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Pseudospectra in Banach Jordan Algebras

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Abstract

The primary focus of this paper is to extend the concept of pseudospectrum from operators and matrices to elements of a unital complex Banach Jordan algebra, thereby moving from the associative to the non-associative setting. We introduce the notion of ε-invertibility in a Banach Jordan algebra $\mathcal{J}$ and establish the invariance of pseudospectra with respect to full subalgebras of $\mathcal{J}$. We further investigate fundamental properties of the pseudospectrum of an element in a Banach Jordan algebra, including its relationship with level sets of analytic functions and pseudospectral bounds. The paper also examines linear maps that preserve pseudospectra in Banach Jordan algebras, as well as decomposition results for certain elements into simpler components within suitable localized subalgebras. Finally, we study an extension of the Roch–Silberman theorem in the setting of JB-algebras.

Keywords: Banach Jordan algebra; ϵ-invertibility; pseudospectrum; index of instability; level set function; spectrum preserving linear mapping

MSC: 46H70; 17A15

1. Introduction

1.1. Motivation from Operator Theory

In operator theory, the spectrum of a bounded or closed operator plays a central role in understanding resolvents, functional calculus, and long-time behavior of associated semigroups. For normal operators, the spectrum provides a robust and geometrically transparent description. However, many operators arising in analysis and mathematical physics are non-normal, and for such operators the spectrum alone may be unstable or misleading.

Pseudospectra were introduced to address this deficiency. They provide a natural extension of spectral theory that captures resolvent growth, spectral instability, and sensitivity to perturbations, phenomena that are ubiquitous in non-self-adjoint operator theory.

1.2. Motivational Questions, History, and Applications of Pseudospectra

The theory of pseudospectra arises from fundamental questions about the stability and reliability of spectral information for linear operators and matrices. Classical spectral theory focuses on eigenvalues as indicators of system behavior, yet in many important problems small perturbations—stemming from modeling errors, discretization, or noise can cause dramatic changes in spectral properties. This leads to the central motivational question behind pseudospectra: *how sensitive is the spectrum of an operator to perturbations, and how well do eigenvalues capture the true behavior of a system?* Closely related questions include understanding transient growth in dynamical systems, diagnosing non-normality, and explaining discrepancies between theoretical predictions and observed behavior.

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Historically, early ideas related to pseudospectra can be traced back to numerical analysis and operator theory in the mid-20th century, particularly in the study of resolvent norms and stability of differential equations. However, the modern theory of pseudospectra was systematically developed in the 1980s and 1990s, most notably through the work of Trefethen, Embree, and collaborators [1]. Their contributions emphasized the geometric and computational aspects of pseudospectra and demonstrated that for non-normal operators, pseudospectral sets can be far more informative than the spectrum alone. This work unified ideas from functional analysis, numerical linear algebra, and applied mathematics, transforming pseudospectra into a central tool for understanding operator behavior.

Pseudospectra have since found wide-ranging applications in mathematics and physics. In mathematics, they play a crucial role in numerical analysis by providing insight into the stability of algorithms, backward error analysis, and the conditioning of eigenvalue problems. They are also important in the study of partial differential equations, where non-self-adjoint operators arise naturally and spectral instability is common. In physics, pseudospectra are used to analyze hydrodynamic and plasma instabilities, quantum mechanics of non-Hermitian operators, wave propagation, and open systems with gain and loss. In these contexts, pseudospectra explain phenomena such as transient energy growth and unexpected system responses that cannot be predicted from eigenvalues alone.

Overall, pseudospectra offer a refined and robust framework for understanding linear operators under perturbations. By capturing both spectral location and sensitivity, they bridge the gap between idealized mathematical models and the complex behavior observed in real-world systems, making them indispensable in modern analysis and applications.

1.3. Pseudospectra of Normal and Non-Normal Operators

When an operator is not self-adjoint, its spectrum is not considered as the right object to study since it is very unstable under perturbation. This suggests that in many cases, the spectrum of an operator alone may not be enough to acquire a sufficient understanding of its behavior. Pseudospectra are sets in the complex plane that are designed to reveal such situations and give more information.

We let $T : \mathcal{H} \longrightarrow \mathcal{H}$ be a closed densely defined operator where $\mathcal{H}$ is a complex Hilbert space. For $\epsilon > 0$, we define the ϵ-pseudospectrum by

$$\mathrm{Sp}_\epsilon(T) = \mathrm{Sp}(T) \cup \left\{ z \in \mathbb{C} \setminus \mathrm{Sp}(T); \|(z\mathbf{1} - T)^{-1}\| > \frac{1}{\epsilon} \right\}, \tag{1}$$

where $\mathrm{Sp}(T)$ denotes the spectrum of T. The main idea about these new subsets is that it is interesting to study not only the points where the resolvent of an operator is not defined (i.e., its spectrum) but also where this resolvent is large in norm. There exists an abundance of literature on pseudospectra. An extensive discussion of theorems and applications together with a review of the history of pseudospectra can be found in [2].

It was shown in [3] that studying the pseudospectra of an operator is exactly studying the level lines of the norm of its resolvent. Studying such level lines gives some information about the spectral stability of the operator. Interestingly, pseudospectra can be defined in an equivalent way in terms of spectra of perturbations of the operator. For instance, we have for any matrix $\mathbf{m} \in \mathcal{M}_n(\mathbb{C})$,

$$\mathrm{Sp}_\epsilon(\mathbf{m}) = \{z \in \mathbb{C} : z \in \mathrm{Sp}(\mathbf{m} + \mathbf{p}), \text{ where } \mathbf{p} \in \mathcal{M}_n(\mathbb{C}), \text{ with } \| \mathbf{p} \| \leq \epsilon\}.$$

Thus, a complex number z belongs to the ϵ-pseudospecrum of a matrix $\mathbf{m}$, if and only if it belongs to the spectrum of one of its perturbations $\mathbf{m} + \mathbf{p}$ with $\| \mathbf{p} \| \leq \epsilon$.

More generally, if T is a closed unbounded linear operator with a dense domain on a complex Hilbert space $\mathcal{H}$, Proposition 4.15 [4] states that pseudospectra can also be defined in terms of perturbation of the spectrum

$$\mathrm{Sp}_\epsilon(T) = \bigcup_{S\in\mathcal{B}(\mathcal{H}),\, \|S\|<\epsilon} \mathrm{Sp}(T+S), \tag{2}$$

where $\mathcal{B}(\mathcal{H})$ stands for the Banach algebra of bounded linear operators on $\mathcal{H}$. Thus, a complex number z is in ϵ-pseudospectrum of T, if and only if it is in the spectrum of some perturbed operator $T+S$ with $\| S \| < \epsilon$. This shows that Sp_ϵ is a region of instability. The importance of this formula is summarized in the following statement: Very large pseudospectra are always associated with eigenvalues, which are very instable with respect to perturbations. Obviously, this is of great importance to numerical analysts [5]: if a spectral problem is unstable enough, no numerical procedure can enable one to find the eigenvalues, whose significance therefore becomes a moot point.

When T is a closed unbounded normal linear operator with a dense domain on a complex Hilbert space $\mathcal{H}$, an exact expression for the norm of its resolvent is given in [6] as

$$\| (z\mathbf{1} - T)^{-1} \| = \frac{1}{d(z, \mathrm{Sp}(T))}, \ \forall z \notin \mathrm{Sp}(T) \tag{3}$$

where $d(z, \mathrm{Sp}(T)))$ stands for the distance between z and the spectrum of T.

This formula proves that the resolvent of a *normal* operator cannot blow up far from its spectrum. It ensures the stability of its spectrum under small perturbations because when T is self-adjoint or more generally normal, we have

$$\mathrm{Sp}_\epsilon(T) = \{z \in \mathbb{C} : d(z, \mathrm{Sp}(T)) \leq \epsilon\}, \tag{4}$$

but in general, $\mathrm{Sp}_\epsilon(T)$ is much larger than the set in the right hand side in the above equation.

Therefore, in the case of non-self-adjoint operators, we may lose the equality and have only

$$\|(z\mathbf{1} - T)^{-1}\| \geq \frac{1}{d(z, \mathrm{Sp}(T))}. \tag{5}$$

The next example, taken from [7], illustrates that the spectrum of non-self-adjoint operators is not sufficient for analyzing the behavior of the associated semigroup. A typical question that is easy in the self-adjoint case but difficult in the non-self-adjoint case is the following: if we consider a semi-bounded self-adjoint operator A on a Hilbert space $\mathcal{H}$, we define the operator e^{-tA} (i.e., $f(A)$ with $f(\lambda) = e^{-t\lambda}$) by the analytic functional calculus; thus, we immediately obtain

$$\|e^{-tA}\|_{\mathcal{L}(\mathcal{H})} = e^{-t \inf \mathrm{Sp}(A)}. \tag{6}$$

On the other hand, in the case where $\mathcal{H} = \mathbb{C}^2$ and we consider the non-symmetric matrix $\mathbf{m} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we have $\mathrm{Sp}(\mathbf{m}) = \{0\}$, but $e^{-t\mathbf{m}} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$ does not satisfy neither (3) nor (6). In fact, $\|(\mathbf{m} - z)^{-1}\|$ behaves like $\frac{1}{|z|^2}$ as $z \longrightarrow 0$.

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1.4. Motivational Questions, History, and Applications of Pseudospectra in JB and JB*-Algebras

The theory of pseudospectra originated in the study of non-normal operators and non-self-adjoint spectral problems in associative Banach and C^*-algebras, with major contributions in numerical analysis, perturbation theory, and operator semigroups (see, e.g., [1]). Traditionally, pseudospectra are defined using resolvent growth, $\sigma_\varepsilon(a) = \{\lambda \in \mathbb{C} : \|(a - \lambda 1)^{-1}\| > \varepsilon^{-1}\}$, where $a - \lambda 1$ is invertible in the ambient Banach algebra.

In the last decades, rich non-associative structures have emerged as natural generalizations of C^*-algebras, most notably JB algebras and JB*-algebras [8–10]. These algebras play a fundamental role in Jordan operator theory, the geometry of state spaces, and mathematical models of quantum mechanics. It is therefore natural to ask to what extent pseudospectral ideas and perturbation techniques extend from the associative setting to the Jordan framework.

Motivated by [11,12], our main aim in this paper is to extend the concepts of pseudospectrum to Banach Jordan algebras. To achieve this, we define ϵ-invertibility in Banach Jordan algebras and give the definitions of pseudospectrum of an element in a Banach Jordan algebra. The elementary properties of the pseudospectrum were studied [13,14]. Because of non-associativity of multiplication, the proofs in the context of Banach Jordan algebras are not just easy modifications of associative ones; often, new proofs and arguments have to be found. In addition, there are also some differences in the statements of the results in the setting of non-associative complete normed algebras compared to their counterparts established for Banach algebras.

We first establish invariance of pseudospectra in full subalgebras (Theorem 1), followed by a summary of properties of the spectrum of an element in a Banach Jordan algebra (Theorem 3). Applying the analytical functional calculus to the pseudospectrum, we obtain a pseudospectral bound on the norm of the image of an element $a \in \mathcal{J}$ under analytic function (Theorem 6) and as a corollary we characterize scalar elements in a Banach Jordan algebra. Then we consider the level sets of functions

$$\Psi_a(z) = \|(a - z\mathbf{1})^{-1}\|^{-1}$$

which also determine the pseudospectrum of an element a of a Banach Jordan algebra. We observe that for real Jordan algebras, Ψ_a is semiconvex as defined in [15].

Pseudospectral technology is well established today and highly successful as a tool for diagnosing situations where eigenvalues may be misleading. When the eigenvalues fail, however, do pseudospectra provide the missing information? Are they a qualitatively helpful tool, or merely a warning device? Many results pertaining to this situation are collected i [2], §§ 14–19, but a general answer is not clear. For example, suppose A and B are two matrices with simple eigenvalues whose ϵ-pseudospectra are identical for all ϵ. Must $\|p(A)\| = \|p(B)\|$ for all polynomials p? Without the hypothesis of simple eigenvalues, the answer is no; see [2], §47. In Proposition 4, we consider the situation of two elements a and b in a Banach Jordan algebra $\mathfrak{J}$ such that $\Psi_a(z) = \Psi_b(z)$.

A new concept of instability index of an isolated spectral value is defined and some of its general properties are outlined in the context of Banach Jordan algebras. Then, we turn to the study of linear maps preserving pseudospectra in Banach Jordan algebras, extending results similar to those in [14].

In the last part of the paper, we examine whether there exists a decomposition of the pseudospectrum $\mathrm{Sp}_\epsilon(\mathfrak{J}, x)$ of an element x in a Banach Jordan algebra $\mathfrak{J}$ in terms of pseudospectra $\mathrm{Sp}_\epsilon(U_p\mathfrak{J}, U_p x)$ and $\mathrm{Sp}_\epsilon(U_q\mathfrak{J}, U_q x)$, where p and q are orthogonal projections in $\mathfrak{J}$ (Theorem 12). Finally, we partially extend to JB algebras, an important theorem

previously proved by Roch–Silbermann in the context of C^*-algebras, stating that the pseudospectrum is a region of spectral instability.

Although pseudospectra are well understood in associative Banach and C^*-algebras, extending this theory to the Jordan setting is far from formal. The lack of associativity fundamentally alters the analytical landscape: arguments that rely implicitly on resolvent identities, multiplicativity, or functional calculus in associative algebras no longer apply verbatim. In particular, many properties that are routine in the associative case require new proofs or entirely different techniques when working with Jordan algebras. Our approach highlights the pivotal role of the quadratic operator U_x and the systematic use of strongly associative Jordan subalgebras as substitutes for associative tools. These methods allow us to recover pseudospectral stability and inversion estimates, but only after substantial conceptual and technical refinements. This contrast underscores that the Jordan framework is not merely a reformulation of known results, but a genuinely nontrivial extension in which pseudospectral control reveals new structural features absent from the associative theory.

2. Preliminaries

2.1. Jordan Algebras

A Jordan algebra $\mathfrak{J}$ is a non-associative algebra whose product $\cdot$ satisfies $a \cdot b = b \cdot a$ and $(a \cdot b) \cdot a^2 = a \cdot (b \cdot a^2)$ for all $a, b \in \mathfrak{J}$, where $a^2 = a \cdot a$. For $x \in \mathfrak{J}$, the operator of left multiplication L_x is defined by $L_x a := x \cdot a$ for all $a \in \mathfrak{J}$. The operator of right multiplication obviously coincides with L_x. As usual, $U_{a,b}$ denotes the operator on J defined by

$$U_{a,b}x = \{a, x, b\} = (L_a L_b + L_b L_a - L_{a \cdot b})x$$

and we set $U_a = U_{a,a}$ for all $a, b, x \in \mathfrak{J}$. The identity element (or the unit) in $\mathfrak{J}$ is an element $\mathbf{1}$ such that $a \cdot \mathbf{1} = a$ for each $a \in \mathfrak{J}$. We let $\mathfrak{J}^1$ denote $\mathfrak{J}$ if $\mathfrak{J}$ is unital, and the Jordan algebra $\mathfrak{J} \oplus \mathbb{C}$ obtained from J by adjoining the unit $\mathbf{1}$ otherwise. An element a in a unital Jordan algebra $\mathfrak{J}$ is invertible if there is $b \in \mathfrak{J}$ such that $a \cdot b = \mathbf{1}$ and $a^2 \cdot b = a$. The inverse is unique and denoted by a^{-1}. The set of all invertible elements of $\mathfrak{J}$ is identified by $\mathrm{Inv}(\mathfrak{J})$. An element $a \in \mathfrak{J}$ is invertible in $\mathfrak{J}$ if and only if U_a is invertible, and in this case $U_a^{-1} = U_{a^{-1}}$. The fundamental identity

$$U_{U_a b} = U_a U_b U_a$$

for all $a, b \in \mathfrak{J}$ shows that $U_a b$ is invertible if and only if a and b are invertible. Taking into account that $a^{-1} = U_{a^{-1}} a = U_a^{-1} a$ for each $a \in \mathrm{Inv}(\mathfrak{J})$, we see from the above that

$$\begin{aligned} (U_a b)^{-1} &= U_{U_a b}^{-1} U_a b \\ &= U_{a^{-1}} U_{b^{-1}} U_{a^{-1}} U_a b \\ &= U_{a^{-1}} b^{-1} \end{aligned} \tag{7}$$

for all $a, b \in \mathrm{Inv}(\mathfrak{J})$. It is easy to check that

$$U_{a-\lambda} = U_a - 2\lambda L_a + \lambda^2$$

for all $a \in \mathfrak{J}$ and $\lambda \in \mathbb{C}$. In particular, $U_1 = U_{-1}$ is the identity operator on $\mathfrak{J}$.

For $x \in \mathfrak{J}$, the spectrum of x (denoted by $\mathrm{Sp}_{\mathfrak{J}}(x)$ or simply $\mathrm{Sp}(x)$) is the set of complex numbers λ for which $\lambda \cdot \mathbf{1} - \mathbf{x}$ is not invertible in $\mathfrak{J}^1$. We have, by definition, $\mathrm{Sp}_{\mathfrak{J}}(x) = \mathrm{Sp}_{\mathfrak{J}^1}(x)$.

A subalgebra $\mathfrak{L}$ of $\mathfrak{J}$ is called *strongly associative* if $[L_a, L_b] = 0$ for any $a, b \in \mathfrak{L}$, *spectral* if $\mathrm{Sp}_{\mathfrak{L}}(a) \cup \{0\} = \mathrm{Sp}_{\mathfrak{J}}(a) \cup \{0\}$ for each $a \in \mathfrak{L}$, and *full* (or *inverse closed*) if $\mathfrak{L}$ contains the

inverse of each element of $\mathfrak{L}$ that is invertible in $\mathfrak{J}$. Note that each strongly associative subalgebra is associative. For any $a \in \mathfrak{J}$ there is a maximal strongly associative subalgebra containing a which is inverse closed whenever $\mathfrak{J}$ is unital; see [16]; this subalgebra is spectral.

The following well-known proposition illustrates the importance of strongly associative algebras in studying problems related to the spectrum in Jordan algebras.

Proposition 1. *We let $\mathfrak{J}$ be a unital Jordan algebra, and let $x, a \in \mathfrak{J}$ be invertible. Then*

(i) $\mathrm{Sp}(a^{-1}) = \{\lambda^{-1} : \lambda \in \mathrm{Sp}(a)\}$.
(ii) $\mathrm{Sp}(U_a x^2) = \mathrm{Sp}(U_x a^2)$.

Proof. (i) We let $\mu \in \mathbb{C}$ be non-zero. As $a - \mu = U_a(a^{-1} - \mu a^{-2})$, we see that $a - \mu$ and $a^{-1} - \mu a^{-1}$ are simultaneously invertible or not. We let $\mathfrak{A}$ be a maximal strongly associative subalgebra of $\mathfrak{J}$ containing a. Then $a^{-1} - \mu a^{-2}$ is invertible in $\mathfrak{J}$ if and only if it is so in $\mathfrak{A}$. As $\mathfrak{A}$ is an associative algebra, $a^{-1} - \mu a^{-2}$ is invertible in $\mathfrak{A}$ if and only if $1 - \mu a^{-1}$ is invertible in $\mathfrak{A}$. Since $\mathfrak{A}$ is an inverse closed subalgebra of $\mathfrak{J}$, it follows that $1 - \mu a^{-1}$ is invertible in $\mathfrak{A}$ if and only if it is so in $\mathfrak{J}$. Hence,

$$\mathrm{Sp}(a^{-1}) = \{\lambda^{-1} : \lambda \in \mathrm{Sp}(a)\}.$$

(ii) It is clear that $U_x a^2$ and $U_a x^2$ are invertible in $\mathfrak{J}$. We let $\lambda \in \mathbb{C}$ be non-zero. Then

$$U_x a^2 - \lambda = U_x(a^2 - \lambda x^{-2}) = U_x U_a(1 - U_{a^{-1}} x^{-2})$$

implies that $U_x a^2 - \lambda$ is invertible in $\mathfrak{J}$ if and only if $1 - \lambda U_{a^{-1}} x^{-2}$ is invertible in $\mathfrak{J}$. As $U_{a^{-1}} x^{-2}$ is the inverse of $U_a x^2$, we have that $\mathrm{Sp}(U_a x^2) = \mathrm{Sp}(U_x a^2)$. □

2.2. Complete Normed Jordan Algebras

A Jordan algebra is called normed if it is a normed space with norm $\|\cdot\|$ and $\|x \cdot y\| \leq \|x\|\|y\|$ for all $x, y \in \mathfrak{J}$. If $\mathfrak{J}$ is complete under the norm, then $\mathfrak{J}$ is called a Banach (or complete normed) Jordan algebra. We let $\mathfrak{J}$ be a unital normed Jordan algebra and $\mathcal{A}^{ic}(x)$ be a closed full subalgebra of $\mathfrak{J}$ generated by $x \in \mathfrak{J}$. It follows from [16] that $\mathcal{A}^{ic}(x)$ is associative and is the closure of the algebra of rational functions $p(x) \cdot q(x)^{-1}$, where p, q are polynomials and $q(x)$ is invertible. By [16], in a Banach Jordan algebra $\mathfrak{J}$, we know that $\mathrm{Sp}_{\mathfrak{J}}(x) = \mathrm{Sp}_{\mathcal{A}^{ic}(x)}(x)$ is a compact non-empty set in $\mathbb{C}$ and the function $\lambda \mapsto (\lambda - x)^{-1}$ is analytic on $\mathbb{C} \backslash \mathrm{Sp}(x)$. By analytic functional calculus, see [16]; for any $\mathbb{C}$-valued function f analytic on some neighborhood V of $\mathrm{Sp}(x)$ there is an element $f(x) \in \mathcal{A}^{ic}(x)$ defined by

$$f(x) = (2\pi i)^{-1} \int_{\Gamma} f(\xi)(\xi \mathbf{1} - x)^{-1} \, d\xi$$

where Γ is a suitable contour in V surrounding $\mathrm{Sp}(x)$ and $f(x)$ does not depend on the choice of V and Γ. The map $f \mapsto f(x)$ is a homomorphism from the algebra $\mathcal{O}(V)$ of functions analytic on V into J; this homomorphism is continuous with respect to convergence of functions on compact subsets of V and satisfies for every function f analytic in some open neighborhood of $\mathrm{Sp}(x)$,

$$f(\mathrm{Sp}(x)) = \mathrm{Sp}(f(x))$$

If $\mathfrak{J}$ is a Banach Jordan algebra, then the map $x \mapsto \mathrm{Sp}(x)$ is upper semicontinuous; see [17] or [16]. The following results were given in [16] as Lemma 4.1.4 and Proposition 4.1.92, respectively.

3. ϵ-Invertibility in Banach Jordan Algebras

3.1. Invariance of Pseudospectra in Full Subalgebras

Let $\mathfrak{J}$ be a unital complete normed Jordan algebra over $\mathbb{C}$. We recall that the spectrum of an element $a \in \mathfrak{J}$ denoted by $\mathrm{Sp}(a, \mathfrak{J})$ or simply $\mathrm{Sp}(a)$ is defined as

$$\mathrm{Sp}(a) := \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathrm{Inv}(\mathfrak{J})\},$$

where $\mathrm{Inv}(\mathfrak{J})$ is the set of all invertible elements of $\mathfrak{J}$. We let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the extended complex plane. As in [18], one can define the function $\mathbf{nr} : \mathfrak{J} \times \mathbb{C} \to [0, \infty]$ (the norm of the resolvent) by the formula

$$\mathbf{nr}(a, \lambda) = \begin{cases} \|(\lambda \mathbf{1} - a)^{-1}\| & \text{if} \quad \lambda \in \mathbb{C} \backslash \mathrm{Sp}(a) \\ +\infty & \text{if} \quad \lambda \in \mathrm{Sp}(a) \\ 0 & \text{if} \quad \lambda = \infty \end{cases}$$

It is easy to see that the function $\mathbf{nr}(a, \cdot)$ is continuous on $\hat{\mathbb{C}}$. In particular,

$$|\mathbf{nr}(a, \mu) - \mathbf{nr}(a, \lambda)| < \frac{|\mu - \lambda| \mathbf{nr}(a, \lambda)^2}{1 - |\mu - \lambda| \mathbf{nr}(a, \lambda)}.$$

Definition 1 (ϵ-pseudospectrum)**.** *If $a \in \mathfrak{J}$ and $\epsilon > 0$, then the ϵ-pseudospectrum of $a \in \mathfrak{J}$ is defined to be the equality* $\mathrm{Sp}_\epsilon(a) = \left\{ \lambda \in \mathbb{C} \mid \boldsymbol{nr}(a, \lambda) \geq \frac{1}{\epsilon} \right\}$.

Unlike the spectrum, which is independent of the norm, the pseudospectrum depends on the norm. We prove that $\mathrm{Sp}_\epsilon(a)$ is a compact subset of $\mathbb{C}$ and contains the spectrum of a.

Proposition 2. *We let $\mathfrak{J}$ be a normed unital Jordan algebra and let a be in $\mathfrak{J}$. Then the closed $\mathfrak{J}$-full subalgebra of $\mathfrak{J}$ generated by a is associative and commutative.*

In associative Banach algebras, *inverse closed subalgebras* parallel the notion of *full subalgebras* defined for Jordan algebras. Indeed, if $\mathfrak{A}$ is a complex unital Banach algebra and $\mathfrak{B}$ is a unital subalgebra of $\mathfrak{A}$, we say that $\mathfrak{B}$ is inverse closed in $\mathfrak{A}$ if every element of $\mathfrak{B}$, which is invertible in $\mathfrak{A}$, is also invertible in $\mathfrak{B}$, that is $\mathfrak{B} \cap \mathrm{Inv}(\mathfrak{A}) \subseteq \mathrm{Inv}(\mathfrak{B})$.

A subalgebra B of a Banach Jordan algebra $\mathfrak{J}$ is a full subalgebra if the unit of $\mathfrak{J}$ belongs to B and $\mathrm{Inv}(B) = B \cap \mathrm{Inv}(\mathfrak{J})$.

Theorem 1. *We let $\mathfrak{J}$ be a complex unital Banach Jordan algebra with unit $\mathbf{1}$ and $\mathfrak{L}$ a full subalgebra of $\mathfrak{J}$. Then, for all $l \in \mathfrak{L}$, $\mathrm{Sp}(l, \mathfrak{L}) = \mathrm{Sp}(l, \mathfrak{J})$. Also, for every $\epsilon > 0$, $\mathrm{Sp}_\epsilon(l, \mathfrak{L}) = \mathrm{Sp}_\epsilon(l, \mathfrak{J})$.*

Proof. Obviously, for each $l \in \mathfrak{L}$, $\mathrm{Sp}(l, \mathfrak{J}) \subseteq \mathrm{Sp}(l, \mathfrak{J})$. Next, we let $l \in \mathfrak{L}$, $\lambda \in \mathbb{C}$ and $\lambda \notin \mathrm{Sp}(l, J)$. Then $\lambda \cdot \mathbf{1} - l$ is invertible in $\mathfrak{J}$ and hence also in $\mathfrak{L}$. Thus, $\lambda \notin \mathrm{Sp}(l, \mathfrak{L})$. Hence, $\mathrm{Sp}(l, \mathfrak{L}) \subseteq \mathrm{Sp}(l, \mathfrak{J})$. This shows that $\mathrm{Sp}(l, \mathfrak{J}) = \mathrm{Sp}(l, \mathfrak{L})$.

Since $\mathfrak{L}$ inherits the norm on $\mathfrak{J}$ and the norms on $\mathfrak{L}$ and $\mathfrak{J}$ are the same, it follows from the definition of pseudospectrum that $\mathrm{Sp}_\epsilon(l, \mathfrak{J}) = \mathrm{Sp}_\epsilon(l, \mathfrak{L})$ for all $\epsilon > 0$. □

3.2. Resolvent and Subharmonic Functions

Recall that a continuous function $\rho : D \longrightarrow \mathbb{R}$ on a domain $D \subset \mathbb{C}$ is *subharmonic* if, for all $z_0 \in D$ and $r > 0$, such that the closed disk $|z - z_0| \leq r$ is contained in D, we have

$$\rho(z) \leq \frac{1}{2\pi} \int_0^{2\pi} \rho(z_0 + re^{it})\, dt.$$

The following results are well known and could be found in either [17] or [16]. We let D be an open set in the complex plane, let E be a complex Banach space, and let f be a *holomorphic* function on D with values in E, i.e., a mapping $f{:}D \longrightarrow E$ such that for every continuous linear functional on E, the complex valued function $\mu \circ f{:}D \longrightarrow \mathbb{C}$ is holomorphic.

Proposition 3 ([16]). *We let $D \subseteq \mathbb{C}$ be an open set, let E be a Banach space, and let $f : D \longrightarrow E$ be holomorphic. Then $\|f\|$ is subharmonic in D.*

Theorem 2. *We let a be an element of a Banach Jordan algebra $\mathfrak{J}$. Then the function*

$$\Psi(z) = \|(a - z\mathbf{1})^{-1}\|$$

is subharmonic on $\mathbb{C} \setminus \mathrm{Sp}(a)$.

Proof. We apply Proposition 3 to the holomorphic function $z \mapsto (a - z)^{-1}$. □

4. Properties of the Pseudospectrum

It is known that $\mathrm{Sp}_\epsilon(a)$ is much larger than $\mathrm{Sp}(a)$; then, small perturbations can move spectral values very far. So, it is important to know whether the pseudospectra are sensitive to small perturbations. If they were, they would be of little value. Fortunately, this is not the case, as shown in the following theorem. Given an element a in a Banach Jordan algebra $\mathfrak{J}$, the pseudospectra $\mathrm{Sp}_\epsilon(a)$ are nested compact subsets in the complex plane whose intersection is $\mathrm{Sp}(a)$. In what follows, $\mathbb{D}(\alpha, r)$ stands for the disk centered at α and radius r in the complex plane.

Theorem 3. *Let $\mathfrak{J}$ be a complex unital Banach Jordan algebra and $a \in \mathfrak{J}$. Then,*

(i)

$$\mathrm{Sp}(a) = \cap_{\epsilon>0}\mathrm{Sp}_\epsilon(a).$$

(ii)

$$\mathrm{Sp}_\epsilon(a + \lambda) = \lambda + \mathrm{Sp}_\epsilon(a)$$

where $\lambda \in \mathbb{C}$.

(iii)

$$\mathrm{Sp}_{\epsilon_1}(a) \subseteq \mathrm{Sp}_{\epsilon_2}(a)$$

with $0 < \epsilon_1 < \epsilon_2$.

(iv)

$$\mathrm{Sp}_\epsilon(\lambda a) = \lambda \mathrm{Sp}_{\frac{\epsilon}{|\lambda|}}(a) \quad (\lambda \in \mathbb{C}\setminus\{0\}) \tag{8}$$

and

$$\alpha\mathrm{Sp}_\epsilon(a) = \mathrm{Sp}_{|\alpha|\epsilon}(\alpha a),\ \epsilon > 0,\ \alpha \neq 0. \tag{9}$$

(v)

$$\mathrm{Sp}_\epsilon(a) \subseteq \mathbb{D}(0; 3\|a\| + \epsilon).$$

(vi) $\mathrm{Sp}_\epsilon(a)$ *is a non-empty compact subset of* $\mathbb{C}$.

Proof. (i) We let $\lambda \in \mathrm{Sp}(a)$. Then $a - \lambda \cdot \mathbf{1}$ is not invertible, hence by the convention in Definition 1, $\|(\lambda \cdot \mathbf{1} - a)^{-1}\| = \infty$, i.e., $\|(\lambda \cdot \mathbf{1} - a)^{-1}\| > \frac{1}{\epsilon}$ for all $\epsilon > 0$. So,

$\lambda \in \cap_{\epsilon>0} \mathrm{Sp}_\epsilon(a)$. On the other hand, if $\lambda \notin \mathrm{Sp}(a)$, then $\lambda \cdot \mathbf{1} - a$ is invertible. Hence there exists $\epsilon_0 > 0$, such that $\|(\lambda \cdot \mathbf{1} - a)^{-1}\| < \frac{1}{\epsilon_0}$. Then, $\lambda \notin \mathrm{Sp}_{\epsilon_0}(a)$.

(ii) It is obvious that if $z \in \lambda + \mathrm{Sp}_\epsilon(a)$ then $z \in \mathrm{Sp}_\epsilon(\lambda \cdot \mathbf{1} + a)$. Let $z \in \mathrm{Sp}_\epsilon(a + \lambda \cdot \mathbf{1})$. Then

$$\|(z \cdot \mathbf{1} - (a + \lambda \cdot \mathbf{1}))^{-1}\| = \|((z - \lambda) \cdot \mathbf{1} - a)^{-1}\| \geq \frac{1}{\epsilon}.$$

Therefore, $z - \lambda \in \mathrm{Sp}_\epsilon(a)$, that is $z \in \lambda + \mathrm{Sp}_\epsilon(a)$.

(iii) We let $0 < \epsilon_1 < \epsilon_2$, and suppose $\lambda \in \mathrm{Sp}_{\epsilon_1}(a)$. If $\lambda \cdot \mathbf{1} - a$ is not invertible, then $\lambda \in \mathrm{Sp}(a) \subseteq \mathrm{Sp}_{\epsilon_2}(a)$. Otherwise, $\lambda \cdot \mathbf{1} - a$ is invertible and

$$\|(\lambda \cdot \mathbf{1} - a)^{-1}\| \geq \frac{1}{\epsilon_1} > \frac{1}{\epsilon_2}.$$

Accordingly, $\lambda \in \mathrm{Sp}_{\epsilon_2}(a)$.

(iv) We let $z \in \mathrm{Sp}_\epsilon(\lambda a)$. Then,

$$\|(z \cdot \mathbf{1} - \lambda \cdot a)^{-1}\| = \|(\frac{z}{\lambda} \cdot \mathbf{1} - a)^{-1}\| |\lambda|^{-1} > \frac{1}{\epsilon}.$$

So

$$|\lambda| \|(z \cdot \mathbf{1} - \lambda \cdot a)^{-1}\| \geq \frac{|\lambda|}{\epsilon}.$$

Hence,

$$\frac{z}{\lambda} \in \mathrm{Sp}_{\frac{\epsilon}{|\lambda|}}(a),$$

that is,

$$z \in \lambda \mathrm{Sp}_{\frac{\epsilon}{|\lambda|}}(a).$$

The second equality follows by taking $\alpha = \frac{1}{\lambda}$ and $b = \frac{1}{\alpha} a$ in (9).

(v) Suppose $|z| > 3\|a\| + \epsilon$. In particular, $z \notin \mathbb{D}(0; \|a\| + \epsilon)$. Then, $|z| > \|a\|$ and $z\mathbf{1} - a$ is invertible. It follows, from Corollary 4.1.5 in [16], that

$$\|(z \cdot \mathbf{1} - a)^{-1}\| \leq \frac{1}{|z| - 3\|a\|} < \frac{1}{\epsilon}.$$

Hence

$$z \notin \mathrm{Sp}_\epsilon(a).$$

So

$$\mathrm{Sp}_\epsilon(a) \subseteq \mathbb{D}(0; 3\|a\| + \epsilon).$$

(vi) By Part (v), $\mathrm{Sp}_\epsilon(a)$ is bounded. It is closed by continuity of the norm and resolvent function $R_a(z) = (z \cdot \mathbf{1} - a)^{-1}$. Hence, it is compact. Since $\mathrm{Sp}_\epsilon(a)$ contains $\mathrm{Sp}(a)$, it is obviously nonempty.

□

4.1. Isolated Points in the Pseudospectrum

It is shown in Theorem 5.1 and Theorem 5.2 in [14] for Banach algebras, that the pseudospectrum has no isolated points and it has a finite number of components. Similar results hold in Banach Jordan algebras and their proofs are the same because we mainly work in the local strongly associative algebra generated by 1 and a.

Theorem 4. *We let $\mathcal{J}$ be a Jordan Banach algebra, $a \in \mathcal{J}$ and $\epsilon > 0$. Then, the pseudospectrum has no isolated points.*

Theorem 5. *We let $\mathcal{J}$ be a Jordan Banach algebra, $a \in \mathcal{J}$ and $\epsilon > 0$. Then, the pseudospectrum $\mathrm{Sp}_\epsilon(a)$ of a has a finite number of components and each component of $\mathrm{Sp}_\epsilon(a)$ contains an element of $\mathrm{Sp}(a)$.*

4.2. Level Sets of Functions

Similarly to [19], we define for an element a in a real Banach Jordan algebra A, the level sets of functions

$$\Psi_a(z) = \begin{cases} \| (a - z\mathbf{1})^{-1} \|^{-1} & \text{if} \quad z \notin \mathrm{Sp}(a) \\ 0 & \text{if} \quad z \in \mathrm{Sp}(a) \end{cases}$$

This function is closely related to the previously well defined ϵ-pseudospectrum of a that is also equally expressed as

$$\mathrm{Sp}_\epsilon(a) = \{z \in \mathbb{C} : \Psi_a(z) \leq \epsilon\}.$$

It is shown in [15] that this function is locally *semiconvex*. We can follow exactly the same proof as in [15] to extend this result to real Jordan algebras in a straightforward way.

4.3. Pseudospectral Bounds on $\|f(a)\|$

It is known that if $\mathbf{m} \in \mathcal{M}_n(\mathbb{C})$ and f is analytic on some open neighborhood of $\mathrm{Sp}_\epsilon(\mathbf{m})$ for some $\epsilon > 0$, then

$$\|f(\mathbf{m})\| \leq \frac{L_\epsilon}{2\pi\epsilon} \sup_{z \in \mathrm{Sp}_\epsilon(\mathbf{m})} |f(z)|,$$

where L_ϵ denotes the contour of the boundary of $\mathrm{Sp}_\epsilon(\mathbf{m})$; see [1].

The following theorem extends this result from matrices to elements of Banach Jordan algebras.

Theorem 6. *We let $\mathfrak{J}$ be a Banach Jordan algebra, $a \in \mathfrak{J}$, and f be analytic on an open neighborhood $\Omega \subset \mathbb{C}$ of $\mathrm{Sp}_\epsilon(a)$. Then*

$$f(a) = \frac{1}{2\pi i} \int_\Gamma (z\mathbf{1} - a)^{-1} f(z)\, dz$$

where Γ be a contour that surrounds $\mathrm{Sp}_\epsilon(a)$ in Ω and

$$\| f(a) \| \leq \frac{Ml}{2\pi\epsilon}$$

where l = length of Γ and $M = \sup\{|f(z)| : z \in \Gamma\}$.

Proof. By the analytical functional calculus Theorem,

$$f(a) = \frac{1}{2\pi i} \int_\Gamma (z\mathbf{1} - a)^{-1} f(z)\, dz$$

is well defined. Hence,

$$\| f(a) \| \leq \frac{1}{2\pi} \int_\Gamma \| (z - a)^{-1} \| \| f(z) \| \, |dz| \leq \frac{Ml}{2\pi\epsilon}$$

since Γ lies outside the interior of $\mathrm{Sp}_\epsilon(a)$. □

The next corollary gives an equivalent condition in terms of the ϵ-pseudospectrum for an element of a Banach Jordan algebra to be a scalar, that is, a multiple of the identity element.

Corollary 1. *We let $\mathfrak{J}$ be a Banach Jordan algebra and $a \in \mathfrak{J}$ and $\mu \in \mathbb{C}$. Then*

$$a = \mu \iff \mathrm{Sp}_\epsilon(a) = \mathbb{D}(\mu, \epsilon) \quad \forall \epsilon > 0.$$

Proof. If $a = \mu$, it is obvious to see that $\mathrm{Sp}_\epsilon(a) = \mathbb{D}(\mu, \epsilon), \forall \epsilon > 0$.

For the converse, by part (ii) of Theorem 3, we may assume that $\mu = 0$. We let $f(z) = z$ and $\Gamma = \{z \in \mathbb{C} : |z| = \epsilon\}$. Then, with the notations of Theorem 6, Γ surrounds $\mathrm{Sp}_\epsilon(a)$, that is in particular, Γ is outside $\mathrm{Sp}_\epsilon(a)$, and we have $M = \epsilon$ and $l = 2\pi\epsilon$. We conclude that $\| a \| \leq \epsilon$. Because this is true for all $\epsilon > 0$, we get $a = 0 = \mu$. □

Using the analytic functional calculus and the level sets functions, we extend the following result proved in [20] for associative Banach algebras to complete normed non-associative algebras. This solves the question on the relation between two elements a and b such that $\Psi_a(z) = \Psi_b(z)$ for $z \in \mathbb{C}$.

Proposition 4. *We let $(\mathfrak{J}, \| \cdot \|)$ be a unital complex Banach Jordan algebra and a, $b \in \mathfrak{J}$ such that*

$$\|(a - z\mathbf{1})^{-1}\|^{-1} = \|(b - z\mathbf{1})^{-1}\|^{-1} \quad (z \in \mathbb{C}).$$

Then

$$\|a\| \leq e\|b\|.$$

Proof. We let $r > \|b\|$ and $n > \frac{\|a\|}{r}$. Integrating the Taylor series associated to the function $\zeta \mapsto (1 - \frac{a}{n\zeta})^{-n}$, we obtain the integral representation

$$a = \frac{1}{2\pi i} \int_{|\zeta|=r} \left(1 - \frac{a}{n\zeta}\right)^{-n} d\zeta.$$

Consequently,

$$\begin{aligned} \|a\| &\leq r \sup_{|\zeta|=r} \| \left(1 - \frac{a}{n\zeta}\right)^{-1} \|^n \\ &= r \sup_{|\zeta|=r} \| \left(1 - \frac{b}{n\zeta}\right)^{-1} \|^n \\ &\leq r \left(1 - \frac{\|b\|}{nr}\right)^{-n}. \end{aligned} \tag{10}$$

Letting $r \longrightarrow \|b\|$ and $n \longrightarrow \infty$, it follows that $\|a\| \leq e\|b\|$. □

4.4. Pseudospectrum and Instability Index

Every isolated spectral value of an element $a \in \mathfrak{J}$ is associated with a quantity called *index of instability* which measures the instability of this spectral value under small perturbations of a. If λ is an isolated point of the spectrum of an element $a \in \mathfrak{J}$, the spectral projection associated with λ is defined (*see* [17] or [21]) by $\mathbf{p} = \frac{1}{2\pi i} \int_\gamma (z\mathbf{1} - a)^{-1} \, dz$, where γ is any sufficiently small closed contour winding around λ, separating λ from the rest of the remaining spectrum of a. In fact, $\mathbf{p}$ does not depend on the contour γ, as long as γ separates λ from the rest of the spectrum. Thus, we can suppose that γ is a small circle with center at λ. The assumption that this projection has *rank* 1 is stronger than the assumption

that λ is an eigenvalue of *multiplicity* 1 (*see* [21,22]) for the definition of spectral rank and spectral multiplicity). We extend the definition of the index of instability of Aslanyan and Davies in [23] from Schrödinger operators to the general context of elements in a Banach Jordan algebra.

Definition 2. *We let a be an element in a Banach Jordan algebra $\mathfrak{J}$ and λ an isolated spectral value of a. The index of instability associated to λ is defined as*

$$\kappa(\lambda) = \|\boldsymbol{p}\|,$$

where $\boldsymbol{p}$ is the spectral projection associated to λ.

The instability index is also related to the notion of pseudospectrum, which is a geometric way of looking at resolvent norm properties. Namely, if $\epsilon > 0$, we recall that

$$\mathrm{Sp}_\epsilon(a) = \left\{ z \in \mathbb{C} : \|(z\mathbf{1} - a)^{-1}\| \geq \epsilon^{-1} \right\}.$$

The sizes of these sets, which all contain the spectrum of a, measure the spectral instability of a. One always has

$$\{z : \mathrm{dist}\{z, \mathrm{Sp}(a)\}\} < \epsilon\} \subseteq \mathrm{Sp}_\epsilon(a),$$

but the RHS is often much larger than the LHS. The following theorem states that if $\kappa(\lambda)$ is large, then the component of $\mathrm{Sp}_\epsilon(a)$ containing λ is large in a related sense. It also establishes a connection between pseudospectra and the norms of the spectral projections.

Theorem 7. *Suppose that the spectral projection $\boldsymbol{p}$ associated with the isolated specral value λ of a has* rank *1. We let γ be a closed contour surrounding the connected component of $\mathrm{Sp}_\epsilon(a)$ which contains λ but does not intersect $\mathrm{Sp}_\epsilon(a)$. Then,*

$$|\gamma| \geq 2\pi\epsilon\kappa(\lambda)$$

where $|\gamma|$ is the length of γ.

Proof. We have from the definition

$$\kappa(\lambda) = \|\mathbf{p}\| = \|\frac{1}{2\pi i}\int_\gamma (z\mathbf{1} - a)^{-1}\, dz\| \leq \frac{1}{2\pi\epsilon}\int_\gamma |dz| = \frac{|\gamma|}{2\pi\epsilon}.$$

□

Remark 1. *It follows from the preceding theorem that one good reason to study the index of instability associated to isolated spectral values of an element a in a Banach Jordan algebra $\mathfrak{J}$ is linked to the size of the pseudospectrum surrounding the corresponding spectral value. Indeed, we let $\mathrm{Sp}_\epsilon^{(\lambda)}$ be the connected component of $\mathrm{Sp}_\epsilon(a)$ containing the isolated spectral value λ, and suppose that this component is bounded and does not contain any other point of the spectrum of a. By choosing any contour γ enclosing $\mathrm{Sp}_\epsilon^{(\lambda)}$ without surronding any other point of $\mathrm{Sp}_\epsilon \setminus \mathrm{Sp}_\epsilon^{(\lambda)}$, and not intersecting $\mathrm{Sp}_\epsilon(a)$, it follows from Theorem 8 that*

$$|\gamma| \geq 2\pi\epsilon\kappa(\lambda).$$

5. Linear Maps Preserving Pseudospectrum in Banach Jordan Algebras

Some results obtained in [13,14] for associative Banach algebras hold true in Banach Jordan algebras. Their proofs, being natural extensions of the associative ones, are given here for the sake of completeness.

Theorem 8. *We let $\mathfrak{J}$ and $\mathcal{L}$ be two complex unital Banach Jordan algebras. Suppose $\Phi : \mathfrak{J} \longmapsto \mathfrak{L}$ is a linear, bijective, unital, i.e., $\Phi(1_{\mathcal{J}}) = 1_{\mathfrak{L}}$, isometric, that is $\|\Phi(x)\|_{\mathcal{J}} = \|x\|_{\mathcal{L}}$ and multiplicative, meaning $\Phi(ab) = \Phi(a)\Phi(b)$ for all $a, b \in \mathcal{J}$. Then*

$$\mathrm{Sp}_\epsilon(a) = \mathrm{Sp}_\epsilon(\Phi(a))$$

for all $a \in \mathcal{J}$ and $\epsilon > 0$.

Proof. We let $a \in \mathcal{J}$ and $\epsilon > 0$. If $\lambda \in \mathrm{Sp}_\epsilon(a)$, then $\|(\lambda\mathbf{1} - a)^{-1}\| \geq \frac{1}{\epsilon}$. Since Φ is an isometry,

$$\|\Phi(\lambda\mathbf{1} - a)^{-1}\| = \|(\lambda - a)^{-1}\| \geq \frac{1}{\epsilon}.$$

Moreover, because Φ is multiplicative and unital, inverses are preserved, hence

$$\Phi(1_{\mathcal{J}}) = \Phi(xx^{-1}) = \Phi(x)\Phi(x^{-1}) = 1_{\mathcal{L}}.$$

Thus,

$$\|(\Phi(\lambda\mathbf{1} - a))^{-1}\| = \|\Phi(\lambda\mathbf{1} - a)^{-1}\| = \|(\lambda\mathbf{1} - a)^{-1}\| \geq \frac{1}{\epsilon}.$$

Consequently, because Φ is unital and linear,

$$\|(\lambda\mathbf{1} - \Phi(a))^{-1}\| = \|(\Phi(\lambda\mathbf{1} - a)^{-1}\| \geq \frac{1}{\epsilon}.$$

Hence $\lambda \in \mathrm{Sp}_\epsilon(\Phi(a))$ and $\mathrm{Sp}_\epsilon(a) \subseteq \mathrm{Sp}_\epsilon(\Phi(a))$. By symmetry we can conversely show that $\mathrm{Sp}_\epsilon(\Phi(a)) \subseteq \mathrm{Sp}_\epsilon(a)$, which completes the proof. □

Theorem 9. *We let $\mathfrak{J}$ and $\mathcal{L}$ be two complex unital Banach Jordan algebras and $\epsilon > 0$. We let $\Phi : \mathfrak{J} \longmapsto \mathfrak{L}$ be a surjective ϵ-pseudospectrum preserving linear map. Then, Φ is spectra-preserving.*

Proof. By assumption $\mathrm{Sp}_\epsilon(a) = \mathrm{Sp}_\epsilon(\Phi(a))$, for all $a \in \mathcal{J}$. Suppose $\lambda \notin \mathrm{Sp}(a)$, and choose $t > \epsilon\|(\lambda\mathbf{1} - a)^{-1}\|$. Then

$$\begin{aligned} \|(t(\lambda\mathbf{1} - a))^{-1}\| &= \frac{1}{t}\|(\lambda\mathbf{1} - a)^{-1}\| \\ &< \frac{1}{t}\frac{t}{\epsilon} = \frac{1}{\epsilon}. \end{aligned} \tag{11}$$

Hence,

$$\begin{aligned} t\lambda \notin \mathrm{Sp}_\epsilon(ta) &= \mathrm{Sp}_\epsilon(\Phi(ta)) \\ &\supseteq \mathrm{Sp}(\Phi(ta)) = t\mathrm{Sp}(\Phi(a)), \end{aligned} \tag{12}$$

hence $\lambda \notin \mathrm{Sp}(\Phi(a))$. Therefore, $\mathrm{Sp}(\Phi(a)) \subseteq \mathrm{Sp}(a)$. In a similar way, we can prove that $\mathrm{Sp}(a) \subseteq \mathrm{Sp}(\Phi(a))$. Hence $\mathrm{Sp}(\Phi(a)) = \mathrm{Sp}(a)$. □

Theorem 10. *We let $\mathfrak{J}$ and $\mathcal{L}$ be two complex unital Banach Jordan algebras and $\Phi : \mathfrak{J} \longmapsto \mathfrak{L}$ be an ϵ-pseudospectrum preserving linear map for some $\epsilon > 0$. Suppose Φ is multiplicative. Then, Φ preserves norms of all invertible elements of $\mathcal{J}$.*

Proof. Suppose there exists an element $a \in \mathrm{Inv}(\mathcal{J})$ such that $\|\Phi(a^{-1})\| \neq \|a^{-1}\|$. Assume $\|a^{-1}\| < \|\Phi(a^{-1})\|$, choose $t > 0$ such that

$$\epsilon\|a^{-1}\| < t \leq \epsilon\|\Phi(a^{-1})\| = \epsilon\|\Phi(a)^{-1}\|.$$

Then $\|(ta)^{-1}\| < \frac{1}{\epsilon}$. Thus, $0 \notin \mathrm{Sp}_{\epsilon}(ta)$. But

$$\|(\Phi(ta))^{-1}\| = \|\frac{1}{t}(\Phi(a))^{-1}\| = \frac{1}{t}\|\Phi(a)^{-1}\| \geq \frac{1}{\epsilon}.$$

This implies $0 \notin \mathrm{Sp}_{\epsilon}(\Phi(ta))$, which contradicts the fact that Φ preserves ϵ-pseudospectrum. □

6. Decomposition of Spectra and Pseudospectra

In a Banach algebra A, it is shown in Theorem 3.3 of [24] that there is a relationship between the spectrum and pseudospectrum of an element $a \in A$ with the spectra and pseudospectra of $pa = pap \in pAp$ and $qa = qaq \in qAq$, where p is a non-trivial idempotent, that is, $p^2 = p, p \neq 0, p \neq 1$, and $q = 1 - p$.

Theorem 11. *We let A be a unital Banach algebra and let $p_1, \cdots, p_n \in A$, such that $p_i^2 = p_i$ for all i and $\sum_{i=1}^{n} p_i = 1$. Suppose $ap_i = p_i a$ for all i. We let $\epsilon > 0$ and $K = \max_i \| p_i \|$. Then*

$$\mathrm{Sp}_{\frac{\epsilon}{n}}(A, a) \subseteq \bigcup_{i=1}^{n} \mathrm{Sp}_{\epsilon}(p_i A p_i, p_i a) \subseteq \mathrm{Sp}_{K\epsilon}(A, a).$$

Our aim is to extend this theorem to Banach Jordan algebras. At the present time, we are able only to obtain a decomposition of the pseudospectrum into the pseudospectra of the corresponding elements in the local Banach Jordan algebras, in the case of two orthogonal idempotents p and $q = 1 - p$.

Contrarily to the associative context, where $ap_i = p_i a$ was enough to prove for instance Theorem 12, in the non-associative situation we request that our elements operator commute. The following propositions can be found in [25].

Proposition 5. *We let $\mathfrak{J}$ be a Jordan algebra with identity, and let $p \in \mathfrak{J}$ be an idempotent element. We let $U_p\mathfrak{J} = \{U_p x \mid x \in \mathfrak{J}\}$ and $\mathfrak{J}_1(p) = \{x \in \mathfrak{J} \mid px = x\}$. Then*

(i) *$\mathfrak{J}_1(p)$ is a subalgebra of $\mathfrak{J}$, with identity p.*
(ii) *$\mathfrak{J}_1(p)$ is semisimple and $\mathfrak{J}_1(p) = U_p\mathfrak{J}$.*
(iii) *If $x \in U_p\mathfrak{J}$, then $\mathrm{Sp}_{U_p\mathfrak{J}}(U_p x) \subset \mathrm{Sp}_{\mathfrak{J}} x$.*

Proposition 6. *We let $a_1, \cdots, a_n \in \mathfrak{J}$ such that $a_i a_j = a_i^2 a_j = a_i a_j^2 = a_i^2 a_j^2 = 0$ for $i \neq j$. Then*

$$\mathrm{Sp}(a_1 + a_2 + \cdots + a_n) \supseteq \mathrm{Sp}(a_1) \cup \mathrm{Sp}(a_2) \cup \cdots \mathrm{Sp}(a_n) \backslash \{0\}.$$

Corollary 2. *We let $\mathfrak{J}$ be a unital complex Banach Jordan algebra, let $p \in \mathfrak{J}$ an idempotent, and let $q = 1 - p$. Let $a \in \mathfrak{J}$. Then*

(i) $\mathrm{Sp}(\mathfrak{J}, a) = \mathrm{Sp}(U_p\mathfrak{J}, U_p a) \cup \mathrm{Sp}(U_q\mathfrak{J}, U_q a)$,
(ii) *The spectral radius* $\rho(a) = \max\{\rho(U_p\mathfrak{J}, U_p a), \rho(U_q\mathfrak{J}, U_q a)\}$.

We recall that two elements a, b in a Jordan algebra $\mathfrak{A}$ are said to operator commute if the operators R_a, R_b commute, i.e., if $(a \circ c) \circ b = a \circ (c \circ b)$ for all $c \in \mathfrak{A}$, where $\circ$ is the

Jordan product in $\mathfrak{A}$. The next lemma is taken from [8] where it appears as Lemma 2.5.5 and it is proven on pages 46–47.

Lemma 1. *We let $\mathfrak{A}$ be a unital Jordan algebra and p an idempotent in $\mathfrak{A}$. For any $a \in \mathfrak{A}$, the following conditions are equivalent:*

(i) *a and p operator commute,*
(ii) *$R_p a = U_p a$,*
(iii) *$a = (U_p + U_{p^\perp})a$,*
(iv) *a and p generate an associative subalgebra of $\mathfrak{A}$.*
Moreover, $U_p\mathfrak{A}$ and $U_{p^\perp}\mathfrak{A}$ are subalgebras of $\mathfrak{A}$, and $a \circ b = 0$ if $a \in U_p\mathfrak{A}$ and $b \in U_{p^\perp}\mathfrak{A}$.

Now, suppose p is a central projection in $\mathfrak{A}$, that is, the p operator commutes with every other element of $\mathfrak{A}$. Then, by Lemma 1, one can write $J = U_p J \oplus U_{p^\perp} J$.

Theorem 12. *We let $\mathfrak{J}$ be a unital complex Banach Jordan algebra, let $p \in \mathfrak{J}$ be a central projection, and $p^\perp = 1 - p$. We let $\epsilon > 0$ and $K = \max\{\|p\|, \|1 - p\|\}$. Then*

$$\mathrm{Sp}_\epsilon(J, a) \subseteq \mathrm{Sp}_\epsilon(U_p J, U_p a) \bigcup \mathrm{Sp}_\epsilon(U_{p^\perp} J, U_{p^\perp} a) \subseteq \mathrm{Sp}_{K\epsilon}(\mathfrak{J}, a).$$

Proof. We let $\epsilon > 0$ and

$$\lambda \in \mathrm{Sp}_\epsilon(U_p\mathfrak{J}, U_p a) \bigcup \mathrm{Sp}_\epsilon(U_{p^\perp}\mathfrak{J}, U_{p^\perp} a).$$

If $\lambda \in \mathrm{Sp}(U_p\mathfrak{J}, U_p a)$, then $\lambda \in \mathrm{Sp}(\mathfrak{J}, a) \subseteq \mathrm{Sp}_\epsilon(\mathfrak{J}, a) \subseteq \mathrm{Sp}_{K\epsilon}(\mathfrak{J}, \mathfrak{a})$, by Proposition 5 (iii).

On the other hand, if $\lambda \cdot p - U_p a$ is invertible, then $\lambda \cdot \mathbf{1} - a$ is invertible by Corollary 2, then it must be the case that $\lambda \in \mathrm{Sp}_\epsilon(U_p J, U_p a)$. Then $\|(\lambda \cdot p - U_p a)^{-1}\| \geq \frac{1}{\epsilon}$. Since p is a projection, then $U_p a = pa$, which leads to

$$\|(\lambda \cdot p - U_p a)^{-1}\| = \|(U_p(\lambda \cdot \mathbf{1} - a))^{-1}\| = \|(p(\lambda \cdot \mathbf{1} - a))^{-1}\| \geq \frac{1}{\epsilon}$$

Now, exactly as in the proof of Theorem 3.3 in [24], this yields a bound for the norm of $(\lambda \cdot \mathbf{1} - a)^{-1}$ as follows:

$$\|(p(\lambda \cdot \mathbf{1} - a))^{-1}\| \geq \frac{1}{\epsilon} \Rightarrow \|(\lambda \cdot \mathbf{1} - a)^{-1}\| \geq \frac{1}{\|p\|\epsilon} > \frac{1}{K\epsilon}$$

Thus,

$$\lambda \in \mathrm{Sp}_\epsilon(U_p\mathfrak{J}, U_p a) \bigcup \mathrm{Sp}_\epsilon(U_{p^\perp}\mathfrak{J}, U_{p^\perp} a)) \subseteq \mathrm{Sp}_{K\epsilon}(\mathfrak{J}, a).$$

Next, suppose

$$\lambda \notin \mathrm{Sp}_\epsilon(U_p\mathfrak{J}, U_p a) \bigcup \mathrm{Sp}_\epsilon(U_{p^\perp}\mathfrak{J}, U_{p^\perp} a).$$

Then $\lambda \cdot p - U_p a$ is invertible with inverse $U_p b$, for some $b \in \mathfrak{J}$, and $\lambda \cdot p^\perp - U_{p^\perp} a$ is invertible with inverse $U_{p^\perp} c$, for some $c \in \mathfrak{J}$. Moreover, $\|U_p b\| < \frac{1}{\epsilon}$ and $\|U_{p^\perp} c\| < \frac{1}{\epsilon}$. Then $\lambda\mathbf{1} - a$ is invertible (by Corollary 2) with inverse $U_p b + U_{p^\perp} c$ (by Lemma 1 (iv)), and further $\|(U_p(\lambda\mathbf{1} - a))^{-1}\| = \|U_p b + U_{p^\perp} c\| < 2\frac{1}{\epsilon}$. Hence, we conclude that

$$\mathrm{Sp}_{\frac{\epsilon}{2}}(J, a) \subseteq \mathrm{Sp}_\epsilon(U_p J, U_p a) \bigcup \mathrm{Sp}_\epsilon(U_{p^\perp} J, U_{p^\perp} a) \subseteq \mathrm{Sp}_{K\epsilon}(\mathfrak{J}, a).$$

□

Corollary 3. *We let $\mathfrak{J}$ be a unital complex Banach Jordan algebra, let $p \in \mathfrak{J}$ an idempotent, and let $q = \mathbf{1} - p$. We let $a \in \mathfrak{J}$ and suppose that $\|p\| = \|q\| = 1$. Then*

$$\mathrm{Sp}_{\frac{\epsilon}{2}}(J, a) \subseteq \mathrm{Sp}_{\epsilon}(U_p J, U_p a) \cup \mathrm{Sp}_{\epsilon}(U_q J, U_q a) \subseteq \mathrm{Sp}_{\epsilon}(\mathfrak{J}, a).$$

Exploring pseudospectra in Banach Jordan algebras is a novel and fertile area of research, bridging non-associative algebra, spectral theory, and stability analysis. There are many interesting questions to answer in this regard; for instance, it will be of great importance to obtain a theorem for a class of Banach Jordan algebras similar to Theorem 12.

7. Roch–Silberman Theorem in a JB Algebra

First, we recall the well-known theorem of Roch and Silberman established in [4] for C^*-algebras. Indeed, we let $\epsilon > 0$ and let $\mathcal{B}$ a C^*-algebra with identity. The ϵ-pseudospectrum $\mathrm{Sp}_{\epsilon}(b)$ of an element $b \in \mathcal{B}$ is the set

$$\mathrm{Sp}_{\epsilon}(b) = \left\{ t \in \mathbb{C}; b - t \cdot \mathbf{1} \text{is not invertible or} \|(b - t \cdot \mathbf{1})^{-1}\| \geq \frac{1}{\epsilon} \right\}. \tag{13}$$

Here is an equivalent characterization of the ϵ-pseudospectrum established as Proposition 4.15 in [4] for C^*-algebras.

Theorem 13. *The ϵ-pseudospectrum $\mathrm{Sp}_{\epsilon}(b)$ of an element $b \in \mathcal{B}$ coincides with the set*

$$\{t \in \mathbb{C};\ \textit{there is a } p \in \mathcal{B} \textit{ with } \|p\| \leq \epsilon \textit{ and } t \in \mathrm{Sp}(b + p)\}. \tag{14}$$

Our aim is to generalize this theorem to a particular class of Banach Jordan algebras, called JB^* algebras. A complex Banach Jordan algebra $\mathcal{J}$ with isometric involution * is called a JB^* algebra if $\|\mathrm{U}_x x^*\| = \|x\|^3$ for all $x \in J$. For instance, every C^*-algebra is a JB^* algebra when equipped with its natural Jordan product, the original norm and involution. The concepts of JB algebras and JB^*-algebras are essentially equivalent as shown in [26]. More precisely, the *self-adjoint* part $\mathcal{J}_s = \{x \in \mathcal{J} : x^* = x\}$ of a JB^* algebra $\mathcal{J}$ is a JB algebra under the retricted norm, whereas the complexification $\mathcal{J}_c = \mathcal{J} \otimes \mathbb{C}$ of a JB algebra $\mathcal{J}$ is a JB^* algebra under a unique norm extending the JB algebra norm on $\mathcal{J}$.

Theorem 14. *(a) If $\mathcal{A}$ is a JB^* algebra, then the set of self-adjoint elements of $\mathcal{A}$ is a JB algebra. (b) If $\mathcal{B}$ is a JB algebra, then under a suitable norm the complexification $\mathcal{C}_{\mathcal{B}}$ of $\mathcal{B}$ is a JB^* algebra.*

The basic examples of JB algebras are of the following types. We let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the Banach algebra of bounded linear operators on $\mathcal{H}$.

(a) The real Banach space $\mathcal{B}(\mathcal{H})$ of all bounded self-adjoint operators on $\mathcal{H}$ is a JB algebra.
(b) Any closed unital subalgebra $\mathcal{S}$ of self-adjoint operators of $\mathcal{B}(\mathcal{H})$ is called a JC algebra. Every JC algebra is a special JB algebra. The converse is also true by the structure theory of [27].

For an element a in a unital JB algebra $\mathfrak{J}$, the spectrum $\mathrm{Sp}(\mathfrak{J}, a)$ is a nonempty and compact subset of $\mathbb{R}$ and coincides with $\mathrm{Sp}(\mathcal{C}(a), a)$, thus the spectrum does not vary with the closed subalgebra of $\mathfrak{J}$ containing a and the unit element.

The main result of Roch–Silbermann is stated and proved in the general setting of C^*-algebras which are a proper subclass of the larger category of JB^* algebras. These constitute nowadays an important field of study as can be seen in [28] and the references cited within pertaining to Quantum Machanics. Similarly to JB algebras, the concepts of invertibility and spectrum are well defined in JB^* algebras. We recall that the Shirshov–Cohn theorem

states that a Jordan algebra, unital or not, with two generators is special. As a consequence, a JB algebra generated by two elements is a JC algebra (see [8], Theorem 7.2.5). Hence, any JB^* subalgebra of a JB^* algebra, that is, generated by two self-adjoint elements and the identity, is a JC^* algebra, that is, a JB^* subalgebra of some $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ [26], Corollary 2.2. We prove the second inclusion of Proposition 4.15 [4], for self-adjoint elements; in this case, the proof of Roch–Silberman works verbatim because the subalgebra generated by $1, a, a^*$ is associative.

Proposition 7. *For self-adjoint element a in a unital JB* algebra $\mathfrak{J}$, the pseudospectrum*

$$\mathrm{Sp}_\epsilon(a) \subseteq \{t \in \mathbb{C} : \exists p \in J \quad \textbf{\textit{with}} \quad ||p|| \leq \epsilon \quad \textbf{\textit{and}} \quad t \in \mathrm{Sp}(a+b)\}$$

Proof. Same as the proof of Proposition 4.15 in [4], because for self-adjoint element a, the subalgebra generated by 1 and a is associative (and commutative), so the C*-algebra proof applies verbatim. □

Final Comments

Is it possible to obtain a more general result than Proposition 7? In other words, does the following theorem hold in Banach Jordan algebras? Of course, new tools and arguments must be used in the proof of such general result since the subalgebra generated by $1, a$ and a^* is not necessarily associative for non-self-adjoint element a.

Theorem 15. *For an element a in a unital JB^* algebra $\mathfrak{J}$, the pseudospectrum*

$$\mathrm{Sp}_\epsilon(a) = \bigcup_{b \in \mathfrak{J},\, \|b\| \leq \epsilon} \mathrm{Sp}(a+b)$$

8. Conclusions

The study of pseudospectra in Banach Jordan algebras is motivated by the need to understand spectral stability in a natural operator-theoretic framework where associativity is absent. Jordan algebras arise intrinsically in the analysis of self-adjoint operator structures and in mathematical models of observables, yet many classical spectral tools rely crucially on associative products. Pseudospectra provide a robust mechanism for quantifying spectral instability, resolvent growth, and sensitivity to perturbations, making them particularly well suited to non-normal and non-associative settings. Extending pseudospectral theory to Banach Jordan algebras is therefore far from routine: arguments based on resolvent identities, factorization, and multiplicativity must be replaced by techniques involving quadratic representations, Peirce decompositions, and strongly associative Jordan subalgebras. This shift not only necessitates new analytical methods, but also reveals structural phenomena that are invisible in the associative theory. Moreover, pseudospectra offer a unifying perspective linking spectral stability with norm-controlled inversion and Hulanicki-type results, allowing classical principles to be reformulated and extended beyond associative Banach algebras. As a consequence, the Jordan pseudospectral framework is not merely a generalization, but a genuinely new setting in which spectral analysis interacts in a subtle and essential way with non-associative algebraic structure.

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