 c_p(\tau) \int_{B_\gamma} x^{\frac{3-m}{m-1}} \left| \frac{\partial \xi_2}{\partial x} \right| dx + d_1 c_p(\tau) a_1 \int_{B_\gamma} x^{\frac{2}{m-1}-1} \left| \frac{\partial \xi_2}{\partial x} \right| dx. \end{aligned} \quad (14)$$

Now considering (11) and using (5) and (6), the following assessments hold:

$$\epsilon_1 \int_{B_\gamma} c_p(\tau) x^{\frac{2}{m-1}} \frac{\partial \xi_2}{\partial x} \eta_{\phi_1} dx \leq \epsilon_1 c_p(\tau) \int_{B_\gamma} x^{\frac{2}{m-1}} \left| \frac{\partial \xi_2}{\partial x} \right| dx; \quad (15)$$

$$\begin{aligned} \int_{B_\gamma} c_p(\tau) x^{\frac{2m}{m-1}} \xi_2 \frac{\partial^2 \eta_{\phi_2}}{\partial x^2} dx &\leq c_p(\tau) \int_{B_\gamma} x^{\frac{2m}{m-1}} \left| \frac{a_1 \xi_2}{\phi_2^2} \right| dx \\ &\leq a_1 c_p(\tau) \int_{B_\gamma} x^{\frac{2m}{m-1}-2} |\xi_2| dx; \end{aligned} \quad (16)$$

$$\begin{aligned} 2 \int_{B_\gamma} c_p(\tau) x^{\frac{2m}{m-1}} \frac{\partial \xi_2}{\partial x} \frac{\partial \eta_{\phi_2}}{\partial x} dx &\leq 2 c_p(\tau) \int_{B_\gamma} x^{\frac{2m}{m-1}} \left| \frac{\partial \xi_2}{\partial x} \right| \left| \frac{a_1}{\phi_2} \right| dx \\ &\leq 2 a_1 c_p(\tau) \int_{B_\gamma} x^{\frac{2m}{m-1}-1} \left| \frac{\partial \xi_2}{\partial x} \right| dx; \end{aligned} \quad (17)$$

$$\begin{aligned} \epsilon_1 \int_{B_\gamma} c_p(\tau) x^{\frac{2}{m-1}} \xi_2 \frac{\partial \eta_{\phi_2}}{\partial x} dx &\leq \epsilon_1 \int_{B_\gamma} c_p(\tau) x^{\frac{2}{m-1}} \xi_2 \left| \frac{\partial \xi_2}{\partial x} \right| \left| \frac{a_1}{\phi_2} \right| dx \\ &\leq \epsilon_1 \int_{B_\gamma} c_p(\tau) x^{\frac{2}{m-1}-1} \xi_2 \left| \frac{\partial \xi_2}{\partial x} \right| dx. \end{aligned} \quad (18)$$

Next, consider the following test function:

$$\xi_2(x, s) = e^{g(s)} (1 + x^2)^{-\mu}, \quad (19)$$

where $g(s) < 0$ and $\mu > 0$ are constants specified in such a way that the integrals in (11) to (18) converge as $\gamma \rightarrow \infty$. Differentiating (19) with regards to x , we get

$$\begin{aligned} \frac{\partial \xi_2}{\partial x} &= \frac{-2\mu e^{g(s)}}{(1 + x^2)^{\mu+1}} x \\ &\leq 2\mu e^{g(s)} x^{-2\mu-1}. \end{aligned} \quad (20)$$

After using (20), the integral in (11) can be obtained as follows:

$$\begin{aligned} d_1 \int_{B_\gamma} c_p(\tau) x^{\frac{2}{m-1}} \frac{\partial^2 \xi_2}{\partial x^2} \eta_{\phi_1} dx &\leq \frac{4d_1\mu}{m-1} c_p(\tau) \int_{B_\gamma} e^{g(s)} x^{\frac{3-m}{m-1}-2\mu-1} dx \\ &\quad + 2\mu a_1 d_1 c_p(\tau) \int_{B_\gamma} e^{g(s)} x^{\frac{2}{m-1}-2\mu-2} dx. \end{aligned} \quad (21)$$

The above integral is converges if $\frac{2}{m-1} - 2\mu - 2 < -1$, which implies that

$$\mu > \frac{3-m}{2(m-1)}. \quad (22)$$

Similarly, after using (20), the integral in (15) is solved as follows:

$$\epsilon_1 \int_{B_\gamma} c_p(\tau) x^{\frac{2}{m-1}} \frac{\partial \xi_2}{\partial x} \eta_{\phi_1} dx \leq 2\mu \epsilon_1 c_p(\tau) \int_{B_\gamma} e^{g(s)} x^{\frac{2}{m-1}-2\mu-1} dx. \quad (23)$$

For the convergence of the above integral, we take $\frac{2}{m-1} - 2\mu - 1 < -1$ and we get

$$\mu > \frac{1}{2(m-1)}. \quad (24)$$

Also, consider

$$\begin{aligned} - \int_\tau^t \int_{B_\gamma} u(1-u) \xi_2 \eta_{\phi_2} dx ds &\leq \int_\tau^t \int_{B_\gamma} u(1-u) \xi_2 \eta_{\phi_2} dx \\ &\leq \int_\tau^t \int_{B_\gamma} u \xi_2 \eta_{\phi_2} dx \\ &\leq \int_{B_\gamma} c_p(\tau) x^{\frac{2}{m-1}} \xi_2 \eta_{\phi_2} dx \\ &\leq \int_{B_\gamma} c_p(\tau) e^{g(s)} x^{\frac{2}{m-1}-2\mu} dx. \end{aligned} \quad (25)$$

The convergence is achieved if we take $\frac{2}{m-1} - 2\mu < -1$; therefore we obtain

$$\mu > \frac{1+m}{2(m-1)}. \quad (26)$$

After using (19), the integral in (11) can be solved as follows:

$$\begin{aligned} \int_\tau^t \int_{B_\gamma} u \frac{\partial \xi_2}{\partial s} \eta_{\phi_2} dx ds &\leq \int_{B_\gamma} c_p(\tau) x^{\frac{2}{m-1}} e^{g(s)} g'(s) \frac{1}{(1+x^2)^\mu} dx \\ &\leq \int_{B_\gamma} c_p(\tau) x^{\frac{2}{m-1}} e^{g(s)} g'(s) x^{-2\mu} dx \\ &= c_p(\tau) \int_{B_\gamma} e^{g(s)} g'(s) x^{\frac{2}{m-1}-2\mu} dx. \end{aligned} \quad (27)$$

The integral converges if we take the same value of μ as that given in (26). After using (19) the integral in (11) can be solved as follows:

$$\begin{aligned} \int_\tau^t \int_{B_\gamma} u \xi_2 \frac{\partial \eta_{\phi_2}}{\partial s} dx ds &\leq \int_{B_\gamma} c_p(\tau) x^{\frac{2}{m-1}} \frac{e^{g(s)}}{(1+x^2)^\mu} \left| \frac{\partial \eta_{\phi_2}}{\partial s} \right| dx \\ &\leq c_p(\tau) \int_{B_\gamma} x^{\frac{2}{m-1}} e^{g(s)} x^{-2\mu} \left| \frac{\partial \eta_{\phi_2}}{\partial s} \right| dx \\ &= c_p(\tau) \int_{B_\gamma} e^{g(s)} x^{\frac{2}{m-1}-2\mu} \left| \frac{\partial \eta_{\phi_2}}{\partial s} \right| dx. \end{aligned} \quad (28)$$

Similarly the integral converges if we take the same value of μ as that given in (26). After using (20) the integral in (18) can be solved as follows:

$$\epsilon_1 \int_{B_\gamma} c_p(\tau) x^{\frac{2}{m-1}-1} \xi_2 dx \leq \epsilon_1 \int_{B_\gamma} c_p(\tau) x^{\frac{2}{m-1}-1-2\mu} dx. \quad (29)$$

The integral converges if we take the same value of μ as that given in (24). After using (20) the integral in (17) can be solved as follows:

$$2a_1 c_p(\tau) \int_{B_\gamma} x^{\frac{2m}{m-1}-1} \left| \frac{\partial \xi_2}{\partial x} \right| dx \leq 2a_1 c_p(\tau) \int_{B_\gamma} x^{\frac{2m}{m-1}-2\mu-2} dx. \quad (30)$$

The integral converges if we take the same value of μ as that given in (26). Also

$$\begin{aligned} \int_{\tau}^t \int_{B_{\gamma}} \frac{\alpha(1-m)uv}{\beta + (1-m)u} \xi_2 \eta_{\phi_2} dx ds &\leq \int_{\tau}^t \int_{B_{\gamma}} \frac{\alpha(1-m)uv}{(1-m)u} \xi_2 \eta_{\phi_2} dx ds \\ &\leq \int_{\tau}^t \int_{B_{\gamma}} \alpha v \xi_2 \eta_{\phi_2} dx ds \\ &\leq \int_{B_{\gamma}} \alpha c_q(\tau) x^{\frac{2}{m-1}-2\mu} \xi_2 \eta_{\phi_2} dx \\ &\leq \alpha c_q \int_{B_{\gamma}} e^{g(s)} x^{\frac{2}{m-1}-2\mu} dx. \end{aligned} \quad (31)$$

The integral converges if we take the same value of μ as that given in (26), and similarly for v . Note that, when $\gamma \gg 1$, then $(\frac{\partial \xi_2}{\partial x} \eta_{\phi_2})_{\partial B_{\gamma}} \ll 1$. Now, using (5), (6), (19) and (20), and after solving the integral in (12), we get

$$\int_{\tau}^t \int_{B_{\gamma}} v \frac{\partial \xi_2}{\partial s} \eta_{\phi_2} dx ds \leq c_q(\tau) \int_{B_{\gamma}} e^{g(s)} g'(s) x^{\frac{2}{m-1}-2\mu} dx. \quad (32)$$

The integral converges if we take the same value of μ as that given in (26).

$$\int_{\tau}^t \int_{B_{\gamma}} v \xi_2 \frac{\partial \eta_{\phi_2}}{\partial s} dx ds \leq c_q(\tau) \int_{B_{\gamma}} e^{g(s)} x^{\frac{2}{m-1}-2\mu} \left| \frac{\partial \eta_{\phi_2}}{\partial s} \right| dx. \quad (33)$$

The integral converges if we take the same value of μ as that given in (26).

$$\begin{aligned} d_2 \int_{\tau}^t \int_{B_{\gamma}} v \frac{\partial^2 \xi_2}{\partial x^2} \eta_{\phi_1} dx ds &\leq \frac{4\mu d_2}{m-1} c_q(\tau) \int_{B_{\gamma}} e^{g(s)} x^{\frac{3-m}{m-1}-2\mu-1} dx \\ &+ 2\mu a_1 d_2 c_q(\tau) \int_{B_{\gamma}} e^{g(s)} x^{\frac{2}{m-1}-2\mu-2} dx. \end{aligned} \quad (34)$$

The integral converges if we take the same value of μ as that given in (22).

$$2a_1 c_q(\tau) \int_{B_{\gamma}} x^{\frac{2m}{m-1}-1} \left| \frac{\partial \xi_2}{\partial x} \right| dx \leq 2a_1 c_q(\tau) \int_{B_{\gamma}} e^{g(s)} x^{\frac{2m}{m-1}-2\mu-2} dx. \quad (35)$$

The integral converges if we take the same value of μ as that given in (26).

$$-\epsilon_1 \int_{\tau}^t \int_{B_{\gamma}} v \xi_2 \frac{\partial \eta_{\phi_2}}{\partial x} dx ds \leq \epsilon_1 a_1 \int_{B_{\gamma}} c_q(\tau) e^{g(s)} x^{\frac{2}{m-1}-1-2\mu} dx. \quad (36)$$

The integral converges if we take the same value of μ as that given in (24).

$$-\epsilon_1 \int_{\tau}^t \int_{B_{\gamma}} v \frac{\partial \xi_2}{\partial x} \eta_{\phi_2} dx ds \leq 2\mu \epsilon_1 c_q(\tau) \int_{B_{\gamma}} e^{g(s)} x^{\frac{2}{m-1}-2\mu-1} dx. \quad (37)$$

The integral converges if we take the same value of μ as that given in (24). Also, consider

$$\begin{aligned} - \int_{\tau}^t \int_{B_{\gamma}} \delta_1 v \xi_2 \eta_{\phi_2} dx ds &\leq \int_{\tau}^t \int_{B_{\gamma}} \delta_1 v \xi_2 \eta_{\phi_2} dx ds \\ &\leq \delta_1 c_q(\tau) \int_{B_{\gamma}} e^{g(s)} x^{\frac{2}{m-1}-2\mu} dx. \end{aligned} \quad (38)$$

The integral converges if we take the same value of μ as that given in (26).

$$\begin{aligned}
 - \int_{\tau}^t \int_{B_{\gamma}} \frac{\gamma(1-m)uv}{\beta + (1-m)u} \xi_2 \eta_{\phi_2} dx ds &\leq \int_{\tau}^t \int_{B_{\gamma}} \frac{\gamma(1-m)uv}{(1-m)u} \xi_2 \eta_{\phi_2} dx ds \\
 &\leq \int_{\tau}^t \int_{B_{\gamma}} \gamma v \xi_2 \eta_{\phi_2} dx ds \\
 &\leq \int_{B_{\gamma}} \gamma c_q(\tau) x^{\frac{2}{m-1}} \xi_2 \eta_{\phi_2} dx \\
 &\leq \gamma(\tau) c_q \int_{B_{\gamma}} e^{g(s)} x^{\frac{2}{m-1}-2\mu} dx.
 \end{aligned} \quad (39)$$

The integral converges if we take the same value of μ as that given in (26). Now consider

$$\begin{aligned}
 - \int_{\tau}^t \int_{B_{\gamma}} \frac{v}{1+\rho v} \xi_2 \eta_{\phi_2} dx ds &\leq \int_{\tau}^t \int_{B_{\gamma}} \frac{v}{1+\rho v} \xi_2 \eta_{\phi_2} dx ds \\
 &\leq \int_{\tau}^t \int_{B_{\gamma}} \frac{v}{\rho v} \xi_2 \eta_{\phi_2} dx ds \\
 &\leq \frac{1}{\rho} \int_{B_{\gamma}} e^{g(s)} x^{-2\mu} dx.
 \end{aligned} \quad (40)$$

The integral is converges if $\mu > \frac{1}{2}$. So for a suitable value of μ , the expressions (7) and (8) become

$$\int_{\tau}^t \int_{B_{\gamma \rightarrow \infty}} u \xi_2 \eta_{\phi_2} dx ds \leq A_1, \quad (41)$$

and

$$\int_{\tau}^t \int_{B_{\gamma \rightarrow \infty}} v \xi_2 \eta_{\phi_2} dx ds \leq A_2. \quad (42)$$

Note that when $B_{\gamma \rightarrow \infty}$, then we can choose sufficiently small A_1 and A_2 . Since ξ_2 and η_{ϕ_2} are bounded, this allows us to establish the boundedness of solutions u and v in $B_{\gamma} \times [0, T]$. $\square$

In the next section, we show that the solution of the system is unique. Let (u_1, v_1) and (u_2, v_2) be the solutions of (4); then we show that $u_1 = u_2$ and $v_1 = v_2$. We introduce this result in the form of a theorem and we introduce the definitions of upper and lower solutions therein.

Theorem 2. Assume $u_1 > 0$ and $v_2 > 0$ are upper solutions of (4). Then both u_1 and v_1 coincide with the lower solutions $u_2 > 0$ and $v_2 > 0$, implying that the solutions are unique.

Proof. Assume (u_1, v_1) are the upper solutions of (4) in $\mathbb{R} \times [0, T]$; then such solutions are defined in accordance with the following problem:

$$\frac{\partial u_1}{\partial t} = d_1 \frac{\partial^2 u_1}{\partial x^2} + \epsilon_1 \frac{\partial u_1}{\partial x} + u_1 - u_1^2 - \frac{\alpha(1-m)u_1 v_1}{\beta + (1-m)u_1}, \quad (43)$$

and

$$\frac{\partial v_1}{\partial t} = d_2 \frac{\partial^2 v_1}{\partial x^2} + \epsilon_1 \frac{\partial v_1}{\partial x} + \frac{\gamma(1-m)u_1 v_1}{\beta + (1-m)u_1} - \delta_1 v_1 + \frac{v_1}{1 + \rho v_1}, \quad (44)$$

with

$$u_1(x, 0) = u_0(x) + \epsilon_2, \quad v_1(x, 0) = v_0(x) + \epsilon_2. \quad (45)$$

The variable ϵ_2 represents a positive perturbation added to the initial conditions to get the upper solutions. Also, the lower solutions (u_2, v_2) of (4) are given as solutions for the following problem:

$$\frac{\partial u_2}{\partial t} = d_1 \frac{\partial^2 u_2}{\partial x^2} + \epsilon_1 \frac{\partial u_2}{\partial x} + u_2 - u_2^2 - \frac{\alpha(1-m)u_2 v_2}{\beta + (1-m)u_2} \quad (46)$$

$$\frac{\partial v_2}{\partial t} = d_2 \frac{\partial^2 v_2}{\partial x^2} + \epsilon_1 \frac{\partial v_2}{\partial x} + \frac{\gamma(1-m)u_2 v_2}{\beta + (1-m)u_2} - \delta_1 v_2 + \frac{v_2}{1 + \rho v_2}, \quad (47)$$

with

$$u_2(x, 0) = u_0(x), \quad v_2(x, 0) = v_0(x). \quad (48)$$

For a test function ξ_2 , the following inequalities are satisfied:

$$\begin{aligned} 0 \leq \int_{\mathbb{R}} (u_1 - u_2)(x, t) \xi_2(x, t) dx &= \int_0^t \int_{\mathbb{R}} \left\{ (u_1 - u_2) \frac{\partial \xi_2}{\partial t} + d_1 (u_1 - u_2) \frac{\partial^2 \xi_2}{\partial x^2} \right. \\ &\quad + \epsilon_1 (u_1 - u_2) \frac{\partial \xi_2}{\partial x} + (u_1 - u_2) \xi_2 - (u_1^2 - u_2^2) \xi_2 \\ &\quad \left. + \left(\frac{\alpha(1-m)u_2 v_2}{\beta + (1-m)u_2} - \frac{\alpha(1-m)u_1 v_1}{\beta + (1-m)u_1} \right) \xi_2 \right\} dx ds, \end{aligned} \quad (49)$$

$$\begin{aligned} 0 \leq \int_{\mathbb{R}} (v_1 - v_2)(x, t) \xi_2(x, t) dx &= \int_0^t \int_{\mathbb{R}} \left\{ (v_1 - v_2) \frac{\partial \xi_2}{\partial t} + d_2 (v_1 - v_2) \frac{\partial^2 \xi_2}{\partial x^2} \right. \\ &\quad + \epsilon_1 (v_1 - v_2) \frac{\partial \xi_2}{\partial x} + \left(\frac{\gamma(1-m)u_1 v_1}{\beta + (1-m)u_1} - \frac{\gamma(1-m)u_2 v_2}{\beta + (1-m)u_2} \right) \xi_2 \\ &\quad \left. - \delta_1 (v_1 - v_2) \xi_2 + \left(\frac{v_1}{1 + \rho v_1} - \frac{v_2}{1 + \rho v_2} \right) \xi_2 \right\} dx ds. \end{aligned} \quad (50)$$

We define the following test function for the evaluation of integrals:

$$\xi(|x|, s) = e^{-Ks} (1 + |x|^2)^{-\mu}, \quad (51)$$

where $K > 0$ and $\mu > 0$ are constants and the following holds:

$$\frac{\partial \xi_2}{\partial s} = -K \xi_2(|x|, s), \quad \left| \frac{\partial \xi_2}{\partial x} \right| \leq k_1(\mu, d_1) \xi_2(|x|, s), \quad \left| \frac{\partial^2 \xi_2}{\partial x^2} \right| \leq k_2(\mu, d_1) \xi_2(|x|, s), \quad (52)$$

so that

$$\begin{aligned} &(u_1 - u_2) \frac{\partial \xi_2}{\partial t} + d_1 (u_1 - u_2) \left| \frac{\partial^2 \xi_2}{\partial x^2} \right| + \epsilon_1 (u_1 - u_2) \left| \frac{\partial \xi_2}{\partial x} \right| \\ &+ (u_1 - u_2) \xi_2 - (u_1 + u_2)(u_1 - u_2) \xi_2 + \left(\frac{\alpha(1-m)u_2 v_2}{\beta + (1-m)u_2} - \frac{\alpha(1-m)u_1 v_1}{\beta + (1-m)u_1} \right) \xi_2 \\ &\leq -(u_1 - u_2) K \xi_2 + d_1 (u_1 - u_2) k_2 \xi_2 + \epsilon_1 (u_1 - u_2) k_1 \xi_2 \\ &+ (1 - (u_1 + u_2))(u_1 - u_2) \xi_2 + \left(\frac{\alpha(1-m)u_2 v_2}{\beta + (1-m)u_2} - \frac{\alpha(1-m)u_1 v_1}{\beta + (1-m)u_1} \right) \xi_2, \end{aligned} \quad (53)$$

and

$$\begin{aligned}
& (v_1 - v_2) \frac{\partial \xi_2}{\partial s} + d_2(v_1 - v_2) \left| \frac{\partial^2 \xi_2}{\partial x^2} \right| + \epsilon_1(v_1 - v_2) \left| \frac{\partial \xi_2}{\partial x} \right| + \left(\frac{\gamma(1-m)u_1v_1}{\beta + (1-m)u_1} - \frac{\gamma(1-m)u_2v_2}{\beta + (1-m)u_2} \right) \xi_2 \\
& - \delta_1(v_1 - v_2) \xi_2 + \left(\frac{v_1}{1 + \rho v_2} - \frac{v_2}{1 + \rho v_1} \right) \xi_2 \\
& \leq -(v_1 - v_2) K \xi_2 + d_2(v_1 - v_2) k_2 \xi_2 + \epsilon_1(v_1 - v_2) k_1 \xi_2 \\
& + \left(\frac{\gamma(1-m)u_1v_1}{\beta + (1-m)u_1} - \frac{\gamma(1-m)u_2v_2}{\beta + (1-m)u_2} \right) \xi_2 \\
& + \delta_1(v_1 - v_2) \xi_2 + \left(\frac{v_2}{1 + \rho v_2} - \frac{v_1}{1 + \rho v_1} \right) \xi_2.
\end{aligned} \tag{54}$$

Now, K is chosen so that the following expressions are satisfied:

$$\begin{aligned}
& - (u_1 - u_2)K + d_1(u_1 - u_2)k_2(1 - (u_1 + u_2))(u_1 - u_2) \\
& + \left(\frac{\alpha(1-m)u_2v_2}{\beta + (1-m)u_2} - \frac{\alpha(1-m)u_1v_1}{\beta + (1-m)u_1} \right) \leq 0,
\end{aligned}$$

and

$$\begin{aligned}
& - (v_1 - v_2)K + d_2(v_1 - v_2)k_2 + \\
& + \left(\frac{\gamma(1-m)u_1v_1}{\beta + (1-m)u_1} - \frac{\gamma(1-m)u_2v_2}{\beta + (1-m)u_2} \right) \\
& + \delta_1(v_1 - v_2) + \left(\frac{v_2}{1 + \rho v_2} - \frac{v_1}{1 + \rho v_1} \right) \leq 0.
\end{aligned} \tag{55}$$

Now, (49) and (50) become

$$\int_{\mathbb{R}} (u_1 - u_2)(x, t) \xi_2(x, t) dx ds \leq \int_0^t \int_{\mathbb{R}} \epsilon_1(u_1 - u_2) \xi_2 dx ds, \tag{56}$$

and

$$\int_{\mathbb{R}} (v_1 - v_2)(x, t) \xi_2(x, t) dx ds \leq \int_0^t \int_{\mathbb{R}} \epsilon_1(v_1 - v_2) \xi_2 dx ds. \tag{57}$$

Differentiating w.r.t t we get

$$\frac{d}{dt} \int_{\mathbb{R}} (u_1 - u_2)(x, t) \xi_2(x, t) dx \leq \epsilon_1 \int_{\mathbb{R}} (u_1 - u_2) \xi_2 dx, \tag{58}$$

and

$$\frac{d}{dt} \int_{\mathbb{R}} (v_1 - v_2)(x, t) \xi_2(x, t) dx \leq \epsilon_1 \int_{\mathbb{R}} (v_1 - v_2) \xi_2 dx. \tag{59}$$

Applying Grownwall's inequality, we get

$$\int_{\mathbb{R}} (u_1 - u_2)(x, t) \xi_2(x, t) dx \leq 0, \tag{60}$$

and

$$\int_{\mathbb{R}} (v_1 - v_2)(x, t) \xi_2(x, t) dx \leq 0. \tag{61}$$

Since (u_1, v_1) are upper solutions and (u_2, v_2) are lower solutions, we have

$$\int_{\mathbb{R}} (u_1 - u_2)(x, t) \xi_2(x, t) dx = 0, \tag{62}$$

and

$$\int_{\mathbb{R}} (v_1 - v_2)(x, t) \xi_2(x, t) dx = 0. \quad (63)$$

Since ξ_2 is a test function, $u_1 - u_2 = 0$ and $v_1 - v_2 = 0$; i.e., $u_1 = u_2$ and $v_1 = v_2$, which shows uniqueness of the solutions of (4). $\square$

3. Stability Analysis

In this section we obtain the global stability for system (4) at equilibrium points $(1, 0)$ and $(0, \frac{1-\delta_1}{\rho\delta_1})$.

Theorem 3. *The equilibrium point $B_0 = (0, \frac{1-\delta_1}{\rho\delta_1})$ is globally asymptotically stable.*

Proof. Since u and v are zero at the boundary, we define the Lyapunov function as follows:

$$L_{B_0}(t) = \int_{\Omega} \left[B_2 u^2 + B_3 \left(v - \frac{1-\delta_1}{\rho\delta_1} \right)^2 \right] dx, \quad (64)$$

where $B_2 > 0$, and $B_3 > 0$ are constants. Differentiating (64) in t , we get

$$\frac{dL_{B_0}(t)}{dt} = \int_{\Omega} \left[2B_2 u \frac{\partial u}{\partial t} + 2B_3 \left(v - \frac{1-\delta_1}{\rho\delta_1} \right) \frac{\partial v}{\partial t} \right] dx. \quad (65)$$

After using (4), we obtain

$$\begin{aligned} \frac{dL_{B_0}(t)}{dt} &= \int_{\Omega} \left[2B_2 u \left(d_1 \frac{\partial^2 u}{\partial x^2} + \epsilon_1 \frac{\partial u}{\partial x} + u(1-u) - \frac{\alpha(1-m)uv}{\beta + (1-m)u} \right) \right. \\ &+ \left. 2B_3 \left(v - \frac{1-\delta_1}{\rho\delta_1} \right) \left(d_2 \frac{\partial^2 v}{\partial x^2} + \epsilon_1 \frac{\partial v}{\partial x} + \frac{\gamma(1-m)uv}{\beta + (1-m)u} - \delta_1 v + \frac{v}{1+\rho v} \right) \right] dx. \end{aligned} \quad (66)$$

Now employing the equilibrium point $(0, \frac{1-\delta_1}{\rho\delta_1})$ in (26), we get

$$\frac{dL_{B_0}(t)}{dt} = \int_{\Omega} \left[2B_2 d_1 u \frac{\partial^2 u}{\partial x^2} + 2B_3 d_2 \left(v - \frac{1-\delta_1}{\rho\delta_1} \right) \frac{\partial^2 v}{\partial x^2} \right] dx. \quad (67)$$

Since u and $\frac{\partial v}{\partial x}$ are zero at the boundary, the following expression holds:

$$\frac{dL_{B_0}(t)}{dt} = -2d_1 B_2 \int_{\Omega} \left| \frac{\partial u}{\partial x} \right|^2 dx - 2d_2 B_3 \int_{\Omega} \left| \frac{\partial v}{\partial x} \right|^2 dx, \quad (68)$$

which implies that $\frac{dL_{B_0}(t)}{dt} \leq 0$. So Theorem 3 is proved. $\square$

Theorem 4. *The equilibrium point $B_0 = (1, 0)$ is globally asymptotically stable.*

Proof. Since u and v are zero at the boundary, we define the Lyapunov function as follows:

$$L_{B_1}(t) = \int_{\Omega} \left[B_2 (u-1)^2 + B_3 v^2 \right] dx, \quad (69)$$

where $B_2 > 0$ and $B_3 > 0$ are constants. Differentiating (64) in t , we get

$$\frac{dL_{B_1}(t)}{dt} = \int_{\Omega} \left[2B_2 (u-1)^2 \frac{\partial u}{\partial t} + 2B_3 v \frac{\partial v}{\partial t} \right] dx. \quad (70)$$

After using (4), we obtain

$$\begin{aligned} \frac{dL_{B_1}(t)}{dt} &= \int_{\Omega} \left[2B_2(u-1) \left(d_1 \frac{\partial^2 u}{\partial x^2} + \epsilon_1 \frac{\partial u}{\partial x} + u(1-u) - \frac{\alpha(1-m)uv}{\beta + (1-m)u} \right) \right. \\ &\quad \left. + 2B_3v \left(d_2 \frac{\partial^2 v}{\partial x^2} + \epsilon_1 \frac{\partial v}{\partial x} + \frac{\gamma(1-m)uv}{\beta + (1-m)u} - \delta_1 v + \frac{v}{1+\rho v} \right) \right] dx. \end{aligned} \quad (71)$$

Now employing the equilibrium point $(1,0)$ in (71), we get

$$\frac{dL_{B_1}(t)}{dt} = \int_{\Omega} \left[2B_2d_1u \frac{\partial^2 u}{\partial x^2} + 2B_3d_2 \left(v - \frac{1-\delta_1}{\rho\delta_1} \right) \frac{\partial^2 v}{\partial x^2} \right] dx. \quad (72)$$

Since u and $\frac{\partial v}{\partial x}$ are zero at the boundary, the following expression holds:

$$\frac{dL_{B_1}(t)}{dt} = -2d_1B_2 \int_{\Omega} \left| \frac{\partial u}{\partial x} \right|^2 dx - 2d_1B_2 \int_{\Omega} u(1-u)^2 dx - 2d_2B_3 \int_{\Omega} \left| \frac{\partial v}{\partial x} \right|^2 dx, \quad (73)$$

which implies that $\frac{dL_{B_1}(t)}{dt} \leq 0$. So Theorem 4 is proved. $\square$

4. Profiles of Solution

The objective of this section is to explore the behavior of solutions to equations of (4) when $u(x, t)$ and $v(x, t)$ are constant at $x \rightarrow \infty$.

Theorem 5. Assume that $u(x, t)$ and $v(x, t)$ are solutions to (4), with $\epsilon_1 \gg \alpha_1$ and $\epsilon_1 \gg \alpha_3$; then it holds that

$$u(x, t) = |u(x, 0)| e^{-\frac{\epsilon_1}{d_1}|x|t^{\alpha_2}} + t^{-\alpha_1} \left[\alpha_1 + 1 - (\alpha_1 + 1) e^{-\frac{\epsilon_1}{d_1}|x|t^{\alpha_2}} \right],$$

and

$$v(x, t) = |v(x, 0)| e^{-\frac{\epsilon_1}{d_2}|x|t^{\alpha_4}} + t^{-\alpha_3} \left[-\frac{\alpha_3 - \delta_1 + 1}{\rho(\alpha_3 - \delta_1)} + \left(\frac{\alpha_3 - \delta_1 + 1}{\rho(\alpha_3 - \delta_1)} \right) e^{-\frac{\epsilon_1}{d_2}|x|t^{\alpha_4}} \right].$$

with

$$\alpha_1 = \alpha_3 = \frac{1}{2}, \quad \alpha_2 = \alpha_4 = -\frac{1}{2}.$$

Proof. To examine the solutions to (4), we introduce self-similar structures of the form

$$u(x, t) = t^{-\alpha_1} f(\zeta), \quad \zeta = |x|t^{\alpha_2}, \quad (74)$$

and

$$v(x, t) = t^{-\alpha_3} g(\eta), \quad \eta = |x|t^{\alpha_4}. \quad (75)$$

Substituting (74) and (75) into (4), we obtain

$$\begin{aligned} -\alpha_1 t^{-\alpha_1-1} f + \alpha_2 \zeta t^{-\alpha_1-1} f' &= d_1 t^{-\alpha_1+2\alpha_2} (f)'' + \epsilon_1 t^{-\alpha_1+\alpha_2} (f)' \\ &\quad + (t^{-\alpha_1} f) (1 - t^{-\alpha_1} f) - \frac{\alpha(1-m)(t^{-\alpha_1} f)(t^{-\alpha_3} g)}{\beta + (1-m)(t^{-\alpha_1} g)}, \end{aligned} \quad (76)$$

and

$$\begin{aligned}
 -\alpha_3 t^{-\alpha_3-1} g + \alpha_4 \eta t^{-\alpha_3-1} g' &= d_2 t^{-\alpha_3+2\alpha_4} (g)'' + \epsilon_1 t^{-\alpha_3+\alpha_4} (g)' \\
 &+ \frac{\gamma(1-m)(t^{-\alpha_1} f)(t^{-\alpha_3} g)}{\beta + (1-m)(t^{-\alpha_1} g)} - \delta_1 t^{-\alpha_3} g + \frac{t^{-\alpha_3} g}{1 + \rho t^{-\alpha_3} g}.
 \end{aligned} \quad (77)$$

Additionally, considering the energy conservation throughout the evolution,

$$\int_{\mathbb{R}} t^{-\alpha_1} f(|x|t^{\alpha_2}) dx = M_1, \quad (78)$$

and

$$\int_{\mathbb{R}} t^{-\alpha_3} g(|x|t^{\alpha_4}) dx = M_2, \quad (79)$$

Using (78),

$$|x| = \zeta t^{-\alpha_2}, \quad (80)$$

and using (79),

$$|x| = \eta t^{-\alpha_4}, \quad (81)$$

The differential unitary elements read

$$dx = d\zeta t^{-\alpha_2}, \quad (82)$$

and

$$dx = d\eta t^{-\alpha_4}, \quad (83)$$

so that

$$\int_{\mathbb{R}} t^{-\alpha_1} f(|x|t^{\alpha_2}) dx = t^{-\alpha_1-\alpha_2} \int_{\mathbb{R}} f(\zeta) d\zeta = M_1, \quad (84)$$

and

$$\int_{\mathbb{R}} t^{-\alpha_3} g(|x|t^{\alpha_4}) dx = t^{-\alpha_3-\alpha_4} \int_{\mathbb{R}} g(\eta) d\eta = M_2, \quad (85)$$

where M_1 and M_2 are positive constants referring to the energy state of the predator–prey model. By invoking the principle of energy conservation across the domain, we assume that $-\alpha_1 - \alpha_2 = 0$ and $-\alpha_3 - \alpha_4 = 0$. This assumption ensures that the total energy remains constant over time, reflecting the conservative nature of the system. Consequently, we can determine the specific values of α_i ($i = 1, 2, 3, 4$) by analyzing the powers of the time variables in (76) and (77):

$$\begin{aligned}
 -\alpha_1 - 1 &= -\alpha_1 + 2\alpha_2 \\
 -\alpha_1 - \alpha_2 &= 0,
 \end{aligned} \quad (86)$$

and

$$\begin{aligned}
 -\alpha_3 - 1 &= -\alpha_3 + 2\alpha_4 \\
 -\alpha_3 - \alpha_4 &= 0.
 \end{aligned} \quad (87)$$

After solving we have

$$\alpha_1 = \alpha_3 = \frac{1}{2}, \quad \alpha_2 = \alpha_4 = -\frac{1}{2}. \quad (88)$$

For simplicity, we take $t = 1$ in (76) and (77), and we have

$$d_1(f)'' + \epsilon_1 f' - \alpha_2 \zeta f' + \alpha_1 f + f(1-f) - \frac{\alpha(1-m)fg}{\beta + (1-m)f} = 0, \quad (89)$$

and

$$d_2(g)'' + \epsilon_1 g' - \alpha_4 \eta g' + \alpha_3 g + \frac{\gamma(1-m)fg}{\beta + (1-m)f} - \delta_1 g + \frac{g}{1 + \rho g} = 0. \quad (90)$$

Choosing $\alpha = \gamma$ and after subtraction of (90) from (89), we have

$$d_1(f)'' + \epsilon_1(f)' - \alpha_2 \zeta f' + \alpha_1 f + f - f^2 - [d_2(g)'' + \epsilon_1(g)' - \alpha_4 \eta g' + \alpha_3 g - \delta_1 g + \frac{g}{1 + \rho g}] = 0, \quad (91)$$

which is possible if

$$d_1(f)'' + \epsilon_1(f)' - \alpha_2 \zeta f' + \alpha_1 f + f - f^2 = 0, \quad (92)$$

and

$$d_2(g)'' + \epsilon_1(g)' - \alpha_4 \eta g' + \alpha_3 g - \delta_1 g + \frac{g}{1 + \rho g} = 0. \quad (93)$$

For the exact solutions to (92) and (93), we take $\alpha_2 = -\alpha_1$ and $\alpha_4 = -\alpha_3$; then the above equations become

$$d_1(f)'' + \epsilon_1(f)' + \alpha_1(\zeta f)' + f - f^2 = 0, \quad (94)$$

and

$$d_2(g)'' + \epsilon_1(g)' + \alpha_3(\eta g)' - \delta_1 g + \frac{g}{1 + \rho g} = 0. \quad (95)$$

Using a perturbation technique with a small parameter d_1 to solve the above Equation (94), neglect the highest derivative term in the outer region:

$$(\epsilon_1 + \alpha_1 \zeta)(f)' + (\alpha_1 + 1)f - f^2 = 0, \quad (96)$$

As for the outer region, ζ is large and f is constant, so assuming slow variation ($f' = 0$), solving the algebraic equation we have a non-trivial solution, which is typically dominant:

$$f_{outer} = \alpha_1 + 1, \quad (97)$$

Now we focus on the boundary layer at $\zeta = 0$. Rescale ζ near the boundary layer

$$\zeta_1 = \frac{\zeta}{d_1} \quad (98)$$

Substitute into original equation and simplify for $d_1 \ll 1$:

$$F'' + \epsilon_1 F' = 0. \quad (99)$$

Solving for the inner region,

$$F(\zeta_1) = A + Be^{-\epsilon_1 \zeta_1}. \quad (100)$$

Now we have matching $\zeta_1 \rightarrow \infty$, $F(\zeta_1) \rightarrow f_{outer}$, so we get $A = \alpha_1 + 1$, and from the boundary condition at $\zeta = 0$, we get $F(0) = f(0)$. Therefore, we obtain $B = f(0) - (\alpha_1 + 1)$. So the inner solution in terms of ζ can be obtained:

$$f_{inner} = \alpha_1 + 1 + (f(0) - (\alpha_1 + 1))e^{-\frac{\epsilon_1 \zeta}{d_1}}. \quad (101)$$

Now for the composite solution:

$$f_{comp}(\zeta) = f_{outer} + f_{inner} - f_{overlap}. \quad (102)$$

After computing we get

$$f(\zeta) = (\alpha_1 + 1) + [f(0) - (\alpha_1 + 1)]e^{-\frac{\epsilon_1 \zeta}{d_1}}. \quad (103)$$

Similarly solving (95) as we solved (94), we get

$$g(\eta) = -\frac{\alpha_3 - \delta_1 + 1}{\rho(\alpha_3 - \delta_1)} + \left[g(0) + \frac{\alpha_3 - \delta_1 + 1}{\rho(\alpha_3 - \delta_1)} \right] e^{-\frac{\epsilon_1 \eta}{d_2}}. \quad (104)$$

After substituting (103) and (104) into (74) and (75), we obtain

$$u(x, t) = |u(x, 0)|e^{-\frac{\epsilon_1}{d_1}|x|t^{\alpha_2}} + t^{-\alpha_1} \left[\alpha_1 + 1 - (\alpha_1 + 1)e^{-\frac{\epsilon_1}{d_1}|x|t^{\alpha_2}} \right],$$

and

$$v(x, t) = |v(x, 0)|e^{-\frac{\epsilon_1}{d_2}|x|t^{\alpha_4}} + t^{-\alpha_3} \left[-\frac{\alpha_3 - \delta_1 + 1}{\rho(\alpha_3 - \delta_1)} + \left(\frac{\alpha_3 - \delta_1 + 1}{\rho(\alpha_3 - \delta_1)} \right) e^{-\frac{\epsilon_1}{d_2}|x|t^{\alpha_4}} \right],$$

as we intended to prove.

These solutions show that if we take an advection coefficient greater than 0.5 then the ecosystem has an initially heterogeneous state, and after an increase in time then the ecosystem becomes stable. The prey and predator populations slowly decrease but keep positive and smooth in the sense of spatial patterns, and the predator population is preserved by supplementary food. So the ecological process is realistic. $\square$

5. Stationary Solutions

The coming idea consists of exploring upper stationary solution to (4). For this, we first look at upper solutions to u and v :

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_1 \frac{\partial^2 u}{\partial x^2} + \epsilon_1 \frac{\partial u}{\partial x} + u(1 - u) - \frac{\alpha(1 - m)uv}{\beta + (1 - m)u} \\ &\leq d_1 \frac{\partial^2 u}{\partial x^2} + \epsilon_1 \frac{\partial u}{\partial x} + u(1 - u) + \frac{\alpha(1 - m)uv}{(1 - m)u} \\ &\leq d_1 \frac{\partial^2 u}{\partial x^2} + \epsilon_1 \frac{\partial u}{\partial x} + u(1 - u) + \alpha v \\ &\leq d_1 \frac{\partial^2 u}{\partial x^2} + \epsilon_1 \frac{\partial u}{\partial x} + u(1 - u) + \alpha \tilde{K}_1 u, \end{aligned} \quad (105)$$

where $\tilde{K}_1 u = \sup\{v\}$ and

$$\begin{aligned}\frac{\partial v}{\partial t} &= d_2 \frac{\partial^2 v}{\partial x^2} + \epsilon_1 \frac{\partial v}{\partial x} + \frac{\gamma(1-m)uv}{\beta + (1-m)u} - \delta_1 v + \frac{v}{1+\rho v} \\ &\leq d_2 \frac{\partial^2 v}{\partial x^2} + \epsilon_1 \frac{\partial v}{\partial x} + \frac{\gamma(1-m)uv}{(1-m)u} - \delta_1 v + \frac{v}{\rho v} \\ &\leq d_2 \frac{\partial^2 v}{\partial x^2} + \epsilon_1 \frac{\partial v}{\partial x} + \gamma v - \delta_1 v + \frac{1}{\rho}.\end{aligned}\quad (106)$$

So, we have

$$\frac{\partial u}{\partial t} = d_1 \frac{\partial^2 u}{\partial x^2} + \epsilon_1 \frac{\partial u}{\partial x} + (\alpha \tilde{K}_1 + 1)u - u^2, \quad (107)$$

and

$$\frac{\partial v}{\partial t} = d_2 \frac{\partial^2 v}{\partial x^2} + \epsilon_1 \frac{\partial v}{\partial x} + (\gamma - \delta_1)v + \frac{1}{\rho}. \quad (108)$$

For the stationary upper solution, the above equations can be written as follows:

$$d_1 \frac{d^2 u}{dx^2} + \epsilon_1 \frac{du}{dx} + (\alpha \tilde{K}_1 + 1)u - u^2 = 0, \quad (109)$$

and

$$d_2 \frac{d^2 v}{dx^2} + \epsilon_1 \frac{dv}{dx} + (\gamma - \delta_1)v + \frac{1}{\rho} = 0. \quad (110)$$

To investigate the solution, we use the definition of a Hamiltonian (see [21]) so that

$$\frac{1}{2}d_1(u')^2 + \epsilon_1 u + \frac{(\alpha \tilde{K}_1 + 1)u^2}{2} - \frac{u^3}{3} = 0, \quad (111)$$

and

$$\frac{1}{2}d_2(v')^2 + \epsilon_1 v + \frac{(\gamma - \delta_1)v^2}{2} + \frac{v}{\rho} = 0, \quad (112)$$

which implies that

$$d_1(u')^2 + 2\epsilon_1 u + (\alpha \tilde{K}_1 + 1)u^2 - \frac{2}{3}u^3 = 0, \quad (113)$$

and

$$d_2(v')^2 + 2\epsilon_1 v + (\gamma - \delta_1)v^2 + \frac{2v}{\rho} = 0. \quad (114)$$

For solving (113), we use Lighthill's technique with small parameter ϵ_1 ; for this we introduce strained coordinates

$$x = s + \epsilon_1 w_1(s) + \epsilon_1^2 w_2(s) + \dots \text{ and } \frac{d}{dx} = (1 - \epsilon_1 \frac{dw_1}{ds} + \dots) \frac{d}{ds}. \quad (115)$$

Substituting this into (113), we have

$$d_1(1 - \epsilon_1 \frac{dw_1}{ds} + \dots)^2 \left(\frac{du}{ds}\right)^2 + 2\epsilon_1 u + (\alpha \tilde{K}_1 + 1)u^2 - \frac{2}{3}u^3 = 0. \quad (116)$$

Now we assume that

$$u(s) = u_0(s) + \epsilon_1 u_1(s) + \epsilon_1^2 u_2(s) + \dots \quad (117)$$

After substituting (117) into (116) and comparing the power of ϵ_1 , we have

$$d_1 \left(\frac{du_0}{ds} \right)^2 + (\alpha \tilde{K}_1 + 1) u_0^2 - \frac{2}{3} u_0^3 = 0, \quad (118)$$

and

$$2d_1 \frac{du_0}{ds} \frac{du_1}{ds} - 2d_1 \frac{dw_1}{ds} \left(\frac{du_0}{ds} \right)^2 + 2u_0 + 2(\alpha \tilde{K}_1 + 1) u_0 u_1 - 2u_0^2 u_1 = 0. \quad (119)$$

After solving (118), we get

$$u_0 = \frac{3}{2} (\alpha \tilde{K}_1 + 1) \sec^2 \left(\frac{1}{2} \sqrt{\frac{\alpha \tilde{K}_1 + 1}{d_1}} s \right). \quad (120)$$

After substituting (120) into (119) and then choosing

$$w_1 = \frac{\sqrt{d_1}}{(\alpha \tilde{K}_1 + 1)^{\frac{5}{2}}} \left[-\frac{4}{3} \cot \left(\frac{1}{2} \sqrt{\frac{\alpha \tilde{K}_1 + 1}{d_1}} s \right) + 3 \sin \left(\sqrt{\frac{\alpha \tilde{K}_1 + 1}{d_1}} s \right) \right] - \frac{s}{(\alpha \tilde{K}_1 + 1)^2}, \quad (121)$$

we obtain

$$\frac{du_1}{ds} + \sqrt{\frac{\alpha \tilde{K}_1 + 1}{d_1}} \left[\frac{\cos \left(\frac{1}{2} \sqrt{\frac{\alpha \tilde{K}_1 + 1}{d_1}} s \right)}{\sin \left(\frac{1}{2} \sqrt{\frac{\alpha \tilde{K}_1 + 1}{d_1}} s \right)} - \frac{3 \sec^2 \left(\frac{1}{2} \sqrt{\frac{\alpha \tilde{K}_1 + 1}{d_1}} s \right)}{2 \tan \left(\frac{1}{2} \sqrt{\frac{\alpha \tilde{K}_1 + 1}{d_1}} s \right)} \right] u_1 = 0. \quad (122)$$

After solving we get

$$u_1(s) = \frac{\left(\sin \left(\frac{1}{2} \sqrt{\frac{\alpha \tilde{K}_1 + 1}{d_1}} s \right) \right)}{\left(\cos \left(\frac{1}{2} \sqrt{\frac{\alpha \tilde{K}_1 + 1}{d_1}} s \right) \right)^3}. \quad (123)$$

Putting (120) and (123) into (117), we get

$$u(s) = \frac{3}{2} (\alpha \tilde{K}_1 + 1) \sec^2 \left(\frac{1}{2} \sqrt{\frac{\alpha \tilde{K}_1 + 1}{d_1}} s \right) + \epsilon_1 \frac{\left(\sin \left(\frac{1}{2} \sqrt{\frac{\alpha \tilde{K}_1 + 1}{d_1}} s \right) \right)}{\left(\cos \left(\frac{1}{2} \sqrt{\frac{\alpha \tilde{K}_1 + 1}{d_1}} s \right) \right)^3} + O(\epsilon_1^2), \quad (124)$$

and

$$x = s + \epsilon_1 \left[\frac{\sqrt{d_1}}{(\alpha \tilde{K}_1 + 1)^{\frac{5}{2}}} \left[-\frac{4}{3} \cot \left(\frac{1}{2} \sqrt{\frac{\alpha \tilde{K}_1 + 1}{d_1}} s \right) + 3 \sin \left(\sqrt{\frac{\alpha \tilde{K}_1 + 1}{d_1}} s \right) \right] - \frac{s}{(\alpha \tilde{K}_1 + 1)^2} \right] + O(\epsilon_1^2), \quad (125)$$

where $s \neq \frac{n\pi\sqrt{d_1}}{(\sqrt{\alpha \tilde{K}_1 + 1})}$. Solving (114) for $\delta_1 > \gamma$ we have

$$d_2 (v')^2 + 2(\epsilon_1 + \frac{1}{\rho}) v - (\delta_1 - \gamma) v^2 = 0, \quad (126)$$

which implies that

$$v' = \sqrt{\frac{\delta_1 - \gamma}{d_2}} \left[\sqrt{v^2 - \frac{2(\epsilon_1 + \frac{1}{\rho})v}{(\delta_1 - \gamma)}} \right]. \quad (127)$$

After solving, we get

$$v = \frac{\epsilon_1 + \frac{1}{\rho}}{\delta_1 - \gamma} + \frac{1}{2} \left[e^{\sqrt{\frac{\delta_1 - \gamma}{d_2}} x} + \left(\frac{\epsilon_1 + \frac{1}{\rho}}{\delta_1 - \gamma} \right)^2 e^{-\sqrt{\frac{\delta_1 - \gamma}{d_2}} x} \right]. \quad (128)$$

These results show that if the direct movement of prey and predator populations are small then the spatial distribution of these populations are high in the main region, and away from this area the prey and predator populations decrease. This slight shift in populations does not change the balance of the system. So the ecosystem remains stable in the sense of spatial patterns.

6. Asymptotic Profiles of Solutions

In this section, we obtain the nonstationary upper solutions of (4). To this end, we consider nonlinear point scaling to develop the solution profile of (107) and (108):

$$u(t, x) = e^{z_2(t, x)}, \quad (129)$$

and

$$v(t, x) = e^{z_3(t, x)}. \quad (130)$$

This type of scaling has been extensively utilized (see [20,22–25]). Hence, u and v satisfy the following Hamilton–Jacobi-type equations:

$$e^{z_2} \frac{\partial z_2}{\partial t} = d_1 e^{z_2} \left(\frac{\partial z_2}{\partial x} \right)^2 + d_1 \left(\frac{\partial^2 z_2}{\partial x^2} \right) e^{z_2} + \epsilon_1 e^{z_2} \frac{\partial z_2}{\partial x} + (1 + \alpha \tilde{K}_1) e^{z_2} - e^{2z_2}, \quad (131)$$

and

$$e^{z_3} \frac{\partial z_3}{\partial t} = d_2 e^{z_3} \left(\frac{\partial z_3}{\partial x} \right)^2 + d_2 \left(\frac{\partial^2 z_3}{\partial x^2} \right) e^{z_3} + \epsilon_1 e^{z_3} \frac{\partial z_3}{\partial x} + (\gamma - \delta_1) e^{z_3} + \frac{1}{\rho}, \quad (132)$$

where $\left| \frac{\partial^2 z_2}{\partial x^2} \right| \ll \left| \frac{\partial z_2}{\partial x} \right|$ and $\left| \frac{\partial^2 z_3}{\partial x^2} \right| \ll \left| \frac{\partial z_3}{\partial x} \right|$. Also $z_2(t, x) \rightarrow 0$ and $z_3(t, x) \rightarrow 0$ at $x \rightarrow \infty$, so $e^{z_2(t, x)} = 1$ and $e^{z_3(t, x)} = 1$. This idea is taken from Galaktionov and Williams [22] (see Proposition 3.1 in this reference), which simplifies to

$$\frac{\partial z_2}{\partial t} = d_1 \left(\frac{\partial z_2}{\partial x} \right)^2 + \epsilon_1 \frac{\partial z_2}{\partial x} + (1 + \alpha \tilde{K}_1), \quad (133)$$

and

$$\frac{\partial z_3}{\partial t} = d_2 \left(\frac{\partial z_3}{\partial x} \right)^2 + \epsilon_1 \frac{\partial z_3}{\partial x} + (\gamma - \delta_1) + \frac{1}{\rho}. \quad (134)$$

We assume

$$z_4(t, x) = z_2 - (1 + \alpha \tilde{K}_1)t, \quad (135)$$

and

$$z_5(t, x) = Z_3 - \left((\gamma - \delta_1) + \frac{1}{\rho} \right) t. \quad (136)$$

After substituting into (133) and (134), we get

$$\frac{\partial z_4}{\partial t} = d_1 \left(\frac{\partial z_4}{\partial x} \right)^2 + \epsilon_1 \frac{\partial z_2}{\partial x}, \quad (137)$$

and

$$\frac{\partial z_5}{\partial t} = d_2 \left(\frac{\partial z_5}{\partial x} \right)^2 + \epsilon_1 \frac{\partial z_5}{\partial x}. \quad (138)$$

For the solution of (137) and (138), we assume $\zeta = x - at$, so the above equations become

$$d_1 \left(\frac{dz_4}{d\zeta} \right)^2 + (\epsilon_1 + a) \frac{dz_4}{d\zeta} = 0, \quad (139)$$

and

$$d_2 \left(\frac{dz_5}{d\zeta} \right)^2 + (\epsilon_1 + a) \frac{dz_5}{d\zeta} = 0. \quad (140)$$

After solving (139) in (140), we get

$$Z_4 = -(1 + \alpha \bar{K}_1)t - \frac{(\epsilon_1 + a)}{d_1} \zeta, \quad (141)$$

and

$$Z_5 = -\left((\gamma - \delta_1) + \frac{1}{\rho} \right) t - \frac{(\epsilon_1 + a)}{d_2} \zeta, \quad (142)$$

where $\frac{(\epsilon_1 + a)}{d_2}$ is small and we can choose $\frac{(\epsilon_1 + a)}{d_1} = e^{-\zeta^2}$ and $\frac{(\epsilon_1 + a)}{d_2} = e^{-\zeta^2}$, which approach zero when $\zeta \rightarrow \infty$. Putting z_4 and z_5 in (135) and (136), we get

$$Z_2 = -\frac{(\epsilon_1 + a)}{d_1} (x - at), \quad (143)$$

and

$$Z_3 = -\frac{(\epsilon_1 + a)}{d_2} (x - at), \quad (144)$$

Finally substituting (143) and (134) into (129) and (130), we get

$$u(t, x) = e^{-\frac{(\epsilon_1 + a)}{d_1} (x - at)}. \quad (145)$$

Similarly, solving (134), we have

$$v(t, x) = e^{-\frac{(\epsilon_1 + a)}{d_2} (x - at)}. \quad (146)$$

These solutions show the migration of prey and predator populations with a constant speed. If we take a large diffusion coefficient the populations spread slowly and widely, and if we take a small diffusion coefficient then this spread is quick. The populations slowly spreading is much more realistic than a quick spread in the ecosystem.

7. Conclusions

In this work, we present a model for describing the dynamics of the predator–prey model with the Holling type II interaction function with diffusion and advection effects. The diffusion effect means the smooth spread of prey and predator populations in space over the time, and advection effect means direct movement of prey and predator populations in space. Firstly, we demonstrate that model (4) exhibits boundedness, uniqueness and stability of solutions. For stability, we use the Lyapunov function and show the global stability at equilibrium points. Subsequently, we obtain the value of the advection coefficient, which is greater than 0.5. With these values, the ecosystem starts in a heterogeneous state and after some time it becomes stable. To obtain these results, we use the singular perturbation technique. Moreover we find that if the advection coefficient is small, this means less direct movement and then the populations in the main region are high, and away from that region populations decrease. To obtain this result, we use the definition of a Hamiltonian. Lastly, we show that for a large diffusion coefficient, the population spreads slowly and wildly, and for a small diffusion coefficient, the population spreads quickly. For this, we use nonlinear point scaling. For future work we can use the p-Laplacian operator instead of diffusion because if the populations are not spread smoothly then the p-Laplacian gives better results compared to diffusion.

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